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ERA, 32(5): 3635–3699.
DOI: 10.3934/era.2024167
Received: 30 October 2023
Revised: 19 March 2024
Accepted: 30 April 2024
Published: 04 June 2024

*Research article*

# Minimal resolutions of threefolds

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**Abstract:** We describe the resolution of singularities of a threefold which has minimal Picard number. We describe the relation between this minimal resolution and an arbitrary resolution of singularities.

**Keywords:** resolution of singularities; complex threefolds; minimal model program

## 1. Introduction

Smooth varieties have certain nice properties, and both algebraic and analytic methods can be applied to them. However, when studying problems in birational geometry, particularly those related to the minimal model program, it becomes necessary to investigate varieties with singularities. Fortunately, as Hironaka's famous theorem states, every variety in characteristic zero has a resolution of singularity. The existence of resolutions of singularities provides a way to study singular varieties. For instance, by comparing a variety to its resolution of singularities, one can measure the complexity of a singularity. This is a fundamental technique in higher-dimensional birational geometry.

Since we want to understand a singularity through its resolution, it is natural to inquire about the difference between two distinct resolutions of singularities. For an algebraic surface $S$, there exists a smooth surface known as the minimal resolution of $S$. This is a resolution of singularities $\bar{S} \to S$ such that $\rho(\bar{S}/S)$ is minimal. The minimal resolution $\bar{S}$ is unique, and any birational morphism $S' \to S$ from a smooth surface $S'$ to $S$ factors through $\bar{S} \to S$.

In this paper, we want to find a higher-dimensional analog of the minimal resolution for surfaces. It is not reasonable to assume the existence of a unique minimal resolution for higher-dimensional singularities. For instance, if $X \dashrightarrow X'$ is a smooth flop over $W$, then $X$ and $X'$ are two different resolutions of singularities for $W$. Since flops are symmetric (at least in dimension three), it appears that $X$ and $X'$ are both minimal. Thus, we need to consider the following issues:

(1) To define "minimal resolutions", which ideally should be resolutions of singularities with some minimal geometric invariants.

(2) To compare two different minimal resolutions. We need some symmetry between them, so that even if minimal resolutions are not unique, it is not necessary to distinguish them.

(3) To compare a minimal resolution with an arbitrary resolution of singularities.

Inspired by the two-dimensional case, it is natural to consider resolutions of singularities with minimal Picard number (we will call such resolutions P-minimal resolutions, see Section 7 for a more precise definition). We know that a fixed singularity may have more than one P-minimal resolution, and two different P-minimal resolutions can differ by a smooth flop. It is also possible that two different P-minimal resolutions "differ by a singular flop": consider $X \dashrightarrow X'$ as a possibly singular flop over $W$. Let $\tilde{X} \to X$ and $\tilde{X}' \to X'$ be P-minimal resolutions of $X$ and $X'$, respectively. Then, because of the symmetry between flops, one may expect that $\tilde{X}$ and $\tilde{X}'$ are two different P-minimal resolutions of $W$. We call the birational map $\tilde{X} \dashrightarrow \tilde{X}'$ a P-desingularization of the flop $X \dashrightarrow X'$ (a precise definition can be found in Section 7). If we consider P-desingularizations of flops as elementary birational maps, then in dimension three, P-minimal resolutions have nice properties.

**Theorem 1.1.** Assume that $X$ is a projective threefold over the complex numbers and $\tilde{X}_1$, $\tilde{X}_2$ are two different P-minimal resolutions of $X$. Then $\tilde{X}_1$ and $\tilde{X}_2$ are connected by P-desingularizations of terminal and $\mathbb{Q}$-factorial flops.

Moreover, if $X$ has terminal and $\mathbb{Q}$-factorial singularities, then the birational map $\tilde{X}_1 \dashrightarrow \tilde{X}_2$ has an $\Omega$-type factorization.

Please see Section 6 for the definition of $\Omega$-type factorizations.

**Theorem 1.2.** Assume that $X$ is a projective threefold over the complex numbers and $W \to X$ is a birational morphism from a smooth threefold $W$ to $X$. Then, for any P-minimal resolution $\tilde{X}$ of $X$, one has a factorization

$$W = \tilde{X}_k \dashrightarrow \ldots \dashrightarrow \tilde{X}_1 \dashrightarrow \tilde{X}_0 = \tilde{X}$$

such that $\tilde{X}_{i+1} \dashrightarrow \tilde{X}_i$ is either a smooth blow-down or a P-desingularization of a terminal $\mathbb{Q}$-factorial flop.

Since three-dimensional terminal flops are topologically symmetric, some topological invariants like Betti numbers will not change after P-desingularizations of terminal flops. Hence, it is easy to see that P-minimal resolutions are the resolution of singularities with minimal Betti numbers.

**Corollary 1.3.** Assume that $X$ is a projective threefold over the complex numbers and $W \to X$ is a birational morphism from a smooth threefold $W$ to $X$. Then, for any P-minimal resolution $\tilde{X}$ of $X$, one has that $b_i(\tilde{X}) \leq b_i(W)$ for all $i = 0, \ldots, 6$.

Although in dimension three P-minimal resolutions behave well, for singularities of dimension greater than three, P-minimal resolutions may not be truly "minimal". A simple example is a smooth flip. If $X \dashrightarrow X'$ is a smooth flip over $W$, then both $X$ and $X'$ are P-minimal resolutions of $W$, but $X'$ is better than $X$. Notice that the only known smooth flips are standard flips [1, Section 11.3], and if $X \dashrightarrow X'$ is a standard flip, then it is easy to see that $b_i(X) \geq b_i(X')$ for all $i$ and the inequality is strict for some $i$. Thus, the resolution of singularities with minimal Betti numbers may be the right minimal resolution for higher-dimensional singularities. Because of Corollary 1.3, in dimension three P-minimal resolutions are exactly those smooth resolutions which have minimal Betti numbers. Therefore, this new definition of minimal resolutions is compatible with our three-dimensional theorems.

We now return to the proof of our main theorems. Let $X$ be a threefold and $W \to X$ be a resolution of singularities. One can run $K_W$-MMP over $X$ as

$$W = X_0 \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_k = X.$$

Let $\tilde{X}_i$ be a P-minimal resolution of $X_i$. Then $\tilde{X}_0 = W$ and it is easy to see that $\tilde{X}_k$ is also a P-minimal resolution of $X$. Thus, our main theorems can be easily proved if we know the relation between $\tilde{X}_i$ and $\tilde{X}_{i+1}$. Since $X_i$ has only terminal and $\mathbb{Q}$-factorial singularities, studying P-minimal resolutions of $X_i$ becomes simpler.

In [2], Chen introduced feasible resolutions for terminal threefolds, which is a resolution of singularities consisting of a sequence of divisorial contractions to points with minimal discrepancies (see Section 2.3.3 for more detail). Given a terminal threefold $X$ and a feasible resolution $\bar{X}$ of $X$, one can define the generalized depth of $X$ to be the integer $\rho(\bar{X}/X)$. The generalized depth is a very useful geometric invariant of a terminal threefold. In our application, the crucial factor is that one can test whether a resolution of singularities $W \to X$ is a feasible resolution or not by comparing $\rho(W/X)$ and the generalized depth of $X$. We need to understand how generalized depths change after steps of the minimal model program. After that, we can prove that for terminal and $\mathbb{Q}$-factorial threefolds, P-minimal resolutions and feasible resolutions coincide.

Now we only need to figure out the following two things: how generalized depths change after a step of minimal model program (MMP), and how P-minimal resolutions change after a step of MMP. To answer those questions, we have to factorize a step of MMP into more simpler birational maps. In [3], Chen and Hacon proved that three-dimensional terminal flips and divisorial contractions to curves can be factorized into a composition of (inverses of) divisorial contractions and flops. In this paper, we construct a similar factorization for divisorial contractions to points. After knowing the factorization, we are able to answer the two questions above and prove our main theorems.

In addition to the above, we introduce the notion of Gorenstein depth for terminal threefolds. The basic idea is as follows: given a sequence of steps of MMP of terminal threefolds

$$X_0 \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_k,$$

one can show that the generalized depth of $X_k$ is bounded above by the integer $k$ and the generalized depth of $X_0$. That is to say, the number of steps of MMP bounds the singularities on the minimal model. One may ask whether there is an opposite bound. Specifically, if we know the singularities of the minimal model $X_k$, can we bound singularities of $X_0$? In this paper, we define the Gorenstein depth of terminal threefolds, which roughly speaking measures only Gorenstein singularities. One can show that the Gorenstein depth is always non-decreasing when running three-dimensional terminal MMP. Our result on Gorenstein depth will have important applications in [4].

This paper is structured as follows: Section 2 is a preliminary section. In Section 3, we develop some useful tools to construct relations between divisorial contractions to points. Those tools, as well as the explicit classification of divisorial contractions, will be used in Section 4 to construct links of different divisorial contractions to points. In Section 5, we prove the property of the generalized depth. The construction of diagrams in Theorem 1.1 will be given in Section 6. All our main theorems will be proved in Section 7. In the last section, we discuss possible higher-dimensional generalizations of the notion of minimal resolutions, and possible applications of our main theorems.

## 2. Preliminaries

### *2.1. Notation and conventions*

In this paper we only consider varieties over complex numbers.

Let $X$ and $Y$ be two algebraic varieties. We say that $X$ and $Y$ are birational if there exists Zariski open sets $U \subset X$ and $V \subset Y$ such that $U$ and $V$ are isomorphic. If $X$ and $Y$ are birational, we say that $\phi : X \dashrightarrow Y$ is a birational map, and we will denote $\phi|_U$ to be the isomorphism $U \to V$. If $\phi : X \to Y$ is a morphism between $X$ and $Y$ and there exists a Zariski open set $U \subset X$ such that $\phi|_U$ is an isomorphism, then we say that $\phi$ is a birational morphism.

For a divisorial contraction, we mean a birational morphism $Y \to X$ which contracts an irreducible divisor $E$ to a locus of codimension at least two, such that $K_Y$ is $\mathbb{Q}$-Cartier and is anti-ample over $X$. We will denote by $v_E$ the valuation that corresponds to $E$.

Let $G$ be a cyclic group of order $r$ generated by $\tau$. For any $\mathbb{Z}$-valued $n$-tuple $(a_1, ..., a_n)$, one can define a $G$-action on $\mathbb{A}^n_{(x_1,...,x_n)}$ by $\tau(x_i) = \xi^{a_i} x_i$, where $\xi = e^{\frac{2\pi i}{r}}$. We will denote the quotient space $\mathbb{A}^n/G$ by $\mathbb{A}^n_{(x_1,...,x_n)}/\frac{1}{r}(a_1, ..., a_n)$.

We say that $w$ is a weight on $W/G = \mathbb{A}^n_{(x_1,...,x_n)}/G$ defined by $w(x_1, ..., x_n) = \frac{1}{r}(b_1, ..., b_n)$ if $w$ is a map $O_W \to \frac{1}{r}\mathbb{Z}_{\geq 0}$ such that

$$w\Big(\sum_{(i_1,...,i_n)\in\mathbb{Z}^n_{\geq 0}} c_{(i_1,...,i_n)} x_1^{i_1}...x_n^{i_n}\Big) = \min\left\{\frac{1}{r}(b_1 i_1 + ... + b_n i_n) \,\middle|\, c_{(i_1,...,i_n)} \neq 0\right\}.$$

Assume that $\phi : X \dashrightarrow Y$ is a birational map. Let $U \subset X$ be the largest open set such that $\phi|_U$ is an isomorphism and $Z \subset X$ be an irreducible subset such that $Z$ intersects $U$ non-trivially. We will denote by $Z_Y$ the closure of $\phi|_U(Z|_U)$.

### *2.2. Weighted blow-ups*

Let $W = \mathbb{A}^n$ and $G$ be a finite cyclic group, such that $\bar{W} = W/G \cong \mathbb{A}^n_{(x_1,...,x_n)}/\frac{1}{r}(a_1, ..., a_n)$. There is an elementary way to construct a birational morphism $W' \to \bar{W}$, so called the weighted blow-up, defined as follows.

We write everything in the language of toric varieties. Let $N$ be the lattice $\langle e_1, ..., e_n, v\rangle_{\mathbb{Z}}$, where $e_1$, ..., $e_n$ is the standard basis of $\mathbb{R}^n$ and $v = \frac{1}{r}(a_1, ..., a_n)$. Let $\sigma = \langle e_1, ..., e_n\rangle_{\mathbb{R}_{\geq 0}}$. We have $\bar{W} \cong \operatorname{Spec} \mathbb{C}[N^\vee \cap \sigma^\vee]$.

Let $w = \frac{1}{r}(b_1, ..., b_n)$ be a vector such that $b_i = \lambda a_i + k_i r$ for $\lambda \in \mathbb{N}$ and $k_i \in \mathbb{Z}$ with $b_i \neq 0$. We define a weighted blow-up of $\bar{W}$ with weight $w$ to be the toric variety defined by the fan consisting of the cones

$$\sigma_i = \langle e_1, ..., e_{i-1}, w, e_{i+1}, ..., e_n\rangle.$$

Let $U_i$ be the toric variety defined by the cone $\sigma_i$ and lattice $N$, namely

$$U_i = \operatorname{Spec} \mathbb{C}[N^\vee \cap \sigma_i^\vee].$$

**Lemma 2.1.** One has that

$$U_i \cong \mathbb{A}^n/\langle \tau, \tau'\rangle$$

where $\tau$ is the action given by

$$x_i \mapsto \xi_{b_i}^{-r} x_i, \quad x_j \mapsto \xi_{b_i}^{b_j} x_j, \;\; j \neq i$$

and $\tau'$ is the action given by

$$x_i \mapsto \xi_{b_i}^{a_i} x_i, \quad x_j \mapsto \xi_{rb_i}^{a_j b_i - a_i b_j} x_j, \; j \neq i.$$

Here, $\xi_k$ denotes a $k$-th roots of unity for any positive integer $k$.

