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Research article

Analyticity estimates for the 3D magnetohydrodynamic equations

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Abstract: This paper was concerned with the Cauchy problem of the 3D magnetohydrodynamic (MHD) system. We first proved that this system was local well-posed with initial data in the Besov space $\dot{B}^s_{p,q}(\mathbb{R}^3)$, in the critical Besov space $\dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)$, and in $L^p(\mathbb{R}^3)$ with $p \in]3,6[$, respectively. We also obtained a new growth rate estimates for the analyticity radius.

Keywords: magnetohydrodynamic equations; the analyticity radius; Besov spaces; Gevrey regularity; global solution

1. Introduction

The system of 3D incompressible MHD equations is given by

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla \Pi = b \cdot \nabla b, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \partial_t b + u \cdot \nabla b - \Delta b = b \cdot \nabla u, \\ \nabla \cdot u = \nabla \cdot b = 0, \\ (u,b)|_{t=0} = (u_0, b_0), \end{cases} \tag{1.1}$$

where $u = u(t,x)$, $b = b(t,x)$, $\Pi = \Pi(t,x)$ denote the velocity of the fluid, the magnetic field and the modified pressure, respectively. The system (1.1) can be widely applied in many fields, including geophysics, astrophysics, and engineering.

In the case of $b = 0$, the system (1.1) reduces to the classical Navier-Stokes system:

$$\begin{cases} \partial_t u + u \cdot \nabla u - \Delta u + \nabla \Pi = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \nabla \cdot u = 0, \\ u|_{t=0} = u_0. \end{cases} \tag{1.2}$$

It is well-known that one of the most challenging problems in applied analysis is establishing the global well-posedness of the system (1.2) when the initial data is large. Thus, many papers have tried to obtain the global existence of solutions with small initial data. Giga [1] proved the local well-posedness of the Navier-Stokes equations with the initial data $u_0 \in L^p(\mathbb{R}^3)$ for some $p \in]3, \infty[$, whereas Kato and Masuda [2] established the endpoint case when $p = 3$. In 1981, Foias and Temam [3] proposed a technique which can be used to derive the analyticity radius of solutions of the system (1.2) and obtained the analyticity of periodic solutions. This result was then generalized by many authors; see [4–7] and its cited references. It should be emphasise that the analyticity radius estimates given above are lower estimates; in certain cases, the true analyticity radius may be significantly higher. Consequently, Biswas and Foias [8] by solving a new auxiliary ODE for the evolution of the analyticity radius involving the Gevrey class norms to obtain a more intimate connection between the radius of analyticity and the dynamics of the Navier-Stokes equations. Bae et al. [9] investigated the analyticity of the system (1.2) with the initial data in the critical Besov spaces. On this basis, Zhang [10] generalized the result to the initial data belonging to general Besov space.

For the MHD system (1.1), Duvaut and Lions [11], Miao and Yuan [12], and Wang and Wang [13] independently obtained the existence, uniqueness, and regularities of generalized solutions. Yu and Li [14] established the time analyticity radius of the 2D MHD system with periodic boundary conditon, and they proved that if the initial data is close enough to a stationary solution, then the radius of the solution at $t = 0$ can be arbitrarily large. Wang et al. [15] utilized the Gevrey class method in [3] to prove the analyticity of the solutions to the system (1.1) with the initial data in the Lei-Lin space $\chi^{-1}(\mathbb{R}^3)$. In [16], the analyticity of periodic solutions to the system (1.1) with initial data in Sobolev spaces $H^s(\mathbb{T}^3)$ with $s > \frac{1}{2}$ had been established. By using the semigroup technique, Xiao and Yuan [17] derived the analytic estimates for small perturbation near the solutions to the generalized MHD system in the critical space $\chi^{1-\alpha}$ with $\frac{1}{2} \leq \alpha \leq 1$. For more results about the Hall-MHD system, the generalized MHD system and magnetohydrodynamics-α system, we may refer to [18–21] and the references cited therein.

Motivated by [10], this paper focuses on the analyticity of solutions to the system (1.1) when the initial data in $\dot{B}^s_{p,q}(\mathbb{R}^3)$ for some $s \in [-1 + \frac{3}{p}, \frac{3}{p}[$, $p \in]1, \infty[$, and $q \in [1, \infty]$. By utilizing the new fixed point theorem, the Mihlin-Hormander multiplier lemma, and the properties of the semigroup operator, lower bounds on the analytic radius of the MHD equations are proven in $\dot{B}^s_{p,q}(\mathbb{R}^3)$ for some $s \in] - 1 + \frac{3}{p}, \frac{3}{p}[$, $p \in]1, \infty[$, and $q \in [1, \infty]$; in $\dot{B}^{-1+\frac{3}{p}}_{p,q}(\mathbb{R}^3)$, $p \in]1, \infty[$, and $q \in [1, \infty]$; and $L^p(\mathbb{R}^3)$ with $p \in]3, 6[$, respectively. In particular, it should be noted that by using the classical Bony decomposition, we estimate the nonlinear term in proving Theorems 1.2 and 1.3. Furthermore, our results correspond to Zhang's findings [10] when $b = 0$. Our conclusion is articulated as follows.

Theorem 1.1. *Suppose* $(u_0, b_0) \in \dot{B}^s_{p,q}(\mathbb{R}^3)$ *for some* $s \in] - 1 + \frac{3}{p}, \frac{3}{p}[$, $p \in]1, \infty[$, $q \in [1, \infty]$, *and* $\Lambda(D) = \sqrt{-\Delta}$. *Then, there exist positive constants* $t_0 = t_0(\varepsilon, \|(u_0, b_0)\|_{\dot{B}^s_{p,q}})$ *and* $c_0 = c_0(\varepsilon)$ *such that for any* $\varepsilon < \varepsilon_0$, *the MHD Eq* (1.1) *admits a unique solution* (u, b) *on* $[0, t_0]$ *satisfying*

$$\|e^{\sqrt{2(1-\varepsilon)(\frac{1+s}{3}-\frac{1}{p}-\varepsilon)}\sqrt{t|\ln t|}\Lambda(D)}(u(t), b(t))\|_{\dot{B}^s_{p,q}} \leq c_0 t^{-\frac{1+s}{2}+\frac{3}{2p}} \qquad \text{for any } t \in [0, t_0], \tag{1.3}$$

and

$$\liminf_{t\to 0} \frac{rad(u(t), b(t))}{\sqrt{t|\ln t|}} \geq \sqrt{2(\frac{1+s}{3} - \frac{1}{p})}. \tag{1.4}$$

Inspired by [10], we consider endpoint cases, i.e., $s = -1 + \frac{3}{p}$.

Theorem 1.2. *Let* $(u_0, b_0) \in \dot{B}_{p,q}^{-1+\frac{3}{p}}(\mathbb{R}^3)$*,* $p \in]1, \infty[$*, and* $q \in [1, \infty[$*. Then, there exist positive constants* $t_1 = t_1(\varepsilon, \|(u_0, b_0)\|_{\dot{B}_{p,q}^{-1+\frac{3}{p}}})$ *and* $c_1 = c_1(\varepsilon)$ *such that for any* $0 < \eta < \varepsilon \le \varepsilon_1$*, the MHD Eq* (1.1) *admits a unique solution* $(u, b) = (u_L, b_L) + (v, w)$ *with* $(v, w) \in \mathscr{B}_r^{\varepsilon}(T)$ *on* $[0, t_1]$ *which satisfies for any* $r \in]1, \infty[$ *with* $\frac{1}{r} + \frac{3}{p} > 1$ *and* $(u_{\eta,L}, b_{\eta,L}) \stackrel{def}{=} (e^{\eta t\Delta}(u_0, b_0))$*,*

$$\|e^{2(1-\varepsilon)\mathcal{A}\sqrt{t}\Lambda(D)}(u(t), b(t))\|_{\dot{B}_{p,q}^{-1+\frac{3}{p}}} \le c_1 \qquad \text{for any } t \in [0, t_1], \tag{1.5}$$

and

$$\liminf_{t\to 0} \frac{rad(u(t), b(t))}{\sqrt{t}\mathcal{A}} \ge 2(1-\varepsilon), \tag{1.6}$$

where $\mathcal{A} = (-\frac{1}{3}\ln \|(u_{\eta,L}, b_{\eta,L})\|_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})})^{\frac{1}{2}}$ *and the norm of* $\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})$ *is defined by Definition A3 in the Appendix.*

Remark 1.1. *When* $(u_0, b_0) \in \dot{B}_{p,q}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ *with* $q \in [1, \infty[$ *and* $0 < r < \infty$*, we can naturally conclude that*

$$\lim_{t\to 0^+} \|(u_{\eta,L}, b_{\eta,L})\|_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})} = 0,$$

which combining with (1.6)*, we can infer that*

$$\liminf_{t\to 0} \frac{rad(u(t), b(t))}{\sqrt{t}} = \infty.$$

Now, we will consider the Gevrey regularity of solutions to the system (1.1) with initial data in $L^p(\mathbb{R}^3)$. To do this, we introduce the following function space:

Definition 1.1. *Let* $0 \le t \le T$*, then for* $p \in]3, 6[$*, the definition of the norm of the function space* $E_p(T)$ *is as follows:*

$$\|f\|_{E_p(T)} = \|f\|_{\widetilde{L}_T^{\infty}(\dot{B}_{p,\frac{p}{2}}^{1-\frac{3}{p}})} + \|f\|_{\widetilde{L}_T^{1}(\dot{B}_{p,\frac{p}{2}}^{3-\frac{3}{p}})}, \tag{1.7}$$

where f *is a homogeneous tempered distribution, and* $\widetilde{L}_T^r(\dot{B}_{p,q}^s)$ *is the Chemin-Lerner type space where its norm is defined by Definition A3.*

Theorem 1.3. *Assume that* $(u_0, b_0) \in L^p(\mathbb{R}^3)$ *with* $p \in]3, 6[$*, then there exists a sufficiently small constant* ξ*, such that for any* T *satisfying*

$$T^{\frac{1}{2\gamma}}\|(u_0, b_0)\|_{L^p} \le \xi \qquad \text{with} \qquad \gamma = \frac{p}{p-3}, \tag{1.8}$$

the MHD Eq (1.1) *admits a unique solution* $(u, b) = (u_L, b_L) + (v, w)$ *with* $(v, w) \in E_p(T)$ *on* $[0, t_2]$ *which satisfies*

$$\|e^{\sqrt{\frac{2(1-\varepsilon)}{3\gamma(1+\varepsilon)}}\sqrt{t|\ln t|}\Lambda(D)}(v, w)\|_{\dot{B}_{p,\frac{p}{2}}^{1-\frac{3}{p}}} \le c_2 t^{-\frac{1}{\gamma}} \qquad \text{for any } t \in [0, t_2], \tag{1.9}$$

and

$$\liminf_{t\to 0} \frac{rad(v,w)}{\sqrt{t|\ln t|}} \geq \sqrt{\frac{2}{3\gamma}}. \tag{1.10}$$

The remainder of the paper is structured as follows: In Section 2, we show the proof of Theorem 1.1. Sections 3 and 4 are devoted to prove Theorems 1.2 and 1.3, respectively. In Section 5, we shall present some basic facts on Littlewood-Paley theory and functional spaces.

