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Research article

Dynamics of a stochastic epidemic model with information intervention and vertical transmission

Feng Wang[1,2,*] **and Taotao Li**[3]

[1] School of Mathematics and Statistics, Central South University, Changsha 410083, China
[2] Department of Mathematics, Pingxiang University, Pingxiang 337055, China
[3] School of Mechanical and Electronic Engineering, Pingxiang University, Pingxiang 337055, China

* **Correspondence:** Email: wangfeng871618@126.com.

Abstract: The dynamic behavior of a stochastic epidemic model with information intervention and vertical transmission was the concern of this paper. The threshold to judge the extinction and persistence of the disease was obtained. Specifically, when $\Delta < 0$ (Δ appears in Section 3), the three classes I_t, M_t, and R_t appearing in the model go extinct at an exponential rate, and the susceptible class S_t almost surely converges to the solution of the boundary equation exponentially. When $\Delta > 0$, the result that the disease in the model is persistent in the mean and the existence of invariant probability measure are proved by constructing a new form of Lyapunov functions, which results in getting sufficient and nearly necessary conditions for different properties. Moreover, one of the main characteristics of this article was the study of the critical case of $\Delta = 0$ under some conditions. Some examples were listed to confirm the obtained results.

Keywords: stochastic epidemic model; information intervention; vertical transmission; extinction; persistence; invariant probability measure

1. Introduction

Outbreaks of infectious diseases do great harm to life and fortune. The construction and research of mathematical models play an extremely important role in the prevention and control of diseases. Scholars have studied various properties of many epidemic models, such as SIR (Susceptible-Infected-Recovered), SEIR (Susceptible-Exposed-Infected-Recovered), SIRS, SIQS (Susceptible-Infected-Quarantined-Recovered), which portray different characteristics of disease transmission [1–6]. The authors in [6] studied the Hopf bifurcation and stability of a delayed SIR model. They established an epidemic model with temporary immunity and specific functional response [7], and got the well-possedness and the threshold to determine different behaviors of the model.

When an epidemic occurs, people can learn about the transmission route of the disease, prevention measures and the government's policy on disease control from media such as the TV or Internet, so that they can take certain measures to slow down the spread of the disease, such as self-isolation, vaccination, and compliance with the government's anti-epidemic regulations. On account of the role of media information on disease control, scholars have studied the different properties of epidemic models with information intervention [8–12]. At present, there are mainly two ways to study the impact of information intervention on the behavior. One is to study the impact of information intervention on the contact rate [8, 9, 13, 14], and the other is to introduce a new class with information awareness [10, 11, 15, 16]. In [15], the following epidemic model with separate information intervention class was established:

$$\begin{cases} \dot{S}_t = \Lambda - d_1 S_t - \beta S_t I_t - d_2 m M_t S_t + \delta R_t, \\ \dot{I}_t = \beta S_t I_t - (d_1 + \gamma + \mu) I_t, \\ \dot{R}_t = d_2 m M_t S_t + \gamma I_t - (d_1 + \delta) R_t, \\ \dot{M}_t = \dfrac{a_1 I_t}{1 + b_1 I_t} - a_2 M_t, \end{cases} \tag{1.1}$$

where S_t, I_t, R_t denote the quantity of the susceptible class, the infected class, and the recovered class at time t, separately. M_t represents the individuals of the class with information awareness. The meanings of the parameters in model (1.1) are shown in Table 1. In addition, a_2 represents the degradation rate of information, which contains the subtraction due to the natural death of M. Thus, the assumption that $a_2 > d_1$ is reasonable. The whole part $d_2 m$ indicates response rate; the detailed meaning of each parameter on the information and schematic diagram of the above model can be seen in [15]. All the above parameters are specified as positive.

Table 1. Meanings of the parameters in model (1.1).

Parameters	Meaning
Λ	The inflow rate in the population
d_1	The natural death rate
μ	Mortality due to disease
β	The contact rate between S and I
γ	The recovery rate of the infected
δ	The loss rate of immunity, turning the recovered into the susceptible
a_1, b_1	The information growth rate and the saturation coefficient
a_2	The degradation rate of information
m	The rate of information interaction
d_2	The response intensity

Reality is not immutable and often full of various uncertainties. The above deterministic model (1.1) can not reflect these uncertain factors. Therefore, the introduction of the model with stochastic noise will better reflect the reality and present more research contexts. For this reason, many scholars have studied epidemic models with various stochastic factors [17–20]. The authors have studied the nontrivial positive periodic solution and condition for extinction of the model with media coverage and white noise [18]. Bao-Shao investigated an SIRS model with Markovian switching, which is used to describe the changes of coefficients in different environments, and discussed the influence of Markovian switching on the behavior of the model. In this paper, we introduce the stochastic perturbation of white noise into the above model, whose intensity is proportional to each class, that is

$$\begin{cases} dS_t = \Big[\Lambda - d_1 S_t - \beta S_t I_t - d_2 m M_t S_t + \delta R_t\Big]dt + \sigma_1 S_t dW_1(t), \\ dI_t = \Big[\beta S_t I_t - (d_1 + \gamma + \mu) I_t\Big]dt + \sigma_2 I_t dW_2(t), \\ dR_t = [d_2 m M_t S_t + \gamma I_t - (d + \delta) R_t]dt + \sigma_4 R_t dW_4(t), \\ dM_t = \Big(\dfrac{a_1 I_t}{1 + b_1 I_t} - a_2 M_t\Big)dt + \sigma_3 M_t dW_3(t). \end{cases} \tag{1.2}$$

Here, $W_i(t), i = 1, 2, 3, 4$ are mutually independent Brownian motions on probability space and $\sigma_i, i = 1, 2, 3, 4$ represent the intensities of the stochastic perturbations. The greater the stochastic perturbations, the deeper the impact on the system, the greater σ_i will be.

In addition to physical or respiratory transmission, there is also a form of vertical transmission from an infected mother to the newborns, such as with hepatitis B and AIDS. Vertical transmission from the mother to the newborn is considered as one of the most important ways of AIDS transmission. Thus, the epidemic models possessing vertical transmission have been extensively investigated [21–23]. The authors in [21] proposed an epidemic model with vertical transmission where the parameter b signifies the birth rate of the population and q stands for the proportion of newborns infected after birth from infectious mothers. $p = 1 - q$ and $d_1 > b \geq 0$ is assumed. Therefore, pb expresses the rate of newborns who have not been infected by their mothers and become susceptible.

Hence, introducing the above factors, including the information intervention and vertical transmission, we can obtain the following stochastic model:

$$\begin{cases} dS_t = \Big[\Lambda - d_1 S_t - \beta S_t I_t - d_2 m M_t S_t + b(S_t + R_t + M_t) \\ \qquad\quad + pbI_t + \delta R_t\Big]dt + \sigma_1 S_t dW_1(t), \\ dI_t = \Big[qbI_t + \beta S_t I_t - (d_1 + \gamma + \mu) I_t\Big]dt + \sigma_2 I_t dW_2(t), \\ dM_t = \Big(\dfrac{a_1 I_t}{1 + b_1 I_t} - a_2 M_t\Big)dt + \sigma_3 M_t dW_3(t), \\ dR_t = [d_2 m M_t S_t + \gamma I_t - (d_1 + \delta) R_t]dt + \sigma_4 R_t dW_4(t). \end{cases} \tag{1.3}$$

The above factors not only make the model more general, but also raise the degree of difficulty of the study. The novelties of this paper are as follows: (i) An epidemic model with information intervention and vertical transmission is established; (ii) a threshold is obtained to determine the different dynamics of the model and the exponential rates of three classes are studied; (iii) the critical case of $\Delta = 0$, which has rarely been discussed in the literature, is investigated here.

This paper is arranged as follows: In Section 2, some estimations of the solution are given, followed by some lemmas to be used later. Section 3 gives a rough illustration of the value for disease extinction and provides the main conclusions of the paper. Part 4 focuses on the proof of Theorem 3.1 and Proposition 3.1 in detail. Part 5 proves the persistence of the model when $\Delta > 0$ and obtains the condition that the model has a stationary distribution. Section 6 studies the critical case when $\Delta = 0$. Section 7 discusses the results of the paper and lists some examples and numerical simulations to check the previous results.

2. Background knowledge

In this paper, $(\Omega, \mathfrak{F}, \{\mathfrak{F}\}_t, \mathbb{P})$ is assumed to be a complete probability space and $\mathbb{R}^4_+ := \{(a, b, c, d)|a \geq 0, b \geq 0, c \geq 0, d \geq 0\}$ and $\mathbb{R}^{4,o}_+ := \{(a, b, c, d)|a > 0, b > 0, c > 0, d > 0\}$. $a \wedge b = \min\{a, b\}$. $\mathbb{P}_{s,i,m_0,r}$ and $\mathbb{E}_{s,i,m_0,r}$ denote the probability and expectation with initial condition (s, i, m_0, r), respectively.

For the general SDE $dx_t = f(x_t)dt + g(x_t)dW(t)$ and the twice-differentiable function $V(x)$, the operator $\mathcal{L}V$ is defined by

$$\mathcal{L}V(x) = f^T V_x(x) + \frac{1}{2}tr(g^T V_{xx}(x)g). \tag{2.1}$$

In addition, the Itô′s formula can be expressed as

$$dV(x) = \mathcal{L}V(x)dt + V_x(x)^T g(x_t)dW(t). \tag{2.2}$$

First of all, we are concerned with the existence and uniqueness as well as approximate scope of the solution. The following lemmas will respond to these problems.

Lemma 2.1.

(i) For the initial condition $(s, i, m_0, r) \in \mathbb{R}^4_+$, the model (1.3) has a global solution (S_t, I_t, M_t, R_t) that possesses Markov-Feller property. Moreover, the solution (S_t, I_t, M_t, R_t) will remain in $\mathbb{R}^4_+$ with probability 1.