In particular, the exceptional divisor of $W' \to \bar{W}$ is $\mathbb{P}(b_1, ..., b_n)/G'$ where $G'$ is a cyclic group of order $m$ where $m$ is an integer that divides $\lambda$.

*Proof.* Let $T_i$ be a linear transformation such that $T_i e_j = e_j$ if $j \neq i$ and $T_i w = e_i$. One can see that

$$T_i e_i = \frac{r}{b_i}(e_i - \sum_{j \neq i} \frac{b_j}{r} e_j)$$

and

$$T_i v = \sum_{j \neq i} \frac{a_j}{r} e_j + \frac{a_i}{r} \frac{r}{b_i}(e_i - \sum_{j \neq i} \frac{b_j}{r} e_j) = \frac{a_i}{b_i} e_i + \sum_{j \neq i} \frac{a_j b_i - a_i b_j}{r b_j} e_j.$$

Under this linear transformation, $\sigma_i$ becomes the standard cone $\langle e_1, ..., e_n \rangle_{\mathbb{R}_{\geq 0}}$. Note that

$$\begin{aligned} k_i T_i e_i + \lambda T_i v &= \frac{k_i r + \lambda a_i}{b_i} e_i + \sum_{j \neq i} \frac{\lambda(a_j b_i - a_i b_j) - k_i b_j r}{r b_i} e_j \\ &= e_i + \sum_{j \neq i} \frac{\lambda a_j b_i - b_i b_j}{r b_i} e_j = e_i - \sum_{j \neq i} k_j e_j. \end{aligned}$$

Hence, $e_i \in T_i N$ and $T_i N = \langle e_1, ..., e_n, T_i e_i, T_i v \rangle_{\mathbb{Z}}$. Now $T_i e_i$ corresponds to the action $\tau$ and $T_i v$ corresponds to the action $\tau'$. This means that $U_i \cong \mathbb{A}^n / \langle \tau, \tau' \rangle$.

The computation above shows that $\tau^{k_i} = \tau'^{\lambda}$. If we glue $(x_i = 0) \subset \mathbb{A}^n / \langle \tau \rangle$ together, then we get a weight projective space $\mathbb{P}(b_1, ..., b_n)$. The relation $\tau^{k_i} = \tau'^{\lambda}$ implies that $(x_i = 0) \subset U_i$ can be viewed as $\mathbb{P}(b_1, ..., b_n)/G'$ where $G'$ is a cyclic group of order $m$ for some factor $m$ of $\lambda$. □

**Corollary 2.2.** Let $x_1$, ..., $x_n$ be the local coordinates of $W$ and let $y_1$, ..., $y_n$ be the local coordinates of $U_i$. The change of coordinates of the morphism $U_i \to \bar{W}$ are given by $x_j = y_j y_i^{\frac{b_j}{r}}$ and $x_i = y_i^{\frac{b_i}{r}}$.

*Proof.* The change of coordinates is defined by $T_i^t$, where $T_i$ is defined as in Lemma 2.1. □

**Corollary 2.3.** Assume that

$$S = (f_1(x_1, ..., x_n) = ... = f_k(x_1, ..., x_n) = 0) \subset \bar{W}$$

is a complete intersection and $S'$ is the proper transform of $S$ on $W'$. Assume that the exceptional locus $E$ of $S' \to S$ is irreducible and reduced. Then

$$a(E, S) = \frac{b_1 + ... + b_n}{r} - \sum_{i=1}^{k} w(f_k) - 1.$$

*Proof.* Assume first that $k = 0$. Denote $\phi : W' \to \bar{W}$. Then, on $U_i$, we have

$$\phi^* dx_1 \wedge ... \wedge dx_n = \frac{b_i}{r} y_i^{\frac{b_i}{r}-1} \left( \prod_{j \neq i} y_i^{\frac{b_j}{r}} \right) dy_1 \wedge ... \wedge dy_n,$$

hence $K_{W'} = \phi^* K_{\bar{W}} + (\frac{b_1+...+b_n}{r} - 1)F$ where $F = exc(W' \to \bar{W})$.

Now the statement follows from the adjunction formula. □

**Corollary 2.4.** Let $F = exc(W' \to \bar{W})$. Then

$$F^n = \frac{(-1)^{n-1} r^{n-1}}{b_1 ... b_n m}.$$

Here, $m$ is the integer in Lemma 2.1.

*Proof.* From the change of coordinate formula in Corollary 2.2, one can see that $F|_F = O_{\mathbb{P}(b_1,...,b_n)/G}(-r)$. It follows that

$$F^n = (F|_F)^{n-1} = \frac{(-1)^{n-1} r^{n-1}}{b_1 ... b_n m}.$$

□

**Definition 2.5.** Let $\phi_i : U_i \to \bar{W}$ be the morphism in Corollary 2.2. For any $G$-semi-invariant function $u \in O_W$, we can define the strict transform of $u$ on $U_i$ by $(\phi_i^{-1})_*(u) = \phi^*(u)/y_i^{w(u)}$.

In this paper, we will consider terminal threefolds which are embedded into a cyclic quotient of $\mathbb{A}^4$ or $\mathbb{A}^5$

$$X \hookrightarrow \mathbb{A}^4_{(x,y,z,u)} / \frac{1}{r}(a, b, c, d) \quad \text{or} \quad X \hookrightarrow \mathbb{A}^5_{(x,y,z,u,t)} / \frac{1}{r}(a, b, c, d, e).$$

We say that $Y \to X$ is a weighted blow-up with weight $w$ if $Y$ is the proper transform of $X$ inside the weighted blow-up of $\mathbb{A}^4_{(x,y,z,u)} / \frac{1}{r}(a, b, c, d)$ or $\mathbb{A}^5_{(x,y,z,u,t)} / \frac{1}{r}(a, b, c, d, e)$ with weight $w$.

**Notation 2.6.** Assume that $X$ is of the above form and let $Y \to X$ be a weighted blow-up. The notation $U_x$, $U_y$, $U_z$, $U_u$ and $U_t$ will stand for $U_1$, ..., $U_5$ in Lemma 2.1.

**Notation 2.7.** Assume that $w$ is a weight on $\mathbb{A}^n_{(x_1,...,x_n)}$ determined by $w(x_1, ..., x_n) = (a_1, ..., a_n)$ and

$$f(x_1, ..., x_n) = \sum_{(i_1,...,i_n) \in \mathbb{Z}^n_{\geq 0}} \lambda_{i_1,...,i_n} x_1^{i_1} ... x_n^{i_n}$$

is a regular function on $\mathbb{A}^n$. We denote

$$f_w = \sum_{a_1 i_1 + ... + a_n i_n = w(f)} \lambda_{i_1,...,i_n} x_1^{i_1} ... x_n^{i_n}.$$

## 2.3. Terminal threefolds

### 2.3.1. Local classification

The local classification of terminal threefolds was done by Reid [5] for Gorenstein cases and Mori [6] for non-Gorenstein cases.

**Definition 2.8.** A compound Du Val point $P \in X$ is a hypersurface singularity which is defined by $f(x, y, z) + tg(x, y, z, t) = 0$, where $f(x, y, z)$ is an analytic function which defines a Du Val singularity.

**Theorem 2.9** ( [5, Theorem 1.1]). Let $P \in X$ be a point of threefold. Then $P \in X$ is an isolated compound Du Val point if and only if $P \in X$ is terminal and $K_X$ is Cartier near $P$.

**Theorem 2.10** ( [6], cf. [7, Theorem 6.1]). Let $P \in X$ be a germ of three-dimensional terminal singularity such that $K_X$ has Cartier index $r > 1$. Then

$$X \cong (f(x, y, z, u) = 0) \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{r}(a_1, ..., a_4)$$

such that $f$, $r$ and $a_i$ are given by Table 1.

**Table 1.** Classification of terminal threefolds.

| Type | $f(x, y, z, u)$ | $r$ | $a_i$ | condition |
|---|---|---|---|---|
| $cA/r$ | $xy + g(z^r, u)$ | any | $(\alpha, -\alpha, 1, r)$ | $g \subset m_P^2$<br>$\alpha$ and $r$ are coprime |
| $cAx/4$ | $xy + z^2 + g(u)$<br>$x^2 + z^2 + g(y, u)$ | 4 | $(1, 1, 3, 2)$ | $g \in m_P^3$ |
| $cAx/2$ | $xy + g(z, u)$ | 2 | $(0, 1, 1, 1)$ | $g \in m_P^4$ |
| $cD/3$ | $x^2 + y^3 + z^3 + u^3$<br>$x^2 + y^3 + z^2u + yg(z, u) + h(z, u)$<br>$x^2 + y^3 + z^3 + yg(z, u) + h(z, u)$ | 3 | $(0, 2, 1, 1)$ | $g \in m_P^4$<br>$h \in m_P^6$ |
| $cD/2$ | $x^2 + y^3 + yzu + g(z, u)$<br>$x^2 + yzu + y^n + g(z, u)$<br>$x^2 + yz^2 + y^n + g(z, u)$ | 2 | $(1, 0, 1, 1)$ | $g \in m_P^4$, $n \geq 4$<br>$n \geq 3$ |
| $cE/2$ | $x^2 + y^3 + yg(z, u) + h(z, u)$ | 2 | $(1, 0, 1, 1)$ | $g, h \in m_P^4$<br>$h_4 \neq 0$ |

Assume that $P \in X$ is a three-dimensional terminal singularity. Then there exists a section $H \in |-K_X|$ which has Du Val singularities (referred to as a general elephant). Please see [7, (6.4)] for details.

### 2.3.2. Classification of divisorial contractions to points

Divisorial contractions to points between terminal threefolds are well-classified by Kawamata [8], Hayakawa [9–11], Kawakita [12–14] and Yamamoto [15].

**Theorem 2.11.** Assume that $Y \to X$ is a divisorial contraction to a point between terminal threefolds. Then there exists an embedding $X \hookrightarrow W$ with $W = \mathbb{A}^4_{(x,y,z,u)}$ or $\mathbb{A}^5_{(x,y,z,u,t)}$ and a weight $w(x, y, z, u) =$

$\frac{1}{r}(a_1, ..., a_4)$ or $w(x, y, z, u, t) = \frac{1}{r}(a_1, ..., a_5)$, respectively, such that $Y \to X$ is a weighted blow-up with respect to $w$.

The defining equation of $X \subset W$ and the weight are given in Table 2 to Table 11.

For the reader's convenience, we put these tables in Section 4. In those tables, we use the following notation: For a non-negative integer $m$, the notation $g_{\geq m}$ represents a function $g \in \mathcal{O}_W$ such that $w(g) = m$. The notation $p_m$ represents a function $p \in \mathcal{O}_W$ which is homogeneous of weight $m$ with respect to the weight $w$.

The reference of each of the cases in Table 2, Table 3, Table 5, ..., Table 7, Table 9, ..., Table 11 is as follows:

- Case A1 is [13, Theorem 1.2 (i)]. Case A2 is [15, Theorem 2.6].
- Case Ax1–Ax4 are [9] Theorems 7.4, 7.9, 8.4 and 8.8 respectively.
- Cases D6 and D7 are [13, Theorem 1.2 (ii)]. Case D8–D11 is [15] Theorem 2.1–2.4. Case D12 is [15, Theorem 2.7].
- Case D13 is [9, Theorems 9.9, 9.14, 9.20]. Case D14 is [9, Theorem 9.25].
- Case D16 is [10, Proposition 4.4]. Case D17 is [10, Proposition 4.7, 4.12]. Case D18 is [10, Proposition 4.9]. Case D18 is [10, Proposition 5.4]. Case D19 is [10, Propositions 5.8, 5.13, 5.22, 5.28, and 5.35]. Case D20 is [10, Propositions 5.18 and 5.25]. Case D21 is [10, Propositions 5.16 and 5.32]. Case D22 is [10, Propositions 5.9 and 5.36].
- Cases D23 and D24 is [13, Theorem 1.2(ii)] and [11, Theorem 1.1 (iii)]. Case D25–D28 is [11] Theorem 1.1 (i), (i'), (ii'), (iii), (ii) respectively. Case D29 is [14, Theorem 2].
- Case E19–E21 is [15] Theorems 2.5, 2.9 and 2.10 respectively.
- Case E22 is [9, Theorems 10.11, 10.17, 10.22, 10.28, 10.33 and 10.41]. Case E23 is [9, Theorems 10.33 and 10.47]. Case E24 is [9, Theorems 10.54 and 10.61]. Case E25 is [9, Theorem 10.67]. Case E26 is [11, Theorem 1.2].

Divisorial contractions to $cD$ points of discrepancy one (Case D1–D5 in Table 4) and divisorial contractions to $cE$ points of discrepancy one (Case E1–E18 in Table 8) was completely classified by Hayakawa in his two unpublished papers "Divisorial contractions to cD points" and "Divisorial contractions to cE points". We will briefly introduce how to derive this classification. For more detail, please contact the author or Professor Takayuki Hayakawa in Kanazawa University.

Let $(o \in X)$ be a germ of three-dimensional Gorenstein terminal singular point with type $cD$ or $cE$.

**Step 1:** Construct a divisorial contraction $X_1 \to X$ which contracts an exceptional divisor of discrepancy one to $o$. We refer to [2, Section 4, Section 6] for the explicit construction. $X_1 \to X$ can be viewed as a weighted blow-up with respect to an explicit embedding and a explicit weight.