Notation: In this paper, the letter C represents different positive and finite constants. The precise values of these constants are not significant and may differ from one line to another. For $A \lesssim B$, what we mean is a universal constant C, such that $A \leq CB$.

2. The analyticity of solutions to system (1.1) with initial data in Besov space

Definition 2.1. *Let* $\Lambda(\xi) = \sum_{j=1}^{3} |\xi_j|$, $\lambda > 0$, *and* $0 \leq t \leq T$. *For* $p, q \in [1, \infty[$, $\varepsilon \in]0, 1[$, *and* $s \in \mathbb{R}$, *the definition of the norm of the function space* $B_s^{\varepsilon}(T)$ *is as follows:*

$$\|f\|_{\mathcal{B}_s^{\varepsilon}(T)} = \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda \frac{t}{\sqrt{T}}\Lambda} f\|_{\widetilde{L}_T^{\infty}(\dot{B}_{p,q}^{s})} + \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda \frac{t}{\sqrt{T}}\Lambda} f\|_{\widetilde{L}_T^{1}(\dot{B}_{p,q}^{s+2})}. \tag{2.1}$$

We recall the following Mihlin-Hörmander multiplier lemma on $\mathbb{R}^n$ from [22].

Lemma 2.1. *([22]) Let* $m(\xi)$ *be a complex-valued bounded function on* $\mathbb{R}^n \setminus 0$ *that obeys for some* $0 < A < \infty$

$$\Big(\int_{R<|\xi|<2R} |\partial_\xi^\alpha m(\xi)|^2 \mathrm{d}\xi\Big)^{\frac{1}{2}} \leq A R^{\frac{n}{2}-|\alpha|} < \infty,$$

for all multi-indices $|\alpha| \leq [\frac{n}{2}] + 1$ *and* $R > 0$. *Then, for all* $1 < p < \infty$, *m lies in* $\mathcal{M}_p(\mathbb{R}^3)$, *and the following estimate is valid:*

$$\forall f \in L^p(\mathbb{R}^n) \qquad \|m(D)f\|_{L^p} \leq C_n \max(p, (p-1)^{-1})(A + \|m\|_{L^\infty})\|f\|_{L^p}.$$

In order to make it easier for the reader, we present the following lemma that outlines the properties of the Heat semigroup operator acting on the initial data. The proof of this lemma can be accessed in [10], and we exclude the details here.

Lemma 2.2. *Let* $0 < \eta < \varepsilon$ *and* $m(t,\xi) = e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda \frac{t}{\sqrt{T}}\Lambda(\xi)} e^{-(1-\eta)t|\xi|^2}$, *then for any* $p \in]1, \infty[$, *there exists a constant* $C_{\varepsilon,\eta}$ *so that*

$$\|m(t,D)f\|_{L^p} \leq C_{\varepsilon,\eta}\|f\|_{L^p} \qquad \forall f \in L^p(\mathbb{R}^3).$$

The following lemmas are also necessary for estimating the nonlinear terms.

Lemma 2.3. *([23]) Let* $B_t(\cdot,\cdot)$ *be the bilinear operator defined by*

$$B_t(f,g) \overset{def}{=} e^{\lambda \frac{t}{\sqrt{T}}\Lambda(D)}\big(e^{-\lambda \frac{t}{\sqrt{T}}\Lambda(D)} f \cdot e^{-\lambda \frac{t}{\sqrt{T}}\Lambda(D)} g\big),$$

and it has the following expansion

$$B_t(f,g) \overset{def}{=} \sum_{(\vec{\phi},\vec{\chi},\vec{\psi})\in\{-1,1\}^3} K_{\phi_1}\otimes K_{\phi_2}\otimes K_{\phi_3}(Z_{t,\vec{\phi},\vec{\chi}}f\cdot Z_{t,\vec{\phi},\vec{\psi}}g).$$

Here, $\vec{\phi}=(\phi_1,\phi_2,\phi_3)$ *and* $\vec{\chi}=(\chi_1,\chi_2,\chi_3)$ *belong to* $\{-1,1\}^3$*, and the operators* $K_\phi = K_{\phi_1}\otimes K_{\phi_2}\otimes K_{\phi_3}$ *and* $Z_{t,\vec{\phi},\vec{\chi}} = \sum_{i=1}^3 K_{\chi_i}L_{t,\phi_i,\chi_i}$ *can be defined by*

$$K_1 f(x_j) \overset{def}{=} \frac{1}{2\pi}\int_0^\infty e^{ix_j\xi_j}\mathcal{F}f(\xi_j)\mathrm{d}\xi_j,\quad K_{-1}f(x_j) \overset{def}{=} \frac{1}{2\pi}\int_{-\infty}^0 e^{ix_j\xi_j}\mathcal{F}f(\xi_j)\mathrm{d}\xi_j,$$

and

$$L_{t,\phi_i,\chi_i}f \overset{def}{=} f \quad \text{if } \phi_i\chi_i = 1,\quad L_{t,\phi_i,\chi_i}f \overset{def}{=} \frac{1}{2\pi}\int_{\mathbb{R}} e^{ix_j\xi_j}e^{-2\lambda\frac{t}{\sqrt{T}}|\xi_j|}\mathcal{F}f(\xi_j)\mathrm{d}\xi_j \quad \text{if } \phi_i\chi_i = -1.$$

Then, for any $p\in]1,\infty[$*, we have*

$$\|B_t(f,g)\|_{L^p} \lesssim \|Z_{t,\vec{\phi},\vec{\chi}}f\cdot Z_{t,\vec{\phi},\vec{\psi}}g\|_{L^p} \quad \text{and} \quad \|Z_{t,\vec{\phi},\vec{\chi}}f\|_{L^p} \lesssim \|f\|_{L^p}.$$

Lemma 2.4. *([10]) Let* $u,v\in\mathcal{B}_s^\varepsilon(T)$ *and* $B(\cdot,\cdot)$ *be the bilinear operator, then for* $p\in]1,\infty[$ *and* $s\in]-1+\frac{3}{p},\frac{3}{p}[$*, there holds*

$$\|B(u,v)\|_{\mathcal{B}_s^\varepsilon(T)} \le C_\varepsilon T^{\frac{1+s}{2}-\frac{3}{2p}}e^{\frac{3\lambda^2}{4(1-\varepsilon)}}\|u\|_{\mathcal{B}_s^\varepsilon(T)}\|v\|_{\mathcal{B}_s^\varepsilon(T)}.$$

The subsequent lemma is crucial in substantiating our findings.

Lemma 2.5. *([24]) Let X be a Banach space, L be a continuous linear map from X to X, and B be a bilinear map from* $X\times X$ *to X. Let us define*

$$\|L\|_{\mathcal{L}(X)} \overset{def}{=} \sup_{\|x\|=1}\|Lx\| \quad \text{and} \quad \|B\|_{\mathcal{B}(x)} \overset{def}{=} \sup_{\|x\|=\|y\|=1}\|B(x,y)\|.$$

If $\|L\|_{\mathcal{L}(X)} < 1$*, then for any* x_0 *in X such that*

$$\|x_0\|_X < \frac{(1-\|L\|_{\mathcal{L}(X)})^2}{4\|B\|_{\mathcal{B}(x)}}, \tag{2.2}$$

the equation

$$x = x_0 + Lx + B(x,x)$$

admits a unique solution in the ball of center 0 *and radius* $\frac{1-\|L\|_{\mathcal{L}(X)}}{2\|B\|_{\mathcal{B}(x)}}$.

Now, we give the complete proof of Theorem 1.1. Before proceeding, we denote $u_L \overset{def}{=} e^{t\Delta}u_0$ and $b_L \overset{def}{=} e^{t\Delta}b_0$. Let $\mathbb{P} = Id - \nabla\Delta^{-1}\mathrm{div}$ be the orthogonal projection of L^2 over divergence-free vector fields. By applying $\mathbb{P}$ to (1.1), we attain

$$\begin{cases}\partial_t u - \Delta u = \mathbb{P}\nabla\cdot(b\otimes b - u\otimes u), & (t,x)\in\mathbb{R}^+\times\mathbb{R}^3,\\ \partial_t b - \Delta b = \mathbb{P}\nabla\cdot(b\otimes u - u\otimes b), & \\ \nabla\cdot u = \nabla\cdot b = 0, & \\ (u,b)|_{t=0} = (u_0,b_0), & \end{cases}$$

then the system (1.1) can be equivalently reformulated as

$$\begin{cases} u = u_L + B(u,u) + B(b,b), \\ b = b_L + B(u,b) + B(b,u), \end{cases} \tag{2.3}$$

with $B(f,g) \overset{def}{=} \int_0^t e^{(t-s)\Delta}\mathbb{P}\nabla\cdot(f\otimes g)\mathrm{d}s$. We deduce from Lemmas 2.2 and A2 that

$$\begin{aligned} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_j u_L\|_{L^p} &= \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}e^{(1-\frac{\varepsilon}{2})t\Delta}\Delta_j e^{\frac{\varepsilon}{2}t\Delta}u_0\|_{L^p} \\ &\leq C_\varepsilon\|\Delta_j e^{\frac{\varepsilon}{2}t\Delta}u_0\|_{L^p} \leq C_\varepsilon e^{-c\varepsilon t 2^{2j}}\|\Delta_j u_0\|_{L^p}, \end{aligned} \tag{2.4}$$

and

$$\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_j b_L\|_{L^p} \leq C_\varepsilon e^{-c\varepsilon t 2^{2j}}\|\Delta_j b_0\|_{L^p}. \tag{2.5}$$

Combining (2.4) and (2.5), we have

$$\begin{aligned} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\left(\Delta_j u_L, \Delta_j b_L\right)\|_{L^\infty_T(L^p)} + 2^{2j}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\left(\Delta_j u_L, \Delta_j b_L\right)\|_{L^1_T(L^p)} \\ \leq C_\varepsilon c_{j,q}2^{-sj}\|(u_0,b_0)\|_{\dot{B}^s_{p,q}}. \end{aligned}$$

Definition 2.1 implies that

$$\|(u_L,b_L)\|_{\mathcal{B}^\varepsilon_s(T)} \leq C_\varepsilon\|(u_0,b_0)\|_{\dot{B}^s_{p,q}}. \tag{2.6}$$

Subsequently, to deduce Theorem 1.1 from Lemma 2.5, let X be $\mathcal{B}^\varepsilon_s(T)$ as in Definition 2.1, L be 0, and x_0 be set to (u_L, b_L), then owing to Lemma 2.4 and (2.6), we get

$$\|x_0\|_X \leq C_\varepsilon\|(u_0,b_0)\|_{\dot{B}^s_{p,q}}, \qquad \|L\|_{\mathcal{L}(X)} = 0, \qquad \|B\|_{\mathcal{B}(X)} \leq C_\varepsilon T^{\frac{1+s}{2}-\frac{3}{2p}}e^{\frac{3\lambda^2}{4(1-\varepsilon)}}.$$

To ensure contraction condition (2.2) holds ture, we take

$$\lambda(T) = \sqrt{2(1-\varepsilon)(\frac{1+s}{3}-\frac{1}{p}-\varepsilon)|\ln T|}, \tag{2.7}$$

so that there exist δ and $T<1$ sufficiently small to have

$$4C_\varepsilon^2 T^{\frac{1+s}{2}-\frac{3}{2p}}e^{\frac{3\lambda^2}{4(1-\varepsilon)}}\|(u_0,b_0)\|_{\dot{B}^s_{p,q}} = 4C_\varepsilon^2\|(u_0,b_0)\|_{\dot{B}^s_{p,q}}T^{\frac{3}{2}\varepsilon} < \delta. \tag{2.8}$$

Then, the system (2.3) admits a unique solution $(u,b)\in\mathcal{B}^\varepsilon_s$ in the ball of center 0 and radius $\frac{1}{2\|B\|_{\mathcal{B}(X)}}$. Moreover, for any $\varepsilon\leq\varepsilon_0$ and $T\leq t_0(\varepsilon,\|(u_0,b_0)\|_{\dot{B}^s_{p,q}})$, we have

$$\|e^{\lambda(T)\sqrt{T}\Lambda(D)}(u(T),b(T))\|_{\dot{B}^s_{p,q}} \leq \frac{1}{2C_\varepsilon}T^{-\frac{1+s}{2}+\frac{3}{2p}}. \tag{2.9}$$

3. The analyticity of solutions to system (1.1) with initial data in critical Besov space

Motivated by [10], this section is to give the proof of Theorem 1.2. In order to do so, we first introduce a functional space.