(ii) Let $\sigma = \max\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$, for $0 < \vartheta < p < \theta < \frac{2(d_1-b)}{\sigma^2}$, there exist constants $N_1 > 0$ and $N_2 > 0$ satisfying

$$\mathbb{E}[(S_t + I_t + M_t + R_t)^{1+\theta} + S_t^{-\vartheta}] \leq [(s + i + m_0 + r)^{1+\theta} + s^{-\vartheta}]e^{-N_1 t} + \frac{N_2}{N_1}. \tag{2.3}$$

Proof. The proof of part (i) is common and omitted. Our main proof is part (ii). Let $V_1(S, I, M, R) := (S + I + M + R)^{1+\theta} + S^{-\vartheta}$. Direct calculation to $V_1(S, I, M, R)$ yields to

$$
\begin{aligned}
\mathcal{L}V_1(S,I,M,R) =&(1+\theta)(S+I+M+R)^{\theta}\Big[\Lambda-(d_1-b)(S+I+M+R)\\
&-\mu I+\frac{a_1 I}{1+b_1 I}-(a_2-d_1)M\Big]\\
&+\frac{\theta(1+\theta)}{2}(S+I+M+R)^{\theta-1}[\sigma_1^2 S^2+\sigma_2^2 I^2+\sigma_3^2 M^2+\sigma_3^2 R^2]\\
&-\vartheta S^{-\vartheta-1}[\Lambda-d_1 S-\beta SI-d_2 mMS+b(S+R+M)\\
&+pbI+\delta R]+\frac{\vartheta(1+\vartheta)\sigma_1^2}{2}S^{-\vartheta}\\
\le&(1+\theta)[\Lambda+\frac{a_1}{b_1}](S+I+M+R)^{\theta}+(1+\theta)(S+I+M+R)^{\theta-1}\Big[\\
&-(d_1-b)(S+I+M+R)^2+\frac{\theta\sigma^2}{2}(S^2+I^2+M^2+R^2)\Big]\\
&-\vartheta\Lambda S^{-\vartheta-1}+\vartheta d_1 S^{-\vartheta}+\vartheta\beta S^{-\vartheta}I+\vartheta d_2 m S^{-\vartheta}M+\frac{\vartheta(1+\vartheta)\sigma_1^2}{2}S^{-\vartheta}.
\end{aligned}
$$

Let $0<\vartheta<p<\theta<\frac{2(d_1-b)}{\sigma^2}$, so $\frac{\vartheta(1+p)}{p}<1+\vartheta$ and $-(d_1-b)+\frac{\theta\sigma^2}{2}<0$ hold true. By using Young's inequality, it has

$$
S^{-\vartheta}I\le\frac{p}{1+p}(S^{-\vartheta})^{\frac{1+p}{p}}+\frac{1}{1+p}I^{1+p}\le S^{-\frac{\vartheta(1+p)}{p}}+\frac{1}{1+p}(S+I+M+R)^{1+p},
$$

and

$$
S^{-\vartheta}M\le\frac{p}{1+p}(S^{-\vartheta})^{\frac{1+p}{p}}+\frac{1}{1+p}M^{1+p}\le S^{-\frac{\vartheta(1+p)}{p}}+\frac{1}{1+p}(S+I+M+R)^{1+p}.
$$

Hence,

$$
\begin{aligned}
\mathcal{L}V_1(S,I,M,R)=&-\left[(d_1-b)-\frac{\theta\sigma^2}{2}\right](1+\theta)(S+I+M+R)^{\theta+1}\\
&+(1+\theta)[\Lambda+\frac{a_1}{b_1}](S+I+M+R)^{\theta}-\vartheta\Lambda S^{-\vartheta-1}\\
&+[\vartheta d_1+\frac{\vartheta(1+\vartheta)\sigma_1^2}{2}]S^{-\vartheta}+[\vartheta\beta+\vartheta d_2 m]S^{-\frac{\vartheta(1+p)}{p}}\\
&+\frac{1}{1+p}[\vartheta\beta+\vartheta d_2 m](S+I+M+R)^{1+p}.
\end{aligned}
$$

Let $N_1=d_1-b-\frac{\theta\sigma^2}{2}$, then $\mathcal{L}V_1(S,I,M,R)+N_1V_1(S,I,M,R)\le N_2$, where

$$
\begin{aligned}
N_2 = \sup_{(S,I,M,R)\in\mathbb{R}_+^4} \Big\{ & - N_1\theta(S + I + M + R)^{\theta+1} \\
& + (1+\theta)[\Lambda + \frac{a_1}{b_1}](S + I + M + R)^{\theta} - \vartheta\Lambda S^{-\vartheta-1} \\
& + [\vartheta d_1 + \frac{\vartheta(1+\vartheta)\sigma_1^2}{2}]S^{-\vartheta} + [\vartheta\beta + \vartheta d_2 m]S^{-\frac{\vartheta(1+p)}{p}} \\
& + \frac{1}{1+p}[\vartheta\beta + \vartheta d_2 m](S + I + M + R)^{1+p} + N_1 S^{-\vartheta}\Big\} < \infty.
\end{aligned}
$$

The rest of the process is standard; one can see Lemma 2.3 in [24]. Thus, (2.3) is obtained.

Lemma 2.2. *For all initial conditions $(s, i, m_0, r) \in \mathbb{R}_+^4$, the solution (S_t, I_t, M_t, R_t) of (1.3) satisfies*

$$\limsup_{t\to\infty}(S_t + I_t + M_t + R_t) < \infty, \quad a.s., \tag{2.4}$$

hence,

$$
\begin{aligned}
&\lim_{t\to\infty}\frac{S_t}{t} = 0, \ \lim_{t\to\infty}\frac{I_t}{t} = 0, \ \lim_{t\to\infty}\frac{M_t}{t} = 0, \ \lim_{t\to\infty}\frac{R_t}{t} = 0, \ a.s., \\
&\lim_{t\to\infty}\frac{\sigma_1}{t}\int_0^t S_s dW_1(s) = 0, \ \lim_{t\to\infty}\frac{\sigma_2}{t}\int_0^t I_s dW_2(s) = 0, \ \lim_{t\to\infty}\frac{\sigma_4}{t}\int_0^t R_s dW_4(s) = 0, \ a.s.
\end{aligned}
\tag{2.5}
$$

Moreover, it has

$$\lim_{t\to\infty}\frac{\ln S_t}{t} = 0, \quad \lim_{t\to\infty}\frac{\ln I_t}{t} \le 0, \quad \lim_{t\to\infty}\frac{\ln M_t}{t} \le 0, \quad \lim_{t\to\infty}\frac{\ln R_t}{t} \le 0, \quad a.s. \tag{2.6}$$

Proof. For (2.4), the proof is analogous to Lemma 3.1 in [22], mainly utilizing the result of Theorem 3.9 in [25], so it is omitted here. The proofs for (2.5) can be derived from (2.4) and strong law of large numbers.

For $\lim_{t\to\infty}\frac{\ln S_t}{t} \le 0$ and other formulas in (2.6), we recommend Lemma 2.3 in [26] to get a detailed proof. In addition, the property that $\mathbb{E}S_t^{-\vartheta} < \infty$ will lead to $\liminf_{t\to\infty}\frac{\ln S_t}{t} \ge 0$, so $\lim_{t\to\infty}\frac{\ln S_t}{t} = 0$ is obtained.

3. The threshold to determine the extinction or permanence of model (1.3)

We will give a value in this section and roughly explain it as the threshold of the extinction or persistence of model (1.3).

Take into account the first equation of model (1.3) on the boundary $I_t = 0$, $M_t = 0$, and $R_t = 0$, it has

$$d\bar{S}_t = [\Lambda - (d_1 - b)\bar{S}_t]dt + \sigma_1\bar{S}_t dW_1(t). \tag{3.1}$$

Let $\bar{S}_t^u$ be the solution to (3.1) with the initial condition $\bar{S}_0 = u$. It should be noted that $S_t \le \bar{S}_t,\ t > 0$ cannot be obtained by using the comparison theorem. Applying the Itô′s formula to the function

$\bar{S}-1-\ln\bar{S}$ and making use of the result in [27], there exists the unique stationary distribution π_0 for (3.1) with the density

$$f^*(x)=\frac{\bar{u}^{\bar{\nu}}}{\Gamma(\bar{\nu})}x^{-\bar{\nu}-1}e^{-\frac{\bar{u}}{x}},\ x>0,$$

where $\bar{\nu}=\frac{2(d_1-b)}{\sigma_1^2}+1$, $\bar{u}=\frac{2\Lambda}{\sigma_1^2}$, $\Gamma(\cdot)$ is the Gamma function. We get from the strong law of large numbers that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\beta\bar{S}_s ds=\int_0^\infty\beta x f^*(x)dx=\frac{\beta\Lambda}{d_1-b},\ a.s. \tag{3.2}$$

Calculating the second formula of model (1.3), it has

$$\frac{\ln I_t}{t}=\frac{\ln I_0}{t}+\frac{1}{t}\int_0^t\beta S_s ds-(d_1+\gamma+\mu-qb+\frac{1}{2}\sigma_2^2)+\frac{\sigma_2 W_2(t)}{t}. \tag{3.3}$$

Intuitively, if $\limsup_{t\to\infty}\frac{\ln I_t}{t}<0$, then $\lim_{t\to\infty}I_t=0$. This leads to the results $\lim_{t\to\infty}M_t=0$, $\lim_{t\to\infty}R_t=0$, which will be explained in detail later. Thus, we have $S_t\approx\bar{S}_t$ if t is sufficiently large, then one can anticipate that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t\beta S_s ds\approx\lim_{t\to\infty}\frac{1}{t}\int_0^t\beta\bar{S}_s ds=\frac{\beta\Lambda}{d_1-b}. \tag{3.4}$$

Defining the value

$$\Delta:=\frac{\beta\Lambda}{d_1-b}-(d_1+\gamma+\mu-qb+\frac{1}{2}\sigma_2^2). \tag{3.5}$$

Hence, $\limsup_{t\to\infty}\frac{\ln I_t}{t}$ will tend to Δ above. If $\Delta<0$, then $\limsup_{t\to\infty}\frac{\ln I_t}{t}$ will be negative, and the disease will die out. Conversely, when $\Delta>0$, no matter how small the initial value I_0 is, I_t tends to be large for a sufficiently long time. The above description seems simple; however, the proof requires careful and rigorous implementation.

Now, we present the main conclusions of this paper, the proof of which will be given in the later section. Let $R_0^S=\frac{\beta\Lambda}{(d_1-b)(d_1+\gamma+\mu-qb+\frac{1}{2}\sigma_2^2)}$.

Theorem 3.1. *When $\Delta<0$, or equivalently $R_0^S<1$, the solution (S_t,I_t,M_t,R_t) with the initial condition $(s,i,m_0,r)\in\mathbb{R}_+^{4,o}$ satisfies*

$$\lim_{t\to\infty}\frac{\ln I_t}{t}=\Delta<0,\quad a.s., \tag{3.6}$$

(3.6) implies that the disease I_t becomes extinct at an exponential rate.

Theorem 3.1 gives a condition to judge the extinction of the disease.

Proposition 3.1. *If $\Delta<0$, let $\overline{\Delta}:=\min\{-\Delta,a_2+\frac{1}{2}\sigma_3^2,d_1+\delta+\frac{1}{2}\sigma_4^2\}>0$ and $\overline{\Delta}_1:=\min\{-\Delta,a_2+\frac{1}{2}\sigma_3^2\}>0$, then*

$$\lim_{t\to\infty}\frac{\ln M_t}{t}=-\overline{\Delta}_1=\max\{\Delta,-(a_2+\frac{\sigma_3^2}{2})\},\quad a.s., \tag{3.7}$$

$$\lim_{t\to\infty}\frac{\ln R_t}{t}=-\overline{\Delta}=\max\{\Delta,-(a_2+\frac{1}{2}\sigma_3^2),-(d_1+\delta+\frac{\sigma_4^2}{2})\},\ \ a.s. \tag{3.8}$$

Furthermore,

$$\lim_{t\to\infty}\frac{\ln|S_t-\bar{S}_t|}{t}\leq\max\{-\overline{\Delta},-(d_1-b+\frac{\sigma_1^2}{2})\},\ \ a.s. \tag{3.9}$$

Definition 3.1. *[28] The disease in model (1.3) is called to be persistent in the mean, if the following inequality holds*

$$\liminf_{t\to\infty}\frac{1}{t}\int_0^t I(u)du>0,\ a.s.$$

Theorem 3.2. *For the solution (S_t,I_t,M_t,R_t) with the initial condition $(s,i,m_0,r)\in\mathbb{R}_+^{4,o}$, when $\Delta>0$, that is, $R_0^S>1$, the disease I_t in model (1.3) is persistent in the mean. Moveover, the solution (S_t,I_t,M_t,R_t) has the invariant probability measure.*

Remark 3.1. *According to Theorems 3.1 and 3.2, we know that the sign of Δ will judge extinction or persistence of model (1.3) and R_0^S can be regarded as the reproduction number, which depicts the number of second-generation infections after a single infected one enters the population.*

4. The case of $\Delta<0$

In this section, we will prove Theorem 3.1 and Proposition 3.1 with the assumption that $\Delta<0$.