**Step 2:** Find all exceptional divisors $E$ over $X$ such that $a(E, X) = 1$ and $\mathrm{Center}_X E = o$. We know that $exc(X_1 \to X)$ is an exceptional divisor of discrepancy one. Assume that $E \neq exc(X_1 \to X)$. Then an easy computation on discrepancies shows that $a(E, X_1) < 1$. In particular, $\mathrm{Center}_{X_1} E$ is a non-Gorenstein point. Since $X_1 \to X$ is an explicit weighted blow-up, all non-Gorenstein points on $X_1$ can be explicitly computed, and all exceptional divisors of discrepancy less than one can be explicitly write down. Say $\mathcal{S}$ is the set of exceptional divisors over $X_1$ with discrepancy less than one. One can compute $a(E, X)$ for $E \in \mathcal{S}$. If $a(E, X) > 1$, then we remove $E$ form $\mathcal{S}$. After that, $\mathcal{S}$ is a set consisting exceptional divisors over $X$ of discrepancy one.

**Step 3:** For any exceptional divisor $E \in \mathcal{S}$, the valuation of $E$ on $X$ can be calculated. One can construct a weighted blow-up $Y_E \to X$ with respect to this valuation. If $Y_E$ do not have terminal singularities, then we remove $E$ form $\mathcal{S}$. Now, $Y_E \to X$ for all $E \in \mathcal{S}$, together with $X_1 \to X$, are all divisorial contractions to $o$ with discrepancy one.

2.3.3. The depth

**Definition 2.12.** Let $Y \to X$ be a divisorial contraction which contracts a divisor $E$ to a point $P$. We say that $Y \to X$ is a *w-morphism* if $a(X, E) = \frac{1}{r_P}$, where $r_P$ is the Cartier index of $K_X$ near $P$.

**Definition 2.13.** The depth of a terminal singularity $P \in X$, $dep(P \in X)$, is the minimal length of the sequence

$$X_m \to X_{m-1} \to \cdots \to X_1 \to X_0 = X,$$

such that $X_m$ is Gorenstein and $X_i \to X_{i-1}$ is a $w$-morphism for all $1 \leq i \leq m$.

The generalized depth of a terminal singularity $P \in X$, $gdep(P \in X)$, is the minimal length of the sequence

$$X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X,$$

such that $X_n$ is smooth and $X_i \to X_{i-1}$ is a $w$-morphism for all $1 \leq i \leq n$. The variety $X_n$ is called a *feasible resolution* of $P \in X$.

The Gorenstein depth of a terminal singularity $P \in X$, $dep_{Gor}(P \in X)$, is defined by $gdep(P \in X) - dep(P \in X)$.

For a terminal threefold we can define

$$dep(X) = \sum_P dep(P \in X),$$

$$gdep(X) = \sum_P gdep(P \in X)$$

and

$$dep_{Gor}(X) = \sum_P dep_{Gor}(P \in X).$$

**Remark 2.14.** In the above definition, the existence of a sequence

$$X_m \to X_{m-1} \to \cdots \to X_1 \to X_0 = X,$$

such that $X_m$ is Gorenstein follows from [10, Theorem 1.2]. The existence of a sequence

$$X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X,$$

such that $X_n$ is smooth follows from [2, Theorem 2].

**Definition 2.15.** Assume that $Y \to X$ is a $w$-morphism such that $gdep(Y) = gdep(X) - 1$. Then we say that $Y \to X$ is a *strict w-morphism*.

**Lemma 2.16.** Assume that $Y \to X$ is a divisorial contraction which is a weighted blow-up with the weight $w(x_1, ..., x_n) = \frac{1}{r}(a_1, ..., a_n)$ with respect to an embedding $X \hookrightarrow \mathbb{A}^n_{(x_1,...,x_n)}/G$ where $G$ is a cyclic group of index $r$. Assume that $E$ is an exceptional divisor over $X$ and $v_E(x_1, ..., x_n) = \frac{1}{r}(b_1, ..., b_n)$. Then $\mathrm{Center}_Y E \cap U_i$ non-trivially if and only if $\frac{b_i}{a_i} \le \frac{b_j}{a_j}$ for all $1 \le j \le n$. Here, $U_1$, ..., $U_n$ denotes the canonical affine chart of the weighted blow-up on $Y$.

*Proof.* Let $y_1$, ..., $y_n$ be the local coordinates of $U_i$. Then we have the following change of coordinates formula:

$$x_i = y_i^{\frac{a_i}{r}}, \quad x_j = y_i^{\frac{a_j}{r}} y_j \text{ if } i \neq j.$$

One can see that

$$v_E(y_i) = \frac{b_i}{a_i}, \quad v_E(y_j) = \frac{b_i}{r} - \frac{a_j b_i}{r a_i} \text{ if } i \neq j.$$

We know that $\mathrm{Center}_E Y$ intersects $U_i$ non-trivially if and only if $\frac{b_i}{r} - \frac{a_j b_i}{r a_i} \geq 0$ for all $j \neq i$, or, equivalently, $\frac{b_j}{a_j} \geq \frac{b_i}{a_i}$ for all $j \neq i$. □

**Corollary 2.17.** Assume that $Y \to X$ and $Y_1 \to X$ are two different $w$-morphisms over the same point. Let $E$ and $F$ be the exceptional divisors of $Y \to X$ and $Y_1 \to X$, respectively. Then there exists $u \in \mathcal{O}_X$ such that $v_E(u) < v_F(u)$.

*Proof.* Let $X \to \mathbb{A}^n_{(x_1,...,x_n)}/G$ be the embedding so that $Y_1 \to X$ can be obtained by the weighted blow-up with respect to the embedding. We may assume that $(x_n = 0)$ defines a Du Val section. Then

$$v_E(x_n) = a(E, X) = a(F, X) = v_F(x_n).$$

It follows that $a_n = b_n = 1$ where $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$ are integers in Lemma 2.16. Now, since $a(F, X) = a(E, X)$, one has that $\mathrm{Center}_{Y_1} E$ is a non-Gorenstein point (if $Y_1$ is generically Gorenstein along $\mathrm{Center}_{Y_1} E$, then an easy computation shows that $a(E, X) > a(F, X)$). It follows that $\mathrm{Center}_{Y_1} E \cap U_n$ is empty since $a_n = 1$ implies that $U_n$ is Gorenstein. Thus, by Lemma 2.16 we know that there exists $j$ so that $\frac{b_j}{a_j} < 1$. Hence, $v_E(u) > v_F(u)$ if $u = x_j$. □

### 2.4. *Chen-Hacon factorizations*

We have the following factorization of steps of three-dimensional terminal MMP by Chen and Hacon [3].

**Theorem 2.18** ( [3, Theorem 3.3]). Assume that either $X \dashrightarrow X'$ is a flip over $V$, or $X \to V$ is a divisorial contraction to a curve such that the exceptional locus contains a non-Gorenstein singular point. Then there exists a diagram

$$\begin{array}{ccccc} Y_1 & \dashrightarrow \cdots \dashrightarrow & Y_k \\ \downarrow & & \downarrow \\ X & & X' \\ & \searrow \quad \swarrow & \\ & V & \end{array}$$

such that $Y_1 \to X$ is a $w$-morphism, $Y_k \to X'$ is a divisorial contraction, $Y_1 \dashrightarrow Y_2$ is a flip or a flop, and $Y_i \dashrightarrow Y_{i+1}$ is a flip for $i > 1$. If $X \to V$ is divisorial, then $Y_k \to X'$ is a divisorial contraction to a curve and $X' \to V$ is a divisorial contraction to a point.

**Remark 2.19.** Notation as in the above theorem. From the construction of the diagram, we can state the following:

(1) Let $C_{Y_1}$ be a flipping/flopping curve of $Y_1 \dashrightarrow Y_2$. Then $C_X$ is a flipping curve of $X \dashrightarrow X'$. Here, we use the notation introduced in Section 2.1, so $C_X$ is the image of $C_{Y_1}$ on $X$.

(2) Assume that the exceptional locus of $X \to V$ contains a non-Gorenstein point $P$ which is not a $cA/r$ or a $cAx/r$ point. Then $Y_1 \to X$ can be chosen to be any $w$-morphism over $P$. This statement follows from the proof of [3, Theorem 3.1].

We have the following properties of the depth [3, Propositions 2.15, 3.8 and 3.9]:

**Lemma 2.20.** Let $X$ be a terminal threefold.

1. If $Y \to X$ is a divisorial contraction to a point, then $dep(Y) \geq dep(X) - 1$.

2. If $Y \to X$ is a divisorial contraction to a curve, then $dep(Y) \geq dep(X)$.

3. If $X \dashrightarrow X'$ is a flip, then $dep(X) > dep(X')$.

### 2.5. *The negativity lemma*

We have the following negativity lemma for flips.

**Lemma 2.21.** Assume that $X \dashrightarrow X'$ is a $(K_X + D)$-flip. Then, for all exceptional divisors $E$, one has that $a(E, X, D) \leq a(E, X', D_{X'})$. The inequality is strict if $\mathrm{Center}_X E$ is contained in the flipping locus.

*Proof.* It is a special case of [16, Lemma 3.38]. □

What we really need is the following corollary of the negativity lemma.

**Corollary 2.22.** Assume that $X \dashrightarrow X'$ is a $(K_X + D)$-flip and $C \subset X$ is an irreducible curve which is not a flipping curve. Then $(K_X + D).C \geq (K_{X'} + D_{X'}).C_{X'}$. The inequality is strict if $C$ intersects the flipping locus non-trivially.

*Proof.* Let $X \xleftarrow{\phi} W \xrightarrow{\phi'} X'$ be a common resolution such that $C$ is not contained in the indeterminacy locus of $\phi$. Then Lemma 2.21 implies that $F = \phi^*(K_X + D) - \phi'^*(K_{X'} + D_{X'})$ is an effective divisor and is supported on exactly those exceptional divisors whose centers on $X$ are contained in the flipping locus. Hence,

$$(K_X + D).C - (K_{X'} + D_{X'}).C_{X'} = (\phi^*(K_X + D) - \phi'^*(K_{X'} + D_{X'})).C_W = F.C_W \geq 0.$$

The last inequality is strict if and only if $C_W$ intersects $F$ non-trivially, or, equivalently, $C$ intersects the flipping locus non-trivially. □

## 3. Factorize divisorial contractions to points

Let $Y \to X$ be a divisorial contraction between $\mathbb{Q}$-factorial terminal threefolds that contracts a divisor $E$ to a point. We construct the diagram

$$\begin{array}{ccccc} Z_1 & \dashrightarrow & \dots & \dashrightarrow & Z_k \\ \downarrow & & & & \downarrow \\ Y & & & & Y_1 \\ & \searrow & & \swarrow & \\ & & X & & \end{array}$$

as follows: Let $Z_1 \to Y$ be a $w$-morphism and let $H \in |-K_X|$ be a Du Val section. According to [3, Lemma 2.7 (ii)], we have $a(E, X, H) = 0$. We run the $(K_{Z_1} + H_{Z_1} + \epsilon E_{Z_1})$-MMP over $X$ for some $\epsilon > 0$ such that $(Z_1, H_{Z_1} + \epsilon E_{Z_1})$ is klt. Notice that a general curve inside $E_{Z_1}$ intersects the pair negatively, and a general curve in $F$ intersects the pair positively where $F = exc(Z_1 \to Y)$. Thus, after finitely many $(K_{Z_1} + H_{Z_1} + \epsilon E_{Z_1})$-flips $Z_1 \dashrightarrow \dots \dashrightarrow Z_k$, the MMP ends with a divisorial contraction $Z_k \to Y_1$ which contracts $E_{Z_k}$, and $Y_1 \to X$ is a divisorial contraction which contracts $F_{Y_1}$.

**Lemma 3.1.** Keeping the above notation, assume that $K_{Z_1}$ is anti-nef over $X$ and $E_{Z_1}$ is not covered by $K_{Z_1}$-trivial curves. Then $Z_i \dashrightarrow Z_{i+1}$ is a $K_{Z_i}$-flip or flop for all $i$ and $Z_k \to Y_1$ is a $K_{Z_k}$-divisorial contraction. In particular, $Y_1, Z_2, \dots, Z_k$ are all terminal.

*Proof.* Assume first that $k = 1$. If $Z_1 \to Y_1$ is a $K_{Z_1}$-negative contraction, then we are done. Otherwise, $Z_1 \to Y_1$ is a $K_{Z_1}$-trivial contraction. In this case, $E_{Z_1}$ is covered by $K_{X_1}$-trivial curves, which contradicts our assumption.

Now assume that $k > 1$. We know that $Z_1 \dashrightarrow Z_2$ is a $K_{Z_1}$-flip or flop. Also, notice that a general curve on $E_{Z_1}$ is $K_{Z_1}$-negative. Hence a general curve on $E_{Z_2}$ is $K_{Z_2}$-negative by Corollary 2.22. Now the relative effective cone $NE(Z_2/X)$ is a two-dimensional cone. One of the boundaries of $NE(Z_2/X)$ corresponds to the flipped/flopped curve of $Z_1 \dashrightarrow Z_2$ and is $K_{Z_2}$-non-negative. Since there is a $K_{Z_2}$-negative curve, we know that the other boundary of $NE(Z_2/X)$ is $K_{Z_2}$-negative. Therefore, if $k = 2$, then $Z_2 \to Y_1$ is a $K_{Z_2}$-divisorial contraction, and for $k > 2$, $Z_2 \dashrightarrow Z_3$ is a $K_{Z_2}$-flip. One can prove the statement by repeating this argument $k - 2$ more times. □

We are going to find the sufficient conditions for the assumptions of Lemma 3.1. Our final results are Lemmas 3.4 and 3.6.

Let $X \hookrightarrow \mathbb{A}^n_{(x_1,\dots,x_n)}/G = W/G$ be the embedding such that $Y \to X$ is a weighted blow-up with respect to the weight $w$ and this embedding. First, we show that after replacing $W$ by a larger affine space, if necessary, we may assume that $Z_1 \to Y \to X$ can be viewed as a sequence of weighted blow-ups with respect to the embedding $X \hookrightarrow W/G$.