Definition 3.1. *Under the conditions of Definition 2.1, let $s = -1 + \frac{3}{p}$, thus the definition of the norm of the function space $\mathcal{B}_r^\varepsilon(T)$ is as follows:*

$$\|f\|_{\mathcal{B}_r^\varepsilon(T)} = \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda} f\|_{\widetilde{L}_T^\infty(\dot{B}_{p,q}^{-1+\frac{3}{p}})} + \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda} f\|_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{3}{p}+\frac{1}{r}})}, \tag{3.1}$$

where $r \in]0,\infty[$ and $\frac{3}{p} + \frac{1}{r} > 1$.

Lemma 3.1. *Let $b_0 \in \dot{B}_{p,q}^{-1+\frac{3}{p}}(\mathbb{R}^3)$ and $b_{\eta,L} = e^{\eta t\Delta} b_0$ for some $\eta \in]0,\varepsilon[$ and $\varepsilon \in]0,1[$. Then, we have*

$$\|B(b_L, b_L)\|_{\mathcal{B}_r^\varepsilon(T)} \lesssim C_{\varepsilon,\eta} e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \|b_{\eta,L}\|^2_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{3}{p}+\frac{1}{r}})}.$$

Proof. By utilizing Bony's decomposition Definition A2 to $b_L \otimes b_L$, we have

$$b_L \otimes b_L = 2T_{b_L} b_L + R(b_L, b_L),$$

where

$$T_{b_L} b_L = \sum_{l' \in \mathbb{Z}} S_{l'-1} b_L \Delta_{l'} b_L, \quad R(b_L, b_L) = \sum_{l' \in \mathbb{Z}} \Delta_{l'} b_L \widetilde{\Delta}_{l'} b_L.$$

For $0 < t \le T$, denote $b_L^\lambda \stackrel{def}{=} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} b_L$. Then, we get

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l(T_{b_L} b_L)\|_{L_T^r(L^p)} \\
\le& \sum_{|l-l'|\le 4} \|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} (e^{-\lambda\frac{t}{\sqrt{T}}\Lambda(D)} S_{l'-1} b_L^\lambda \otimes e^{-\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_{l'} b_L^\lambda)\|_{L_T^r(L^p)} \\
\le& C \sum_{|l-l'|\le 4} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\chi}} S_{l'-1} b_L^\lambda\|_{L_T^{2r}(L^\infty)} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\psi}} \Delta_{l'} b_L^\lambda\|_{L_T^{2r}(L^p)},
\end{aligned} \tag{3.2}$$

where the operators $Z_{t,\vec{\phi},\vec{\chi}}$ and $Z_{t,\vec{\phi},\vec{\psi}}$ are defined by Lemma 2.3 and $(\vec{\phi},\vec{\chi},\vec{\psi}) \in \{1,-1\}^3$. By using Lemmas 2.2 and 2.3, we deduce

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\psi}} \Delta_{l'} b_L^\lambda\|_{L_T^{2r}(L^p)} \\
\le& C\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_{l'} e^{t\Delta} b_0\|_{L_T^{2r}(L^p)} \\
=& \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} e^{(1-\eta)t\Delta} \Delta_{l'} e^{\eta t\Delta} b_0\|_{L_T^{2r}(L^p)} \\
\le& C_{\varepsilon,\eta} \|\Delta_{l'} b_{\eta,L}\|_{L_T^{2r}(L^p)} \\
\le& C_{\varepsilon,\eta} c_{l',q} 2^{-l'(-1+\frac{1}{r}+\frac{3}{p})} \|b_{\eta,L}\|_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})}.
\end{aligned} \tag{3.3}$$

For the case $r > 1$, we can conclude from Lemmas 2.3 and A1 that

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\chi}} S_{l'-1} b_L^\lambda\|_{L_T^{2r}(L^\infty)}\\
&\le C \sum_{k\le l'-2} 2^{\frac{3}{p}k} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\chi}} \Delta_k b_L^\lambda\|_{L_T^{2r}(L^p)}\\
&\le C \sum_{k\le l'-2} 2^{\frac{3}{p}k} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} e^{(1-\eta)t\Delta} \Delta_k e^{\eta t\Delta} b_0\|_{L_T^{2r}(L^p)}\\
&\le C_{\varepsilon,\eta} \sum_{k\le l'-2} 2^{\frac{3}{p}k} \|\Delta_k b_{\eta,L}\|_{L_T^{2r}(L^p)}\\
&\le C_{\varepsilon,\eta} \sum_{k\le l'-2} c_{k,q} 2^{(1-\frac{1}{r})k} \|b_{\eta,L}\|_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})}\\
&\le C_{\varepsilon,\eta} 2^{(1-\frac{1}{r})l'} \|b_{\eta,L}\|_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})}.
\end{aligned} \tag{3.4}$$

By substituting estimates (3.3) and (3.4) into (3.2), we get

$$\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l(T_{b_L} b_L)\|_{L_T^r(L^p)} \le C_{\varepsilon,\eta} c_{l,q} 2^{l(2-\frac{2}{r}-\frac{3}{p})} \|b_{\eta,L}\|^2_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})}. \tag{3.5}$$

Along the same lines of the proof of (3.5), for $p \in [2, \infty[$, we get

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l(R(b_L, b_L))\|_{L_T^r(L^p)}\\
&\le \sum_{l'\ge l-3} 2^{\frac{3}{p}l} \|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} (e^{-\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_{l'} b_L^\lambda \cdot e^{-\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \widetilde{\Delta}_{l'} b_L^\lambda)\|_{L_T^r(L^{\frac{p}{2}})}\\
&\le C \sum_{l'\ge l-3} 2^{\frac{3}{p}l} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\chi}} \Delta_{l'} b_L^\lambda\|_{L_T^{2r}(L^p)} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\psi}} \widetilde{\Delta}_{l'} b_L^\lambda\|_{L_T^{2r}(L^p)}\\
&\le C_{\varepsilon,\eta} 2^{\frac{3}{p}l} \sum_{l'\ge l-3} c_{l',q} 2^{-2l'(-1+\frac{3}{p}+\frac{1}{r})} \|b_{\eta,L}\|^2_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})}\\
&\le C_{\varepsilon,\eta} c_{l,q} 2^{l(2-\frac{3}{p}-\frac{2}{r})} \|b_{\eta,L}\|^2_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})},
\end{aligned} \tag{3.6}$$

where we used the fact that $\frac{3}{p} + \frac{1}{r} > 1$ in the last step. For $p \in]1, 2[$, taking advantage of Lemmas 2.2, 2.3 and A1, we get

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l(R(b_L, b_L))\|_{L_T^r(L^p)}\\
&\le 2^{3l(1-\frac{1}{p})} \|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l(R(b_L, b_L))\|_{L_T^r(L^1)}\\
&\le 2^{3l(1-\frac{1}{p})} \sum_{l'\ge l-3} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\chi}} \Delta_{l'} b_L^\lambda\|_{L_T^{2r}(L^{\frac{p}{p-1}})} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\psi}} \widetilde{\Delta}_{l'} b_L^\lambda\|_{L_T^{2r}(L^p)}\\
&\le 2^{3l(1-\frac{1}{p})} \sum_{l'\ge l-3} C_{\varepsilon,\eta} 2^{l'(\frac{6}{p}-3)} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} e^{(1-\eta)t\Delta} \Delta_{l'} b_{\eta,L}\|_{L_T^{2r}(L^p)}\\
&\qquad \times \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} e^{(1-\eta)t\Delta} \widetilde{\Delta}_{l'} b_{\eta,L}\|_{L_T^{2r}(L^p)}\\
&\le C_{\varepsilon,\eta} 2^{3l(1-\frac{1}{p})} \sum_{l'\ge l-3} 2^{l(-1-\frac{2}{r})} c_{l',q} \|b_{\eta,L}\|^2_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})}
\end{aligned}$$

$$\le C_{\varepsilon,\eta} c_{l,q} 2^{l(2-\frac{1}{p}-\frac{2}{r})} \|b_{\eta,L}\|^2_{\widetilde{L}^{2r}_T(\dot{B}^{-1+\frac{1}{r}+\frac{3}{p}}_{p,q})}. \tag{3.7}$$

By summing up the estimates (3.5)–(3.7), for $p \in]1,\infty[$, we obtain

$$\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l(b_L \otimes b_L)\|_{L^r_T(L^p)} \le C_{\varepsilon,\eta} c_{l,q} 2^{l(2-\frac{2}{r}-\frac{3}{p})} \|b_{\eta,L}\|^2_{\widetilde{L}^{2r}_T(\dot{B}^{-1+\frac{1}{r}+\frac{3}{p}}_{p,q})}. \tag{3.8}$$

Meanwhile, for $t \le T$ and $\varepsilon \in]0,1[$, by using Lemmas 2.2 and A1, we get that

$$\begin{aligned}
&\|\int_0^t e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} e^{(t-s)\Delta} \Delta_l \nabla(b_L \otimes b_L)(s) \mathrm{d}s\|_{L^p} \\
&\le \int_0^t \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t-s}{T}} e^{\lambda\frac{t-s}{\sqrt{T}}\Lambda(D)} e^{(1-\frac{\varepsilon}{2})(t-s)\Delta} \\
&\qquad \times e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{s}{T}} e^{\lambda\frac{s}{\sqrt{T}}\Lambda(D)} e^{\frac{\varepsilon}{2}(t-s)\Delta} \Delta_l \nabla(b_L \otimes b_L)(s)\|_{L^p} \mathrm{d}s \\
&\le C_\varepsilon 2^l e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \int_0^t e^{-c\varepsilon(t-s)2^{2l}} \|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{s}{T}} e^{\lambda\frac{s}{\sqrt{T}}\Lambda(D)} \Delta_l(b_L \otimes b_L)(s)\|_{L^p} \mathrm{d}s.
\end{aligned} \tag{3.9}$$