4.1. Proof of Theorem 3.1

First, consider the equation

$$d\bar{S}_t^{\varepsilon}=[\Lambda-(d_1-b)\bar{S}_t^{\varepsilon}+\varepsilon]dt+\sigma_1\bar{S}_t^{\varepsilon}dW_1(t). \tag{4.1}$$

Similar to the Eq (3.1), a suitable Lyapunov function can be proved to obtain the invariant measure π^{ε} with density

$$f^{\varepsilon}(s)=\frac{(\frac{2(\Lambda+\varepsilon)}{\sigma_1^2})^{\frac{2(d_1-b)}{\sigma_1^2}+1}}{\Gamma(\frac{2(d_1-b)}{\sigma_1^2}+1)}x^{-\frac{2(d_1-b)}{\sigma_1^2}-2}e^{-\frac{2(\Lambda+\varepsilon)}{\sigma_1^2x}},\ x>0.$$

Lemma 4.1. *Provided that $\Delta<0$, for any $\epsilon>0$ and $H>0$, there is a constant $\delta_1>0$ such that for any $(s,i,m_0,r)\in[0,H]\times[0,\delta_1]^3$ (where $[0,\delta_1]^3$ represents $[0,\delta_1]\times[0,\delta_1]\times[0,\delta_1]$), it has*

$$\mathbb{P}_{s,i,m_0,r}\{\lim_{t\to\infty}I_t=0,\ \lim_{t\to\infty}M_t=0,\ \lim_{t\to\infty}R_t=0\}\geq1-\epsilon. \tag{4.2}$$

Proof. Let $(s,i,m_0,r)\in[0,H]\times[0,\delta_1]^3$. For the $0<\varepsilon_1<\frac{(d_1-b)\bar{\Delta}}{8\beta}$ ($\bar{\Delta}$ is defined in Proposition 3.1), consider the Eq (4.1) with ε replaced by ε_1, the ergodicity of solution with the initial data s denoted by $\bar{S}_t^{\varepsilon_1,s}$ leads to

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t \bar{S}_u^{\varepsilon_1,s}du=\int_0^\infty xf^{\varepsilon_1}(x)dx=\frac{\Lambda+\varepsilon_1}{d_1-b}.$$

If the initial value is not emphasized later, we still use $\bar{S}_t^{\varepsilon_1}$ to express the solution of the equation. Thus, there is a constant T_1 such that $\mathbb{P}(\Omega_1^H)>1-\frac{\epsilon}{5}$, where

$$\Omega_1^s=\{\omega:\frac{1}{t}\int_0^t \bar{S}_u^{\varepsilon_1,s}du\le\frac{\Lambda+\varepsilon_1}{d_1-b}+\frac{\bar{\Delta}}{8\beta},\forall t\ge T\}.$$

Due to $\bar{S}_t^{\varepsilon_1,s}\le\bar{S}_t^{\varepsilon_1,H}$ for $s\le H$, it yields $\mathbb{P}_s(\Omega_1)>1-\frac{\epsilon}{5}$ for $s\in[0,H]$.

Because $\lim_{t\to\infty}\frac{\sigma_iW_i(t)}{t}=0,\ a.s.,\ i=1,2,3,4$, it has for some constant T_2, $\mathbb{P}(\Omega_2)>1-\frac{\epsilon}{5}$, where

$$\Omega_2=\{\omega:\frac{\sigma_iW_i(t)}{t}\le\min\{\frac{\bar{\Delta}}{8},\varepsilon_1\},\forall t\ge T_2\ and\ i=1,2,3,4\}.$$

Assume that $T=\max\{T_1,T_2\}$; thanks to Lemmas 2.1 and 2.2, then for some positive constants C_1 and K, it has $\mathbb{P}(\Omega_3)>1-\frac{\epsilon}{5}$ and $\mathbb{P}(\Omega_4)>1-\frac{\epsilon}{5}$, where

$$\Omega_3=\{\omega:S_t(\omega)\le C_1,t>0\},\tag{4.3}$$

and

$$\Omega_4=\{\omega:\int_0^T\beta S_u du\le K\}.$$

Let K above be sufficiently large such that $\mathbb{P}(\Omega_5)>1-\frac{\epsilon}{5}$. Here,

$$\Omega_5=\{\omega:|\sigma_iW_i(t)|\le K,\ for\ i=1,2,3,4\ and\ t\le T\}.$$

Choose δ_1 sufficiently small so that

$$\delta_1(e^{2K}+a_1e^{4K}T+d_2m(e^{3K}+a_1e^{6K}T)K/\beta+\gamma e^{4K}T)<\frac{\varepsilon_1}{2b+pb+\delta},\tag{4.4}$$

and

$$\delta_1\max\{C_2,C_3\}<\frac{\varepsilon_1}{2b+pb+\delta}.\tag{4.5}$$

Here, C_2 and C_3 will be determined in (4.11) and (4.12), respectively, later.

Let the stopping time be defined as

$$\tau=\inf\left\{t:\max_{t>0}\{I_t,M_t,R_t\}\ge\frac{\varepsilon_1}{2b+pb+\delta}\right\}.$$

From the second equation, we get

$$I_t=I_0\exp\{\int_0^t\beta S_u du-(d_1+\gamma+\mu-qb+\frac{1}{2}\sigma_2^2)t+\sigma_2W_2(t)\}.\tag{4.6}$$

Similarly, the third and fourth equations of (1.3) result in

$$M_t = \Psi_1(t)[M_0 + \int_0^t \frac{a_1 I_u}{1 + b_1 I_u} \Psi_1^{-1}(u) du] \tag{4.7}$$

and

$$R_t = \Psi_2(t)[R_0 + \int_0^t (d_2 m S_u M_u + \gamma I_u) \Psi_2^{-1}(u) du], \tag{4.8}$$

where $\Psi_1(t) = \exp\{-(a_2 + \frac{\sigma_3^2}{2})t + \sigma_3 W_3(t)\}$ and $\Psi_2(t) = \exp\{-(d_1 + \delta + \frac{\sigma_4^2}{2})t + \sigma_4 W_4(t)\}$.

Hence, we get from (4.6) that for almost every $\omega \in \Omega_4 \cap \Omega_5$ and $t \in [0, T \wedge \tau]$, it has

$$I_t \le I_0 e^{\int_0^T \beta S_u du + \sigma_2 W_2(u)} \le I_0 e^{2K} \le \delta_1 e^{2K} \le \frac{\varepsilon_1}{2b + pb + \delta}.$$

Moreover, the expression of $\Psi_1(t)$ leads to that for $\omega \in \Omega_5$,

$$\exp\{-(a_2 + \frac{\sigma_3^2}{2})t - K\} \le \Psi_1(t) \le \exp\{-(a_2 + \frac{\sigma_3^2}{2})t + K\}, \ \forall t \in [0, T \wedge \tau].$$

Thus, when $\omega \in \cap_{i=4}^5$,

$$\begin{aligned}
&\Psi_1(t) \int_0^t \frac{a_1 I_u}{1 + b_1 I_u} \Psi_1^{-1}(u) du \\
\le &\int_0^t a_1 I_0 e^{2K} e^{-(a_2 + \frac{\sigma_3^2}{2})t + \sigma_3 W_3(t)} e^{(a_2 + \frac{\sigma_3^2}{2})u - \sigma_3 W_3(u)} du \\
\le &\delta_1 a_1 e^{4K} T.
\end{aligned} \tag{4.9}$$

Due to (4.4), it has

$$\begin{aligned}
M_t = &\Psi_1(t) M_0 + \Psi_1(t) \int_0^t \frac{a_1 I_u}{1 + b_1 I_u} \Psi_1^{-1}(u) du \\
\le &e^K \delta_1 + \delta_1 a_1 e^{4K} T \le \frac{\varepsilon_1}{2b + pb + \delta}.
\end{aligned} \tag{4.10}$$

In the same way, for $t \in [0, T \wedge \tau]$ on $\cap_{i=4}^5$,

$$\begin{aligned}
&\Psi_2(t) \int_0^t d_2 m S_u M_u \Psi_2^{-1}(u) du \\
\le &\int_0^t d_2 m \delta_1 [e^K + a_1 e^{4K} T] S_u e^{-(d_1 + \delta + \frac{\sigma_4^2}{2})(t-u) + \sigma_3 W_3(t) - \sigma_3 W_3(u)} du \\
\le &d_2 m \delta_1 [e^K + a_1 e^{4K} T] e^{2K} K/\beta,
\end{aligned}$$

and

$$\Psi_2(t) \int_0^t \gamma I_u \Psi_2^{-1}(u) du \le \int_0^T \gamma I_0 e^{2K} e^{2K} du \le \delta_1 \gamma e^{4K} T.$$

Thus, it yields from (4.8) and (4.4) that

$$\begin{aligned} R_t \leq & e^K R_0 + d_2 m\delta_1[e^K + a_1 e^{4K}T]e^{2K}K/\beta + \delta_1\gamma e^{4K}T \\ \leq & \delta_1[e^K + d_2 m(e^{3K} + a_1 e^{6K}T)K/\beta + \gamma e^{4K}T] \\ \leq & \frac{\varepsilon_1}{2b + pb + \delta}. \end{aligned}$$

Hence, for $(s, i, m_0, r) \in [0, H] \times [0, \delta_1]^3$ on almost every $\omega \in \cap_{i=4}^5$, we have

$$\tau \geq T.$$

Next, we shall prove the assertion $\tau = \infty$ for almost every $\omega \in \cap_{k=1}^5 \Omega_k$.

First, when $t \in [T, \tau]$, it has $S_v \leq \bar{S}_v^{\varepsilon_1}$ for all $v \in [0, t]$ by comparison principle; then for almost $\omega \in \cap_{k=1}^5 \Omega_k$,

$$\begin{aligned} I_t = & I_0 \exp\{\int_0^t \beta S_u du - (d_1 + \gamma + \mu - qb + \frac{1}{2}\sigma_2^2)t + \sigma_2 W_2(t)\} \\ \leq & I_0 \exp\{\int_0^t \beta \bar{S}_u^{\varepsilon_1} du - (d_1 + \gamma + \mu - qb + \frac{1}{2}\sigma_2^2)t + \sigma_2 W_2(t)\} \\ \leq & I_0 \exp\{\beta(\frac{\Lambda + \varepsilon_1}{d_1 - b})t + \frac{\bar{\Delta}}{8}t - (d_1 + \gamma + \mu - qb + \frac{1}{2}\sigma_2^2)t + \frac{\bar{\Delta}}{8}t\} \\ \leq & I_0 \exp\{\Delta t + \frac{2\bar{\Delta}}{8}t + \frac{\beta\varepsilon_1}{d_1 - b}t\} \\ \leq & I_0 e^{(\Delta + \frac{3\bar{\Delta}}{8})t} \leq \delta_1 e^{-\frac{5\bar{\Delta}}{8}t}, \ (s, i, m_0, r) \in [0, H] \times [0, \delta_1]^3. \end{aligned}$$

One can rewrite M_t on $t \geq T$ that

$$M_t = \Psi_1(t)[M_0 + \int_0^T \frac{a_1 I_u}{1 + b_1 I_u}\Psi_1^{-1}(u)du + \int_T^t \frac{a_1 I_u}{1 + b_1 I_u}\Psi_1^{-1}(u)du].$$