Let $V$ be a suitable open set which contains $P = \mathrm{Center}_Y F$ such that $Z_1 \to Y$ can be viewed as a weighted blow-up with respect to an embedding $V \hookrightarrow \mathbb{A}^{n_1}_{(y'_1,\dots,y'_{n_1})}/G'$. For $j = 1, \dots, n_1$, we define $D_j \subset V$ to be the Weil divisor corresponding to $y'_j = 0$. Then $D_{j,X} = \phi_* D_j$ is a Weil divisor on $X$. Since $X$ is $\mathbb{Q}$-factorial, $D_{j,X}$ corresponds to a $G$-semi-invariant function $s_j \in \mathcal{O}_W$. We can consider the embedding

$$X \hookrightarrow W/G \hookrightarrow (x_{n+j} - s_j = 0)_{j=1,\dots,n_1} \subset \mathbb{A}^n_{(x_1,\dots,x_{n+n_1})}/G = \bar{W}/G.$$

Then, $Y \to X$ is also a weighted blow-up with respect the weight $\bar{w}$ which is defined by $\bar{w}(x_j) = w(x_j)$ if $j < n$, and $\bar{w}(x_{n+j}) = w(s_j)$. The embedding $X \hookrightarrow \bar{W}$ is exactly what we need.

Now, let $W' \to W$ be the first weighted blow-up. We may assume $P = \text{Center}_Y F$ is the origin of $U_i \subset W'$. Let $y_1, ..., y_n$ be the local coordinate system of $U_i$ that is mentioned in Corollary 2.2. We know that $E|_{U_i} = (y_i = 0)$. Let $f_4, ..., f_n$ be the defining equation of $X \subset W/G$. Then $f'_4, ..., f'_n$ define $Y|_{U_i}$ where $f'_i = (\phi|_{U_i}^{-1})_*(f_i)$. Since $Y$ has terminal singularities, the weighted embedding dimension of $Y|_{U_i}$ near $P$ is less than 4. For $5 \leq j \leq n$, we may write $f'_j = \xi_j y_j + f'_j(y_1, ...y_4)$ for some $\xi_j$ which does not vanish on $P$. One can always assume that $H_Y = (y_3 = 0)$ and so $i \neq 3$.

Let $f'^{\circ}_j = f'_j|_{y_i=y_3=0}$. Then $f'^{\circ}_4, ..., f'^{\circ}_n$ defines $H \cap E$ near $P$. If $f'^{\circ}_j$ is irreducible as a $G'$-semi-invariant function, then we let $\eta'_j = f'^{\circ}_j$. Otherwise, let $\eta'_j$ be a $G'$-semi-invariant irreducible factor of $f'^{\circ}_j$.

**Lemma 3.2.** Assume that $Y \to X$ can be viewed as a four-dimensional weighted blow-up. Then $\eta'_4 = ... = \eta'_n = 0$ defines an irreducible component of $H_Y \cap E$.

*Proof.* Since $Y \to X$ can be viewed as a four-dimensional weighted blow-up, we know that $i \leq 4$ and $f'^{\circ}_j = y_j + f'_j|_{y_3=y_i=0}$ for all $j > 4$. Hence, we have $\eta'_j = f'^{\circ}_j$. One can see that the projection

$$(\eta'_4 = ... = \eta'_n = 0)|_{H_Y \cap E} \subset \mathbb{P}(a_1, ..., a_n) \to \mathbb{P}(a_1, ..., a_4) \supset (\eta'_4 = 0)|_{H_Y \cap E}$$

is an isomorphism. Since $\eta'_4$ is an irreducible function, it defines an irreducible curve. □

Notice that $\eta'_j$ is a polynomial in $y_1, ..., y_n$. There exists $\eta_j \in \mathcal{O}_W$ such that $\eta'_j = (\phi|_{U_i}^{-1})_*(\eta_j)$. We assume that $Y \to X$ is a weighted blow-up with the weight $\frac{1}{r}(a_1, ..., a_n)$ and $Z_1 \to Y$ is a weighted blow-up with the weight $\frac{1}{r'}(a'_1, ..., a'_n)$.

**Lemma 3.3.** Let $\Gamma = (\eta'_4 = ... = \eta'_n = 0)$ and assume that $\Gamma$ is an irreducible and reduced curve. Then

$$K_{Z_1}.\Gamma_{Z_1} = -\frac{a_3^2 v_E(\eta_4)...v_E(\eta_n) r^{n-3}}{m a_1 ... a_n} + \frac{a'_i v_F(\eta'_4)...v_F(\eta'_n) r'^{n-4}}{a'_1 ... a'_n}.$$

Here, $m$ is the integer in Lemma 2.1 corresponding to the weighted blow-up $Y \to X$.

*Proof.* Since $\Gamma \subset E$ and $K_{Z_1} + H_{Z_1}$ is numerically trivial over $X$, we only need to show that

$$H_{Z_1}.\Gamma_{Z_1} = \frac{a_3^2 v_E(\eta_4)...v_E(\eta_n) r^{n-3}}{m a_1 ... a_n} - \frac{a'_i v_F(\eta'_4)...v_F(\eta'_n) r'^{n-4}}{a'_1 ... a'_n}. \quad (3.1)$$

We know that $H_{Z_1}.\Gamma_{Z_1} = H.\Gamma - v_F(H_Y)F.\Gamma_{Z_1}$. We need to show that the first term of (3.1) equals $H.\Gamma$ and the second term of (3.1) equals $v_F(H_Y)F.\Gamma_{Z_1}$.

We have an embedding $Y \subset W' \subset \mathbb{P}_W(a_1, ..., a_n)$. Let $D_j$ be the divisor on $W'$ which corresponds to $\eta'_j$. Then $\Gamma = D_4. \cdots .D_n.E.H$ is a weighted complete intersection, so $\Gamma_{Z_1} = D_{4,Z_1}. \cdots .D_{n,Z_1}.E_{Z_1}.H_{Z_1}$. To compute $H.\Gamma$, we view $\Gamma$ as a curve inside $\mathbb{P}(a_1, ..., a_n)$ which is defined by $H = D_4 = ... = D_n = 0$. It follows that

$$H.\Gamma = \frac{a_3^2 v_E(\eta_4)...v_E(\eta_n) r^{n-3}}{m a_1 ... a_n}.$$

To compute $F.\Gamma_{Z_1}$, one writes

$$\begin{aligned}F.\Gamma_{Z_1} &= F.(\psi^*D_4 - v_F(\eta'_4)F).\cdots.(\psi^*D_n - v_F(\eta'_n)F).(\psi^*E - v_F(E)F).(\phi^*H_Y - v_F(H_Y)F)\\ &= (-1)^{n-1}v_F(\eta'_4)...v_F(\eta'_n)v_F(E)v_F(H_Y)F^n.\end{aligned}$$

Since $Z_1 \to Y$ is a $w$-morphism, the integer $\lambda$ in Section 2.2 is 1. Hence, we know that $F^n = \frac{(-1)^{n-1}r'^{n-1}}{a'_1...a'_n}$. Now, $v_F(E) = \frac{a'_i}{r'}$ and $v_F(H_Y) = a(Y, F) = \frac{1}{r'}$, so

$$v_F(H_Y)F.\Gamma_{Z_1} = \frac{a'_i v_F(\eta'_4)...v_F(\eta'_n)r'^{n-4}}{a'_1...a'_n}.$$

□

**Lemma 3.4.** Notation and assumption as in Lemma 3.3. Assume that:

(i) For all $4 \le j \le n$, there exists an integer $\delta_j$ so that $x_{\delta_j}^{k_j}$ appears in $\eta_j$ as a monomial for some positive integer $k_j$. Moreover, the integers $\delta_4$, ..., $\delta_n$ are all distinct.

(ii) If $j \ne i, 3, \delta_4, ..., \delta_n$, then $a_3a'_j \ge a_j$.

Then $K_{Z_1}.\Gamma_{Z_1} \le 0$.

*Proof.* Fix $j \ge 4$. From the construction and our assumption we know that that $i$, $\delta_4$, ..., $\delta_n$ are all distinct. One can see that $rv_E(\eta_j) = k_j a_{\delta_j}$ and $r'v_F(\eta'_j) \le k_j a'_{\delta_j}$. Thus, we have a relation

$$\frac{rv_E(\eta_j)}{a_{\delta_j}} \ge \frac{r'v_F(\eta'_j)}{a'_{\delta_j}}.$$

One can always assume that if $j > 4$, $j \ne i$, then $\delta_j = j$. By interchanging the order of $y_1$, ..., $y_4$, we may assume that $\delta_4 = 4$. Now, if $i < 4$, then we may assume that $i = 1$. If $i > 4$, then we may assume that $\delta_i = 1$. We can write

$$\frac{a_3^2 v_E(\eta_4)...v_E(\eta_n)r^{n-3}}{ma_1...a_n} = \frac{1}{ma_i}\frac{a_3}{a_2}\frac{rv_E(\eta_4)}{a_{\delta_4}}...\frac{rv_E(\eta_n)}{a_{\delta_n}}$$

and

$$\frac{a'_i v_F(\eta_4)...v_F(\eta'_n)r'^{n-4}}{a'_1...a'_n} = \frac{1}{r'}\frac{1}{a'_2}\frac{r'v_F(\eta'_4)}{a'_{\delta_4}}...\frac{r'v_F(\eta'_n)}{a'_{\delta_n}}.$$

Since $ma_i = r'$, $\frac{a_3}{a_2} \ge \frac{1}{a'_2}$ and $\frac{rv_E(\eta_j)}{a_{\delta_j}} \ge \frac{r'v_F(\eta'_j)}{a'_{\delta_j}}$, we know that

$$\frac{a_3^2 v_E(\eta_4)...v_E(\eta_n)r^{n-3}}{ma_1...a_n} \ge \frac{a'_i v_F(\eta'_4)...v_F(\eta'_n)r'^{n-4}}{a'_1...a'_n},$$

So $K_{Z_1}.\Gamma_{Z_1} \le 0$. □

**Remark 3.5.**

(1) From the construction we know that if $j \geq 5$, $j \neq i$, then one can choose $\delta_j = j$.

(2) If $Y \to X$ can be viewed as a four-dimensional weighted blow-up, then condition (i) of Lemma 3.4 always holds. Indeed, in this case one has $i \leq 4$, so $\eta'_4$ is a two-variable irreducible function, hence there exists $\delta_4 \leq 4$ such that $y_{\delta_4}^{k_4} \in \eta'_4$ for some positive integer $k_4$. One also has $\delta_j = j$ for all $j > 4$. Thus, condition (i) of Lemma 3.4 holds.

(3) If for $j \neq i$, 3, $\delta_4$, ..., $\delta_n$ one has that $a_j \leq a_i$, then condition (ii) of Lemma 3.4 holds. Indeed, by Lemma 2.1 we know that $U_i \cong \mathbb{A}^n/\langle\tau, \tau'\rangle$ where $\tau$ corresponds to the vector $v = \frac{1}{a_i}(a_1, ..., a_{i-1}, -r, a_{i+1}, a_n)$. Let $\bar{v}$ be the vector corresponding to the cyclic action near $P \in U_i$. Then $v \equiv m\bar{v} \pmod{\mathbb{Z}^n}$ and $r' = ma_i$. Since $Z_1 \to Y$ is a $w$-morphism, and since $H_Y$ is defined by $y_3 = 0$, we know that $a'_3 = 1$ and $\bar{v} \equiv a_3\frac{1}{r'}(a'_1, ..., a'_n) \pmod{\mathbb{Z}^n}$. One can see that $a_3a'_j \equiv a_j \pmod{r'}$. This implies that $a_3a'_j \geq a_j$ since

$$a_j \leq a_i \leq ma_i = r'.$$

**Lemma 3.6.** Assume that $K_{Z_1}$ is anti-nef over $X$ and there exists $u \in \mathcal{O}_X$ such that $v_E(u) < \frac{a(E,X)}{a(F,X)}v_F(u)$. Then $Z_i \dashrightarrow Z_{i+1}$ is a $K_{Z_i}$-flip or flop for all $1 \leq i \leq k-1$ and $Z_k \to Y_1$ is a terminal divisorial contraction.

In particular, if there exists $j \neq i$ such that $a_3a'_j > a_j$, then the conclusion of this lemma holds.

*Proof.* We only need to show that $E_{Z_1}$ is not covered by $K_{Z_1}$-trivial curves. Then the conclusion follows from Lemma 3.1.

Assume that $E_{Z_1}$ is covered by $K_{Z_1}$-trivial curves. Since $K_{Z_1}$ is anti-nef, those $K_{Z_1}$-trivial curves are contained in the boundary of the relative effective cone $NE(Z_1/X)$. Hence, $k = 1$ and $Z_1 \to Y_1$ is a $K_{Z_1}$-trivial divisorial contraction. Notice that if $C_Y \subset E$ is a curve which does not contain $P$, then $K_{Z_1}.C_{Z_1} = K_Y.C_Y < 0$, hence the curve $C_{Z_1}$ is not contracted by $Z_1 \to Y_1$. Thus, $Z_1 \to Y_1$ is a divisorial contraction to the curve $C_{Y_1}$. Notice that, in this case, $a(E, Y_1) = 0$.

By computing the discrepancy, one can see that the pull-back of $F_{Y_1}$ on $Z_1$ is $F_{Z_1} + \frac{a(E,X)}{a(F,X)}E_{Z_1}$. It follows that for all $u \in \mathcal{O}_X$, one has that

$$v_E(u) \geq \frac{a(E,X)}{a(F,X)}v_F(u).$$

Hence, if there exists $u$ such that $v_E(u) < \frac{a(E,X)}{a(F,X)}v_F(u)$, then $E_{Z_1}$ is not covered by $K_{Z_1}$-trivial curves, so $Z_k \to Y_1$ is a terminal divisorial contraction.