Combining (3.9) with (3.8), we have

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l B(b_L, b_L)\|_{L^\infty_T(L^p)} \\
&\le C_\varepsilon 2^l e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \|e^{-c\varepsilon t 2^{2l}}\|_{L_T^{\frac{r}{r-1}}} \|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{s}{T}} e^{\lambda\frac{s}{\sqrt{T}}\Lambda(D)} \Delta_l(b_L \otimes b_L)(s)\|_{L^r_T(L^p)} \\
&\le C_{\varepsilon,\eta} c_{l,q} 2^{l(1-\frac{3}{p})} e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \|b_{\eta,L}\|^2_{\widetilde{L}^{2r}_T(\dot{B}^{-1+\frac{1}{r}+\frac{3}{p}}_{p,q})}.
\end{aligned} \tag{3.10}$$

Along the same lines, we obtain

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l B(b_L, b_L)\|_{L^r_T(L^p)} \\
&\le C_\varepsilon 2^l e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \|e^{-c\varepsilon t 2^{2l}}\|_{L^1_T} \|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{s}{T}} e^{\lambda\frac{s}{\sqrt{T}}\Lambda(D)} \Delta_l(b_L \otimes b_L)(s)\|_{L^r_T(L^p)} \\
&\le C_{\varepsilon,\eta} c_{l,q} 2^{l(1-\frac{3}{p}-\frac{2}{r})} e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \|b_{\eta,L}\|^2_{\widetilde{L}^{2r}_T(\dot{B}^{-1+\frac{1}{r}+\frac{3}{p}}_{p,q})}.
\end{aligned} \tag{3.11}$$

We deduce from the time-interpolation formula that

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l B(b_L, b_L)\|_{L^{2r}_T(L^p)} \\
&\le \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l B(b_L, b_L)\|^{\frac{1}{2}}_{L^\infty_T(L^p)} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l B(b_L, b_L)\|^{\frac{1}{2}}_{L^r_T(L^p)} \\
&\le C_{\varepsilon,\eta} c_{l,q} 2^{l(1-\frac{3}{p}-\frac{2}{r})} e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \|b_{\eta,L}\|^2_{\widetilde{L}^{2r}_T(\dot{B}^{-1+\frac{1}{r}+\frac{3}{p}}_{p,q})},
\end{aligned}$$

which together with Definition 3.1 and (3.10) ensures Lemma 3.1. This proof is complete. □

Lemma 3.2. *Let $u, b \in \mathscr{B}^\varepsilon_r(T)$, then we have*

$$\|B(u,b)\|_{\mathscr{B}^\varepsilon_r(T)} \lesssim C_\varepsilon e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \|u\|_{\mathscr{B}^\varepsilon_r(T)} \|b\|_{\mathscr{B}^\varepsilon_r(T)}.$$

Proof. Similar to the proof of Lemma 3.1, let $u^\lambda \overset{def}{=} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}u(t)$, and we obtain for $0 \le t \le T$

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(T_u b)\|_{L^r_T(L^p)}\\
&\le C\sum_{|l-l'|\le 4}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\chi}}S_{l'-1}u^\lambda\|_{L^{2r}_T(L^\infty)}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\psi}}\Delta_{l'}b^\lambda\|_{L^{2r}_T(L^p)},
\end{aligned}\tag{3.12}$$

where $(\vec{\phi},\vec{\chi},\vec{\psi}) \in \{1,-1\}^3$. Then, by using Lemmas 2.3 and A1, we infer that

$$\begin{aligned}
\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\chi}}S_{l'-1}u^\lambda\|_{L^{2r}_T(L^\infty)} &\le C\sum_{k\le l'-2}2^{\frac{3}{p}k}C\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\chi}}\Delta_k u^\lambda\|_{L^{2r}_T(L^p)}\\
&\le C\sum_{k\le l'-2}2^{\frac{3}{p}k}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_k u^\lambda\|_{L^{2r}_T(L^p)}\\
&\le C_\varepsilon 2^{(1-\frac{1}{r})l'}\|u\|_{\mathcal{B}^\varepsilon_r(T)}.
\end{aligned}\tag{3.13}$$

Thus, we have

$$\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(T_u b)\|_{L^r_T(L^p)} \le C_\varepsilon c_{l,q}2^{l(2-\frac{2}{r}-\frac{3}{p})}\|u\|_{\mathcal{B}^\varepsilon_r(T)}\|b\|_{\mathcal{B}^\varepsilon_r(T)}.\tag{3.14}$$

Along the same lines of the proof of (3.14), we can show that

$$\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(T_b u)\|_{L^r_T(L^p)} \le C_\varepsilon c_{l,q}2^{l(2-\frac{2}{r}-\frac{3}{p})}\|u\|_{\mathcal{B}^\varepsilon_r(T)}\|b\|_{\mathcal{B}^\varepsilon_r(T)}.$$

By using a method similar to get (3.7) and Lemma A1, for $p \in [2,\infty[$, we obtain

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(R(u,b))\|_{L^r_T(L^p)}\\
&\le \sum_{l'\ge l-3}2^{\frac{3}{p}l}\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}(e^{-\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_{l'}u^\lambda\cdot e^{-\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\widetilde{\Delta}_{l'}b^\lambda)\|_{L^r_T(L^{\frac{p}{2}})}\\
&\le C\sum_{l'\ge l-3}2^{\frac{3}{p}l}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\chi}}\Delta_{l'}u^\lambda\|_{L^{2r}_T(L^p)}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\psi}}\widetilde{\Delta}_{l'}b^\lambda\|_{L^{2r}_T(L^p)}\\
&\le C_\varepsilon 2^{\frac{3}{p}l}\sum_{l'\ge l-3}c_{l',q}2^{-2l'(-1+\frac{3}{p}+\frac{1}{r})}\|u\|_{\mathcal{B}^\varepsilon_r(T)}\|b\|_{\mathcal{B}^\varepsilon_r(T)}\\
&\le C_\varepsilon c_{l,q}2^{l(2-\frac{3}{p}-\frac{2}{r})}\|u\|_{\mathcal{B}^\varepsilon_r(T)}\|b\|_{\mathcal{B}^\varepsilon_r(T)},
\end{aligned}\tag{3.15}$$

where we used the fact that $\frac{3}{p}+\frac{1}{r} > 1$ in the last step. The case for $p \in]1,2[$ is similar to that of (3.7). Hence, for $p \in]1,\infty[$, we have

$$\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(u\otimes b)\|_{L^r_T(L^p)} \le c_{l,q}2^{l(2-\frac{2}{r}-\frac{3}{p})}\|u\|_{\mathcal{B}^\varepsilon_r(T)}\|b\|_{\mathcal{B}^\varepsilon_r(T)}.\tag{3.16}$$

Using a similar approach to (3.10), we derive

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l B(u,b)\|_{L^\infty_T(L^p)}\\
&\le C_\varepsilon 2^l e^{\frac{3\lambda^2}{4(1-\varepsilon)}}\|e^{-c\varepsilon t2^{2l}}\|_{L^{\frac{r}{r-1}}_T}\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(u\otimes b)\|_{L^r_T(L^p)}
\end{aligned}$$

$$\leq C_\varepsilon c_{l,q} 2^{l(1-\frac{3}{p})} e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \|u\|_{\mathcal{B}_r^\varepsilon(T)} \|b\|_{\mathcal{B}_r^\varepsilon(T)}, \tag{3.17}$$

and

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l B(u,b)\|_{L_T^r(L^p)} \\
&\leq C_\varepsilon 2^l e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \|e^{-c\varepsilon t 2^{2l}}\|_{L_T^1} \|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{s}{T}} e^{\lambda\frac{s}{\sqrt{T}}\Lambda(D)} \Delta_l(u\otimes b)\|_{L_T^r(L^p)} \\
&\leq C_\varepsilon c_{l,q} 2^{l(1-\frac{3}{p}-\frac{2}{r})} e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \|u\|_{\mathcal{B}_r^\varepsilon(T)} \|b\|_{\mathcal{B}_r^\varepsilon(T)}.
\end{aligned} \tag{3.18}$$

Combined with Definition 3.1, this proof is complete. □

Lemma 3.3. *Let $u \in \mathcal{B}_r^\varepsilon(T)$ and $b_0 \in \dot{B}_{p,q}^{-1+\frac{3}{p}}$, then the following inequality holds*

$$\|B(u,b_L)\|_{\mathcal{B}_r^\varepsilon(T)} \lesssim C_{\varepsilon,\eta} e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \|u\|_{\mathcal{B}_r^\varepsilon(T)} \|b_{\eta,L}\|_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})}.$$

Proof. According to the Bony decomposition of Definition A2, we have

$$ub_L = T_u b_L + T_{b_L} u + R(u,b_L).$$

By applying an argument similar to the one used to prove (3.2) shows that

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l(T_u b_L)\|_{L_T^r(L^p)} \\
&\leq C \sum_{|l-l'|\leq 4} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\chi}} S_{l'-1} u^\lambda\|_{L_T^{2r}(L^\infty)} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\psi}} \Delta_{l'} b_L^\lambda\|_{L_T^{2r}(L^p)},
\end{aligned}$$

where $(\vec{\phi},\vec{\chi},\vec{\psi}) \in \{1,-1\}^3$. Then, we can conclude from (3.3) and (3.13) that

$$\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l(T_u b_L)\|_{L_T^r(L^p)} \leq C_{\varepsilon,\eta} c_{l,q} 2^{l(2-\frac{2}{r}-\frac{3}{p})} \|u\|_{\mathcal{B}_r^\varepsilon(T)} \|b_{\eta,L}\|_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})}. \tag{3.19}$$

Along the same lines of the proof of (3.19), it is easy to verify that

$$\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l(T_{b_L} u)\|_{L_T^r(L^p)} \leq C_{\varepsilon,\eta} c_{l,q} 2^{l(2-\frac{2}{r}-\frac{3}{p})} \|u\|_{\mathcal{B}_r^\varepsilon(T)} \|b_{\eta,L}\|_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})}. \tag{3.20}$$

By using a method similar to get (3.7) and Lemma A1, for $p \in [2,\infty[$, we obtain

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l(R(u,b_L))\|_{L_T^r(L^p)} \\
&\leq C \sum_{l'\geq l-3} 2^{\frac{3}{p}l} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\chi}} \Delta_{l'} u^\lambda\|_{L_T^{2r}(L^p)} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\psi}} \widetilde{\Delta}_{l'} b_L^\lambda\|_{L_T^{2r}(L^p)} \\
&\leq C_{\varepsilon,\eta} 2^{\frac{3}{p}l} \sum_{l'\geq l-3} c_{l',q} 2^{-2l'(-1+\frac{3}{p}+\frac{1}{r})} \|b_{\eta,L}\|_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})} \|u\|_{\mathcal{B}_r^\varepsilon(T)} \\
&\leq C_{\varepsilon,\eta} c_{l,q} 2^{l(2-\frac{3}{p}-\frac{2}{r})} \|u\|_{\mathcal{B}_r^\varepsilon(T)} \|b_{\eta,L}\|_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})}.
\end{aligned} \tag{3.21}$$

