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*Research article*

# A sub-super solution method to continuous weak solutions for a semilinear elliptic boundary value problems on bounded and unbounded domains

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**Abstract:** In this paper, we prove the existence of solutions for an elliptic system. More precisely, we combine the potential theory with the sub-super solution method and use the properties of the well-known Kato class to justify our existence results. The novelty of our study is that we consider either the bounded or the exterior domain; Also, the nonlinearities may be singular near the boundary. Some examples are presented to validate our main results.

**Keywords:** Green's function; elliptic systems; Schauder fixed point theorem; sub-supersolutions method

## 1. Introduction and main results

In recent years, semilinear elliptic boundary value problems have attracted more attention. This is due to their importance in several fields such as chemical reactions, population evolution, and pattern formation, see, for instance, [1] for other related applications. Due to their importance, several researchers have concentrated on the development of problems involving semilinear elliptic operators. In this paper, we will continue in this direction, so we fix a $C^{1,1}$ domain $\Lambda$ in $\mathbb{R}^N$, $(N \geq 3)$, with a nonempty compact boundary $\partial\Lambda$, and we consider the following problem:

$$\begin{cases} -\Delta u = \psi(x, u) \quad \text{in } \Lambda, \\ u = f, \quad \text{on } \partial\Lambda, \end{cases} \tag{1.1}$$

and in the case when $\Lambda$ is unbounded, we assume the following supplementary condition

$$\lim_{|x|\to\infty} u(x) = c. \tag{1.2}$$

Also, we consider the following semilinear elliptic system

$$\begin{cases} -\Delta u = \psi_1(x,u,v) & \text{in } \Lambda, \\ -\Delta v = \psi_2(x,u,v) & \text{in } \Lambda, \\ u = f_1, \; v = f_2 \;, & \text{on } \partial\Lambda, \end{cases} \tag{1.3}$$

and in the case when $\Lambda$ is unbounded, we assume the following supplementary condition

$$\lim_{|x|\to\infty} u(x) = c_1 \text{ and } \lim_{|x|\to\infty} v(x) = c_2, \tag{1.4}$$

where $c$, $c_1$, and $c_2$ are real numbers, and the functions $f$, $f_1$, and $f_2$ are continuous on $\partial\Lambda$. Problems like (1.1) or like (1.3) are extensively studied by several authors and by different methods. For interested readers, we refer to the works of, Akô [2] (Schauder's estimates in the Banach space of Hölder continuous functions), Alsaedi et al. [3] (combination of the Karamata regular variation theory with a related comparison principle), Amann [4] (fixed point index), Clément and Sweers [5] (monotone iterative and sub-super solution methods), Cui [6] (sub-super solution method and Sobolev-Morrey's inequality), Keller [7] (strong Maximum principle), Montenegro and Ponce [8] (method of sub-supersolutions combined with Schauder's fixed point theorem), Montenegro and Suárez [9] (adequate sub-super solution method for singular systems), [10] (iterative method combined with Schauder's type and Sobolev inequalities), Noussair and Swanson [11] (Atkinson's theorem), Ogata [12] (monotone arguments combined with iterative methods), Rădulecu and Repovš [13] (monotone argument combined with Variational method). Several works in the literature treat problems like (1.1) or problems like (1.3) in the case where the nonlinearities $\psi$, $\psi_1$, and $\psi_2$ are continuous concerning $x$ (in the case of classical solutions) or Caratheodory functions (in the case of weak solutions in the Sobolev spaces); Moreover, these problems are generally considered in regular bounded domains. Our goal in this paper is to ensure the validity of the sub-super solution method for continuous distributional solutions in the cases where $f$, $\psi_1$, and $\psi_2$ may be singular concerning $x$ near the boundary. This will be done using a fixed-point argument based on the compacity property of a class of potential functions defined in $\Lambda$. To be more precise, these singularities are related to the Kato class $K(\Lambda)$ which is introduced and studied by Bachar et al. [14] for the exterior domain, and by Mâagli and Zribi [15] for the bounded domain. This class is used to find solutions for several semilinear elliptic problems. We cite, for example, the papers of Alsaedi et al. [16], Bachar et al. [14], Ghanmi et al. [17], Mâagli and Zribi [15], and Zeddini and Sari [18]. Our results for (1.1) apply to prove the existence of continuous solutions for singular nonlinearities such that $|\psi(x,u)| \leq \frac{1}{(\delta(x)^\lambda)}|h(u)|$ on a $C^{1,1}$-bounded domain with $\delta(x) = dist(x,\partial\Lambda)$ and $\lambda < 2$, and in the case where $\Lambda$ is a $C^{1,1}$-exterior domain, our results for (1.1) apply to nonlinearities satisfying $|\psi(x,u)| \leq \frac{1}{(1+|x|)^{\mu-\lambda}(\delta(x)^\lambda)}|h(u)|$ with $\lambda < 2 < \mu$. Our results concerning sub-super-solution methods involving nonlinearities that may be singular near the boundary are new and have not been discussed before.

In this paper, we continue to study such problems using the Kato class. The novelty of our study is that we consider either the bounded or the exterior domains; moreover, the nonlinearities used in our problems can be singular, which means more complicated manipulation of our study. More precisely, we transform our problem to an equivalent integral equation, and after that, we define an associated operator, which is (using the properties of the Kato class) relatively compact. Finally, we prove that the fixed points of the associated operator are weak solutions for the studied problem.

Before giving the main results of this paper, we assume the following hypothesis:

**(H$_1$)** $\psi \in B(\Lambda \times \mathbb{R})$, such that for almost every $x \in \Lambda$, the function $t \to \psi(x, t)$ is continuous. Moreover, for all $C > 0$, there exists a function $p_C \in K(\Lambda)$ such that for any $t \in$ and $x \in \Lambda$, we have

$$|\psi(x, t)| \leq p_C(x), \;\; \forall (x, t) \in \Lambda \times [-C, C],$$

where $B(D \times \mathbb{R})$ and $K(\Lambda)$ are introduced in Section 2.

**(H$_2$)** For $i = 1, 2$, the function $\psi_i \in B(\Lambda \times \mathbb{R}$ and the map $(s, t) \to \psi_i(x, s, t)$ are continuous on $\mathbb{R} \times \mathbb{R}$ for almost every $x \in \Lambda$.