Now, by Lemma 3.7, we know that

$$\frac{a(E,X)}{a(F,X)} = \frac{r'a_3}{r + a_3a'_i}.$$

Consider $u = x_j$. For $j \neq i$ we know that $v_E(x_j) = \frac{a_j}{r}$ and

$$v_F(x_j) = v_F(y_jy_i^{\frac{a_j}{r}}) = \frac{a'_j}{r'} + \frac{a_ja'_i}{rr'} = \frac{ra'_j + a_ja'_i}{rr'}.$$

The inequality $v_E(u) \geq \frac{a(E,X)}{a(F,X)}v_F(u)$ becomes

$$\frac{a_j}{r} \geq \frac{r'a_3}{r + a_3a'_i}\frac{ra'_j + a_ja'_i}{rr'} = \frac{1}{r}\frac{a_3(ra'_j + a_ja'_i)}{r + a_3a'_i},$$

or, equivalently,

$$a_j(r + a_3 a'_i) \geq a_3(r a'_j + a_j a'_i).$$

This is equivalent to

$$a_j \geq a_3 a'_j.$$

Hence, the condition $a_3 a'_j > a_j$ implies that $v_E(u) < \frac{a(E,X)}{a(F,X)} v_F(u)$. □

**Lemma 3.7.** One has that

$$a(E, X) = \frac{a_3}{r}, \quad a(F, X) = \frac{r + a_3 a'_i}{r r'}.$$

*Proof.* Since $a(E, X, H) = 0$, we know that $a(E, X) = v_E(H) = \frac{a_3}{r}$. Then

$$a(F, X) = \frac{1}{r'} + \frac{a_3}{r}\frac{a'_i}{r'} = \frac{r + a_3 a'_i}{r r'}.$$

□

**Remark 3.8.** Note that the assumption in Lemma 3.4 depends only on the first weighted blow-up $Y \to X$. In other words, we can check whether the assumption holds or not by simply considering the embedding which defines the weighted blow-up $Y \to X$ instead of considering the (possibly) larger embedding which defines both $Y \to X$ and $Z_1 \to Y$. Likewise, to apply Lemma 3.6, we can simply look at the embedding that defines $Y \to X$, if condition $a_3 a'_j > a_j$ already holds under this embedding.

**Notation 3.9.**

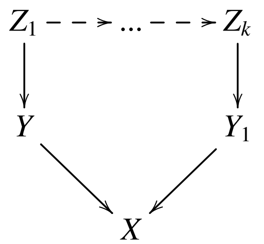
(1) We say that the condition ($\Xi$) holds if conditions (i) and (ii) in Lemma 3.4 hold for all possible choices of $\Gamma$. We say that the condition ($\Xi'$) holds if conditions (2) and (3) in Remark 3.5 hold for all possible choices of $\Gamma$. As explained in Remark 3.5, we know that the condition ($\Xi'$) implies the condition ($\Xi$).

(2) We say that the condition ($\Xi_-$) (resp. ($\Xi'_-$)) holds if the condition ($\Xi$) (reps. ($\Xi'$)) holds and the inequality in Lemma 3.4 is strict for all possible choices of $\Gamma$. Using the notation in Lemma 3.4, it is equivalent to say that either there exists $j \neq i, 3, \delta_4, ..., \delta_n$ such that $a_3 a'_j > a_j$, or there exists $j \geq 4$ such that

$$\frac{r v_E(\eta_j)}{a_{\delta_j}} > \frac{r' v_F(\eta'_j)}{a'_{\delta_j}}.$$

(3) We say that the condition ($\Theta_u$) holds for some function $u$ if $v_E(u) < \frac{a(E,X)}{a(F,X)} v_F(u)$. We say that the condition ($\Theta_j$) holds for some index $j$ if $a_3 a'_j > a_j$. In either case, Lemma 3.6 can be applied.

**Notation 3.10.** We say that a divisorial contraction $Y \to X$ is linked to another divisorial contraction $Y_1 \to X$ if the diagram



exists, where $Z_1 \to Y$ is a strict $w$-morphism over a non-Gorenstein point, $Z_k \to Y_1$ is a divisorial contraction, and $Z_i \dashrightarrow Z_{i+1}$ is a flip or a flop for all $1 \le i \le k-1$. We use the notation $Y \underset{X}{\Rightarrow} Y_1$ if $Y \to X$ is linked to $Y_1 \to X$.

Furthermore, if all $Z_i \dashrightarrow Z_{i+1}$ are all flips, or $k = 1$, then we say that $Y$ is negatively linked to $Y_1$, and use the notation $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1$.

**Remark 3.11.** At this point, it is not clear why $Z_1 \to Y$ should be a divisorial contraction to a non-Gorenstein point. In fact, from the classification of divisorial contractions between terminal threefolds (cf. Tables in Section 4), one can see that if there are two different divisorial contractions $Y \to X$ and $Y_1 \to X$, then $Y$ or $Y_1$ always contain a non-Gorenstein point. It is natural to construct the diagram starting with the most singular point, which is always a non-Gorenstein point.

**Remark 3.12.**

(1) If $(\Xi)$ or $(\Xi')$ holds and $(\Theta_u)$ or $(\Theta_j)$ holds for some function $u$ or index $j$, then by Lemmas 3.4 and 3.6 one has that $Y \underset{X}{\Rightarrow} Y_1$.

(2) Assume that $(\Xi_-)$ or $(\Xi'_-)$ holds and $(\Theta_u)$ or $(\Theta_j)$ holds for some function $u$ or index $j$. Then one has that $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1$.

**Lemma 3.13.** Assume that

$$X \cong (x_1(x_1 + p(x_2, ..., x_4)) + g(x_2, ..., x_4) = 0) \subset \mathbb{A}^4/G,$$

such that

(1) $v_E(g) = \frac{a_1}{r} + v_E(p) = \frac{2a_1}{r} - 1$.

(2) $i = 1$, $a_2 + a_4 = a_1$ and $a_3 = 1$.

Then $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1$.

*Proof.* We know that $a_1 > a_j$ for $j = 2, ..., 4$, so $(\Xi')$ holds. Consider the embedding

$$X \cong (x_1x_5 + g(x_2, ..., x_4) = x_5 - x_1 - p(x_2, ..., x_4) = 0) \subset \mathbb{A}^5_{(x_1,...,x_5)}/G.$$

Then $Y \to X$ can be viewed as a weighted blow-up with the weight $\frac{1}{r}(a_1, ..., a_5)$ with respect to this embedding, where $a_5 = rv_E(p)$. The origin of $U_1$ is a cyclic quotient point of type $\frac{1}{a_1}(-r, a_2, ..., a_5)$. The only $w$-morphism is the weighted blow-up that corresponds to the weight $w(y_2, ...y_4) = \frac{1}{a_1}(a_2, ..., a_4)$. One can see that $a'_5 = rv_E(g) > rv_E(p) = a_5$, hence $(\Theta_5)$ holds. Moreover, one can see that $\eta'_5 = y_5 - p(y_2, 0, y_4)$, so $r'v_F(\eta'_5) = rv_E(\eta_5) = rv_E(p(x_2, 0, x_4))$, hence

$$\frac{rv_E(\eta_5)}{a_5} > \frac{r'v_F(\eta'_5)}{a'_5}.$$

Thus, $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1$. ∎

**Lemma 3.14.** Assume that $Y \to X$ and $Y_1 \to X$ are two divisorial contractions such that $Y \underset{X}{\Rightarrow} Y_1$. Let $E$ and $F$ be the exceptional divisors of $Y \to X$ and $Y_1 \to X$, respectively. Assume that there exists $u \in O_X$ such that $v_F(u) = \frac{1}{r}$ and $v_F(u') > 0$ where $r$ is the Cartier index of $\mathrm{Center}_X E$ and $u'$ is the strict transform of $u$ on $Y$. Then $a(F, X) < a(E, X)$ if $a(E, X) > 1$.

*Proof.* Notice that we have

$$v_F(u) = v_F(\tilde{u}) + v_E(u)v_F(E).$$

Since $v_E(u) \geq \frac{1}{r}$ and $v_F(\tilde{u}) > 0$, we know that $v_F(E) < 1$. Thus

$$a(F, X) = \frac{1}{r'} + a(E, X)v_F(E) \leq \frac{1}{r'} + a(E, X)\frac{r' - 1}{r'} = a(E, X) + \frac{1 - a(E, X)}{r'}$$

where $r'$ is the Cartier index of $\mathrm{Center}_Y F$. Hence, $a(F, X) < a(E, X)$ if $a(E, X) > 1$. □

## 4. Constructing links

The aim of this section is to prove the following proposition:

**Proposition 4.1.** Let $X$ be a terminal threefold and $Y \to X$, $Y' \to X$ be two different divisorial contractions to points over $X$. Then there exists $Y_1$, ..., $Y_k$, $Y'_1$, ..., $Y'_{k'}$ such that

$$Y = Y_1 \underset{X}{\Rightarrow} Y_2 \underset{X}{\Rightarrow} \dots \underset{X}{\Rightarrow} Y_k = Y'_{k'} \underset{X}{\Leftarrow} \dots \underset{X}{\Leftarrow} Y'_1 = Y'.$$

*Proof.* We need a case-by-case discussion according to the type of the singularity on $X$. Please see Propositions 4.2, 4.3, 4.5–4.10 and 4.15–4.18. □

We keep the notation in Section 3.

### *4.1. Divisorial contractions to cA/r points*

In this subsection, we assume that $X$ has $cA/r$ singularities. Divisorial contractions over $X$ are listed in Table 2.

**Table 2.** Divisorial contractions to $cA/r$ points.

| No. | defining equations | $(r; a_i)$ / weight | type / $a(X, E)$ | condition |
|---|---|---|---|---|
| A1 | $xy + z^{rk} + g_{\geq ka}(z, u)$ | $(r; \beta, -\beta, 1, r)$ / $\frac{1}{r}(b, c, a, r)$ | $cA/r$ / $a/r$ | $b \equiv a\beta \pmod r, b + c = rka$ |
| A2 | $x^2 - y^2 + z^3 + xu^2 + g_{\geq 6}(x, y, z, u)$ | $(1; -)$ / $(4, 3, 2, 1)$ | $cA_2$ / $3$ | $xz \notin g(x, y, z, u)$ |

**Proposition 4.2.**

(1) If $Y \to X$ is of type A1 with $a > 1$, then $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1$ for some $Y_1 \to X$ which is of type A1 with the discrepancy less than $a$.

(2) If $Y \to X$ is of type A1 with $a = 1$ and $b > r$, then $Y \underset{X}{\Rightarrow} Y_1$ where $Y_1$ is an A1 type weighted blow-up with the weight $\frac{1}{r}(b - r, c + r, 1, r)$. One also has $Y_1 \underset{X}{\Rightarrow} Y$ if we begin with $Y_1 \to X$ and interchange the role of $x$ and $y$. Moreover, $Y \underset{X}{\overset{-}{\not\Rightarrow}} Y_1$ if and only if $\eta_4 = y$.

(3) If $Y \to X$ is of type A2, then $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1$ where $Y_1 \to X$ is a divisorial contraction of type A1.

*Proof.* Assume first that $Y \to X$ is of type A1. We are going to prove (1) and (2). If both $b$ and $c$ are less than $r$, then $a = k = 1$. In this case, there is exactly one divisorial contraction of type A1, so there is nothing to prove. Thus, we may assume that one of $b$ or $c$, say $b > r$.

The origin of the chart $U_x \subset Y$ is a cyclic quotient point. On this chart, one can choose $(y_1, ..., y_4) = (x, u, z, y)$ with $i = 1$ and $\delta_4 = 4$. One can see that $(\Xi')$ holds. Now the two action in Lemma 2.1 is given by

$$\tau = \frac{1}{b}(-r, c, a, r), \quad \tau' = \frac{1}{b}(\beta, \frac{-\beta(b+c)}{r}, \frac{b - a\beta}{r}, b - \beta).$$

Since $U_x$ is terminal, there exists a vector $\tau'' = \frac{1}{b}(b - \delta, \epsilon, 1, \delta)$ such that $\tau \equiv a\tau'' \pmod{\mathbb{Z}^4}$ and $\tau' \equiv \lambda'\tau'' \pmod{\mathbb{Z}^4}$ for some integer $\lambda'$. There is exactly one $w$-morphism over the origin of $U_x$ which extracts the exceptional divisor $F$ so that $v_F$ corresponds to the vector $\tau''$. One can also see that

$$\frac{\epsilon}{b} = v_F(y) = v_F(g') \geq \frac{rk}{b}$$

where $g'$ is the strict transform of $g$ on $U_x$, since if $z^{rp}u^q \in g'$, then $ap + q \geq ak$ and

$$v_F(z^{rp}u^q) = \frac{1}{b}(rp + \delta q) = \frac{1}{ab}(rap + \delta aq) \geq \frac{rka}{ab} = \frac{rk}{b}$$

for $\delta a \geq r$ because $\delta a \equiv r (\mathrm{mod}\ b)$ and $b > r$. Thus,

$$a_4 = c < rka \leq a\epsilon = a_3 a'_4,$$

hence $(\Theta_4)$ holds, and there exists $Y_1 \to X$ such that $Y \underset{X}{\Rightarrow} Y_1$.

We need to check whether $(\Xi'_-)$ holds or not. We have that $f_4'^{\circ} = \eta'_4 = y_4 + g'^{\circ}$. One always has that

$$\frac{r' v_F(\eta'_4)}{a'_4} = \frac{b\frac{\epsilon}{b}}{\epsilon} = 1.$$

Now, $g'^{\circ} = 0$ if and only if

$$\frac{r v_E(\eta_4)}{a_4} = \frac{r\frac{c}{r}}{c} = 1,$$

and $\delta a = r$ if and only if

$$a_3 a'_2 = a\delta = r = a_2.$$

Thus, $(\Xi'_-)$ holds if and only if $g'^{\circ} \neq 0$ or $a$ do not divide $r$.