For almost every $\omega \in \Omega_2$, $\exp\{-(a_2 + \frac{\sigma_3^2}{2})t - \frac{1}{8}\bar{\Delta}t\} \leq \Psi_1(t) \leq \exp\{-(a_2 + \frac{\sigma_3^2}{2})t + \frac{1}{8}\bar{\Delta}t\}$ is obtained and

$$\begin{aligned} & \Psi_1(t)\int_T^t \frac{a_1 I_u}{1 + b_1 I_u}\Psi_1^{-1}(u)du \\ \leq & e^{-(a_2 + \frac{\sigma_3^2}{2})t + \frac{1}{8}\bar{\Delta}t}\int_T^t a_1 I_0 e^{-\frac{5\bar{\Delta}}{8}u} e^{(a_2 + \frac{\sigma_3^2}{2} + \frac{1}{8}\bar{\Delta})u} du \\ \leq & \frac{\delta_1 a_1}{a_2 + \frac{\sigma_3^2}{2} - \frac{1}{2}\bar{\Delta}} e^{-\frac{3}{8}\bar{\Delta}t}, \ on \ \cap_{i=1}^2 \Omega_i. \end{aligned}$$

Therefore, there exists a positive constant C_2 satisfying

$$\begin{aligned} M_t \leq & e^{-(a_2 + \frac{\sigma_3^2}{2} - \frac{1}{8}\bar{\Delta})t}\Big[m_0 + \int_0^T a_1 I_0 e^{2K} e^{(a_2 + \frac{\sigma_3^2}{2})u + K} du\Big] + \frac{\delta_1 a_1}{a_2 + \frac{\sigma_3^2}{2} - \frac{1}{2}\bar{\Delta}} e^{-\frac{3}{8}\bar{\Delta}t} \\ \leq & (\delta_1 + a_1\delta_1 e^{3K} e^{(a_2 + \frac{\sigma_3^2}{2})T} T) e^{-(a_2 + \frac{\sigma_3^2}{2} - \frac{1}{8}\bar{\Delta})t} + \frac{\delta_1 a_1}{a_2 + \frac{\sigma_3^2}{2} - \frac{1}{2}\bar{\Delta}} e^{-\frac{3}{8}\bar{\Delta}t} \\ \leq & \delta_1 C_2 e^{-\frac{3}{8}\bar{\Delta}t}, \end{aligned}$$

where

$$C_2 = \frac{a_1}{a_2 + \frac{\sigma_3^2}{2} - \frac{1}{2}\bar{\Delta}} + 1 + a_1 e^{3K} e^{(a_2+\frac{\sigma_3^2}{2})T} T. \tag{4.11}$$

Similarly, rewriting the expression of R_t (4.8) yields

$$R_t = \Psi_2(t)[R_0 + \int_0^T (d_2 m S_u M_u + \gamma I_u)\Psi_2^{-1}(u)du] + \Psi_2(t)\int_T^t (d_2 m S_u M_u + \gamma I_u)\Psi_2^{-1}(u)du.$$

For almost all $\omega \in \Omega_2$, we have

$$e^{-(d_1+\delta+\frac{\sigma_4^2}{2})t - \frac{1}{8}\bar{\Delta}t} \le \Psi_2(t) \le e^{-(d_1+\delta+\frac{\sigma_4^2}{2})t + \frac{1}{8}\bar{\Delta}t}.$$

For almost all $\omega \in \cap_{i=1}^3 \Omega_i$ and $t > T$, it has

$$\begin{aligned}
&\Psi_2(t)\int_T^t d_2 m S_u M_u \Psi_2^{-1}(u)du \\
&\le e^{-(d_1+\delta+\frac{\sigma_4^2}{2})t + \frac{1}{8}\bar{\Delta}t}\int_T^t d_2 m C_1 \delta_1 C_2 e^{-\frac{3}{8}\bar{\Delta}u} e^{(d_1+\delta+\frac{\sigma_4^2}{2}+\frac{1}{8}\bar{\Delta})u} du \\
&\le \frac{\delta_1 d_2 m C_1 C_2}{d_1 + \delta + \frac{\sigma_4^2}{2} - \frac{1}{4}\bar{\Delta}} e^{-\frac{1}{8}\bar{\Delta}t}.
\end{aligned}$$

Similar to the proof of M_t, it has

$$\Psi_2(t)\int_T^t \gamma I_u \Psi_2^{-1}(u)du \le \frac{\gamma I_0}{d_1 + \delta + \frac{\sigma_4^2}{2} - \frac{1}{2}\bar{\Delta}} e^{-\frac{3}{8}\bar{\Delta}t}.$$

Thus,

$$\begin{aligned}
R_t \le & e^{-(d_1+\delta+\frac{\sigma_4^2}{2})t + \frac{1}{8}\bar{\Delta}t}[\delta_1 + d_2 m \delta_1 (e^{3K} + a_1 e^{6K} T)K/\beta + \delta_1 \gamma e^{4K} T] \\
&+ \frac{\delta_1 d_2 m C_1 C_2}{d_1 + \delta + \frac{\sigma_4^2}{2} - \frac{1}{4}\bar{\Delta}} e^{-\frac{1}{8}\bar{\Delta}t} + \frac{\gamma\delta_1}{d_1 + \delta + \frac{\sigma_4^2}{2} - \frac{1}{2}\bar{\Delta}} e^{-\frac{3}{8}\bar{\Delta}t} \\
\le & \delta_1 C_3 e^{-\frac{1}{8}\bar{\Delta}t},
\end{aligned}$$

where

$$C_3 = \frac{d_2 m C_1 C_2}{d_1 + \delta + \frac{\sigma_4^2}{2} - \frac{1}{4}\bar{\Delta}} + \frac{\gamma}{d_1 + \delta + \frac{\sigma_4^2}{2} - \frac{1}{2}\bar{\Delta}} + 1 + d_2 m (e^{3K} + a_1 e^{6K} T)K/\beta + \gamma e^{4K} T. \tag{4.12}$$

Assume that n is an integer with $n > T$. We get from the estimation of I_t, M_t and R_t for $t \in [0, n \wedge \tau]$ and almost every $\omega \in \cap_{k=1}^5 \Omega_k$ that

$$I_t \le \delta_1 \le \frac{\varepsilon_1}{2b + pb + \delta}, \quad M_t \le \delta_1 C_2 \le \frac{\varepsilon_1}{2b + pb + \delta},$$

and

$$R_t \le \delta_1 C_3 \le \frac{\varepsilon_1}{2b + pb + \delta}.$$

These results imply $n \le \tau$ on almost every $\omega \in \cap_{k=1}^5 \Omega_k$. On the basis of arbitrariness of n, the assertion $\tau = \infty$ is obtained. In addition, from the estimation of I_t, M_t, and R_t, one has

$$\lim_{t\to\infty} \frac{\ln I_t}{t} \le -\frac{5\bar{\Delta}}{8}, \ \lim_{t\to\infty} \frac{\ln I_t}{t} \le -\frac{3\bar{\Delta}}{8}, \ \lim_{t\to\infty} \frac{\ln I_t}{t} \le -\frac{\bar{\Delta}}{8}.$$

This will lead to the (4.2) for any initial condition $(s, i, m_0, r) \in [0, H] \times [0, \delta_1]^3$ on almost $\cap_{k=1}^5 \Omega_k$ with $\mathbb{P}(\cap_{k=1}^5 \Omega_k) \ge 1 - \epsilon$.

The subsequent proof of Theorem 1 is analogue to that of Theorem 2.2 in [29], so it is omitted here.

4.2. Proof of Proposition 3.1

Now, we provide the proof of the Proposition 3.1. From (3.6), we get that for any small ϵ_1, there exist positive random variables ξ_1, ξ_2 satisfying that for $t > 0$,

$$\xi_1 e^{(\Delta-\epsilon_1)t} \le I_t \le \xi_2 e^{(\Delta+\epsilon_1)t}.$$

Due to the expression of $\Psi_1(t)$, it yields

$$\lim_{t\to\infty} \frac{\ln \Psi_1(t)}{t} = -(a_2 + \frac{\sigma_3^2}{2}), \ \ a.s.$$

This means there exist random variables $\xi_3 > 0$, $\xi_4 > 0$ satisfying that for $t > 0$,

$$\xi_3 e^{-(a_2+\frac{\sigma_3^2}{2}+\epsilon_1)t} \le \Psi_1(t) \le \xi_4 e^{-(a_2+\frac{\sigma_3^2}{2}-\epsilon_1)t}.$$

By virtue of (4.7) and the increasing property of $\frac{a_1 I}{1+b_1 I}$ with respect to I, it yields

$$\begin{aligned} M_t \ge & \xi_3 e^{-(a_2+\frac{\sigma_3^2}{2}+\epsilon_1)t} m_0 + \xi_3 e^{-(a_2+\frac{\sigma_3^2}{2}+\epsilon_1)t} \int_0^t \frac{a_1 \xi_1 e^{(\Delta-\epsilon_1)u}}{\xi_4(1 + b_1 \xi_1 e^{(\Delta-\epsilon_1)u})} e^{(a_2+\frac{\sigma_3^2}{2}-\epsilon_1)u} du \\ \ge & \xi_3 e^{-(a_2+\frac{\sigma_3^2}{2}+\epsilon_1)t} m_0 + \frac{\xi_3 a_1 \xi_1}{\xi_4(1 + b_1\xi_1)(\Delta + a_2 + \frac{\sigma_3^2}{2} - 2\epsilon_1)} [e^{(\Delta-3\epsilon_1)t} - e^{-(a_2+\frac{\sigma_3^2}{2}+\epsilon_1)t}]. \end{aligned}$$

Thus,

$$\liminf_{t\to\infty} \frac{\ln M_t}{t} \ge -\overline{\Delta}_1 - 4\epsilon_1.$$

Similarly, we can get that

$$\limsup_{t\to\infty} \frac{\ln M_t}{t} \le -\overline{\Delta}_1 + 4\epsilon_1.$$

Therefore, one obtains that for almost every $\omega \in \cap_{k=1}^5 \Omega_k$, (3.7) holds true by the arbitrariness of ϵ_1.

Due to Lemma 2.2 and (3.7), it has $\lim_{t\to\infty}\frac{\ln S_tM_t}{t} = -\overline{\Delta}_1$, which leads to there being random variables $\xi_5 > 0$ and $\xi_6 > 0$ satisfying

$$\xi_5 e^{(-\overline{\Delta}_1-\epsilon_1)t} \le S_tM_t \le \xi_6 e^{(-\overline{\Delta}_1+\epsilon_1)t}.$$

Moreover, there are random variables $\xi_7 > 0$ and $\xi_8 > 0$ such that

$$\xi_7 e^{-(d_1+\delta+\frac{\sigma_4^2}{2}+\epsilon_1)t} \le \Psi_2(t) \le \xi_8 e^{-(d_1+\delta+\frac{\sigma_4^2}{2}-\epsilon_1)t}.$$

Thus, we get from (4.8) that

$$\begin{aligned} &\xi_7 e^{-(d_1+\delta+\frac{\sigma_4^2}{2}+\epsilon_1)t}\Big[R_0 + \frac{1}{\xi_8}\int_0^t (d_2m\xi_5 e^{(-\overline{\Delta}_1-\epsilon_1)u} + \gamma\xi_1 e^{(\Delta-\epsilon_1)u})e^{(d_1+\delta+\frac{\sigma_4^2}{2}-\epsilon_1)u}du\Big] \le \\ &R_t \le \xi_8 e^{-(d_1+\delta+\frac{\sigma_4^2}{2}-\epsilon_1)t}\Big[R_0 + \frac{1}{\xi_7}\int_0^t (d_2m\xi_6 e^{(-\overline{\Delta}_1+\epsilon_1)u} + \gamma\xi_2 e^{(\Delta+\epsilon_1)u})e^{(d_1+\delta+\frac{\sigma_4^2}{2}+\epsilon_1)u}du\Big]. \end{aligned} \tag{4.13}$$

Hence, it has

$$\begin{aligned} &\limsup_{t\to\infty}\frac{\ln R_t}{t} \le \max\{-\overline{\Delta}_1, -(d_1+\delta+\frac{\sigma_4^2}{2})\} + 4\epsilon_1, \\ &\liminf_{t\to\infty}\frac{\ln R_t}{t} \ge \max\{-\overline{\Delta}_1, -(d_1+\delta+\frac{\sigma_4^2}{2})\} - 4\epsilon_1. \end{aligned}$$

This implies that (3.8) holds.