**(H$_3$)** For every $C > 0$ there exists a nonnegative function $p_{i,C}$ $(i = 1, 2)$, such that

$$|\psi_i(x, s, t)| \leq p_{i,C}(x), \;\; \forall \, (x, s, t) \in \Lambda \times [-C, C] \times [-C, C].$$

Our main results from this work are the following theorems:

**Theorem 1.1.** *Assume that the function $f$ is continuous on $\partial\Lambda$ and the hypothesis* (**H$_1$**) *is satisfied. If* (1.1) *and* (1.2) *have a continuous sub solution $\underline{u}$ and a continuous super solution $\overline{u}$ with $\underline{u} \leq \overline{u}$ in $\Lambda$, then problems* (1.1) *and* (1.2) *admit a continuous weak solution $u \in C(\Lambda \cup \partial^\infty\Lambda)$ satisfying*

$$\underline{u} \leq u \leq \overline{u}, \;\; in \;\; \overline{\Lambda}.$$

**Theorem 1.2.** *Assume that the functions $f_i (i = 1, 2)$ are continuous on $\partial\Lambda$ and the hypotheses* (**H$_2$**) *and* (**H$_3$**) *are satisfied. If* (1.3) *and* (1.4) *have a double pair of continuous sub-super solution $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$, then, problems* (1.3) *and* (1.4) *admit a continuous weak solution $(u, v) \in C(\Lambda \cup \partial^\infty\Lambda) \times C(\Lambda \cup \partial^\infty\Lambda)$ satisfying in addition*

$$\underline{u} \leq u \leq \overline{u}, \;\; and \;\; \underline{v} \leq v \leq \overline{v}, \;\; in \; \overline{\Lambda},$$

*where $\partial^\infty\Lambda$ and the notion of sub-super solution are introduced in Section 2.*

Next, in Section 2, we introduce some notations and we present several properties of the Kato class $K(\Lambda)$. Section 3 is devoted to the proofs of our main results. As applications of our main results, four examples are presented in Section 4 to validate the above theorems.

## 2. Notations and preliminaries on the Kato class

In this section, we begin by giving some notations, which will be used later in Section 3. We denote by $B(\Lambda)$ the set of all Borel measurable functions in $\Lambda$, by $B^+(\Lambda)$ the subset of all nonnegative functions of $B(\Lambda)$, and by $B_b(\Lambda)$ the subset of the bounded ones. Also, we denote by $C_0(\Lambda)$ the set of continuous functions in $\Lambda$ that tend to zero near $\partial\Lambda$ and satisfy, in addition, $\lim\limits_{|x|\to\infty} u(x) = 0$ in the case of unbounded domain. Now, let us denote by $C(\overline{\Lambda})$ the subset of $B(\Lambda)$ composed by continuous functions in $\overline{\Lambda}$ in the case when $\Lambda$ is bounded and in the case when $\Lambda$ is unbounded, $C(\overline{\Lambda} \cup \{\infty\})$ will denotes the subset of $B(\Lambda)$ composed of continuous functions in $\overline{\Lambda}$ for which the limit as $|x| \to \infty$ exists and is finite.

$\partial^\infty\Lambda$ will denotes $\partial\Lambda$ in the case when $\Lambda$ is bounded and $\partial\Lambda \cup \{\infty\}$ in the case when $\Lambda$ is unbounded. Consequently, $C\,(\Lambda \cup \partial^\infty\Lambda)$ will denote $C(\overline{\Lambda} \cup \{\infty\})$ if $\Lambda$ is unbounded and $C(\overline{\Lambda})$ if $\Lambda$ is bounded. Also, we denote by $\mathcal{D}(\Lambda)$ the set of all $C^\infty$-functions in $\Lambda$ with compact support in $\Lambda$. Now, we recall that the supremum norm is defined for $u \in C\,(\Lambda \cup \partial^\infty D)$ by

$$||u||_\infty = \sup_{x \in \overline{\Lambda}} |u(x)|,$$

furthermore it is well known that the normed space $(C_0(\Lambda), \|.\|\infty)$ is also a Banach space. The Green function of the Dirichlet Laplacian in $\Lambda$ will be denoted by $G^\Lambda$: moreover, for a given function $p$ in $B^+(\Lambda)$, the Green potential $Vp$ of a function $p$ is defined on $\Lambda$ as follows:

$$Vp(x) = \int_\Lambda G^\Lambda(x,y)p(y)\,dy.$$

It is well known (see [19] p.52) that if $p \in L^1_{loc}(\Lambda)$ is such that $Vp \in L^1_{loc}(\Lambda)$, then ( in the sense of distributions) we have

$$\Delta(Vp) = -p \text{ in } \Lambda. \tag{2.1}$$

Let $f$ be a nonnegative continuous function on $\partial\Lambda$, then, $H_\Lambda f$ will denotes the unique solution in $C^2(\Lambda) \cap C(\overline{\Lambda})$ of the following problem

$$\begin{cases} \Delta u = 0 & \text{in } \Lambda \\ u = f & \text{on } \partial\Lambda, \end{cases}$$

and in the case of an unbounded domain, the above problem is subject to the following condition:

$$\lim_{|x|\to\infty} u(x) = 0.$$

Hereafter, $\xi$ will denote the following function

$$\xi = 1 - H_\Lambda 1.$$

It is not difficult to see that the function $\xi$ is harmonic and equal to zero at the boundary of $\Lambda$; moreover, if $\Lambda$ is unbounded, then $\lim\limits_{|x|\to\infty} \xi(x) = 1$.

Since we use the potential theory, it is natural to define the Kato class, which is defined in the following definition.

**Definition 2.1.** (See [14, 15].) A function $p \in B(\Lambda)$ is said to be in the Kato class $K(\Lambda)$ if we have

$$\lim_{\sigma\to 0} \sup_{x\in\Lambda} \int_{\Lambda\cap D(x,\sigma)} \frac{\rho(y)}{\rho(x)} G^\Lambda(x,y)|p(y)|dy = 0,$$

in the case when $\Lambda$ is bounded, and in addition, in the case of an unbounded domain, we have

$$\lim_{C\to\infty} \sup_{x\in\Lambda} \int_{\Lambda\cap\{|y|\geq C\}} \frac{\rho(y)}{\rho(x)} G^\Lambda(x,y)|p(y)|dy = 0,$$

where $D(x,\alpha)$ is the open ball with center $x$ and radius $\alpha$, $\delta(x) = d(x, \partial\Lambda)$, and $\rho(x) = \min(1, \delta(x))$.