One can compute the discrepancy of $Y_1 \to X$ using Lemma 3.7. We know that $a'_i = b - \delta$ and $a_3 = a$, so

$$a(F, X) = \frac{r + a_3 a'_i}{rr'} = \frac{r - \delta a + ba}{rb} \leq \frac{a}{r} = a(E, X).$$

If $a = 1$, then $r = \delta$, so $a(F, X) = a(E, X) = \frac{1}{r}$. One can verify that $Y_1 \to X$ is the weighted blow-up with the weight $\frac{1}{r}(b - r, c + r, 1, r)$. In this case, $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$ if and only if $g'^{\circ} \neq 0$. Hence, $Y \overset{-}{\underset{X}{\not\Rightarrow}} Y_1$ if and only if $\eta_4 = y$. This proves (2).

Now assume that $a > 1$. We already know that $\delta a \geq r$. If $\delta a > r$, then $a(F, X) < a(E, X)$ and $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$, so (1) holds. Hence, one only needs to show that $\delta a \neq r$. If $\delta a = r$, then

$$b = a\beta + \lambda' r = a(\beta + \lambda'\delta)$$

where $\lambda' = \frac{b - a\beta}{r}$. One can see that $b - \beta = (a - 1)\beta + \lambda' a\delta$. On the other hand, since $\tau' \equiv \lambda'\tau''$ (mod $\mathbb{Z}^n$), we know that $b - \beta \equiv \lambda'\delta$ (mod $b$). Hence, $b$ divides

$$b - \beta - \lambda'\delta = (a - 1)(\beta + \lambda'\delta).$$

This is impossible since $(a - 1)(\beta + \lambda'\delta)$ is a positive integer and is less than $b$.

Finally, assume that $Y \to X$ of type A2. In this case, one needs to look at the chart $U_x \subset Y$, and we choose $(y_1, ..., y_4) = (x, z, y, u)$ with $i = 1$ and $\delta_4 = 4$. One can see that $(\Xi')$ holds. The origin of the chart $U_x$ is a $cAx/4$ point of the form

$$(x^2 - y^2 + z^3 + u^2 + g'(x, y, z, u) = 0) \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{4}(1, 1, 2, 3).$$

From [9, Theorem 7.9], we know that there are exactly two $w$-morphisms over this point which are weighted blow-ups with the weights $w_{\pm}(x \pm y, x \mp y, z, u) = \frac{1}{4}(5, 1, 2, 3)$. For both these two $w$-morphisms, one has that $a_2 = 2$, $a_3 = 3$ and $a_2' = 2$, so $(\Theta_2)$ holds and $(\Xi_-)$ holds since $a_3 a_2' > a_2$. Thus, there exists $Y_1 \to X$ so that $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$. One can compute that the discrepancy of $Y_1 \to X$ is one, so $Y_1 \to X$ is of type A1. This proves (3). □

### *4.2. Divisorial contractions to cAx/r points*

In this subsection, we assume that $X$ has $cAx/r$ singularities with $r = 2$ or 4. Divisorial contractions over $X$ are listed in Table 3.

**Proposition 4.3.** (1) Assume that $Y \to X$ is of type Ax1 or Ax3. Then $Y \to X$ is the only divisorial contraction over $X$.

(2) Assume that $Y \to X$ is of type Ax2 or Ax4. Then there are exactly two divisorial contractions over $X$. Let $Y_1 \to X$ be another divisorial contraction. Then $Y_1 \to X$ has the same type of $Y \to X$, and one has that $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1 \overset{-}{\underset{X}{\Rightarrow}} Y$.

*Proof.* The number of divisorial contractions follows from [9, Section 7,8]. So we can assume that $Y \to X$ is of type Ax2 or Ax4, and Lemma 3.13 implies that $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$. □

**Table 3.** Divisorial contractions to $cAx/r$ points

| No. | defining equations | $(r; a_i)$ / weight | type / $a(X, E)$ | condition |
|---|---|---|---|---|
| Ax1 | $x^2 + y^2 + g_{\geq \frac{2k+1}{2}}(z, u)$ | $(4; 1, 3, 1, 2)$ / $\frac{1}{4}(b, c, 1, 2)$ | $cAx/4$ / $1/4$ | $(b, c) = (2k+1, 2k+3)$ or $(2k+3, 2k+1)$ |
| Ax2 | $x^2 + y^2 + (\lambda x + \mu y)p_{\frac{2k+1}{4}}(z, u)$ $+g_{\geq \frac{2k+3}{2}}(z, u)$ | $(4; 1, 3, 1, 2)$ / $\frac{1}{4}(b, c, 1, 2)$ | $cAx/4$ / $1/4$ | $(b, c, \lambda, \mu) =$ $(2k+5, 2k+3, 1, 0)$ or $(2k+3, 2k+5, 0, 1)$ |
| Ax3 | $x^2 + y^2 + g_{\geq k}(z, u)$ | $(2; 0, 1, 1, 1)$ / $\frac{1}{2}(b, c, 1, 1)$ | $cAx/2$ / $1/2$ | $(b, c) = (k, k+1)$ or $(k+1, k)$ |
| Ax4 | $x^2 + y^2 + (\lambda x + \mu y)p_{\frac{k}{2}}(z, u)$ $+g_{\geq k+1}(z, u)$ | $(2; 0, 1, 1, 1)$ / $\frac{1}{2}(b, c, 1, 1)$ | $cAx/2$ / $1/2$ | $(b, c, \lambda, \mu) =$ $(k+2, k+1, 1, 0)$ or $(k+1, k+2, 0, 1)$ |

### *4.3. Divisorial contractions to cD points*

In this subsection, we assume that $X$ has $cD$ singularities. At first, we consider $w$-morphisms over $X$, which are listed in Table 4. Notice that for types D1, D2 or D5 in Table 4 there is at most one divisorial contraction over $X$ which is of the given type. This is because the equations of type D1, D2 and D5 come from the normal form of $cD$-type singularities, which are unique, and the blowing-up weights are determined by the defining equations.

**Lemma 4.4.** Assume that there exists two different divisorial contractions with discrepancy one over $X$. Then, one of the following holds:

(1) One of the divisorial contractions is of type D1.

(2) The two morphisms are of type D2 and D5, respectively.

(3) Both of the divisorial contractions are of type D3.

(4) Both of the divisorial contractions are of type D4.

*Proof.* Assume that $Y \to X$ and $Y_1 \to X$ are the two given divisorial contractions. It is enough to prove the following statements:

(i) If $Y \to X$ is of type D4, then $Y_1 \to X$ is of type D1 or D4.

(ii) If $Y \to X$ is of type D2, then $Y_1 \to X$ is of type D1 or D5.

(iii) If $Y \to X$ is of type D5, then $Y_1 \to X$ is not of type D3.

Let $E$ and $F$ be the exceptional divisors of $Y \to X$ and $Y_1 \to X$, respectively. Then $a(F, Y) < 1$, since otherwise $a(F, X) > 1$. Thus, $P = \mathrm{Center}_F Y$ is a non-Gorenstein point.

First, assume that $Y \to X$ is of type D4. In this case, $P$ may be the origin of $U_x$ or the origin of $U_y$, and they are both cyclic quotient points. Exceptional divisors over $P$ with discrepancy less than one are described in [4, Proposition 3.1]. The origin of $U_x$ is a $\frac{1}{b+1}(b, 1, 1)$ point. If $P$ is this point, then, since $z = 0$ defines a Du Val section, we have that $v_F(z) = a(F, X) = 1$. One can verify that

$v_F(u) = v_F(z) = 1$ and $v_F(x) = v_F(y) = b$. Now, $v_F(x) = b$ only when $xp_b(z,u) \in g(x,z,u)$ for some homogeneous polynomial $p(z,u)$ of degree $b$. One can check that $v_F(x + p(z,u)) = b + 1$. In this case, $Y_1 \to X$ is also of type D4 after a change of coordinates $x \mapsto x - p(z,u)$. If $P$ is the origin of $U_y$, then it is a $\frac{1}{b}(1,-1,1)$ point. One can verify that $v_F(u) > 1$. This implies that $Y_1 \to X$ is of type D1.

Now, assume $Y \to X$ is of type D2. Then $P$ is the origin of $U_y \subset Y$, which is a $cA/b$ point. Exceptional divisors of discrepancy less than one over $P$ are described in [4, Proposition 3.4]. One can verify that if $\lambda \neq 0$ and $k = b$, then $v_F(u) = 1$. In this case, $Y_1 \to X$ is of type D5. Otherwise, $v_F(u) = 2$, and so $Y_1 \to X$ is of type D1.

Finally, assume that $Y \to X$ is of type D5. One can see that $Y_1 \to X$ cannot have type D3 since $z^b \in p(z,u)$. This finishes the proof. □

**Table 4.** Divisorial contractions to $cD$ points with discrepancy one.

| No. | defining equations | weight | type / $a(X,E)$ | condition |
|---|---|---|---|---|
| D1 | $x^2 + y^2u + \lambda yz^k + g_{\geq l}(z,u)$ | $(b, b-1, 1, 2)$ | $\frac{cD}{1}$ | $b = \min\{k-1, \lfloor \frac{l}{2} \rfloor\}$ |
| D2 | $x^2 + y^2u + \lambda yz^k + g_{\geq 2l}(z,u)$ | $(b, b, 1, 1)$ | $\frac{cD}{1}$ | $b = \min\{k, l\}$ |
| D3 | $\begin{cases} x^2 + ut + \lambda yz^k + g_{\geq 2b+2}(z,u) \\ y^2 + p_{2b}(x,z,u) + t \end{cases}$ | $(b+1, b, 1, 1, 2b+1)$ | $\frac{cD}{1}$ | $k \geq b+2$ |
| D4 | $x^2 + y^2u + yh_{\geq k}(z,u) + g_{\geq 2b+1}(x,z,u)$ | $(b+1, b, 1, 1)$ | $\frac{cD}{1}$ | $k \geq b+1$ |
| D5 | $\begin{cases} x^2 + yt + g_{\geq 2b}(z,u) \\ yu + p_b(z,u) + t \end{cases}$ | $(b, b-1, 1, 1, b+1)$ | $\frac{cD}{1}$ | $z^b \in p(z,u)$ |

**Proposition 4.5.** Assume that there exist two different divisorial contractions with discrepancy one over $X$, say $Y \to X$ and $Y_1 \to X$.

(1) If $Y \to X$ and $Y_1 \to X$ are both of type D4, then $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1 \overset{-}{\underset{X}{\Rightarrow}} Y$.

(2) If $Y \to X$ is of type D3 and $Y_1 \to X$ is of type D1, then $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$. If $Y_1 \to X$ is of type D3, then there exists another divisorial contraction $Y_2 \to X$ which is of type D1 so that $Y \overset{-}{\underset{X}{\Rightarrow}} Y_2 \overset{-}{\underset{X}{\Leftarrow}} Y_1$.

(3) If $Y \to X$ is of type D2 and $Y_1 \to X$ is of type D5, then $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$.

(4) If $Y \to X$ is of type D1 and $Y_1 \to X$ is not of type D3, then $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$.

*Proof.* Assume first that $Y \to X$ and $Y_1 \to X$ are both of type D4. Notice that, in this case, $xp_b(z,u) \in g(x,z,u)$. Thus, $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$ by Lemma 3.13.

Now assume that $Y \to X$ is of type D3. Consider the chart $U_t \subset Y$ which is defined by

$$(x^2 + u + \lambda yz^k t^{b+k-2b-2} + g'(z,u,t) = y^2 - p(x,z,u) + t = 0) \subset$$
$$\mathbb{A}^5_{(x,y,z,u,t)}/\frac{1}{2b+1}(b+1,b,1,1,-1).$$

Notice that, using the notation in Section 3, we know that

$$f'^{\circ}_4 = \eta'_4 = x^2 + u + g'(0,u,0), \quad f'^{\circ}_5 = y^2 + p(x,0,u).$$

$\eta'_5$ can be $y \pm \mu u^b$ if $p(x,0,u) = -\mu^2 u^{2b}$ for some $\mu \in \mathbb{C}$, and otherwise $\eta'_5 = f'^{\circ}_5$. One can see that $\eta'_4 = \eta'_5 = 0$ defines an irreducible and reduced curve. There is only one $w$-morphism over the origin of $U_t$ which is defined by weighted blowing up the weight $w(x,y,z,u,t) = \frac{1}{2b+1}(b+1,b,1,2b+2,2b)$. Now, in this case we choose $(y_1,...,y_5) = (x,y,z,u,t)$ with $i = 5$, $\delta_4 = 4$, $\delta_5 = 2$. One can see that $(\Xi)$ holds and $(\Theta_4)$ holds. Also, one has that $\frac{rv_E(\eta_4)}{a_4} = 2b+2$ while $\frac{r'v_F(\eta'_4)}{a'_4} = 1$. Thus, $(\Xi_-)$ holds and so there exists $Y_2 \to X$ such that $Y \overset{-}{\underset{X}{\Rrightarrow}} Y_2$. One can compute that $Y_2 \to X$ is of type D1. If $Y_1 \to X$ is of type D1, then $Y_2 = Y_1$ since there is at most one divisorial contraction with type D1. This proves statement (2).