The case for $p \in]1,2[$ is similar to that of (3.7). Hence, (3.19)–(3.21) implies that for $p \in]1,\infty[$,

$$\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l(u\otimes b_L)\|_{L_T^r(L^p)} \leq c_{l,q} 2^{l(2-\frac{2}{r}-\frac{3}{p})} \|u\|_{\mathcal{B}_r^\varepsilon(T)} \|b_{\eta,L}\|_{\widetilde{L}_T^{2r}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})}. \tag{3.22}$$

The remainder of the argument is analogous to that in Lemma 3.2 and is left to the reader. □

We are now in a position to prove Theorem 1.2. We shall use Lemma 2.5 and denote $(v, w) \stackrel{def}{=} (u - u_L, b - b_L)$; thus, from (2.3), we can express

$$\begin{cases} v = & B(v, v) + B(v, u_L) + B(u_L, v) + B(u_L, u_L) \\ & +B(w, w) + B(w, b_L) + B(b_L, w) + B(b_L, b_L), \\ w = & B(v, w) + B(v, b_L) + B(u_L, w) + B(u_L, b_L) \\ & +B(w, v) + B(w, u_L) + B(b_L, v) + B(b_L, u_L). \end{cases} \tag{3.23}$$

Let $X \stackrel{def}{=} \mathcal{B}_r^\varepsilon(T)$ be defined by Definition 3.1,

$$\begin{aligned} L(v, w) &= B(v, u_L) + B(u_L, v) + B(w, b_L) + B(b_L, w) \\ &\quad + B(v, b_L) + B(u_L, w) + B(w, u_L) + B(b_L, v), \\ x_0 &= B(u_L, u_L) + B(b_L, u_L) + B(u_L, b_L) + B(b_L, b_L). \end{aligned} \tag{3.24}$$

Based on Lemmas 3.1–3.3, it turns out that

$$\begin{aligned} &\|x_0\|_X \le C_{\varepsilon,\eta} e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \|(u_{\eta,L}, b_{\eta,L})\|^2_{\widetilde{L}^{2r}_T(\dot{B}_{p,q}^{-1+\frac{3}{p}+\frac{1}{r}})}, \\ &\|B\|_{\mathcal{B}(X)} \le C_\varepsilon e^{\frac{3\lambda^2}{4(1-\varepsilon)}}, \quad \text{and} \quad \|L\|_{\mathcal{L}(X)} \le C_{\varepsilon,\eta} e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \|(u_{\eta,L}, b_{\eta,L})\|_{\widetilde{L}^{2r}_T(\dot{B}_{p,q}^{-1+\frac{3}{p}+\frac{1}{r}})}. \end{aligned}$$

Let us now examine the conditions (2.2) of Lemma 2.5. Since $(u_0, b_0) \in \dot{B}_{p,q}^{-1+\frac{3}{p}}$, we can choose δ and $t_1(\varepsilon, \eta, \|(u_0, b_0)\|_{\dot{B}_{p,q}^{-1+\frac{3}{p}}})$ sufficiently small, such that

$$\|(u_{\eta,L}, b_{\eta,L})\|_{\widetilde{L}^{2r}_{t_1}(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})} \le \left(\frac{\delta}{4C_\varepsilon C_{\varepsilon,\eta}}\right)^{\frac{1}{2\varepsilon}}. \tag{3.25}$$

For $T \le t_1$, we set

$$\lambda(T) \stackrel{def}{=} 2(1 - \varepsilon) \sqrt{-\frac{1}{3} \ln \|(u_{\eta,L}, b_{\eta,L})\|_{\widetilde{L}^{2r}_T(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})}}. \tag{3.26}$$

From (3.25) and (3.26), it suffices to have

$$4C_{\varepsilon,\eta} C_\eta e^{\frac{3\lambda^2}{4(1-\varepsilon)}} e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \|(u_{\eta,L}, b_{\eta,L})\|^2_{\widetilde{L}^{2r}_T(\dot{B}_{p,q}^{-1+\frac{3}{p}+\frac{1}{r}})} \le \delta. \tag{3.27}$$

Using (3.27), we can show that (2.2) holds, and, therefore, the system (3.23) admits a unique solution $(v, w) \in \mathcal{B}_r^\varepsilon(T)$ satisfying

$$\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)} \frac{t}{T}} e^{\lambda \frac{t}{\sqrt{T}} \Lambda} (v, w)\|_{\widetilde{L}^\infty_T(\dot{B}_{p,q}^{-1+\frac{3}{p}})} \le \frac{1}{2C_\varepsilon} e^{-\frac{3\lambda^2}{4(1-\varepsilon)}}. \tag{3.28}$$

Consequently, from (3.26) and (3.28), we have

$$\liminf_{t \to 0} \frac{rad(v(t), w(t))}{\sqrt{-\frac{1}{3} t \ln \|(u_{\eta,L}, b_{\eta,L})\|_{\widetilde{L}^{2r}_T(\dot{B}_{p,q}^{-1+\frac{1}{r}+\frac{3}{p}})}}} \ge 2(1 - \varepsilon).$$

4. The Gevrey regularity of solutions to system (1.1) with initial data in L^p

Following [10], we shall study the Gevrey regularity of solutions to system (1.1) and the instantaneous radius of space analyticity of $(v, w) = (u - u_L, b - b_L)$ with the initial data in L^p for $p \in]3, 6[$.

4.1. The Gevrey regularity of solutions to system (1.1)

Lemma 4.1. *Let $b_0 \in L^p(\mathbb{R}^3)$ and $b_L = e^{t\Delta}b_0$. Then, for $p \in [2, \infty[$, we have*

$$\|B(b_L, b_L)\|_{E_p(T)} \lesssim \|b_0\|^2_{L^p}.$$

Proof. By definition of the para-product of b_L and b_L in Definition A2, we have

$$\begin{aligned}\|\Delta_l(T_{b_L}b_L)\|_{L^1_T(L^p)} &\leq C\sum_{|l-l'|\leq 4}\|S_{l'-1}b_L\|_{L^\infty_T(L^\infty)}\|\Delta_{l'}b_L\|_{L^1_T(L^p)}\\ &\leq C\sum_{|l-l'|\leq 4}\sum_{k\leq l'-1}2^{\frac{3}{p}k}\|\Delta_k b_L\|_{L^\infty_T(L^p)}\|\Delta_{l'}b_L\|_{L^1_T(L^p)}\\ &\leq c_{l,\frac{p}{2}}2^{-(2-\frac{3}{p})l}\|b_0\|^2_{L^p},\end{aligned}\tag{4.1}$$

where we use the fact that $L^p \hookrightarrow \dot{B}^0_{p,p}$, and by virtue of Lemma A2, we infer that

$$\|\Delta_l b_L\|_{L^\infty_T(L^p)} + 2^{2l}\|\Delta_l b_L\|_{L^1_T(L^p)} \lesssim \|\Delta_l b_0\|_{L^p} \leq c_{l,p}\|b_0\|_{L^p}.\tag{4.2}$$

By using Lemma A1 and (4.2), for $p \in [2, \infty[$, we obtain

$$\begin{aligned}\|\Delta_l(R(b_L, b_L))\|_{L^1_T(L^p)} &\leq C2^{\frac{3}{p}l}\sum_{l'\geq l-3}\|\Delta_{l'}b_L\|_{L^\infty_T(L^p)}\|\widetilde{\Delta}_{l'}b_L\|_{L^1_T(L^p)}\\ &\leq C2^{\frac{3}{p}l}\sum_{l'\geq l-3}c_{l',\frac{p}{2}}2^{-2l'}\|b_0\|^2_{L^p}\\ &\leq c_{l,\frac{p}{2}}2^{-(2-\frac{3}{p})l}\|b_0\|^2_{L^p}.\end{aligned}\tag{4.3}$$

Using (4.1) and (4.3), we can show that

$$\|\Delta_l(b_L \otimes b_L)\|_{L^1_T(L^p)} \leq c_{l,\frac{p}{2}}2^{-(2-\frac{3}{p})l}\|b_0\|^2_{L^p}.\tag{4.4}$$

We infer an estimate similar to (3.17) and (3.18) such that

$$\begin{aligned}\|\Delta_l B(b_L, b_L)\|_{L^\infty_T(L^p)} + 2^{2l}\|\Delta_l B(b_L, b_L)\|_{L^1_T(L^p)} &\leq 2^l\|\Delta_l(b_L \otimes b_L)\|_{L^1_T(L^p)}\\ &\leq mc_{l,\frac{p}{2}}2^{-(1-\frac{3}{p})l}\|b_0\|^2_{L^p},\end{aligned}$$

which combined with Definition 1.1 completes the proof of Lemma 4.1. □

Lemma 4.2. *Let $u, b \in E_p(T)$ with $p \in]3, 6[$ and $\frac{1}{\gamma} = 1 - \frac{3}{p}$, then we have*

$$\|B(u, b)\|_{E_p(T)} \lesssim T^{\frac{1}{\gamma}}\|u\|_{E_p(T)}\|b\|_{E_p(T)}.$$

Proof. In a similar way as in the proof of Lemma 4.1, we obtain

$$\begin{aligned}\|\Delta_l(T_u b)\|_{L_T^{\frac{p}{3}}(L^p)} &\le C\sum_{|l-l'|\le 4}\|S_{l'-1}u\|_{L_T^\infty(L^\infty)}\|\Delta_{l'}b\|_{L_T^{\frac{p}{3}}(L^p)}\\ &\le C\sum_{|l-l'|\le 4}\sum_{k\le l'-1}2^{\frac{3}{p}k}\|\Delta_k u\|_{L_T^\infty(L^p)}\|\Delta_{l'}b\|_{L_T^{\frac{p}{3}}(L^p)}\\ &\le c_{l,\frac{p}{2}}2^{-(2-\frac{3}{p})l}\|u\|_{E_p(T)}\|b\|_{E_p(T)},\end{aligned} \tag{4.5}$$

where we use the fact that

$$\|\Delta_{l'}b\|_{L_T^{\frac{p}{3}}(L^p)} \lesssim \|\Delta_{l'}b\|_{L_T^\infty(L^p)}^{1-\frac{3}{p}}\|\Delta_{l'}b\|_{L_T^1(L^p)}^{\frac{3}{p}} \le c_{l',\frac{p}{2}}2^{-(1+\frac{3}{p})l'}\|b\|_{E_p(T)}, \tag{4.6}$$

and

$$\sum_{k\le l'-1}2^{\frac{3}{p}k}\|\Delta_k u\|_{L_T^\infty(L^p)} \le c_{l',\frac{p}{2}}2^{-(1-\frac{6}{p})l'}\|u\|_{E_p(T)}.$$