We note that if $\Lambda$ is bounded, then we will use the following elementary inequality:

$$\frac{1}{1+d}\delta(x) \leq \rho(x) \leq \delta(x),$$

where $d$ is the diameter of $\Lambda$. This means that we can replace $\rho$ by $\delta$ in Definition 2.1. Moreover, Zeddini and Sari [18] proved that in the case when $\Lambda$ is a $C^{1,1}$-bounded domain, then, this definition is equivalent to

$$\lim_{\sigma\to 0}\left( \sup_{(x,y)\in\Lambda\times\Lambda} \int_{\Lambda\cap(D(x,\sigma)\cup D(y,\sigma))} \frac{G^\Lambda(x,z)G^\Lambda(z,y)}{G^\Lambda(x,y)}|p(z)|dz\right) = 0.$$

The Kato class $K(\Lambda)$ is very important in the manipulation of the Green potential, it is quite rich and it contains several functions, as shown in the following example.

**Examples.** *(see [14, 15].)*

1) *If $\Lambda$ is bounded, then the function $x \mapsto \frac{1}{(\delta(x))^{\lambda}}$ is in $K(\Lambda)$ if and only if $\lambda \in (-\infty, 2)$.*
2) *In the case when $\Lambda$ is the open unit ball, then a radial function $p$ is in $K(\Lambda)$ if and only if $\int_0^1 r(1-r)|p(r)|\,dr < \infty$.*
3) *If $\Lambda$ is an exterior domain. Then the function $x \to \frac{1}{(|x|+1)^{\mu-\lambda}\,\delta(x)^{\lambda}}$ is in $K(\Lambda)$ if and only if $\lambda < 2 < \mu$.*
4) *If $\Lambda$ is the exterior of the unit closed ball, a radial function $p$ is in $K(\Lambda)$ if and only if $\int_1^{\infty}(r-1)\,|p(r)|\,dr < \infty$.*

**Remark 2.2.** *If $\Lambda$ is a $C^{1,1}$ exterior domain and a function $q$ is nontrivial and nonnegative in $K(\Lambda)$. Then its Green potential, $Vq$, is positive in $\Lambda$. Indeed, $q \in L^1_{loc}(\Lambda)$, moreover, there exists a compact subset $F$ of $\Lambda$ such that*

$$0 < \int_F q(y)\,dy < \infty.$$

*Without loss of generality, we can assume that $0 \notin \Lambda$. Then it has been proved in Bachar et al. [14], that there exists $C > 0$ such that*

$$C\frac{\delta(x)}{|x|^{N-1}}\,\frac{\delta(y)}{|y|^{N-1}} \le G^{\Lambda}(x,y),\ \forall\ (x,y) \in \Lambda^2.$$

*Hence, for every $x \in \Lambda$, we have*

$$\begin{aligned} Vq(x) &= \int_{\Lambda} G^{\Lambda}(x,y)q(y)\,dy \\ &\ge C\frac{\delta(x)}{|x|^{N-1}}\int_F \frac{\delta(y)}{|y|^{N-1}}q(y)\,dy \\ &\ge C\frac{\delta(x)}{|x|^{N-1}}\inf_{z\in F}\left(\frac{\delta(z)}{|z|^{N-1}}\right)\int_F q(y)\,dy > 0. \end{aligned}$$

Next, to prove the existence of solutions, we use the sub-super solution method, so in the following, we define such a notion.

**Definition 2.3.** A function $\underline{u} \in C(\Lambda \cup \partial^{\infty}\Lambda)$ is said to be a continuous sub-solution of the problems (1.1) and (1.2) if the following statements are true:

$(i)$ $\lim\limits_{x \longrightarrow \xi \in \partial\Lambda} \underline{u}(x) \le f(\xi)$, and in the case of an unbounded domain, it satisfies, in addition, $\lim\limits_{|x|\longrightarrow\infty} \underline{u}(x) \le c$.

$(ii)$ For any nonnegative function $\varphi \in \mathcal{D}(\Lambda)$, we get

$$\int_{\Lambda} \underline{u}(x)\Delta\varphi(x) + \psi(x, \underline{u}(x))\varphi(x)dx \ge 0.$$

The definition of a super-solution of the problems (1.1) and (1.2) is obtained similarly by reversing the inequality in the last definition.

An analog definition is adopted for the problems (1.3) and (1.4). This type of definition is utilized by Gfaifia et al. [20], and Pao [21]. Next, we recall this general definition.

**Definition 2.4.** $((\underline{u},\underline{v}),(\overline{u},\overline{v})) \in (C(D \cup \partial^{\infty}\Lambda))^2 \times (C(D \cup \partial^{\infty}\Lambda))^2$ is said to be a sub-super solution of problems (1.3) and (1.4) if the following statements hold:

(*i*) For all $x \in \Lambda$, we have

$$\underline{u}(x) \leq \overline{u}(x), \text{ and } \underline{v}(x) \leq \overline{v}(x).$$

(*ii*) We have

$$\begin{cases} \lim\limits_{x \longrightarrow \xi \in \partial\Lambda} \underline{u}(x) \leq f_1(\xi) \leq \lim\limits_{x \longrightarrow \xi \in \partial\Lambda} \overline{u}(x) \\ \lim\limits_{x \longrightarrow \xi \in \partial\Lambda} \underline{v}(x) \leq f_2(\xi) \leq \lim\limits_{x \longrightarrow \xi \in \partial\Lambda} \overline{v}(x), \end{cases}$$

and if in addition $\Lambda$ is unbounded, then we get

$$\begin{cases} \lim\limits_{|x| \longrightarrow \infty} \underline{u}(x) \leq c_1 \leq \lim\limits_{|x| \longrightarrow \infty} \overline{u}(x), \\ \lim\limits_{|x| \longrightarrow \infty} \underline{v}(x) \leq c_2 \leq \lim\limits_{|x| \longrightarrow \infty} \overline{v}(x). \end{cases}$$

(*iii*) For any nonnegative function $\varphi \in \mathcal{D}(\Lambda)$ and any $(u, v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$, we have

$$\begin{cases} \int_D \underline{u}(x)\Delta\varphi(x) + \psi_1(x, \underline{u}(x), v(x))\varphi(x)dx \geq 0, \\ \int_D \underline{v}(x)\Delta\varphi(x) + \psi_2(x, u(x), \underline{v}(x))\varphi(x)dx \geq 0, \end{cases}$$

moreover, the last inequalities hold by reversing the inequality and replacing $\underline{u}$ and $\underline{v}$ by $\overline{u}$ and $\overline{v}$, respectively.