For the convergence rate of the solution S_t with initial data (s, i, m_0, r) of (1.3) to the solution $\bar{S}_t$ with initial data s of (3.1), take into account the equation

$$\begin{aligned} d(\bar{S}_t - S_t) =\Big[&-(d_1-b)(\bar{S}_t - S_t) + \beta S_tI_t + d_2mM_tS_t \\ &- pbI_t - bM_t - (b+\delta)R_t\Big]dt + \sigma_1(\bar{S}_t - S_t)dW_1(t). \end{aligned} \tag{4.14}$$

Let $\Psi_3(t) := \exp\{-(d_1 - b + \frac{\sigma_1^2}{2})t + \sigma_1W_1(t)\}$, then utilizing the constant variation method, we can obtain

$$\bar{S}_t - S_t = \Psi_3(t)\int_0^t \Big[\beta S_uI_u + d_2mM_uS_u - pbI_u - bM_u - (b+\delta)R_u\Big]\Psi_3^{-1}(u)du. \tag{4.15}$$

This means

$$\begin{aligned} &-\Psi_3(t)\int_0^t [pbI_u + bM_u + (b+\delta)R_u]\Psi_3^{-1}(u)du \le \bar{S}_t - S_t \\ &\le \Psi_3(t)\int_0^t \Big[\beta S_uI_u + d_2mM_uS_u\Big]\Psi_3^{-1}(u)du. \end{aligned} \tag{4.16}$$

By (3.6) and Lemma 2.2, there is a random variable $\xi_9 > 0$ such that

$$S_tI_t \le \xi_9 e^{(\Delta+\epsilon_1)t}. \tag{4.17}$$

Let $\Delta_1 := \max\{-\bar{\Delta}, -(d_1 - b + \frac{\sigma_1^2}{2})\}$, and $\Delta_2 > \Delta_1$. For a sufficiently small ϵ_1, $\Delta_2 > \Delta_1 + 3\epsilon_1$ can be satisfied. The expression of $\Psi_3(t)$ implies that there exist random variables $\xi_{10} > 0$ and $\xi_{11} > 0$ such that

$$\xi_{10} e^{-(d_1 - b + \frac{\sigma_1^2}{2} + \epsilon_1)t} \leq \Psi_3(t) \leq \xi_{11} e^{-(d_1 - b + \frac{\sigma_1^2}{2} - \epsilon_1)t}.$$

Similar to the method in (4.12), we can get from (4.16) that

$$\bar{S}_t - S_t \leq \frac{\xi_{11}}{\xi_{10}} e^{-(d_1 - b + \frac{\sigma_1^2}{2} - \epsilon_1)t} \int_0^t [\beta \xi_9 e^{(\Delta + d_1 - b + \frac{\sigma_1^2}{2} + 2\epsilon_1)u} + d_2 m \xi_6 e^{(-\overline{\Delta}_1 + d_1 - b + \frac{\sigma_1^2}{2} + 2\epsilon_1)u}] du.$$

Applying L'Hospital rule means

$$\lim_{t \to \infty} \frac{\bar{S}_t - S_t}{e^{\Delta_2 t}} \leq 0.$$

Likewise, we can get that $\lim_{t \to \infty} \frac{\bar{S}_t - S_t}{e^{\Delta_2 t}} \geq 0$. Thus, (3.9) is obtained.

5. The case of $\Delta > 0$

In this section, we shall prove Theorem 3.2, which studies the condition of persistence in the mean and the invariant probability measure of the model (1.3). First, we prove the disease persistence in Subsection 5.1.

5.1. Persistence of the disease

For C_1 in (4.3), define $V_2(I) = -\ln I$, $V_3 = \bar{S} - S$ and

$$V_4(S, I, M) = V_2(I) + \frac{\beta}{d_1 - b} V_3 + \frac{\beta d_2 m C_1}{(d_1 - b) a_2} M. \tag{5.1}$$

Direct calculation by Ito's formula yields that

$$dV_4 = \mathcal{L} V_4 dt + \frac{\beta \sigma_1}{d_1 - b} (\bar{S} - S) dW_1(t) - \sigma_2 dW_2(t) + \frac{\beta d_2 m C_1 \sigma_3}{(d_1 - b) a_2} M dW_3(t), \tag{5.2}$$

where

$$
\begin{aligned}
\mathcal{L}V_4 =& -\frac{1}{I}[\beta SI - (d_1+\gamma+\mu-qb)I] + \frac{\sigma_2^2}{2} + \frac{\beta}{d_1-b}[-(d_1-b)(\bar{S}-S)+\beta SI] \\
&+ \frac{\beta}{d_1-b}[d_2mMS-(b+\delta)R-bM-pbI] + \frac{\beta d_2mC_1}{(d_1-b)a_2}(\frac{a_1I}{1+b_1I}-a_2M) \\
\le& -\beta S + (d_1+\gamma+\mu-qb+\frac{\sigma_2^2}{2}) - \beta(\bar{S}-S) + \frac{\beta^2}{d_1-b}SI \\
&+ \frac{\beta d_2mMS}{d_1-b} + \frac{\beta d_2mC_1a_1}{(d_1-b)a_2}I - \frac{\beta d_2mC_1}{d_1-b}M \\
\le& -\beta\bar{S} + (d_1+\gamma+\mu-qb+\frac{\sigma_2^2}{2}) + \frac{\beta^2C_1}{d_1-b}I + \frac{\beta d_2mC_1a_1}{(d_1-b)a_2}I \\
\le& -\Delta + \frac{\beta^2C_1}{d_1-b}I + \frac{\beta d_2mC_1a_1}{(d_1-b)a_2}I - \beta(\bar{S}-\frac{\Lambda}{d_1-b}).
\end{aligned}
\tag{5.3}
$$

Here, we use (4.3) and (4.14). Hence, integrating for (5.2) from 0 to t and dividing it by t, we have

$$
\begin{aligned}
&\frac{V_4(S_t,I_t,M_t)-V_4(S_0,I_0,M_0)}{t} \\
\le& -\Delta + \Big(\frac{\beta^2C_1}{d_1-b} + \frac{\beta d_2mC_1a_1}{(d_1-b)a_2}\Big)\frac{1}{t}\int_0^t I_s ds - \frac{\beta}{t}\int_0^t \bar{S}_s ds + \frac{\beta\Lambda}{d_1-b} - \frac{\sigma_2W_2(t)}{t} \\
&+ \frac{\beta\sigma_1}{d_1-b}\frac{1}{t}\int_0^t(\bar{S}_s-S_s)dW_1(s) + \frac{\beta d_2mC_1\sigma_3}{(d_1-b)a_2}\frac{1}{t}\int_0^t M_s dW_3(s).
\end{aligned}
$$

Then, taking the limit, for $\Delta > 0$, using the results in Lemma 2.2 and the expression of V_4 as well as the ergodicity of $\bar{S}$, yields

$$
\Big(\frac{\beta^2C_1}{d_1-b} + \frac{\beta d_2mC_1a_1}{(d_1-b)a_2}\Big)\lim_{t\to\infty}\frac{1}{t}\int_0^t I_s ds \ge \Delta, \tag{5.4}
$$

which signifies the disease in model (1.3) is persistent in the mean.

Next, we prove the invariant probability measure of the model (1.3) under the condition of $\Delta > 0$.

5.2. Invariant probability measure of model (1.3)

Let $\sigma^2 = \max\{\sigma_1^2, \sigma_2^2, \sigma_3^2, \sigma_4^2\}$. Define a function by

$$
F(S,I,M,R) := H_2V_4(S,I,M) + V_5(S,I,M,R) - \ln S - \ln M - \ln R, \tag{5.5}
$$

where the function V_4 is defined in (5.1), $V_5(S,I,M,R) = \frac{1}{1+a}(S+I+M+R)^{1+a}$, $a \in (0,1)$ satisfies

$$
d_1 - b - \frac{a\sigma^2}{2} > 0, \tag{5.6}
$$

and the constant H_2 is to be explained later.

Notice that the function $F(S,I,M,R)$ is continuous, and thus in the interior of $\mathbb{R}_+^4$, it has the minimum value $F(S_0,I_0,M_0,R_0)$. So, a nonnegative function $\widetilde{F}(S,I,M,R)$ can be defined by

$$
\widetilde{F}(S,I,M,R) = F(S,I,M,R) - F(S_0,I_0,M_0,R_0).
$$

By calculation to V_5, it has

$$\begin{aligned}\mathcal{L}V_5 \leq&(S+I+M+R)^a\Big[\Lambda-(d_1-b)S-(d_1-b)I-\mu I-(a_2-b)M\\&-(d_1-b)R+\frac{a_1}{b_1}\Big]+\frac{a(S+I+M+R)^{a-1}}{2}(\sigma_1^2S^2+\sigma_2^2I^2+\sigma_3^2M^2+\sigma_4^2R^2)\\\leq&(\Lambda+\frac{a_1}{b_1})(S+I+M+R)^a-(d_1-b-\frac{a\sigma^2}{2})(S+I+M+R)^{a+1}\\\leq&N_1-\frac{1}{2}(d_1-b-\frac{a\sigma^2}{2})(S+I+M+R)^{a+1},\end{aligned}\tag{5.7}$$

where $N_1=\sup_{(S,I,M,R)\in\mathbb{R}_+^4}[(\Lambda+\frac{a_1}{b_1})(S+I+M+R)^a-\frac{1}{2}(d_1-b-\frac{a\sigma^2}{2})(S+I+M+R)^{a+1}]<\infty$ by (5.6).