The term $\|\Delta_l(T_b u)\|_{L_T^{\frac{p}{3}}(L^p)}$ has the same bound as (4.5). By using Lemma A1 and (4.6), for $p\in[2,\infty[$, we obtain

$$\begin{aligned}\|\Delta_l(R(u,b))\|_{L_T^{\frac{p}{3}}(L^p)} &\le C2^{\frac{3}{p}l}\sum_{l'\ge l-3}\|\Delta_{l'}u\|_{L_T^{\frac{p}{3}}(L^p)}\|\widetilde{\Delta}_{l'}b\|_{L_T^\infty(L^p)}\\ &\le C2^{\frac{3}{p}l}\sum_{l'\ge l-3}c_{l',\frac{p}{2}}2^{-2l'}\|u\|_{E_p(T)}\|b\|_{E_p(T)}\\ &\le c_{l,\frac{p}{2}}2^{-(2-\frac{3}{p})l}\|u\|_{E_p(T)}\|b\|_{E_p(T)}.\end{aligned} \tag{4.7}$$

By a suitable modification of the proof of Lemma 4.1, we can show that

$$\begin{aligned}\|\Delta_l B(u,b)\|_{L_T^\infty(L^p)} + 2^{2l}\|\Delta_l B(u,b)\|_{L_T^1(L^p)} &\le 2^l\|\Delta_l(u\otimes b)\|_{L_T^1(L^p)}\\ &\le T^{\frac{1}{\gamma}}2^l\|\Delta_l(u\otimes b)\|_{L_T^{\frac{p}{3}}(L^p)}\\ &\le c_{l,\frac{p}{2}}2^{-(1-\frac{3}{p})l}T^{\frac{1}{\gamma}}\|u\|_{E_p(T)}\|b\|_{E_p(T)}.\end{aligned}$$

This combined with Definition 1.1 ensures Lemma 4.2. □

Lemma 4.3. *Let $u\in E_p(T)$ and $b_0\in L^p$ with $p\in]3,6[$, then we get*

$$\|B(u,b_L)\|_{E_p(T)} \lesssim T^{\frac{1}{2\gamma}}\|u\|_{E_p(T)}\|b_0\|_{L^p}.$$

Proof. By applying an argument similar to the one used to prove (4.5), as well as Hölder's inequality and (4.2), one shows that

$$
\begin{aligned}
\|\Delta_l(T_u b_L)\|_{L_T^{\frac{2p}{p+3}}(L^p)} &\le C\sum_{|l-l'|\le 4}\|S_{l'-1}u\|_{L_T^\infty(L^\infty)}\|\Delta_{l'}b_L\|_{L_T^{\frac{2p}{p+3}}(L^p)}\\
&\le C\sum_{|l-l'|\le 4}\sum_{k\le l'-1}2^{\frac{3}{p}k}\|\Delta_k u\|_{L_T^\infty(L^p)}\|\Delta_{l'}b_L\|^{1-\frac{p+3}{2p}}_{L_T^\infty(L^p)}\|\Delta_{l'}b_L\|^{\frac{p+3}{2p}}_{L_T^1(L^p)}\\
&\le C\sum_{|l-l'|\le 4}c_{l',\frac{p}{2}}2^{-(2-\frac{3}{p})l'}\|u\|_{E_p(T)}\|\Delta_{l'}b_L\|^{1-\frac{p+3}{2p}}_{L_T^\infty(L^p)}(2^{2j'}\|\Delta_{l'}b_L\|_{L_T^1(L^p)})^{\frac{p+3}{2p}}\\
&\le c_{l,\frac{p}{2}}2^{-(2-\frac{3}{p})l}\|u\|_{E_p(T)}\|b_0\|_{L^p}. \qquad (4.8)
\end{aligned}
$$

The term with $T_{b_L}u$ has the same bound as (4.8). Owing to Lemma A1, we have

$$
\begin{aligned}
\|\Delta_l(R(u,b_L))\|_{L_T^{\frac{2p}{p+3}}(L^p)} &\le C2^{\frac{3}{p}l}\sum_{l'\ge l-3}\|\Delta_{l'}u\|_{L_T^{\frac{2p}{p+3}}(L^p)}\|\widetilde{\Delta}_{l'}b_L\|_{L_T^\infty(L^p)}\\
&\le C2^{\frac{3}{p}l}\sum_{l'\ge l-3}c_{l',\frac{p}{2}}2^{-2l'}\|u\|_{E_p(T)}\|b_0\|_{L^p}\\
&\le c_{l,\frac{p}{2}}2^{-(2-\frac{3}{p})l}\|u\|_{E_p(T)}\|b_0\|_{L^p}. \qquad (4.9)
\end{aligned}
$$

From estimates (4.8) and (4.9), it is evident that

$$
\begin{aligned}
\|\Delta_l B(u,b_L)\|_{L_T^\infty(L^p)} + 2^{2l}\|\Delta_l B(u,b_L)\|_{L_T^1(L^p)} &\le 2^l\|\Delta_l(u\otimes b_L)\|_{L_T^1(L^p)}\\
&\le T^{\frac{1}{2\gamma}}2^l\|\Delta_l(u\otimes b_L)\|_{L_T^{\frac{2p}{p+3}}(L^p)}\\
&\le c_{l,\frac{p}{2}}2^{-(1-\frac{3}{p})l}T^{\frac{1}{2\gamma}}\|u\|_{E_p(T)}\|b_0\|_{L^p},
\end{aligned}
$$

which together with Definition 1.1 ensures Lemma 4.3. □

Now, let us present the existence part of Theorem 1.3 by using Lemma 2.5. We take $X \overset{def}{=} E_p(T)$ according to Definition 1.1, and use (3.24) to choose L and x_0. Then, we deduce from Lemmas 4.1 to 4.3 that

$$
\|x_0\|_X \le \|(u_0,b_0)\|^2_{L^p}, \quad \|B\|_{\mathcal{B}(X)} \le T^{\frac{1}{\gamma}}, \quad \text{and} \quad \|L\|_{\mathcal{L}(X)} \le T^{\frac{1}{2\gamma}}\|(u_0,b_0)\|_{L^p}.
$$

To ensure contraction condition (2.2) holds true, we take $T^{\frac{1}{2\gamma}}\|(u_0,b_0)\|_{L^p} \le \xi$ for a sufficiently small ξ, then by using Lemma 2.5, we get that the system (3.23) admits a unique solution $(v,w) \in E_p(T)$. We thus obtain the unique solution $(u,b) = (u_L + v, b_L + w)$ of the system (1.1) on $[0,T]$.

The aim of the next subsection is to study the instantaneous radius of analyticity of (v,w) under the assumptions that the initial data $(u_0,b_0) \in L^p(\mathbb{R}^3)$ for $p \in]3,6[$.

4.2. The radius of analyticity of (v,w)

Definition 4.1. *Under the conditions of Definition 1.1, let $p \in [2,\infty[$, then the definition of the norm of the function space $\mathcal{B}_p^\varepsilon(T)$ is as follows:*

$$
\|f\|_{\mathcal{B}_p^\varepsilon(T)} \overset{def}{=} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda}f\|_{E_p(T)}. \qquad (4.10)
$$

Lemma 4.4. *Let $b_0 \in L^p(\mathbb{R}^3)$ with $2 \le p \le \infty$ and $b_L = e^{t\Delta} b_0$ for some $\eta \in]0, \varepsilon[$ and $\varepsilon \in]0,1[$. Then, we have*

$$\|B(b_L, b_L)\|_{\mathcal{B}^\varepsilon_p(T)} \lesssim C_\varepsilon e^{\frac{3\lambda^2}{4(1-\varepsilon)}} \|b_0\|^2_{L^p}.$$

Proof. An argument similar to Lemma 3.1 shows that

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda \frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l(T_{b_L} b_L)\|_{L^1_T(L^p)} \\
&\le \sum_{|l-l'|\le 4} \|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda \frac{t}{\sqrt{T}}\Lambda(D)} (e^{-\lambda \frac{t}{\sqrt{T}}\Lambda(D)} S_{l'-1} b^\lambda_L \otimes e^{-\lambda \frac{t}{\sqrt{T}}\Lambda(D)} \Delta_{l'} b^\lambda_L)\|_{L^1_T(L^p)} \\
&\le C \sum_{|l-l'|\le 4} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\chi}} S_{l'-1} b^\lambda_L\|_{L^\infty_T(L^\infty)} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\psi}} \Delta_{l'} b^\lambda_L\|_{L^1_T(L^p)},
\end{aligned} \tag{4.11}$$

for the operators $Z_{t,\vec{\phi},\vec{\chi}}$ and $Z_{t,\vec{\phi},\vec{\psi}}$ defined by Lemma 2.3 and $(\vec{\phi},\vec{\chi},\vec{\psi}) \in \{1,-1\}^3$. By virtue of $L^p \hookrightarrow \dot{B}^0_{p,p}(\mathbb{R}^3)$ in Theorem 2.40 of [25] and (2.5), we have

$$\begin{aligned}
\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\psi}} \Delta_{l'} b^\lambda_L\|_{L^1_T(L^p)} &\le C \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda \frac{t}{\sqrt{T}}\Lambda(D)} \Delta_{l'} e^{t\Delta} b_0\|_{L^1_T(L^p)} \\
&= \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda \frac{t}{\sqrt{T}}\Lambda(D)} e^{(1-\eta)t\Delta} \Delta_{l'} e^{\eta t\Delta} b_0\|_{L^1_T(L^p)} \\
&\le C_\varepsilon \| \|e^{-c\eta t 2^{2l'}} \Delta_{l'} b_0\|_{L^p}\|_{L^1_T} \\
&\le C_\varepsilon c_{l',p} 2^{-2l'} \|b_0\|_{L^p}.
\end{aligned} \tag{4.12}$$

By Lemmas 2.3 and A1, for $r > 1$, the following inequality holds true:

$$\begin{aligned}
\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\chi}} S_{l'-1} b^\lambda_L\|_{L^\infty_T(L^\infty)} &\le C \sum_{k\le l'-2} 2^{\frac{3}{p}k} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} Z_{t,\vec{\phi},\vec{\chi}} \Delta_k b^\lambda_L\|_{L^\infty_T(L^p)} \\
&\le C \sum_{k\le l'-2} 2^{\frac{3}{p}k} \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}} e^{\lambda \frac{t}{\sqrt{T}}\Lambda(D)} e^{(1-\eta)t\Delta} \Delta_k e^{\eta t\Delta} b_0\|_{L^\infty_T(L^p)} \\
&\le C_\varepsilon \sum_{k\le l'-2} 2^{\frac{3}{p}k} \|\Delta_k e^{\eta t\Delta} b_0\|_{L^\infty_T(L^p)} \\
&\le C_\varepsilon \sum_{k\le l'-2} 2^{\frac{3}{p}k} c_{k,p} \|b_0\|_{L^p} \\
&\le C_\varepsilon 2^{\frac{3}{p}l'} c_{l',p} \|b_0\|_{L^p}.
\end{aligned} \tag{4.13}$$

By substituting estimates (4.12) and (4.13) into (4.11), we get

$$\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda \frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l(T_{b_L} b_L)\|_{L^1_T(L^p)} \le C_\varepsilon c_{l,\frac{p}{2}} 2^{-l(2-\frac{3}{p})} \|b_0\|^2_{L^p}. \tag{4.14}$$