## 3. Proofs of the main results

In this section, we present the proofs of our main results ( Theorems 1.1 and 1.2). Firstly, let us introduce the following key result, which can be found in Mâagli and Zribi [15] in the case when $\Lambda$ is bounded, and in Bachar et al. [14] in the case when $\Lambda$ is unbounded.

**Proposition 3.1.** *For a given function $p$ in $K(\Lambda)$. The following statements hold:*

1) *$Vp$ is a continuous functions in $\Lambda$ and tends to zero on $\partial\Lambda$.*
2) *The family of functions*

$$\{Vq \text{ such that } |q| \leq |p|\},$$

*is equicontinuous in $\Lambda \cup \partial^\infty\Lambda$.*

3) $p \in L^1_{loc}(\Lambda)$.

We note that from the well-known Ascoli's theorem and (2) in the previous proposition, we deduce that $\{Vq$ such that $|q| \leq |p|\}$ is relatively compact in $C_0(\Lambda)$.

### *3.1. Proof of Theorem 1.1*

In order to prove Theorem 1.1, we begin by defining the auxiliary function $\widetilde{\psi}$ on $\Lambda \times \mathbb{R}$ by

$$\widetilde{\psi}(x, u(x)) = \begin{cases} \psi(x, \underline{u}(x)) & \text{if } u(x) < \underline{u}(x), \\ \psi(x, u(x)) & \text{if } \underline{u}(x) \leq u(x) \leq \overline{u}(x), \\ \psi(x, \overline{u}(x)) & \text{if } \overline{u}(x) < u(x). \end{cases} \tag{3.1}$$

Now, let us consider the following associated problem

$$\begin{cases} -\Delta u = \widetilde{\psi}(x,u) \quad \text{in } \Lambda, \\ u = f\ , \quad \text{on } \partial\Lambda \\ \lim\limits_{|x|\to\infty} u(x) = c. \end{cases} \tag{3.2}$$

We begin by proving that problem (3.2) admits a weak solution $u$ in $C(\Lambda \cup \partial^{\infty}\Lambda)$. Since the function $t \mapsto \psi(x,t)$ is continuous, the function $t \mapsto \widetilde{\psi}(x,t)$ is also continuous. On the other hand, if we put $C = \|\underline{u}\|_{\infty} + \|\overline{u}\|_{\infty}$, then from hypothesis $(\mathbf{H_1})$ there exists a nonnegative function $p_C \in K(\Lambda)$ such that

$$|\psi(x,t)| \leq p_C(x),\ \forall\ (x,t) \in \Lambda \times [-C, C].$$

Therefore, we deduce that

$$|\widetilde{\psi}(x,u(x))| \leq p_C(x),\ \forall\, (x,u) \in \Lambda \times C(\Lambda \cup \partial^{\infty}\Lambda).$$

So, Proposition 3.1 implies that

$$\widetilde{\psi}(.,c\,\xi(.) + H_{\Lambda}\phi(.) + v(.)) \in K(\Lambda),$$

moreover, for each $v \in C_0(\Lambda)$ the family

$$\left\{V\left(\widetilde{\psi}(.,c\,\xi + H_{\Lambda}\phi + v)\right) : v \in C_0(\Lambda)\right\},$$

is relatively compact in $(C_0(\Lambda), \|.\|_{\infty})$.

Now we define the operator $T : C_0(\Lambda) \to C_0(\Lambda)$ by

$$Tv(x) = V\left(\widetilde{\psi}(.,c\,\xi + H_{\Lambda}\phi + v)\right)(x).$$

It is clear from the above information that $T(C_0(\Lambda))$ is relatively compact in $(C_0(\Lambda), \|.\|_{\infty})$. On the other hand, the operator $T$ is continuous. Indeed, let $\{v_n\}$ be a sequence in $C_0(\Lambda)$ that converges uniformly to $v \in C_0(\Lambda)$. Since we have

$$|\widetilde{\psi}(y, c\,\xi + H_{\Lambda}\phi + v_n(y))| \leq p_M(y),\ \forall\, (y,n) \in \Lambda \times \mathbb{N},$$

then, by combining the dominated convergence theorem and Proposition 3.1 with the continuity of $\psi$ concerning the second variable, we deduce

$$\begin{cases} \lim\limits_{n\to\infty} Tv_n(x) = Tv(x),\ \forall\, x \in \Lambda, \\ \\ \lim\limits_{\|x\|\to\infty} Tv_n(x) = 0, \quad \text{uniformly in } n. \end{cases}$$

From the equicontinuity of $T(C_0(\Lambda))$, we deduce that the pointwise convergence implies uniform convergence. This fact implies that $Tv_n$ converges uniformly to $Tv$ in $C_0(\Lambda)$. Which implies that $T$ is continuous. By combining this fact and the fact that $T(C_0(\Lambda))$ is relatively compact in $(C_0(\Lambda), \|.\|_{\infty})$

with the Schauder fixed point theorem, we deduce that $T$ has a fixed point $v \in C_0(\Lambda)$. Now if we put $u = c\xi + H_\Lambda f + v$, then we get $u \in C(\Lambda \cup \partial^\infty \Lambda)$ and

$$\begin{cases} \lim\limits_{x \to \sigma \in \partial\Lambda} u(x) = f(\sigma), \\ \lim\limits_{|x| \to \infty} u(x) = c, \end{cases}$$

moreover, $u$ satisfies the following integral equation:

$$u = c\xi + H_\Lambda f + V\left(\widetilde{\psi}(., u)\right) \text{ in } \Lambda.$$

Next, let us prove that the function $u$ satisfies the following inequality

$$\underline{u}(x) \le u(x) \le \overline{u}(x), \;\forall\, x \in \overline{\Lambda}.$$