Now assume that $Y \to X$ is of type D2 and $Y_1 \to X$ is of type D5. In this case, we consider the embedding corresponding to $Y_1 \to X$. Under this embedding, $Y \to X$ is given by the weighted blow-up with the weight $(b,b,1,1,b)$ and the chart $U_y \subset Y$ is given by

$$U_y = (x^2 - t + g'(y,z,u) = yu + z^b + t = 0) \subset \mathbb{A}^5_{(x,y,z,u,t)}/\frac{1}{b}(0,-1,1,1,0).$$

We take $(y_1,...,y_5) = (y,u,z,x,t)$ with $\delta_4 = 4$ and $\delta_5 = 5$. Then $(\Xi)$ holds. The origin of $U_y$ is a $cA/b$ point and the weight $w(y_1,...,y_5) = \frac{1}{b}(b-1,1,1,b,2b)$ defines a $w$-morphism over $U_y$. One can see that $(\Theta_5)$ holds. Moreover, since $a'_5 = 2b > b = a_5$, we know that $(\Xi_-)$ holds. Thus, $Y \overset{-}{\underset{X}{\Rrightarrow}} Y_1$.

Finally, assume that $Y \to X$ is of type D1 and $Y_1 \to X$ is not of type D3. Let $b$ and $b_1$ be the integers in Table 4 corresponding to $Y \to X$ and $Y_1 \to X$, respectively. First, we claim that $b \leq b_1$. Indeed, if $Y_1 \to X$ is of type D5, then $z^{b_1} \in h(z,u)$, which implies that $b_1 \geq b+1$. If $Y_1 \to X$ is of type D2 or D4, then the inequality $b \leq b_1$ follows from Corollary 2.17. Now, the origin of the chart $U_u \subset Y$ is defined by

$$(x^2 + y^2 + \lambda yz^k u^{k-b-1} + g'(z,u) = 0) \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{2}(b,b-1,1,1),$$

which is a $cAx/2$ point. We can take $(y_1,...,y_4) = (u,y,z,x)$ with $i = 1$ and $\delta_4 = 4$. $w$-morphisms over this point are fully described in [9, Section 8]. Since $b \leq b_1$, we know that $2k - b - 1 > b$ and the multiplicity of $g'(z,u)$ is greater than or equal to $2b$. Hence, if $F$ is the exceptional divisor of a $w$-morphism over $Y$, then

$$v_F(y) \geq b > b - 1 = v_E(y).$$

Thus, $(\Xi_-)$ and $(\Theta_2)$ holds and one has $Y \overset{-}{\underset{X}{\Rrightarrow}} Y_1$. □

Now we study divisorial contractions of discrepancy greater than one. All such divisorial contractions are listed in Table 5.

**Table 5.** Divisorial contractions to $cD$ points with discrepancies greater than one.

| No. | defining equations | weight | type / $a(X,E)$ | condition |
|---|---|---|---|---|
| D6 | $x^2+y^2u+z^k+g_{\geq 2b+1}(x,y,z,u)$ | $(b+1,b,a,1)$ | $\frac{cD}{a}$ | $ak=2b+1$ |
| D7 | $\begin{cases} x^2+yt+g_{\geq 2b+2}(y,z,u) \\ yu+z^k+p_{b+1}(z,u)+t \end{cases}$ | $(b+1,b,a,1,b+2)$ | $\frac{cD}{a}$ | $ak=b+1$ |
| D8 | $\begin{cases} x^2+ut+\lambda z^{\frac{b+1}{4}}+g_{\geq b+1}(y,z,u) \\ y^2+\mu z^{\frac{b-1}{4}}+p_{b-1}(x,z,u)+t \end{cases}$ | $(\frac{b+1}{2},\frac{b-1}{2},4,1,b)$ | $\frac{cD}{4}$ | $\frac{b+1}{4}\in\mathbb{N}$, $\lambda=1$, $\mu=0$, or $\frac{b-1}{4}\in\mathbb{N}$, $\mu=1$, $\lambda=0$. |
| D9 | $\begin{cases} x^2+ut+z^{\frac{b+1}{2}}+g_{\geq b+1}(y,z,u) \\ y^2+p_{b-1}(x,z,u)+t \end{cases}$ | $(\frac{b+1}{2},\frac{b-1}{2},2,1,b)$ | $\frac{cD}{2}$ | |
| D10 | $x^2+y^2u+z^b+g_{\geq 2b}(y,z,u)$ | $(b,b,2,1)$ | $\frac{cD}{2}$ | |
| D11 | $x^2+y^2u+yp_3(z,u)+u^3+g_{\geq 6}(z,u)$ | $(3,3,1,2)$ | $\frac{cD_4}{2}$ | $z^3\in p(z,u)$ |
| D12 | $x^2+y^2u+z^3+yu^2+g_{\geq 6}(y,z,u)$ | $(3,4,2,1)$ | $\frac{cD_4}{3}$ | |

**Proposition 4.6.** Assume that $Y\to X$ is a divisorial contraction with discrepancy $a>1$. Then there exists a divisorial contraction $Y_1\to X$ such that $Y \underset{X}{\Rrightarrow} Y_1$, and $a(F,X)<a$ where $F=exc(Y_1\to X)$.

*Proof.* First, notice that $(\Theta_u)$ holds in cases D6–D10 or D12, and $(\Theta_z)$ holds in case D11. This is because $v_E(u)$ or $v_E(z)=1$ in those cases and $\frac{a(E,X)}{a(F,X)}=a>1$.

Now we list all cases in Table 5, write down the chart on $Y$ we are looking at, and write down the variables $y_1,\dots,y_n$. One can easily see that $(\Xi)$ holds in all cases.

(1) Assume that $Y\to X$ is of type D6. Consider the chart $U_x\subset Y$ and take $(y_1,\dots,y_4)=(x,y,z,u)$ with $\delta_4=4$.

(2) Assume that $Y\to X$ is of type D7. Consider the chart $U_t\subset Y$ and take $(y_1,\dots,y_5)=(y,u,z,x,t)$, $\delta_4=4$ and $\delta_5=1$ or $2$.

(3) Assume that $Y\to X$ is of type D8 or D9. We consider the chart $U_t\subset Y$ and take $(y_1,\dots,y_5)=(x,y,z,u,t)$ with $\delta_4=4$ and $\delta_5=2$.

(4) Assume that $Y\to X$ is of type D10. We consider the chart $U_y\subset Y$ and take $(y_1,\dots,y_4)=(y,u,z,x)$ with $\delta_4=4$.

(5) Assume that $Y\to X$ is of type D11. We consider the chart $U_y\subset Y$ and $(y_1,\dots,y_4)=(y,z,u+\lambda y,x)$ for some $\lambda\in\mathbb{C}$ with $\delta_4=4$.

(6) Assume that $Y\to X$ is of type D12. We consider the chart $U_y\subset Y$ and $(y_1,\dots,y_4)=(y,z,x+\lambda y,u)$ for some $\lambda\in\mathbb{C}$ with $\delta_4=4$.

Now we know that there exists $Y_1\to X$ so that $Y \underset{X}{\Rrightarrow} Y_1$. Then $Y_1\to X$ is of one of types in Table 4 or Table 5. One can see that $v_F(z)=1$ if $Y_1\to X$ is of types D1–D5, D7–D9 or D11 and $v_F(u)=1$ if

$Y_1 \to X$ is of type D6, D10 or D12. Since $\mathrm{Center}_Y F$ is the origin of the chart $U_x$, $U_y$ or $U_t$, one can apply Lemma 3.14 to say that $a(F, X) < a$. This finishes the proof. □

## *4.4. Divisorial contractions to $cD/r$ points with $r > 1$*

In this subsection, we assume that $X$ has $cD/r$ singularities with $r = 2$ or 3. We first study $w$-morphisms over $X$.

**Table 6.** Divisorial contractions to $cD/r$ points with discrepancy one.

| No. | defining equations | $(r; a_i)$ / weight | type / $a(X,E)$ | condition |
|---|---|---|---|---|
| D13 | $x^2 + y^3 + g_{\geq k}(y, z, u)$ | $(3; 0, 2, 1, 1)$ / $\frac{1}{3}(3, 2, 4, 1)$ | $cD/3$ / $1/3$ | $k = 2$ and $zu^2$ or $z^3 \in g$, or $k = 3$ and $z^2u \in g$ |
| D14 | $x^2 + y^3 + z^3 + g_{\geq 4}(y, z, u)$ | $(3; 0, 2, 1, 1)$ / $\frac{1}{3}(6, 5, 4, 1)$ | $cD/3$ / $1/3$ | |
| D15 | $x^2 + yzu + g_{\geq 2}(y, z, u)$ | $(2; 1, 1, 1, 0)$ / $\frac{1}{2}(3, 1, 1, 2)$ | $cD/2$ / $1/2$ | |
| D16 | $x^2 + yzu + g_{\geq 3}(y, z, u)$ | $(2; 1, 1, 1, 0)$ / $\frac{1}{2}(3, b, c, d)$ | $cD/2$ / $1/2$ | $(b, c, d) = (3, 1, 2)$, $(1, 1, 4)$ |
| D17 | $\begin{cases} x^2 + yt + g_{\geq 3}(z, u) \\ zu + y^3 + t \end{cases}$ | $(2; 1, 1, 1, 0, 1)$ / $\frac{1}{2}(3, 1, 1, 2, 5)$ | $cD/2$ / $1/2$ | |
| D18 | $x^2 + y^2u + \lambda yz^k + g_{\geq l}(z, u)$ | $(2; 1, 1, 1, 0)$ / $\frac{1}{2}(b, b-2, 1, 4)$ | $cD/2$ / $1/2$ | $b = \min\{k-2, \lceil \frac{l}{2} \rceil - 1\}$ |
| D19 | $x^2 + y^2u + \lambda yz^k + g_{\geq l}(z, u)$ | $(2; 1, 1, 1, 0)$ / $\frac{1}{2}(b, b, 1, 2)$ | $cD/2$ / $1/2$ | $b = \min\{k, l\}$ |
| D20 | $\begin{cases} x^2 + ut + \lambda yz^k + g_{\geq b+2}(z, u) \\ y^2 + p_b(x, z, u) + t \end{cases}$ | $(2; 1, 1, 1, 0, 0)$ / $\frac{1}{2}(b+2, b, 1, 2, 2b+2)$ | $cD/2$ / $1/2$ | $k \geq b + 4$ |
| D21 | $x^2 + y^2u + yh_{\geq k}(z, u) + g_{\geq b+1}(x, z, u)$ | $(2; 1, 1, 1, 0)$ / $\frac{1}{2}(b+2, b, 1, 2)$ | $cD/2$ / $1/2$ | $k \geq b + 2$ |
| D22 | $\begin{cases} x^2 + yt + g_{\geq 2b}(z, u) \\ yu + z^b + t \end{cases}$ | $(2; 1, 1, 1, 0, 1)$ / $\frac{1}{2}(b, b-2, 1, 2, b+2)$ | $cD/2$ / $1/2$ | |

**Proposition 4.7.** Assume that $X$ has $cD/3$ singularities.

(1) If $Y \to X$ is of type D14 or if $Y \to X$ is of type D13 and both $zu^2$ and $z^2u \notin g(y, z, u)$, then there is only one $w$-morphism over $X$.

(2) If $Y \to X$ is of type D13 and $zu^2$ or $z^2u \in g(y, z, u)$, then there are two or three $w$-morphisms over $X$. Say $Y_1 \to X$, ..., $Y_k \to X$ are other $w$-morphisms with $k = 1$ or 2. Then $Y \overset{-}{\underset{X}{\Rightarrow}} Y_i \overset{-}{\underset{X}{\Rightarrow}} Y$ for all $1 \leq i \leq k$.

*Proof.* The statement about the number of $w$-morphisms follows from [9, Section 9]. Now we may assume that $Y \to X$ is of type D13 and $zu^2$ or $z^2u \in g(y, z, u)$. The chart $U_z \subset Y$ is defined by

$$(x^2 + y^3 + g'(y, z, u) = 0) \subset \mathbb{A}^4_{(x,y,z,u)} / \frac{1}{4}(3, 2, 1, 1)$$

with $u^2$ or $zu \in g'(y,z,u)$. We can take $(y_1,...,y_4) = (z, x, u + \lambda z, y)$ for some $\lambda \in \mathbb{C}$ with $\delta_4 = 4$. Now the $w$-morphism over $U_z$ is a weighted blow-up with the weight $w(y_1,...y_4) = \frac{1}{4}(3,5,1,2)$. One can see that $(\Theta_2)$ and $(\Xi'_-)$ hold. Hence, we can get a divisorial contraction $Y_1 \to X$ such that $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$. One can compute that $Y_1 \to X$ is also a $w$-morphism.

If there are three $w$-morphisms over $X$, then the defining equation of $X$ is of the form $x^2 + y^3 + zu(z+u)$ as in [9, Section 9.A], so $g'(y,z,u) = u(z+u)$. One can make a change of coordinates $u \mapsto u - z$ and again consider the weighted blow-up with the same weight $\frac{1}{4}(3,2,1,5)$. In this way, we can get a divisorial contraction $Y_2 \to X$ which is different to $Y_1$, and we also have that $Y \overset{-}{\underset{X}{\Rightarrow}} Y_2$. This finishes the proof. □

**Proposition 4.8.** Assume that $X$ has $cD/2$ singularities and $Y \to X$ is of type D15, D16 or D17.

(1) If $Y \to X$ is of type D15, then there is only one $w$-morphism over $X$.

(2) If $Y \to X$ is of type D17, then there exists exactly two $w$-morphisms over $X$. The other one, $Y_1 \to X$, is of type D16, and one has that $Y \underset{X}{\Rightarrow} Y_1$.

(3) If $Y \to X$ is of type D16 and there are no $w$-morphisms over $X$ with type D17, then there are exactly three $w$-morphisms over $X$. They are all of type D16 and are negatively linked to each other.

*Proof.* The statement about the number of $w$-morphisms follows from [10, Section 4]. Assume that $Y \to X$ is of type D17. Consider the chart $U_t \subset Y$ with $(y_1,...,y_5) = (y, u, y+z, x, t)$ with $\delta_4 = 4$ and $\delta_5 = 1$. One can see that $(\Xi)$ holds. Now the origin of $U_t$ is a cyclic quotient point. Let $F$ be the exceptional divisor of the $w$-morphism over $U_t$. Then one has that $v_F(y_1,...y_5) = \frac{1}{5}(6,2,1,3,3)$. One can see that $(\Theta_1)$ holds.