Meanwhile,

$$\begin{aligned}\mathcal{L}(-\ln S)=&-\frac{1}{S}[\Lambda-d_1S-\beta SI-d_2mMS]\\&-\frac{1}{S}[b(S+R+M)+pbI+\delta R]+\frac{1}{2}\sigma_1^2\\\leq&-\frac{\Lambda}{S}+d_1+\frac{\sigma_1^2}{2}+\beta I+d_2mM,\\\mathcal{L}(-\ln M)=&-\frac{1}{M}[\frac{a_1I}{1+b_1I}-a_2M]+\frac{\sigma_3^2}{2}\\\leq&-\frac{a_1I}{M(1+b_1I)}+a_2+\frac{\sigma_3^2}{2},\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}(-\ln R)=&-\frac{1}{R}[d_2mMS+\gamma I-(d_1+\delta)R]+\frac{\sigma_4^2}{2}\\\leq&-\frac{d_2mMS}{R}-\frac{\gamma I}{R}+d_1+\delta+\frac{\sigma_4^2}{2}\\\leq&-\frac{\gamma I}{R}+d_1+\delta+\frac{\sigma_4^2}{2}.\end{aligned}$$

Hence, let $N_2=2d_1+a_2+\delta+\frac{\sigma_1^2+\sigma_3^2+\sigma_4^2}{2}$,

$$\begin{aligned}\mathcal{L}\widetilde{F}(S,I,M,R,\bar{S})\leq&H_2[-\Delta+\frac{\beta^2C_1}{d_1-b}I+\frac{\beta d_2mC_1a_1}{(d_1-b)a_2}I]+N_1+N_2\\&-\frac{1}{2}(d_1-b-\frac{a\sigma^2}{2})(S+I+M+R)^{a+1}-\frac{\Lambda}{S}+\beta I\\&+d_2mM-\frac{a_1I}{M(1+b_1I)}-\frac{\gamma I}{R}-H_2[\beta(\bar{S}-\frac{\beta}{d_1-b})]\\=:&\widetilde{F}_1(S,I,M,R)-H_2[\beta(\bar{S}-\frac{\beta}{d_1-b})].\end{aligned}\tag{5.8}$$

We define the function $f_1(S,I,M,R):=\frac{1}{2}(d_1-b-\frac{a\sigma^2}{2})(S+I+M+R)^{a+1}$ for convenience. Let the constants $N_i(i=3,4,5)$ be defined as follows,

$$N_3 := \sup_{(S,I,M,R)\in\mathbb{R}_+^4} \Big\{ - f_1 + H_2[-\Delta + \frac{\beta^2 C_1}{d_1 - b} I + \frac{\beta d_2 m C_1 a_1}{(d_1 - b)a_2} I] + \beta I + d_2 m M \Big\},$$
$$N_4 = \sup_{(S,I,M,R)\in\mathbb{R}_+^4} \Big\{ - f_1(S, I, M, R) + \beta I + d_2 m M \Big\},$$

and

$$N_5 := \sup_{\{S\geq\frac{1}{\epsilon_2}\}\times(I,M,R)\in\mathbb{R}_+^3} \Big\{ H_2[-\Delta + \frac{\beta^2 C_1}{d_1 - b} I + \frac{\beta d_2 m C_1 a_1}{(d_1 - b)a_2} I] - \frac{1}{2} f_1 + \beta I + d_2 m M \Big\}.$$

It is easy to see that $N_i < \infty$ $(i = 3, 4, 5)$. Let the constant ϵ_2 be sufficiently small and H_2 sufficiently large such that

$$-\Delta + \frac{\beta^2 C_1}{d_1 - b}\epsilon_2 + \frac{\beta d_2 m C_1 a_1}{(d_1 - b)a_2}\epsilon_2 < 0, \tag{5.9}$$

$$-\frac{\Lambda}{\epsilon_2} + N_1 + N_2 + N_3 \leq -1, \tag{5.10}$$

$$H_2\Big(- \Delta + \frac{\beta^2 C_1}{d_1 - b}\epsilon_2 + \frac{\beta d_2 m C_1 a_1}{(d_1 - b)a_2}\epsilon_2\Big) + N_1 + N_2 + N_4 \leq -1, \tag{5.11}$$

$$-\frac{a_1}{\epsilon_2(1 + b_1\epsilon_2)} + N_1 + N_2 + N_3 \leq -1, \tag{5.12}$$

$$-\frac{\gamma}{\epsilon_2} + N_1 + N_2 + N_3 \leq -1, \tag{5.13}$$

and

$$-\frac{d_1 - b - \frac{a\sigma^2}{2}}{4}\frac{1}{\varepsilon_2^{a+1}} + N_1 + N_2 + N_5 \leq -1. \tag{5.14}$$

For the ϵ_2 above, define the bounded set E as the following form,

$$E := \{(S, I, M, R) : \epsilon_2 \leq S \leq \frac{1}{\epsilon_2}, \epsilon_2 \leq I \leq \frac{1}{\epsilon_2}, \epsilon_2^2 \leq R \leq \frac{1}{\epsilon_2^2}, \epsilon_2^2 \leq M \leq \frac{1}{\epsilon_2^2}\}.$$

The following will suffice to prove $\mathcal{L}\widetilde{F}_1(S, I, M, R) \leq -1$ in the domain $\mathbb{R}_+^4 \setminus E$. Note that $\mathbb{R}_+^4 \setminus E$ could be divided into eight sub-regions E_i^c, $i = 1, ..., 8$:

$$E_1^c = \{(S, I, M, R) \in \mathbb{R}_+^4 : S < \epsilon_2\},\ \ E_2^c = \{(S, I, M, R) \in \mathbb{R}_+^4 : I < \epsilon_2\},$$
$$E_3^c = \{(S, I, M, R) \in \mathbb{R}_+^4 : M < \epsilon_2^2, I \geq \epsilon_2\},\ E_4^c = \{(S, I, M, R) \in \mathbb{R}_+^4 : R < \epsilon_2^2, I \geq \epsilon_2\},$$
$$E_5^c = \{(S, I, M, R) \in \mathbb{R}_+^4 : S > \frac{1}{\epsilon_2}\},\ E_6^c = \{(S, I, M, R) \in \mathbb{R}_+^4 : I > \frac{1}{\epsilon_2}\},$$
$$E_7^c = \{(S, I, M, R) \in \mathbb{R}_+^4 : M > \frac{1}{\epsilon_2^2},\ E_8^c = \{(S, I, M, R) \in \mathbb{R}_+^4 : R > \frac{1}{\epsilon_2^2}\}.$$

(i) When $(S, I, M, R) \in E_1^c$, it follows from (5.8) and (5.10) that

$$\widetilde{F}_1(S, I, M, R) \leq -\frac{\Lambda}{S} + N_1 + N_2 + N_3 \leq -\frac{\Lambda}{\epsilon_2} + N_1 + N_2 + N_3 \leq -1.$$

(ii) When $(S, I, M, R) \in E_2^c$, it yields from (5.8) and (5.11) that

$$\widetilde{F}_1(S, I, M, R) \leq H_2\Big(- \Delta + \frac{\beta^2 C_1}{d_1 - b}\epsilon_2 + \frac{\beta d_2 m C_1 a_1}{(d_1 - b)a_2}\epsilon_2\Big) + N_1 + N_2 + N_4 \leq -1.$$

(iii) When $(S, I, M, R) \in E_3^c$, (5.8) and (5.12) lead to

$$\begin{aligned}\widetilde{F}_1(S, I, M, R) &\leq -\frac{a_1 I}{M(1 + b_1 I)} + N_1 + N_2 + N_3 \\ &\leq -\frac{a_1}{\epsilon_2(1 + b_1\epsilon_2)} + N_1 + N_2 + N_3 \leq -1.\end{aligned}$$

(iv) When $(S, I, M, R) \in E_4^c$, it follows from (5.8) and (5.13) that

$$\widetilde{F}_1(S, I, M, R) \leq -\frac{\gamma I}{R} + N_1 + N_2 + N_3 \leq -\frac{\gamma}{\epsilon_2} + N_1 + N_2 + N_3 \leq -1.$$

(v) When $(S, I, M, R) \in E_5^c$, we know from (5.8) and (5.14) that

$$\begin{aligned}\widetilde{F}_1(S, I, M, R) &\leq - \frac{d_1 - b - \frac{a\sigma^2}{2}}{4} S^{a+1} + N_1 + N_2 + N_5 \\ &\leq - \frac{d_1 - b - \frac{a\sigma^2}{2}}{4}\frac{1}{\varepsilon_2^{a+1}} + N_1 + N_2 + N_5 \leq -1.\end{aligned}$$

The cases in E_6^c, E_7^c, E_8^c are similar to that in E_5^c, which we will omit here. Hence, the assertion that $\mathcal{L}\widetilde{F}_1(S, I, M, R) \leq -1$ in $\mathbb{R}_+^4 \setminus E$ is obtained.

Meanwhile, by the continuity of $\widetilde{F}_1(S, I, M, R)$ and the compactness of E, there is a constant $H_3 > 0$ such that $\widetilde{F}_1(S, I, M, R) \leq H_3$ for $(S, I, M, R) \in E$. Thus, it yields

$$\begin{aligned}&- \frac{\mathbb{E}(\widetilde{F}(S_0, I_0, M_0, R_0))}{t} \\ \leq& \frac{\mathbb{E}(\widetilde{F}(S_t, I_t, M_t, R_t)) - \mathbb{E}(\widetilde{F}(S_0, I_0, M_0, R_0))}{t} \\ =&\frac{1}{t}\int_0^t \mathbb{E}[\mathcal{L}\widetilde{F}(S_u, I_u, M_u, R_u)]du \\ \leq&\frac{1}{t}\int_0^t \widetilde{F}_1(S_u, I_u, M_u, R_u)du - H_2\beta\frac{1}{t}\int_0^t [(\widetilde{S}_u - \frac{\Lambda}{d_1 - b})]du.\end{aligned}$$

By using the ergodicity of $\bar{S}_t$, it has

$$
\begin{aligned}
0 &\leq \liminf_{t\to\infty} \frac{1}{t}\int_0^t \widetilde{F}_1(S_u, I_u, M_u, R_u)du \\
&= \liminf_{t\to\infty} \frac{1}{t}\int_0^t (\widetilde{F}_1(S_u, I_u, M_u, R_u)\mathbb{I}_{\{(S,I,M,R)\in E\}} + \widetilde{F}_1(S_u, I_u, M_u, R_u)\mathbb{I}_{\{(S,I,M,R)\in E^c\}})du \\
&\leq \liminf_{t\to\infty} \frac{1}{t}\int_0^t H_3\mathbb{P}(\{(S_u, I_u, M_u, R_u)\in E\}) + (-1)\mathbb{P}(\{(S_u, I_u, M_u, R_u)\in E^c\})du \\
&\leq \liminf_{t\to\infty} \frac{1}{t}\int_0^t [(1+H_3)\mathbb{P}\{(S_u, I_u, M_u, R_u)\in E\} - 1]du.
\end{aligned}
$$

This means

$$
\liminf_{t\to\infty} \frac{1}{t}\int_0^t \mathbb{P}(u, (S_0, I_0, M_0, R_0), E)du \geq \frac{1}{1+H_3}. \tag{5.15}
$$

Therefore, due to the compactness of E and (5.15), model (1.3) has the invariant probability measure by exploiting Theorem 2 in [30].

6. The case of $\Delta = 0$

This section will deal with the case of $\Delta = 0$, which is a critical one that has been less investigated in literature.