To do this, we proceed by contradiction, and we suppose that there exists $x_0 \in \Lambda$ such that $u(x_0) < \underline{u}(x_0)$. In this case, we define the following set:

$$E = \{x \in \Lambda \text{ such that } u(x) < \underline{u}(x)\}.$$

Then, from our assumption, the set $E$ is nonempty; moreover, from the fact that the functions $u$ and $\underline{u}$ are continuous, we see that $E$ is an open set in $\Lambda$. Moreover, we have $u - \underline{u} = 0$ on $\partial E$. On the other hand, from Eq (3.1), we can see that for all $x \in E$, we have $\widetilde{\psi}(x, u(x)) = \widetilde{\psi}(x, \underline{u}(x))$. Hence, for any nonnegative function $\varphi$ in $\mathcal{D}(E)$, we have

$$\int_E (u(x) - \underline{u}(x))(-\Delta\varphi)(x)dx \ge \int_E \left(\widetilde{\psi}(x, u(x)) - \widetilde{\psi}(x, \underline{u}(x))\right)\varphi(x)dx = 0.$$

The above information shows that $u - \underline{u}$ is a continuous superharmonic function in $E$ with a boundary value equal to zero. On the other hand, from the definition of the sub-solution, in the case when $E$ is unbounded, we have

$$\lim_{|x| \to \infty,\, x \in E} \left(u(x) - \underline{u}(x)\right) = c - \lim_{|x| \to \infty,\, x \in E} \underline{u}(x) \;\ge 0.$$

Hence the maximum principle (see [22, p.397–398]) can be applied, and we conclude that $u - \underline{u} \ge 0$ in $E$. This contradicts the fact that $E$ is nonempty. So $\underline{u} \le u$ in $\overline{\Lambda}$. Similar arguments can be used to show that $u \le \overline{u}$ in $\overline{\Lambda}$. Now, the fact that $\underline{u}(x) \le u(x) \le \overline{u}(x)$, implies that for any $x \in \Lambda$ we have $\widetilde{\psi}(x, u(x)) = \psi(x, u(x))$. Finally, since $u$ is a solution for the problem (3.1) and since $\widetilde{\psi}(x, u(x)) = \psi(x, u(x))$, then $u$ is also a solution for the problem (1.1). The proof is now finished.

### *3.2. Proof of Theorem 1.2*

We suppose that problems (1.3) and (1.4) admit a double pair of continuous sub-supersolutions $(\underline{u}, \underline{v})$ and $(\overline{u}, \overline{v})$. As in the proof of Theorem 1.1, for $y \in C(\overline{\Lambda} \cup \{\infty\})$ and $x \in \overline{\Lambda}$, we define

$$\theta_1(y)(x) = \begin{cases} \underline{u}(x) & \text{if } y(x) < \underline{u}(x), \\ y(x) & \text{if } \underline{u}(x) \le y(x) \le \overline{u}(x), \\ \overline{u}(x) & \text{if } \overline{u}(x) < y(x), \end{cases}$$

and

$$\theta_2(y)(x) = \begin{cases} \underline{v}(x) & \text{if } y(x) < \underline{v}(x), \\ y(x) & \text{if } \underline{v}(x) \le y(x) \le \overline{v}(x), \\ \overline{u}(x) & \text{if } \overline{v}(x) < y(x). \end{cases}$$

Clearly

$$\theta_1(y) \in [\underline{u}, \overline{u}], \text{ and } \theta_2(y) \in [\underline{v}, \overline{v}].$$

On the other hand, by the fact that the functions $\underline{u}, \underline{v}, \overline{u}$, and $\overline{v}$ are in $C\left(\overline{\Lambda} \cup \{\infty\}\right)$, we deduce the existence of a positive constant $C$, such that for any $x \in \overline{\Lambda}$, we have

$$\begin{cases} -C \le \underline{u}(x) \le \overline{u}(x) \le C, \\ -C \le \underline{v}(x) \le \overline{v}(x) \le C. \end{cases}$$

So, from $(\mathbf{H_3})$, there exist two nonnegative functions $p_{1,C}$, $p_{2,C}$ in $K(\Lambda)$ such that for all $(x, y) \in \Lambda \times C(\overline{\Lambda} \cup \{\infty\})$, we have

$$\begin{cases} |\psi_1(x, \theta_1(y)(x), \theta_2(y)(x))| \le p_{1,C}(x), \\ |\psi_2(x, \theta_1(y)(x), \theta_2(y)(x))| \le p_{2,C}(x). \end{cases}$$

Hence, from Proposition 3.1, we deduce that the map $\omega \to \psi_i(\omega, \theta_1(y)(\omega), \theta_2(y)(\omega))$ is in $K(\Lambda)$, moreover, for $i \in \{1, 2\}$, the set

$$\left\{V\left(\psi_i(., \theta_1(y), \theta_2(y))\right) : y \in C(\overline{\Lambda} \cup \{\infty\})\right\},$$

is relatively compact in $(C_0(\Lambda), ||.||_\infty)$.

Now, we consider the Banach space $C_0(\Lambda) \times C_0(\Lambda)$, which is equipped with the following norm

$$||(\chi_1, \chi_2)|| = ||\chi_1||_\infty + ||\chi_2||_\infty,$$

and we define the operator $T : C_0(\Lambda) \times C_0(\Lambda) \to C_0(\Lambda) \times C_0(\Lambda)$ by

$$T(\chi_1, \chi_2) = \left(T_1(\chi_1, \chi_2), T_2(\chi_1, \chi_2)\right),$$

where $T_1(\chi_1, \chi_2)$ and $T_2(\chi_1, \chi_2)$ are defined by

$$\begin{cases} T_1(\chi_1, \chi_2) = V\psi_1\left(., \theta_1[c_1\xi + H_\Lambda f_1 + \chi_1], \theta_2[c_2\xi + H_\Lambda f_2 + \chi_2]\right)(x), \\ T_2(\chi_1, \chi_2) = V\psi_2\left(., \theta_1[c_1\xi + H_\Lambda f_1 + \chi_1], \theta_2[c_2\xi + H_\Lambda f_2 + \chi_2]\right)(x). \end{cases}$$