Along the same lines as in the proof of (4.14), we get

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda \frac{t}{\sqrt{T}}\Lambda(D)} \Delta_l(R(b_L, b_L))\|_{L^1_T(L^p)} \\
&\le \sum_{l'\ge l-3} 2^{\frac{3}{p}l} \|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}} e^{\lambda \frac{t}{\sqrt{T}}\Lambda(D)} (e^{-\lambda \frac{t}{\sqrt{T}}\Lambda(D)} \Delta_{l'} b^\lambda_L \cdot e^{-\lambda \frac{t}{\sqrt{T}}\Lambda(D)} \widetilde{\Delta}_{l'} b^\lambda_L)\|_{L^1_T(L^{\frac{p}{2}})}
\end{aligned}$$

$$
\begin{aligned}
&\le C\sum_{l'\ge l-3} 2^{\frac{3}{p}l}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\chi}}\Delta_{l'}b_L^\lambda\|_{L^\infty_T(L^p)}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\psi}}\tilde{\Delta}_{l'}b_L^\lambda\|_{L^1_T(L^p)}\\
&\le C_\varepsilon 2^{\frac{3}{p}l}\sum_{l'\ge l-3} c_{l',q}2^{-2l'}\|b_0\|^2_{L^p}\\
&\le C_\varepsilon c_{l,\frac{p}{2}}2^{-l(2-\frac{3}{p})}\|b_0\|^2_{L^p}.
\end{aligned}\tag{4.15}
$$

By summing up the estimates (4.14) and (4.15), for $p\in[2,\infty[$, we obtain

$$
\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(b_L\otimes b_L)\|_{L^1_T(L^p)}\le C_\varepsilon c_{l,q}2^{-l(2-\frac{3}{p})}\|b_0\|^2_{L^p}.\tag{4.16}
$$

We deduce from Lemmas 2.2 and A1 that for any $t\le T$ and $\varepsilon\in]0,1[$,

$$
\begin{aligned}
&\|\int_0^t e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}e^{(t-s)\Delta}\Delta_l\nabla(b_L\otimes b_L)(s)\mathrm{d}s\|_{L^p}\\
&\le\int_0^t \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t-s}{T}}e^{\lambda\frac{t-s}{\sqrt{T}}\Lambda(D)}e^{(1-\frac{\varepsilon}{2})(t-s)\Delta}\\
&\qquad\times e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{s}{T}}e^{\lambda\frac{s}{\sqrt{T}}\Lambda(D)}e^{\frac{\varepsilon}{2}(t-s)\Delta}\Delta_l\nabla(b_L\otimes b_L)(s)\|_{L^p}\mathrm{d}s\\
&\le C_\varepsilon 2^l e^{\frac{3\lambda^2}{4(1-\varepsilon)}}\int_0^t e^{-c\varepsilon(t-s)2^{2l}}\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}e^{\lambda\frac{s}{\sqrt{T}}\Lambda(D)}\Delta_l(b_L\otimes b_L)(s)\|_{L^p}\mathrm{d}s.
\end{aligned}\tag{4.17}
$$

Combining (4.16) with (4.17), we have

$$
\begin{aligned}
&\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l B(b_L,b_L)\|_{L^\infty_T(L^p)}+2^{2l}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l B(b_L,b_L)\|_{L^1_T(L^p)}\\
&\le C_\varepsilon 2^l e^{\frac{3\lambda^2}{4(1-\varepsilon)}}\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{s}{T}}e^{\lambda\frac{s}{\sqrt{T}}\Lambda(D)}\Delta_l(b_L\otimes b_L)(s)\|_{L^1_T(L^p)}\\
&\le C_\varepsilon c_{l,q}2^{-l(1-\frac{3}{p})}e^{\frac{3\lambda^2}{4(1-\varepsilon)}}\|b_0\|^2_{L^p},
\end{aligned}
$$

which together with Definition A1, we get the desired result. □

Lemma 4.5. *Let $v,w\in\mathcal{B}^\varepsilon_p(T)$ with $p\in]3,6[$, then for $\frac{1}{\gamma}=1-\frac{3}{p}$, we have*

$$
\|B(w,v)\|_{\mathcal{B}^\varepsilon_p(T)}\lesssim C_\varepsilon e^{\frac{3\lambda^2}{4(1-\varepsilon)}}T^{\frac{1}{\gamma}}\|w\|_{\mathcal{B}^\varepsilon_p(T)}\|v\|_{\mathcal{B}^\varepsilon_p(T)}.
$$

Proof. We recall $v^\lambda\overset{def}{=}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}v(t)$, and follow the same technique as in the proof of Lemma 4.4 to obtain for $0\le t\le T$,

$$
\begin{aligned}
&\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(T_w v)\|_{L^{\frac{p}{3}}_T(L^p)}\\
&\le C\sum_{|l-l'|\le 4}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\chi}}S_{l'-1}w^\lambda\|_{L^\infty_T(L^\infty)}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\psi}}\Delta_{l'}v^\lambda\|_{L^{\frac{p}{3}}_T(L^p)},
\end{aligned}
$$

where $(\vec{\phi},\vec{\chi},\vec{\psi})\in\{1,-1\}^3$. By Lemmas 2.3 and A1, we deduce

$$\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\chi}}S_{l'-1}w^\lambda\|_{L^\infty_T(L^\infty)} \leq C\sum_{k\leq l'-2}2^{\frac{3}{p}k}C\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\chi}}\Delta_k w^\lambda\|_{L^\infty_T(L^p)}$$
$$\leq C\sum_{k\leq l'-2}2^{\frac{3}{p}k}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_k w^\lambda\|_{L^\infty_T(L^p)}$$
$$\leq C_\varepsilon c_{l',\frac{p}{2}}2^{(\frac{6}{p}-1)l'}\|w\|_{\mathcal{B}^\varepsilon_p(T)}. \tag{4.18}$$

We infer from the interpolation inequality that

$$\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\psi}}\Delta_{l'}v^\lambda\|_{L^{\frac{p}{3}}_T(L^p)} \leq \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_{l'}v^\lambda\|_{L^{\frac{p}{3}}_T(L^p)}$$
$$\leq \|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_{l'}v^\lambda\|^{1-\frac{3}{p}}_{L^\infty_T(L^p)}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\Delta_{l'}v^\lambda\|^{\frac{3}{p}}_{L^1_T(L^p)}$$
$$\leq C_\varepsilon c_{l',\frac{p}{2}}2^{-l'(1+\frac{3}{p})}\|v\|_{\mathcal{B}^\varepsilon_p(T)}. \tag{4.19}$$

Combining (4.18) and (4.19), we get

$$\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(T_w v)\|_{L^r_T(L^p)} \leq C_\varepsilon c_{l',\frac{p}{2}}2^{-l'(2-\frac{3}{p})}\|w\|_{\mathcal{B}^\varepsilon_p(T)}\|v\|_{\mathcal{B}^\varepsilon_p(T)}. \tag{4.20}$$

Along the same lines of the proof of (4.20), the term $\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(T_v w)\|_{L^{\frac{p}{3}}_T(L^p)}$ admits the same estimate. Moreover, from the same lines to prove (4.15) and Lemma A1, we get that

$$\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(R(w,v))\|_{L^{\frac{p}{3}}_T(L^p)}$$
$$\leq \sum_{l'\geq l-3}2^{\frac{3}{p}l}\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}(e^{-\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_{l'}w^\lambda\cdot e^{-\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\widetilde{\Delta}_{l'}v^\lambda)\|_{L^{\frac{p}{3}}_T(L^{\frac{p}{2}})}$$
$$\leq C\sum_{l'\geq l-3}2^{\frac{3}{p}l}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\chi}}\Delta_{l'}w^\lambda\|_{L^\infty_T(L^p)}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\psi}}\widetilde{\Delta}_{l'}v^\lambda\|_{L^{\frac{p}{3}}_T(L^p)}$$
$$\leq C_\varepsilon 2^{\frac{3}{p}l}\sum_{l'\geq l-3}c_{l',\frac{p}{2}}2^{-2l'}\|w\|_{\mathcal{B}^\varepsilon_p(T)}\|v\|_{\mathcal{B}^\varepsilon_p(T)}$$
$$\leq C_\varepsilon c_{l',\frac{p}{2}}2^{-l'(2-\frac{3}{p})}\|w\|_{\mathcal{B}^\varepsilon_p(T)}\|v\|_{\mathcal{B}^\varepsilon_p(T)}. \tag{4.21}$$

As a consequence, we obtain

$$\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(w\otimes v)\|_{L^{\frac{p}{3}}_T(L^p)} \leq C_\varepsilon c_{l,\frac{p}{2}}2^{-l(2-\frac{3}{p})}\|w\|_{\mathcal{B}^\varepsilon_p(T)}\|v\|_{\mathcal{B}^\varepsilon_p(T)}. \tag{4.22}$$

Then, we get

$$\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l B(w,v)\|_{L^p}$$
$$\leq C_\varepsilon c_{l,\frac{p}{2}}2^l e^{\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}\int_0^t e^{-c\varepsilon(t-s)2^{2l}}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(w\otimes v)(s)\|_{L^p}\mathrm{d}s.$$

which combines with (4.22) to ensure that

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l B(w,v)\|_{L^\infty_T(L^p)}+2^{2l}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l B(w,v)\|_{L^1_T(L^p)}\\
&\le C_\varepsilon 2^l e^{\frac{3\lambda^2}{4(1-\varepsilon)}}\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(w\otimes v)\|_{L^1_T(L^p)}\\
&\le C_\varepsilon 2^l e^{\frac{3\lambda^2}{4(1-\varepsilon)}}T^{\frac{1}{\gamma}}\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(w\otimes v)\|_{L_T^{\frac{p}{3}}(L^p)}\\
&\le C_\varepsilon c_{l,\frac{p}{2}}2^{-l(1-\frac{3}{p})}e^{\frac{3\lambda^2}{4(1-\varepsilon)}}T^{\frac{1}{\gamma}}\|w\|_{\mathcal{B}^\varepsilon_p(T)}\|v\|_{\mathcal{B}^\varepsilon_p(T)},
\end{aligned}$$

which together with Definition A1 ensures Lemma 4.5. □

Lemma 4.6. *Let $w\in\mathcal{B}^\varepsilon_p(T)$ and $b_0\in L^p(\mathbb{R}^3)$ with $p\in]3,6[$, then the following inequality holds*

$$\|B(w,b_L)\|_{\mathcal{B}^\varepsilon_p(T)}\lesssim C_\varepsilon T^{\frac{1}{2\gamma}}e^{\frac{3\lambda^2}{4(1-\varepsilon)}}\|w\|_{\mathcal{B}^\varepsilon_p(T)}\|b_0\|_{L^p}.$$

Proof. According to the Bony decomposition of Definition A2, we have

$$wb_L=T_w b_L+T_{b_L}w+R(w,b_L).$$

By an argument similar to the one used to prove Lemma 4.5, we get that

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(T_w b_L)\|_{L_T^{\frac{2p}{p+3}}(L^p)}\\
&\le C\sum_{|l-l'|\le 4}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\chi}}S_{l'-1}w^\lambda\|_{L^\infty_T(L^\infty)}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\psi}}\Delta_{l'}b_L^\lambda\|_{L_T^{\frac{2p}{p+3}}(L^p)},
\end{aligned}$$

where $(\vec{\phi},\vec{\chi},\vec{\psi})\in\{1,-1\}^3$. Then, by using (3.3) and (4.18), we infer that

$$\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(T_w b_L)\|_{L_T^{\frac{2p}{p+3}}(L^p)}\le C_\varepsilon c_{l,\frac{p}{2}}2^{-l(2-\frac{3}{p})}\|w\|_{\mathcal{B}^\varepsilon_p(T)}\|b_0\|_{L^p}. \tag{4.23}$$