Theorem 6.1. *For the model (1.3) with initial condition $(s, i, m_0, r) \in \mathbb{R}_+^{4,o}$, if $\Delta = 0$ and $d_1 - b + \mu - \frac{ba_1}{a_2} > 0$, then*

$$
\lim_{t\to\infty} \frac{1}{t}\int_0^t I(u)du = 0, \ a.s. \tag{6.1}
$$

Proof. We prove it by contradiction. Assume that (S_t, I_t, M_t, R_t) has the invariant measure $\mathfrak{m}$ on $\mathbb{R}_+^{4,o}$. Thus, it can be concluded by the ergodicity that for any $\mathfrak{m}$ measurable function g,

$$
\lim_{t\to\infty} \frac{1}{t}\int_0^t g(S_u, I_u, M_u, R_u)du = \int_{\mathbb{R}_+^{4,o}} g(s, i, m, r)\mathfrak{m}(ds, di, dm, dr). \tag{6.2}
$$

Hence, there exist positive constants h_1 and h_2 such that

$$
\lim_{t\to\infty} \frac{1}{t}\int_0^t I_u du = \int_{\mathbb{R}_+^{4,o}} i\mathfrak{m}(ds, di, dm, dr) = h_1 > 0,
$$

and

$$
\lim_{t\to\infty} \frac{1}{t}\int_0^t R_u du = \int_{\mathbb{R}_+^{4,o}} r\mathfrak{m}(ds, di, dm, dr) = h_2 > 0.
$$

Integrating for the second equation of (1.3) and using (2.4) as well as Lemma 2.2 leads to

$$
\lim_{t\to\infty} \frac{1}{t}\int_0^t (\beta S_u I_u - (d_1 + \gamma + \mu - qb)I_u)du + \lim_{t\to\infty} \frac{1}{t}\int_0^t \sigma_2 I_u dW_2(u) = \lim_{t\to\infty} \frac{I_t - i}{t} = 0. \tag{6.3}
$$

Thus, $\lim_{t\to\infty}\frac{1}{t}\int_0^t \beta S_u I_u du = (d_1+\gamma+\mu-qb)\lim_{t\to\infty}\frac{1}{t}\int_0^t I_u du$.
Utilizing the same method yields to

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t d_2 m M_u S_u du = \lim_{t\to\infty}\frac{1}{t}\int_0^t (-\gamma I_u + (d_1+\delta)R_u)du, \tag{6.4}$$

and

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t a_2 M_u du = \lim_{t\to\infty}\frac{1}{t}\int_0^t \frac{a_1 I_u}{1+b_1 I_u}du \le \lim_{t\to\infty}\frac{1}{t}\int_0^t a_1 I_u du. \tag{6.5}$$

For (4.14), it has

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t &[-(d_1-b)(\bar{S}_u - S_u) + \beta S_u I_u + d_2 m M_u S_u \\ &- pbI_u - bM_u - (b+\delta)R_u]du = \lim_{t\to\infty}\frac{\bar{S}_t - S_t}{t} = 0,\end{aligned} \tag{6.6}$$

Substituting (6.3)–(6.5) into the above equation yields

$$\begin{aligned}0 =& \lim_{t\to\infty}\frac{1}{t}\int_0^t [-(d_1-b)(\bar{S}_u - S_u) + \beta S_u I_u + d_2 m M_u S_u - pbI_u - bM_u - (b+\delta)R_u]du \\ \ge& \lim_{t\to\infty}\frac{1}{t}\int_0^t [-(d_1-b)(\bar{S}_u - S_u) + (d_1 - b + \mu - \frac{ba_1}{a_2})I_u + (d_1-b)R_u]du.\end{aligned}$$

Then,

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t}\int_0^t (\bar{S}_u - S_u)du \ge& \frac{1}{d_1-b}\lim_{t\to\infty}\frac{1}{t}\int_0^t [(d_1 - b + \mu - \frac{ba_1}{a_2})I_u + (d_1-b)R_u]du \\ \ge& \frac{1}{d_1-b}[(d_1 - b + \mu - \frac{ba_1}{a_2})h_1 + (d_1-b)h_2 =: h_3 > 0.\end{aligned} \tag{6.7}$$

Therefore, we have

$$\begin{aligned}\lim_{t\to\infty}\frac{\ln I_t}{t} =& \lim_{t\to\infty}\frac{\ln I_0 + \sigma_2 W_2(t)}{t} + \lim_{t\to\infty}\frac{1}{t}\int_0^t (\beta S_u - (d_1+\gamma+\mu-qb+\frac{\sigma_2^2}{2}))du \\ =& \lim_{t\to\infty}\frac{1}{t}\int_0^t (\beta\bar{S}_u - (d_1+\gamma+\mu-qb+\frac{\sigma_2^2}{2}))du - \lim_{t\to\infty}\frac{1}{t}\int_0^t \beta(\bar{S}_u - S_u)du \\ \le& \Delta - \beta h_3 = -\beta h_3 < 0, \ a.s.\end{aligned}$$

Consequently, $\mathbb{P}\{\lim_{t\to\infty} I_t = 0\} = 1$, which contradicts the hypothesis at the beginning of this proof.

7. Discussion and simulations

From the analysis above, we see that Δ could be used to determine the different dynamical behavior of the model. In the deterministic model without stochastic noise, if $\Delta > 0$, then the disease will persist. When the noise σ_2 is large enough, $\Delta < 0$ can always hold, which means the disease shall be extinct by Theorem 3.1. This reveals that stochastic noise contributes to the extinction of disease.

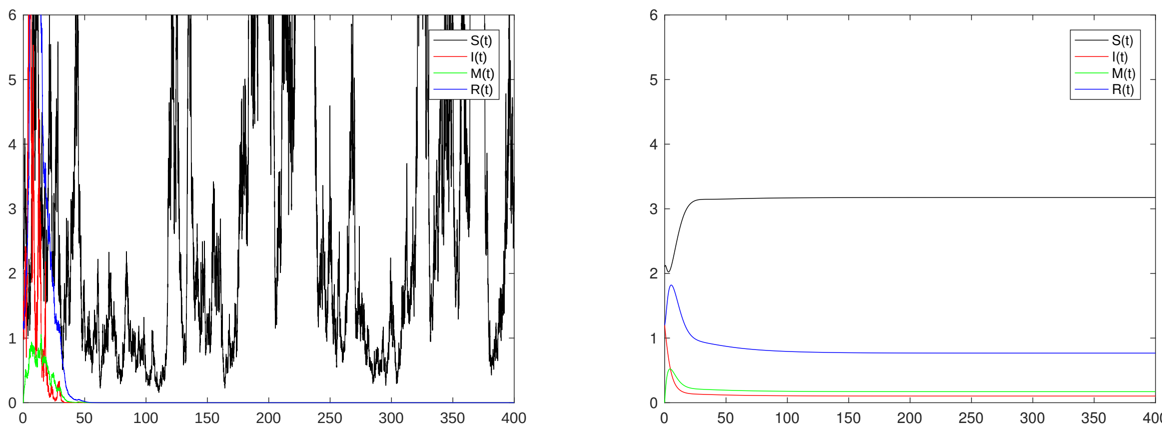
If $b = 0$ and $q = 0$, that is, there is no vertical transmission, the model (1.3) we discuss in this paper becomes the one in [16]. The authors in [16] obtained the value $\Re_0^S$ to decide the persistence and stationary distribution of the model with information intervention. However, compared with the value R_1^S in this paper, there is an additional term $\frac{\mu_1 ma}{a_0 b}$ in $\Re_0^S$, which makes the value $\Re_0^S$ smaller. In fact, this term can be eliminated by establishing appropriate functions. Due to the two terms $\frac{\mu_1 ma}{a_0 b}$ in $\Re_0^S$ and $\frac{1}{2}\sigma^2$, there is a greater gap between the value $\Re_S$ to judge extinction and the value $\Re_0^S$ to determine persistence. This article gets the same value that determines both behaviors. In addition, we also obtain more detailed estimates of I, M, and R when $\Delta < 0$.

In addition, our model is more complex and general than that in [22]. If there is no M_t and R_t class, model (1.3) will degenerate into the model similar to that in [22]. For the two-dimensional model there, the author obtained the values R_0^S and $\tilde{R}_0^S$ for deciding the extinction ($\tilde{R}_0^S < 1$) and the existence of stationary distribution ($R_0^S > 1$), which are different. In our paper, a new function is built to acquire the same threshold that determines different properties.

Moreover, we see from the expression of Δ that qb will make Δ larger, and when $\Delta > 0$, the disease will spread by Theorem 3.2. Therefore, it is proposed that women should avoid to get pregnant during the period of infection for the sake of maternal and child health, which will reduce the vertical transmission rate and be beneficial for disease control.

In what follows, we will enumerate some examples to check the conclusions reached in the previous section.

Example 7.1. In order to verify the conclusion of Theorem 3.1, let $\Lambda = 0.12$, $d_1 = 0.05$, $\beta = 0.16$, $a_1 = 0.12$, $b_1 = 0.12$, $a_2 = 0.5$, $b = 0.02$, $p = 0.4$, $\mu = 0.02$, $\delta = 0.35$, $\gamma = 0.45$, $m = 1.2$, $\sigma_1 = 0.4$, $\sigma_2 = 0.6$, $\sigma_3 = 0.2$ and $\sigma_4 = 0.1$, thus, the parameter $\Delta = -0.048 < 0$, the disease will die out; see Figure 1(a) with the initial condition $S_0 = 2.1$, $I_0 = 1.2$, $M_0 = 0$, $R_0 = 1.2$. While in the deterministic model without noise, the value $\Delta = 0.132 > 0$, the disease is persistent; see Figure 1(b). Figure 1(b) shows the trajectories of various parts of the model, indicating that the disease is persistent, while Figure 1(a) shows that the disease is extinct without noise.



(a) (b)

Figure 1. (a) The trajectory of model (1.3) taking values in Example 7.1; (b) the trajectory of model (1.3) with values in Example 6.1 without stochastic noise.

Example 7.2. Let $\Lambda = 0.12$, $d_1 = 0.06$, $d_2 = 0.8$, $\beta = 0.08$, $a_1 = 0.12$, $b_1 = 1$, $a_2 = 0.25$, $b = 0.03$, $p = 0.4$, $\mu = 0.04$, $\delta = 0.35$, $\gamma = 0.55$, $m = 0.15$, $\sigma_1 = 0.3$, $\sigma_2 = 0.4$, $\sigma_3 = 0.2$ and $\sigma_4 = 0.1$ such that $\Delta < -0.392$. According to the results of Proposition 3.1, $\lim_{t\to\infty} \frac{\ln R_t}{t} = -0.27$, $\lim_{t\to\infty} \frac{\ln R_t}{t} = -0.27$ and $\lim_{t\to\infty} \frac{\ln |S_t - \bar{S}_t|}{t} \leq -0.075$; see Figure 2.