We note that $T_1(\chi_1, \chi_2)$ and $T_2(\chi_1, \chi_2)$ are the unique pair of solutions to the following problem:

$$\begin{cases} -\Delta y = \psi_1\left(x, \theta_1[c_1\xi + H_\Lambda f_1 + \chi_1], \theta_2[c_2\xi + H_\Lambda f_2 + \chi_2]\right) & \text{in } \Lambda, \\ -\Delta z = \psi_1\left(x, \theta_1[c_1\xi + H_\Lambda f_1 + \chi_1], \theta_2[c_2\xi + H_\Lambda f_2 + \chi_2]\right) & \text{in } \Lambda, \\ y = 0, \ z = 0, \quad \text{on } \partial\Lambda \\ \lim\limits_{|x|\to\infty} y(x) = 0 \text{ and } \lim\limits_{|x|\to\infty} z(x) = 0. \end{cases}$$

The same arguments as in the proof of Theorem 1.1, show that the set $T\left(C_0(\Lambda) \times C_0(\Lambda)\right)$ is relatively compact in $C_0(\Lambda) \times C_0(\Lambda)$, and moreover, the operator $T$ is continuous. So, the Schauder fixed point theorem implies that $T$ has a fixed point $(\chi_1, \chi_2) \in C_0(\Lambda) \times C_0(\Lambda)$.

Now, if we put

$$\begin{cases} u = c_1 \xi + H_\Lambda f_1 + \chi_1, \\ v = c_2 \xi + H_\Lambda f_2 + \chi_2, \end{cases}$$

then, $u\,,\ v \in C\left(\overline{\Lambda} \cup \{\infty\}\right)$, moreover, we have

$$\begin{cases} \lim\limits_{x\to\sigma\in\partial\Lambda} u(x) = f_1(\sigma), \\ \lim\limits_{x\to\sigma\in\partial\Lambda} v(x) = f_2(\sigma), \end{cases}$$

and

$$\begin{cases} \lim\limits_{|x|\to\infty} u(x) = c_1, \\ \lim\limits_{|x|\to\infty} v(x) = c_2. \end{cases}$$

Also, we have

$$\begin{cases} u(x) = c_1\, \xi(x) + H_\Lambda f_1(x) + V\psi_1\left(.,\theta_1(u),\theta_2(v)\right)(x), \\ v(x) = c_2\, \xi(x) + H_\Lambda f_2(x) + V\psi_2\left(.,\theta_1(u),\theta_2(v)\right)(x). \end{cases}$$

Finally, we will prove that $\underline{u} \leq u \leq \overline{u}$ and $\underline{v} \leq v \leq \overline{v}$ in $\overline{\Lambda}$. Since the proofs are similar for the cases $\underline{u} \leq u$, $u \leq \overline{u}$, $\underline{v} \leq v$, and $v \leq \overline{v}$, we will only prove that $\underline{u} \leq u$. By contradiction, we assume that this is not true, so the set $E_1 = \{x \in \Lambda \text{ such that } u(x) < \underline{u}(x)\}$ is nonempty. Since $u$ and $\underline{u}$ are continuous, then $E_1$ is open; $u - \underline{u} = 0$ on $\partial E_1$, and in the case when $E_1$ is unbounded, we have

$$\lim_{|x|\to\infty\,,x\in E_1} (u(x) - \underline{u}(x)) = c - \lim_{|x|\to\infty\,,x\in E_1} \underline{u}(x) \geq 0.$$

Moreover since $\theta_1(u)(x) = \underline{u}(x)$ for every $x \in E_1$ and $\theta_2(v) \in \left[\underline{v}, \overline{v}\right]$, then for every $\varphi \in C_c^\infty(E_1)$ with $\varphi \geq 0$ we have

$$\begin{aligned} \int_{E_1} \left[u(x) - \underline{u}(x)\right] \Delta\varphi(x)dx &= \int_{E_1} u(x)\Delta\varphi(x)dx - \int_{E_1} \underline{u}(x)\Delta\varphi(x)dx \\ &= -\int_{E_1} \psi_1\left(x, \theta_1(u)(x), \theta_2(v)(x)\right)\varphi(x)dx \\ &\quad -\int_{E_1} \underline{u}(x)\Delta\varphi(x)dx \\ &\geq -\int_{E_1} \psi_1\left(x, \underline{u}(x), \theta_2(v)(x)\right)\varphi(x)dx \\ &\quad +\int_{E_1} \psi_1\left(x, \underline{u}(x), \theta_2(v)(x)\right)\varphi(x)dx = 0. \end{aligned}$$

The above information shows that $u - \underline{u}$ is a continuous super-harmonic function in $E_1$ and satisfies $u - \underline{u} \geq 0$ on $\partial E_1$ and $\lim\limits_{|x|\to\infty\,,x\in E_1} u(x) - \underline{u}(x) \geq 0$, if $E_1$ is unbounded. Hence, from the maximum principle [22, p.397–398] we deduce that $u - \underline{u} \geq 0$ in $E_1$. This contradicts the definition of $E_1$ and so $E_1$ is empty. Which proves that $\underline{u} \leq u$ in $\Lambda$. Consequently, we conclude that $\theta_1(u) = u$, $\theta_2(v) = v$, and $(u, v)$ is a continuous weak solution of problem (1.3). The proof of Theorem 1.2 is now completed.

## 4. Applications

In this section, we present several applications of the main results. These applications approve and validate the main results of this paper. Since the case of the unbounded domain is more general than the bounded domain, throughout this section, we assume that $\Lambda$ is a $C^{1,1}$ exterior domain, and the letter $i$ will denote the integer 1 or 2.

**Application 1.** Let $\alpha > 0$ and $a$ be a nontrivial nonnegative function in $B(K(\Lambda))$, and consider the following problem:

$$\begin{cases} -\Delta u = \lambda a(x)\,(1-u^{\alpha}) & \text{in } \Lambda\,, \\ u = 0\,, \quad \text{on } \partial\Lambda\,, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases} \tag{4.1}$$

By the fact that $a \in K(\Lambda)$, we deduce that $Va \in C_0(\Lambda)$. So if we put $\overline{u} = \lambda Va$, then it is not difficult to see that for $\lambda > 0$, the function $\overline{u}$ is a continuous super-solution of problem (4.1). On the other hand, if we define the function $f$ on $\left[0, \frac{1}{\|Va\|_\infty}\right)$ by

$$f(t) = \frac{t}{1 - \|Va\|_\infty^{\alpha}\, t^{\alpha}}.$$

Then it is not difficult to see that the function $f$ is differentiable and increasing on $\left(0, \frac{1}{\|Va\|_\infty}\right)$, and since $f(0) = 0$ and $\lim\limits_{t\to\frac{1}{\|Va\|_\infty}} f(t) = \infty$, then the function $f : \left[0, \frac{1}{\|Va\|_\infty}\right) \to [0,\infty)$ is a bijection.