Along the same lines as in the proof of (4.23), it is easy to verify that

$$\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(T_{b_L}w)\|_{L_T^{\frac{2p}{p+3}}(L^p)}\le C_\varepsilon c_{l,\frac{p}{2}}2^{-l(2-\frac{3}{p})}\|w\|_{\mathcal{B}^\varepsilon_p(T)}\|b_0\|_{L^p}. \tag{4.24}$$

Taking advantage of Lemma A1 and a similar way to get (4.15), we have

$$\begin{aligned}
&\|e^{-\frac{3\lambda^2}{2(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda(D)}\Delta_l(R(w,b_L))\|_{L_T^{\frac{2p}{p+3}}(L^p)}\\
&\le C\sum_{l'\ge l-3}2^{\frac{3}{p}l}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\chi}}\Delta_{l'}w^\lambda\|_{L^\infty_T(L^p)}\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}Z_{t,\vec{\phi},\vec{\psi}}\widetilde{\Delta}_{l'}b_L^\lambda\|_{L_T^{\frac{2p}{p+3}}(L^p)}\\
&\le C_\varepsilon 2^{\frac{3}{p}l}\sum_{l'\ge l-3}c_{l',q}2^{-2l'}\|b_0\|_{L^p}\|w\|_{\mathcal{B}^\varepsilon_p(T)}\\
&\le C_\varepsilon c_{l,\frac{p}{2}}2^{-l(2-\frac{3}{p})}\|w\|_{\mathcal{B}^\varepsilon_p(T)}\|b_0\|_{L^p}.
\end{aligned}$$

The remainder of the argument is analogous to that in Lemma 3.2 and is left to the reader. □

We now apply Lemma 2.5 to prove Theorem 1.3. Let $X \overset{def}{=} \mathcal{B}_p^{\varepsilon}(T)$ according to Definition A1, and use (3.24) to choose L and x_0. Based on Lemmas 4.4–4.6, it turns out that

$$\|x_0\|_X \leq C_\varepsilon e^{\frac{3\lambda^2}{4(1-\varepsilon)}}\|(u_0,b_0)\|_{L^p}^2, \ \|B\|_{\mathcal{B}(X)} \leq C_\varepsilon e^{\frac{3\lambda^2}{4(1-\varepsilon)}}T^{\frac{1}{\gamma}},$$
$$\text{and} \quad \|L\|_{\mathcal{L}(X)} \leq C_\varepsilon e^{\frac{3\lambda^2}{4(1-\varepsilon)}}T^{\frac{1}{2\gamma}}\|(u_0,b_0)\|_{L^p}.$$

We set

$$\lambda(T) \overset{def}{=} \sqrt{\frac{2(1-\varepsilon)}{3\gamma(1+\varepsilon)}|\ln T|}, \tag{4.25}$$

and we deduce from (4.25) such that

$$2C_\varepsilon e^{\frac{3\lambda^2}{4(1-\varepsilon)}}T^{\frac{1}{2\gamma}}\|(u_0,b_0)\|_{L^p} = 2C_\varepsilon T^{\frac{\varepsilon}{2\gamma(1+\varepsilon)}}\|(u_0,b_0)\|_{L^p}. \tag{4.26}$$

We take t_2 so small such that

$$2C_\varepsilon t_2^{\frac{\varepsilon}{2\gamma(1+\varepsilon)}}\|(u_0,b_0)\|_{L^p} \leq \delta,$$

where δ being sufficiently small, then we have

$$4C_\varepsilon^2 e^{\frac{3\lambda^2}{2(1-\varepsilon)}}T^{\frac{1}{\gamma}}\|(u_0,b_0)\|_{L^p} \leq \delta^2,$$

from which we can show that (2.2) is true. Thus, Lemma 2.5 leads us to conclude that the system (3.23) admits a unique solution $(v,w) \in \mathcal{B}_p^{\varepsilon}(T)$ satisfying

$$\|e^{-\frac{3\lambda^2}{4(1-\varepsilon)}\frac{t}{T}}e^{\lambda\frac{t}{\sqrt{T}}\Lambda}(v,w)\|_{\tilde{L}_T^\infty(\dot{B}^{1-\frac{3}{p}}_{p,\frac{p}{2}})} \leq \frac{1}{2C_\varepsilon}e^{-\frac{3\lambda^2}{4(1-\varepsilon)}}T^{-\frac{1}{\gamma}},$$

which implies

$$\|e^{\lambda\frac{t}{\sqrt{T}}\Lambda}(v,w)\|_{\dot{B}^{1-\frac{3}{p}}_{p,\frac{p}{2}}} \leq \frac{1}{2C_\varepsilon}T^{-\frac{1}{\gamma}}. \tag{4.27}$$

This is (1.9) of Theorem 1.3. (1.10) follows from (1.9) and, thus, we complete the proof of Theorem 1.3.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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Appendix: Littlewood-Paley decomposition and functional spaces

We start with the classical dyadic decomposition in $\mathbb{R}^3$; see [25]. Let $\chi, \varphi \in [0, 1]$ be two nonnegative radial functions which are supported, respectively, in the ball $\mathcal{B} = \{\xi \in \mathbb{R}^3, |\xi| \leq \frac{4}{3}\}$ and the annulus $C = \{\xi \in \mathbb{R}^3, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$ such that

$$\chi(\xi) + \sum_{j\geq 0} \varphi(2^{-j}\xi) = 1, \quad \forall\, \xi \in \mathbb{R}^3 \quad \textit{and} \quad \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\xi) = 1, \quad \forall\, \xi \in \mathbb{R}^3\backslash\{0\}.$$

The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operator $\dot{S}_j$ are defined for all $j \in \mathbb{Z}$ by

$$\dot{\Delta}_j u = \varphi(2^{-j}D)u = 2^{jd} \int_{\mathbb{R}^3} h(2^j y) u(x-y) \mathrm{d}y,$$

$$\dot{S}_j u = \chi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^3}\widetilde{h}(2^j y)u(x-y)\mathrm{d}y,$$

where $h \overset{def}{=} \mathcal{F}^{-1}\varphi$ and $\widetilde{h} \overset{def}{=} \mathcal{F}^{-1}\chi$. Formally, $\dot{\Delta}_j = \dot{S}_{j+1} - \dot{S}_j$ is a frequency projection to annulus $|\xi| \sim 2^j$, and $\dot{S}_j$ is a frequency projection to the ball $\{|\xi| \le C2^j\}$. The homogeneous dyadic blocks Δ_j are defined by

$$\Delta_j u = 0 \quad if \quad j \le -2, \quad \Delta_{-1}u = \chi(D)u = \int_{\mathbb{R}^3}\widetilde{h}(y)u(x-y)\mathrm{d}y,$$
$$and \quad \Delta_j u = \varphi(2^{-j}D)u = 2^{jd}\int_{\mathbb{R}^3} h(2^j y)u(x-y)\mathrm{d}y \quad if \quad j \ge 0.$$

The homogeneous low-frequency cut-off operator S_j is defined by

$$S_j u = \sum_{j' \le j-1}\Delta_{j'}u.$$

First, we recall the definition of the homogeneous Besov space.

Definition A1. *(Besov Spaces) Let $\mathcal{S}'$ be the space of all tempered distributions. For s is a real number and (p,q) is in $[1,\infty]^2$, the homogeneous Besov space $\dot{B}^s_{p,q}$ is defined by*

$$\dot{B}^s_{p,q} = \{f \in \mathcal{S}'_h(\mathbb{R}^3) |\ \|f\|_{\dot{B}^s_{p,q}} < \infty\},$$

where

$$\mathcal{S}'_h = \{f \in \mathcal{S}'(\mathbb{R}^3) : f = \sum_{j\in\mathbb{Z}}\dot{\Delta}_j f\},$$

and

$$\|f\|_{\dot{B}^s_{p,q}} = \begin{cases} (\sum_{j\in\mathbb{Z}} 2^{sqj}\|\dot{\Delta}_j f\|^q_{L^p})^{\frac{1}{q}}, & \text{if} \quad q < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{js}\|\dot{\Delta}_j f\|_{L^p}, & \text{if} \quad q = \infty. \end{cases}$$

Now, we give the definition of the following so-called Bony decomposition.

Definition A2. *(Bony's para-product) The para-product of f and g is defined by*

$$T_g f = \sum_{j\in\mathbb{Z}} S_{j-1}g\Delta_j f = \sum_{j\in\mathbb{Z}}\sum_{i\le j-2}\Delta_i g\Delta_j f.$$

The remainder of f and g is defined by

$$R(f,g) = \sum_{j\in\mathbb{Z}}\Delta_i g\widetilde{\Delta}_j f, \quad where \quad \widetilde{\Delta}_j f = \sum_{j'=j-1}^{j'=j+1}\Delta_{j'}f.$$

Then, Bony's decomposition reads

$$fg = T_g f + T_f g + R(f,g).$$

Lemma A1. *(Bernstein's inequalities; see [25]) Let $\mathcal{B}$ be a ball and C be an annulus of $\mathbb{R}^3$. A constant C exists such that for positive real number λ and nonnegative integer k, any smooth homogeneous function σ of degree m, and any couple of real numbers (p, q) in $[1, \infty]^2$ with $q \geq p \geq 1$, we have*

$$Supp\hat{u} \subset \lambda\mathcal{B} \Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^q} \leq C^{k+1}\lambda^{k+3(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p},$$
$$Supp\hat{u} \subset \lambda C \Rightarrow C^{-(k+1)}\lambda^k\|u\|_{L^p} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p} \leq C^{k+1}\lambda^k\|u\|_{L^p},$$
$$Supp\hat{u} \subset \lambda C \Rightarrow \|\sigma(D)u\|_{L^q} \leq C_{\sigma,m}\lambda^{k+3(\frac{1}{p}-\frac{1}{q})}\|u\|_{L^p},$$

with $\sigma(D)u \stackrel{def}{=} \mathcal{F}^{-1}(\sigma\hat{u})$.

Definition A3. *(Chemin-Lerner type space; see [25]) For $T > 0$, $s \in \mathbb{R}$, and $0 \leq r, q \leq \infty$, we set*

$$\|u\|_{\widetilde{L}^r(0,T;\dot{B}^s_{p,q})} \stackrel{def}{=} \|2^{js}\|\dot{\Delta}_j u\|_{L^r_T(L^p)}\|_{l^q(\mathbb{Z})}.$$

We also need the following lemma to describe the action of the semigroup of the heat equation on distributions with Fourier transforms supported in an annulus.

Lemma A2 ([25])**.** *Let C be an annulus. Then, there exist positive constants c and C such that for any p in $[1, \infty]$ and any couple (t, λ) of positive real numbers, we have*

$$Supp\hat{u} \subset \lambda C \Rightarrow \|e^{t\Delta}u\|_{L^p} \leq Ce^{-ct\lambda^2}\|u\|_{L^p}.$$