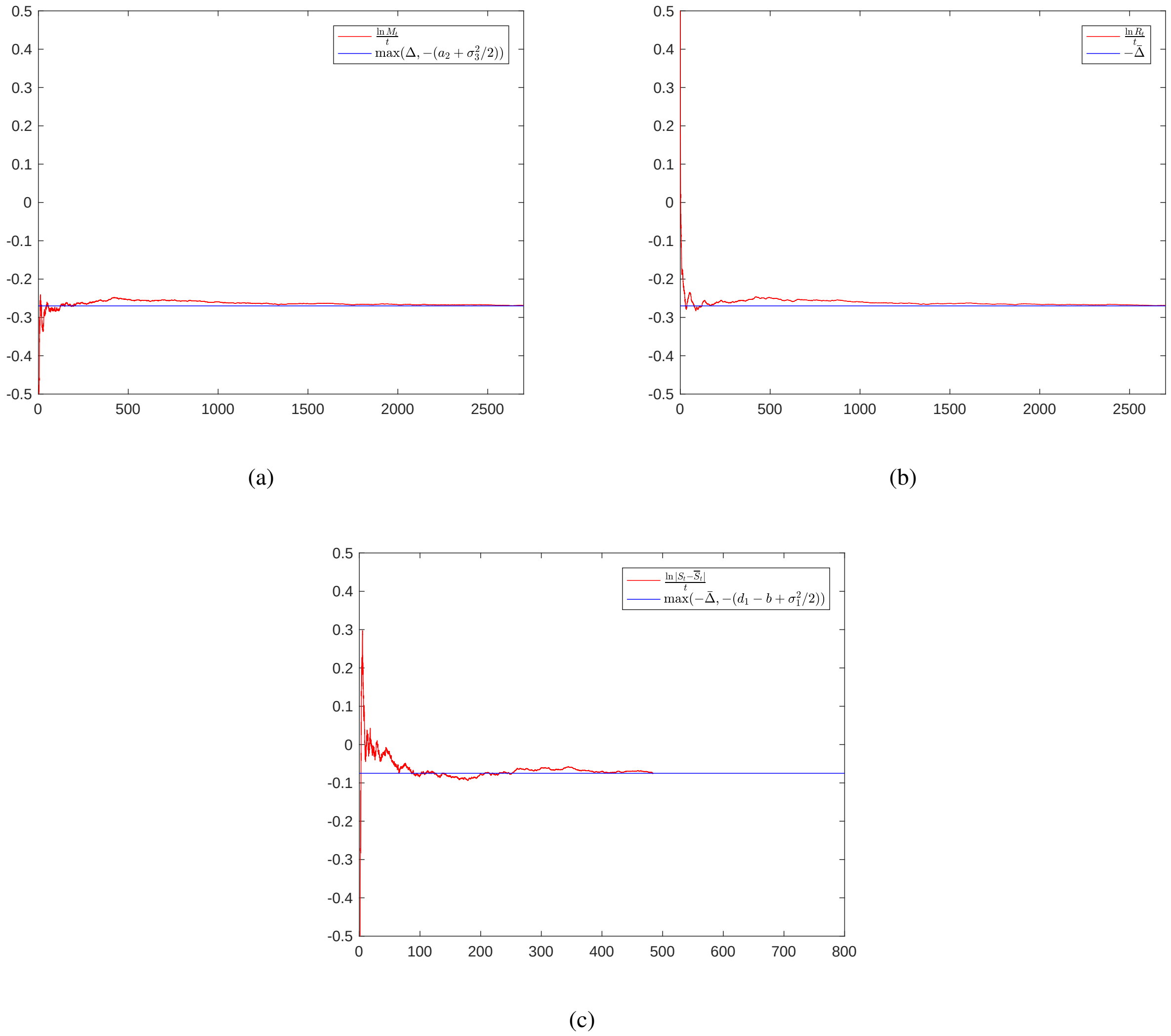
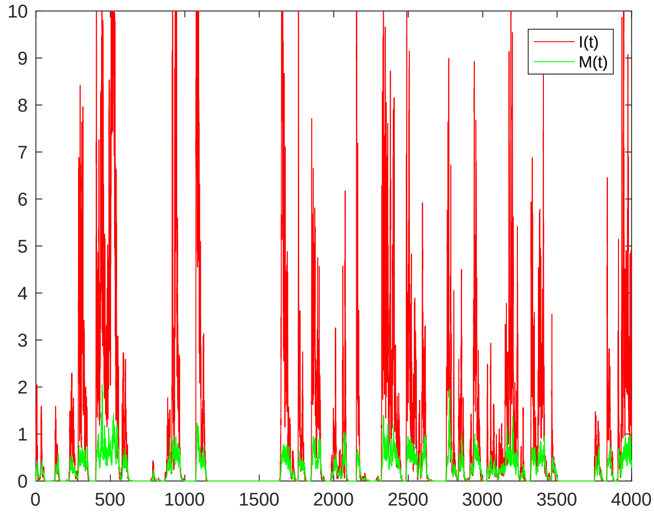


Figure 2. The trajectory of model (1.3) with values in Example 7.2 and the same initial values as in Example 7.1: (a) the trajectory of $\frac{\ln R_t}{t}$, (b) the trajectory of $\frac{\ln M_t}{t}$, and (c) the trajectory of $\frac{\ln |S_t - \bar{S}_t|}{t}$.

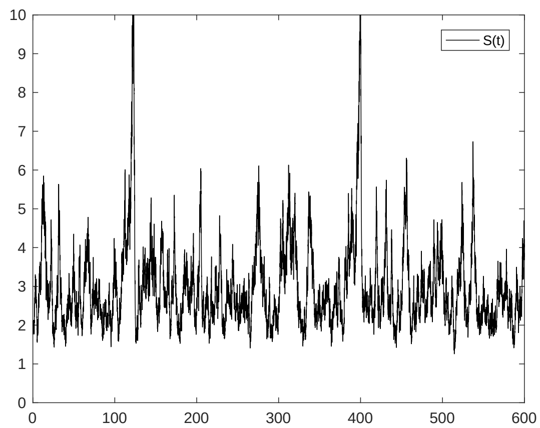
Example 7.3. Let $\Lambda = 0.12$, $\beta = 0.2$ and $\gamma = 0.45$; other parameters are the same as those in Example 7.2. Thus, $\Delta = 0.7213 > 0$. We know from Theorem 3.2 that the disease in model (1.3) is persistent in the mean; see Figure 3.



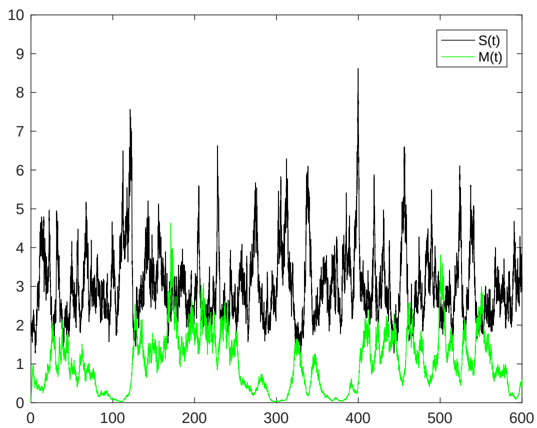
(a)

Figure 3. The trajectory of I_t and M_t in model (1.3) with values in Example 7.3.

Example 7.4. Now, we discuss the influence of information intervention factor on the behavior of model (1.3). Suppose that $m = 0.5$, $d_2 = 0.8$, and other parameters are the same as those in Example 6.2. First, let $a_1 = 0$ $a_2 = 0$, that is, there is no M_t class, then $\Delta = 0.7563 > 0$ and the disease will spread. The trajectory of susceptible class S_t is shown in Figure 4(a). When $a_1 = 0.9$, $a_2 = 0.3$, and $\sigma_3 = 0.2$, the class M_t affected by the information intervention exists and makes themselves as uninfected as possible through different measures, such as self-isolation or vaccination, which will reduce the size of the susceptible population to varying degrees. Figure 4(b) shows the trajectory of M_t and S_t, where the trajectory of S_t is slightly smaller than that of S_t in Figure 4(a).



(a)



(b)

Figure 4. (a) The trajectory of S_t in model (1.3) taking values in Example 7.4; (b) the trajectory of S_t and M_t in model (1.3) with values in Example 7.4.

Example 7.5. Next, to verify the conclusion of Theorem 6.1, let $\Lambda = 0.12$, $d_1 = 0.06$, $\beta = 0.2$, $a_1 = 0.2$, $b_1 = 1$, $a_2 = 0.25$, $b = 0.02$, $p = 0.4$, $\mu = 0.047$, $\delta = 0.4$, $\gamma = 0.5$, $m = 0.15$, $\sigma_1 = 0.08$, $\sigma_2 = 0.1$, $\sigma_3 = 0.5$ and $\sigma_4 = 0.05$; thus, the parameter $\Delta = 0$ and $d_1 - b + \mu - \frac{ba_1}{a_2} > 0$, and the disease will not be persistent in the mean. See Figure 5 with the initial condition $S_0 = 2.1$, $I_0 = 1.2$, $M_0 = 0$, and $R_0 = 1.2$. (a) is the sample path of model (1.3) and (b) represents the trajectory of $\frac{1}{t}\int_0^t I_s ds$.

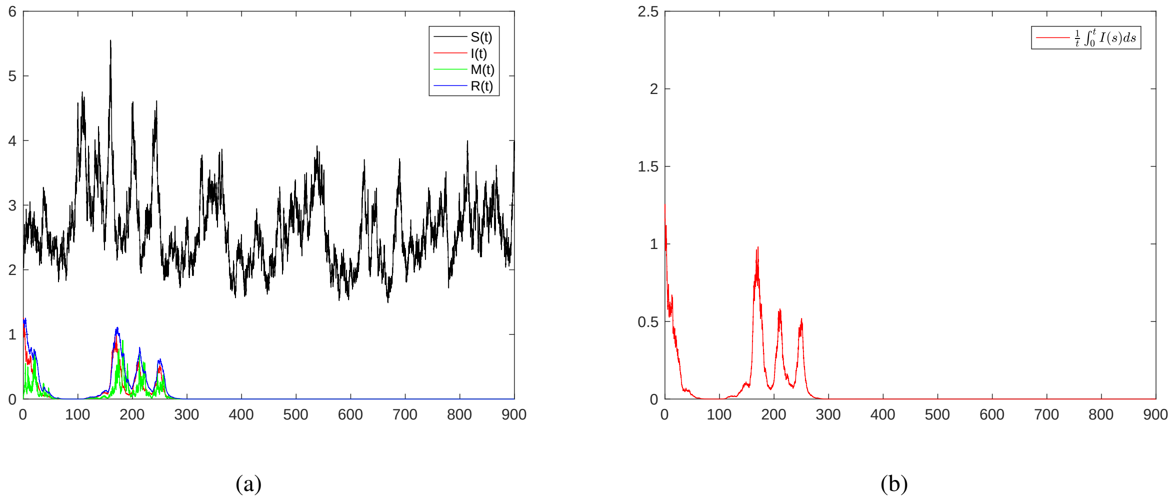


Figure 5. (a) The sample path of (1.3) with values in Example 7.5; (b) the trajectory of $\frac{1}{t}\int_0^t I_s ds$ in (1.3) with values in Example 7.5.

8. Conclusions and future research

The dynamic behavior of a stochastic epidemic model with information intervention and vertical transmission was the concern of this paper. The threshold to judge the extinction and persistence of the disease is obtained. When $\Delta = \frac{\beta\Lambda}{d_1 - b} - (d_1 + \gamma + \mu - qb + \frac{1}{2}\sigma_2^2) < 0$, the three classes I_t, M_t, and R_t appearing in the model go extinct at an exponential rate, and the susceptible class S_t almost surely converges to the solution of the boundary equation exponentially. When $\Delta > 0$, the disease in the model is persistent in the mean. Besides, the existence of invariant probability measure under this condition is proved by constructing proper Lyapunov functions. In addition, the critical case of $\Delta = 0$ is also investigated and it is found that the disease will not be persistent in the mean under some conditions. Several discussions are presented to explain the results and some numerical examples are proposed to verify the obtained results.

A few other issues are worth further studies. This paper analyzes the model with a bilinear incidence rate, while a nonlinear one can be applied to a wider range of circumstances. Therefore, it will be more generic to generalize the model to one with nonlinear incidence. We consider, in this paper, that the stochastic noise is continuously characterized by white noise, and the introduction of more noises such as Markovian switching and Lévy noise will enable the model to be more realistic. Further research can be conducted on optimizing strategies for some control and prevention measures. We leave these issues for future investigations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no competing interests.

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