Now, if $\lambda > 0$, then there exists $\varepsilon \in \left(0, \frac{1}{\|Va\|_\infty}\right)$ such that $0 < \varepsilon < f(\varepsilon) < \lambda$, which implies that

$$\varepsilon < \lambda\,(1 - \varepsilon^{\alpha}\|Va\|_\infty^{\alpha}) < \lambda\,(1 - (\varepsilon Va)^{\alpha})\,.$$

So, $\varepsilon\, a \le \lambda\, a\,(1 - (\varepsilon Va)^{\alpha})$ in $\Lambda$, and the function $\underline{u} = \varepsilon Va$ becomes a continuous weak sub-solution of (4.1) satisfying $\underline{u} \le \overline{u}$. Hence, it follows from Theorem 1.1 that problem (4.1) has a continuous weak solution $u$ satisfying $\varepsilon\, Va \le u \le \lambda Va$.

**Application 2.** Let $\alpha > 0$ and let $a$ and $b$ be two nontrivial nonnegative functions in $B(K(\Lambda))$, and we consider the following problem

$$\begin{cases} -\Delta u = \lambda\,(a(x) + b(x)u^{\alpha}) & \text{in } \Lambda\,, \\ u = 0\,, \quad \text{on } \partial\Lambda\,, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases} \tag{4.2}$$

We begin by remarking that, similar to Application 1, for any $\lambda > 0$, the function $\underline{u} = \lambda Va$ is a continuous sub-solution of (4.2). So to use the main theorems of this paper, we will find a positive continuous weak supersolution of (4.2).

Next, if we define the function $g$ on $[0,\infty)$ by

$$g(t) = \frac{t}{1 + \|V(a+b)\|_\infty^{\alpha}\, t^{\alpha}}.$$

Then a simple calculation shows that $g$ is differentiable on $(0, \infty)$ and

$$g'(t) = \frac{1 + (1 - \alpha)\|V(a + b)\|_\infty^\alpha t^\alpha}{(1 + \|V(a + b)\|_\infty^\alpha t^\alpha)^2}. \tag{4.3}$$

To discuss the monotonicity of the function $g$, we distinguish two cases.

**Case 1:** In this case, we consider the sublinear case, which means that $\alpha \in (0, 1)$. In this case, we see from (4.3) that $g$ is increasing, and so, it is a bijection from $[0, \infty)$ into $[0, \infty)$. Hence, for any $\lambda > 0$, there exists $M \in (0, \infty)$ such that $0 < \lambda < g(M) < M$. Now, it is not difficult to see that the function $\overline{u} = MV(a + b)$ is a continuous super-solution of (4.2), which satisfies in addition $\underline{u} \leq \overline{u}$. Finally, Theorem 1.1 implies that (4.2) has a continuous weak solution $u$; moreover, we have

$$\lambda V(a) \leq u \leq MV(a + b).$$

**Case 2:** In this case, we consider the super-solution case,which is the case when $\alpha \geq 1$. In this case, easily, the function $g$ is increasing on $[0, t_0]$ and decreasing on to $[t_0, \infty)$ and satisfies $g(0) = 0$ and $\lim_{t\to\infty} g(t) = 0$, where

$$t_0 = \frac{1}{(\alpha - 1)^{\frac{1}{\alpha}} \|V(a + b)\|_\infty}.$$

Put

$$\lambda_* = g(t_0) = \frac{(\alpha - 1)^{1-\frac{1}{\alpha}}}{\alpha\|V(a + b)\|_\infty}.$$

Then, for any $\lambda$ in $(0, \lambda_*]$, we see that $\lambda < t_0$. So the function $\overline{u} = t_0V(a + b)$ is a continuous super-solution to problem (4.2), which satisfies in addition $\underline{u} \leq \overline{u}$. Again, Theorem 1.1 implies that (4.2) has a continuous weak solution satisfying

$$\lambda Va \leq u \leq t_0V(a + b).$$

**Application 3.** For a positive real numbers $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, and a nontrivial nonnegative functions $a$, $b$ in $B(K(\Lambda))$, we consider the following system:

$$\begin{cases} -\Delta u = \lambda a(x)\left(1 - u^{\alpha_1}v^{\beta_1}\right) & \text{in } \Lambda\,, \\ -\Delta v = \mu b(x)\left(1 - u^{\alpha_2}v^{\beta_2}\right) & \text{in } \Lambda\,, \\ u = v = 0\,, \quad \text{on } \partial\Lambda\,, \\ \lim\limits_{|x|\to\infty} u(x) = \lim\limits_{|x|\to\infty} v(x) = 0 = 0. \end{cases} \tag{4.4}$$

We begin by remarking that for every $\lambda > 0$, $\mu > 0$, the function

$$f_i : \left[0, \frac{1}{\|Va\|_\infty^{\frac{\alpha_i}{\alpha_i+\beta_i}} \|Vb\|_\infty^{\frac{\beta_i}{\alpha_i+\beta_i}}}\right) \to [0, \infty),$$

defined by:

$$f_i(t) = \frac{t}{1 - \|Va\|_\infty^{\alpha_i}\|Vb\|_\infty^{\beta_i}\, t^{\alpha_i+\beta_i}},$$

is an increasing bijection. So, there exists $\varepsilon > 0$ such that

$$0 < \varepsilon < f_i(\varepsilon) < \min(\lambda, \mu).$$

From which we deduce that

$$\varepsilon a \leq \lambda a\left(1 - (\varepsilon Va)^{\alpha_1}(\varepsilon Vb)^{\beta_1}\right), \text{ and } \varepsilon b \leq \mu b\left(1 - (\varepsilon Va)^{\alpha_2}(\varepsilon Vb)^{\beta_2}\right).$$

This allows us to prove that the double pair $(\underline{u}, \underline{v}) = (\varepsilon Va, \varepsilon Vb)$, $(\overline{u}, \overline{v}) = (\lambda Va, \mu Vb)$ is a continuous sub-supersolution of system (4.4). Hence, it follows from Theorem 1.2 that (4.4) has a continuous weak solution $(u, v)$ satisfying $\varepsilon Va \leq u \leq \lambda Va$ and $\varepsilon Vb \leq v \leq \mu Vb$.

**Application 4.** For a positive real numbers $\alpha_1$, $\alpha_2$, $\beta_1$, $\beta_2$, and a nontrivial nonnegative functions $a$, $b$ in $B(K(\Lambda))$, we consider the following system:

$$\begin{cases} -\Delta u = \lambda\left(a_1(x) + b_1(x)u^{\alpha_1}v^{\beta_1}\right) & \text{in } \Lambda, \\ -\Delta v = \mu\left(a_2(x) + b_2(x)u^{\alpha_2}v^{\beta_2}\right) & \text{in } \Lambda, \\ u = v = 0, \quad \text{on } \partial\Lambda, \\ \lim_{|x|\to\infty} u(x) = \lim_{|x|\to\infty} v(x) = 0. \end{cases} \tag{4.5}$$

We define the function $g_i$ on $[0, \infty)$ by

$$g_i(t) = \frac{t}{1 + \|V(a_1 + b_1)\|_\infty^{\alpha_i} \|V(a_2 + b_2)\|_\infty^{\beta_i} t^{\alpha_i+\beta_i}}.$$

Clearly, $g_i$ is differentiable on $[0, \infty)$ and

$$g_i'(t) = \frac{1 + (1 - \alpha_i - \beta_i)\|V(a_1 + b_1)\|_\infty^{\alpha_i} \|V(a_2 + b_2)\|_\infty^{\beta_i} t^{\alpha_i+\beta_i}}{\left(1 + \|V(a_1 + b_1)\|_\infty^{\alpha_i} \|V(a_2 + b_2)\|_\infty^{\beta_i} t^{\alpha_i+\beta_i}\right)^2}.$$

Thus we will discuss four cases.

**Case 1:** $0 < \alpha_i + \beta_i < 1$. In this case, each $g_i$ is an increasing bijection from $[0, \infty)$ to $[0, \infty)$. Hence, for every $0 < \lambda$ and $0 < \mu$, there exists $M \in (0, \infty)$ such that $0 < \max(\lambda, \mu) < \min(g_1(M), g_2(M)) < M$. The double pair of continuous functions $(\underline{u}, \underline{v}) = (\lambda Va_1, \mu Va_2)$ and $(\overline{u}, \overline{v}) = (MV(a_1 + b_1), MV(a_2 + b_2))$ is a continuous sub-supersolution of system (4.5). Hence, it follows from Theorem 1.2 that (4.5) has a continuous weak solution $(u, v)$ satisfying $\lambda Va_1 \leq u \leq MV(a_1 + b_1)$ and $\mu Va_2 \leq v \leq MV(a_2 + b_2)$.

**Case 2**: $\alpha_i + \beta_i \geq 1$. If we put

$$t_i = \frac{1}{(\alpha_i + \beta_i - 1)^{\frac{1}{\alpha_i+\beta_i}} \|V(a_1 + b_1)\|_\infty^{\frac{\alpha_i}{\alpha_i+\beta_i}} \|V(a_2 + b_2)\|_\infty^{\frac{\beta_i}{\alpha_i+\beta_i}}}.$$

Then we see that $g_i$ is increasing on $[0, t_i]$ and decreasing on $[t_i, \infty)$ and satisfies $g_i(0) = 0$ and $\lim_{t\to\infty} g_i(t) = 0$. So, if we take $\lambda_* = g_1(t_1)$ and $\mu_* = g_2(t_2)$. Then for every $0 < \lambda \leq \lambda_*$ and any $0 < \mu \leq \mu_*$, the double pair

$$(\underline{u}, \underline{v}) = (\lambda Va_1, \mu Va_2) \text{ and } (\overline{u}, \overline{v}) = (t_1 V(a_1 + b_1), t_2 V(a_2 + b_2)),$$

is a continuous sub-supersolution of the system (4.5). Hence, it follows from Theorem 1.2 that (4.5) has a continuous weak solution $(u, v)$ satisfying $\lambda Va_1 \leq u \leq t_1 V(a_1 + b_1)$ and $\mu Va_2 \leq v \leq t_2 V(a_2 + b_2)$.

**Case 3**: $0 < \alpha_1 + \beta_1 < 1$ and $\alpha_2 + \beta_2 \geq 1$. In this case, we obtain that for every $0 < \lambda$ and $0 < \mu \leq \mu_* = g_2(t_2)$ the double pair $(\underline{u}, \underline{v}) = (\lambda V a_1, \mu V a_2)$ and $(\overline{u}, \overline{v}) = (MV(a_1 + b_1), t_2 V(a_2 + b_2))$ is a continuous sub-supersolution to system (4.5), where $M$ is chosen so that $0 < \lambda < g_1(M)$. Hence, it follows from Theorem 1.2 that (4.5) has a continuous weak solution $(u, v)$; moreover, we have

$$\lambda V a_1 \leq u \leq MV(a_1 + b_1) \text{ and } \mu V a_2 \leq v \leq t_2 V(a_2 + b_2).$$

**Case 4:** $0 < \alpha_2 + \beta_2 < 1$, and $\alpha_1 + \beta_1 \geq 1$. Inspired by cases 2 and 3, we can prove that for every $0 < \lambda \leq \lambda_* = g_1(t_1)$, and for every $\mu > 0$, the double pair

$$(\underline{u}, \underline{v}) = (\lambda V a_1, \mu V a_2) \text{ and } (\overline{u}, \overline{v}) = (t_1 V(a_1 + b_1), MV(a_2 + b_2)),$$

is a continuous sub-supersolution of system (4.5), where $M$ is chosen so that $0 < \mu < g_2(M)$. Hence, it follows from Theorem 1.2 that (4.5) has a continuous weak solution $(u, v)$ satisfying in addition

$$\lambda V a_1 \leq u \leq t_1 V(a_1 + b_1) \text{ and } \mu V a_2 \leq v \leq MV(a_2 + b_2).$$

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there is no conflict of interest.

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