Assume that $Y \to X$ is of type D16 and there are no $w$-morphisms of type D17 over X. By [10, Section 4], we know that neither $y^4$ nor $z^4 \in g(y,z,u)$. Assume first that $(b,c,d) = (1,1,4)$. Consider the chart $U_u \subset Y$ which has a $cAx/4$ singular point at the origin. We choose $(y_1,...,y_4) = (y, u, y+z, x)$ with $\delta_4 = 4$. One can see that $(\Xi')$ holds. Let $w$ be the weight on $U_u$ so that $w(y_1,...,y_4) = \frac{1}{4}(5,2,1,3)$. Then the weighted blow-up with weight $w$ gives a $w$-morphism. It follows that $(\Theta_1)$ holds and also $(\Xi'_-)$ holds since $a'_1 = 5 > 3 = a_1$. Hence, there exists a $w$-morphism $Y_1 \to X$ so that $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$. If we interchange the roles of $y$ and $z$, we can get another $w$-morphism $Y_2 \to X$ with $Y \overset{-}{\underset{X}{\Rightarrow}} Y_2$.

Now assume that $(b,c,d) = (3,1,2)$. Consider the chart $U_y \subset Y$ which is defined by

$$(x^2 + zu + g'(y,z,u) = 0) \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{3}(0,1,1,2).$$

Taking $(y_1,...,y_4) = (y, u, y+z, x)$, then $(\Xi')$ holds. Let $w$ be the weight $w(y_1,...,y_4) = \frac{1}{3}(1,5,1,3)$. Then the weighted blow-up with the weight $w$ gives a $w$-morphism over $U_y$ and $(\Theta_2)$ and $(\Xi'_-)$ holds. If we take $w$ to be another weight $w(x,y,z,u) = \frac{1}{3}(3,1,4,2)$, then we get another $w$-morphism over $U_y$ and $(\Theta_z)$ holds. Thus, we can get two different $w$-morphisms over $X$ and $Y$ is negatively linked to both of them. □

**Proposition 4.9.** Assume that $X$ has $cD/2$ singularities, $Y \to X$ is of type D18–D22, and assume that there are two $w$-morphisms $Y \to X$ and $Y_1 \to X$.

(1) Assume that $Y \to X$ is of type D18 and $Y_1 \to X$ is not of type D20. Then $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$.

(2) Assume that both $Y \to X$ and $Y_1 \to X$ are not of type D18. Then, one of the following holds:

(2–1) $Y \to X$ is of type D19 and $Y_1 \to X$ is of type D22. One has that $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1$.

(2–2) Both $Y \to X$ and $Y_1 \to X$ are of type D21 and $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1 \underset{X}{\overset{-}{\Rightarrow}} Y$.

(2–3) Both $Y \to X$ and $Y_1 \to X$ are of type D20 and there exists another $w$-morphism $Y_2 \to X$ which is of type D18 so that $Y \underset{X}{\overset{-}{\Rightarrow}} Y_2 \underset{X}{\overset{-}{\Leftarrow}} Y_1$.

*Proof.* The computation is similar to the proof of Proposition 4.5 after replacing the types D1–D5 by D18–D22, so we will omit the proof. Notice that an analog result of Lemma 4.4 can be proved by a similar computation, or can be directly followed by [10, Section 5]. □

Now we consider non-$w$-morphisms over $X$. Notice that there is no divisorial contraction with discrepancy greater than $\frac{1}{3}$ over $cD/3$ points. Divisorial contractions of discrepancy greater than $\frac{1}{2}$ over $cD/2$ points are listed in Table 7.

**Table 7.** Divisorial contractions to $cD/r$ points with large disprepancies.

| No. | defining equations | $(r; a_i)$ / weight | type / $a(X,E)$ | condition |
|---|---|---|---|---|
| D23 | $x^2+y^2u+z^m+g_{\geq b+1}(x,y,z,u)$ | $(2;1,1,1,0)$ / $\frac{1}{2}(b+2,b,a,2)$ | $cD/2$ / $a/2$ | $ma=2b+2$, $a$ and $b$ are odd |
| D24 | $\begin{cases} x^2+yt+g_{\geq b+2}(z,u) \\ yu+z^m+p_{\frac{b}{2}+1}(z,u)+t \end{cases}$ | $(2;1,1,1,0,1)$ / $\frac{1}{2}(b+2,b,a,2,b+4)$ | $cD/2$ / $a/2$ | $ma=b+2$, $a\equiv b(\text{mod } 2)$ |
| D25 | $x^2+y^2u+z^{4b}+g_{\geq 4b}(y,z,u)$ | $(2;1,1,1,0)$ / $(2b,2b,1,1)$ | $cD/2$ / $1$ | |
| D26 | $x^2+yzu+y^4+z^b+u^c$ | $(2;1,1,1,0)$ / $(2,1,2,1)$ | $cD/2$ / $1$ | $b,c\geq 4$, $b$ is even |
| D27 | $\begin{cases} x^2+ut+y^4+z^4 \\ yz+u^2+t \end{cases}$ | $(2;1,1,1,0,0)$ / $(2,1,1,1,3)$ | $cD/2$ / $1$ | |
| D28 | $\begin{cases} x^2+ut+g_{\geq 2b+2}(y,z,u) \\ y^2+p_{2b}(x,z,u)+t \end{cases}$ | $(2;1,1,1,0,0)$ / $(b+1,b,1,1,2b+1)$ | $cD/2$ / $1$ | Either $b$ is odd, or $b$ is even and $xz^{b-1}$ or $z^{2b}\in p$ |
| D29 | $\begin{cases} x^2+ut+g_{\geq 2b+2}(y,z,u) \\ y^2+p_{2b}(x,z,u)+t \end{cases}$ | $(2;1,1,1,0,0)$ / $(b+1,b,2,1,2b+1)$ | $cD/2$ / $2$ | $xz^{\frac{b-1}{2}}$ or $z^b\in p$ |

**Proposition 4.10.** Assume that $r=2$ and $Y\to X$ is a divisorial contraction with the discrepancy $\frac{a}{2}>1$. Then there exists a divisorial contraction $Y_1\to X$ such that $Y \underset{X}{\Rightarrow} Y_1$, and $a(F,X)<\frac{a}{2}$ where $F=exc(Y_1\to X)$.

*Proof.* First, assume that $Y\to X$ is of type D23. Consider the chart $U_x\subset Y$ which is defined by

$$(x+y^2u+z^m+g'(x,y,z,u)=0)\subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{b+2}(-2,b,a,2).$$

We take $(y_1,...,y_4)=(x,y,z,u)$ with $\delta_4=2$ or 4. One can see that $(\Xi')$ holds. Since $a_4=2$ and $a\geq 3$, we know that $(\Theta_4)$ holds. Thus, there exists $Y_1\to X$ such that $Y\underset{X}{\Rrightarrow}Y_1$. The origin of $U_x$ is a cyclic quotient point. Let $F$ be the exceptional divisor of the $w$-morphism over this point. Then $v_F(x)\leq\frac{m}{b+2}$. It follows that

$$a(F,X)=\frac{1}{b+2}+\frac{a}{2}v_F(x)\leq\frac{2+ma}{2b+4}=1<\frac{a}{2}.$$

Assume that $Y\to X$ is of type D24. Consider the chart $U_t\subset Y$ which is defined by

$$(x^2+y+g'(z,u,t)=yu+z^m+p(z,u)+t=0)\subset\mathbb{A}^4_{(x,y,z,u,t)}/\frac{1}{b+4}(c_1,...,c_5)$$

where $(c_1,...,c_5)=(b+2,b,a,2,-2)$ if $a$, $b$ are odd, and $(c_1,...,c_5)=(b+3,b+2,\frac{a}{2},1,-1)$ if $a$, $b$ are even. We take $(y_1,...,y_5)=(y,u,z,x,t)$ with $\delta_4=4$ and $\delta_5=1$ or 2. Then $(\Xi)$ holds. Now the origin of $U_t$ is a cyclic quotient point. Let $F$ be the exceptional divisor over this point. Then $v_F(y_1,...,y_5)=\frac{1}{b+4}(a'_1,...,a'_5)$ with $a'_2+a'_4=b+4$. It follows that $a(a'_2+a'_4)>b+4=a_2+a_4$, hence $(\Theta_j)$ holds for $j=2$ or 4. Thus, there exists $Y_1\to X$ so that $Y\underset{X}{\Rrightarrow}Y_1$. One has that

$$a(F,X)=\frac{1}{b+4}+\frac{a}{2}v_F(x)\leq\frac{2+ma}{2b+8}\leq\frac{1}{2}.$$

Hence, $Y_1\to X$ is a $w$-morphism.

Assume that $Y\to X$ is of type D25. The chart $U_y\subset Y$ is given by

$$(x^2+yu+z^{4b}+g'(y,z,u)=0)\subset\mathbb{A}^4_{(x,y,z,u)}/\frac{1}{4b}(0,2b-1,1,2b+1).$$

We take $(y_1,...,y_4)=(y,u,z,x)$ with $\delta_4=4$. One can see that $(\Xi')$ holds. The origin of $U_y$ is a $cA/4b$ point and there is only one $w$-morphism over this point. Let $F$ be the exceptional divisor of the $w$-morphism. Then $v_F(y_1,...,y_4)=\frac{1}{4b}(2b-1,2b+1,1,4b)$. Hence, $(\Theta_2)$ holds. One can also compute that $a(F,X)=\frac{1}{2}$, hence there exists a $w$-morphism $Y_1\to X$ such that $Y\underset{X}{\Rrightarrow}Y_1$.

Assume that $Y\to X$ is of type D26. The chart $U_z\subset Y$ is a $cA/4$ point given by

$$(x^2+yu+y^4+z^{2b-4}+u^cz^{c-4}=0)\subset\mathbb{A}^4_{(x,y,z,u)}/\frac{1}{4}(0,1,1,3).$$

We take $(y_1,...,y_4)=(y,u,y+z,x)$ with $\delta_4=4$. One can see that $(\Xi')$ holds. Now let $w$ be the weight such that $w(y_1,...,y_4)=\frac{1}{4}(1,3,1,4)$ if $b=4$ and $w(y_1,...,y_4)=\frac{1}{4}(1,7,1,4)$ if $b\geq 6$. Hence, $(\Theta_2)$ holds, and there exists $Y_1\to X$ such that $Y\underset{X}{\Rrightarrow}Y_1$. One can compute that $a(F,X)=\frac{1}{2}$.

Assume that $Y\to X$ is of type D27. Consider the chart $U_t\subset Y$ which is defined by

$$(x^2+u+y^4+z^4=yz+u^2+t=0)\subset\mathbb{A}^5_{(x,y,z,u,t)}/\frac{1}{6}(5,1,1,4,2).$$

We take $(y_1,...,y_5)=(u,y,y+z,x,t)$ with $\delta_4=4$ and $\delta_5=1$. In this case, $(\Xi)$ holds. Now the origin of $U_t$ is a cyclic quotient point. Let $F$ be the exceptional divisor over this point which corresponds to a $w$-morphism. Then $v_F(y_1,...,y_5)=\frac{1}{6}(5,1,1,4,2)$. One can see that $(\Theta_1)$ holds. Thus, there exists $Y_1\to X$ which extracts $F$ so that $Y\underset{X}{\Rrightarrow}Y_1$. One can compute that $a(F,X)=\frac{1}{2}$.

Finally, assume that $Y \to X$ is of type D28 or D29. The chart $U_t \subset Y$ is defined by

$$(x^2 + u + g'(y,z,u,t) = y^2 + p(x,z,u) + t = 0) \subset \mathbb{A}^4_{(x,y,z,u,t)}/\frac{1}{4b+2}(1,-1,a-2b-1,2,-2),$$

where $a = 2$ in case D28 and $a = 4$ in case D29. We take $(y_1,...,y_5) = (x,y,z,u,t)$ with $\delta_4 = 4$ and $\delta_5 = 2$. Then ($\Xi$) holds. The origin of $U_t$ is a cyclic quotient point. Let $F$ be the exceptional divisor corresponding to the $w$-morphism over this point. Then $v_F(y_1,...,y_5) = \frac{1}{4b+2}(a'_1,...,a'_5)$ with $a'_1 + a'_2 = 4b+2$, $a'_2(2b+1-a) \equiv 1 \pmod{4b+2}$ and $a'_3 = 1$. From the defining equation one can see that $a'_4 > 1$, hence ($\Theta_4$) holds. Thus, there exists a divisorial contraction $Y_1 \to X$ which extracts $F$ so that $Y \underset{X}{\Rrightarrow} Y_1$. We only need to show that $a(F,X) < \frac{a}{2}$.

Assume that $Y \to X$ is of type D28. In this case, $a'_2$ is the integer such that $a'_2(2b-1) \equiv 1 \pmod{4b+2}$. If $b$ is odd, then $a'_2 = b$ since

$$b(2b-1) = 2b^2 - b = (4b+2)\frac{b-1}{2} + 1.$$

One can see that $a'_5 \leq 2b$. If $b$ is even, then $a'_2 = 3b+1$ since

$$(3b+1)(2b-1) = 6b^2 - b - 1 = (4b+2)(\frac{3}{2}b - 1) + 1.$$

Hence, $a'_1 = b+1$. Now, since $xz^{b-1}$ or $z^{2b} \in p(x,z,u)$, we also have that $a'_5 \leq 2b$. In either case we have

$$a(F,X) = \frac{1}{4b+2} + \frac{a'_5}{4b+2} \leq \frac{2b+1}{4b+2} = \frac{1}{2} < 1 = \frac{a}{2}.$$

Finally, assume that $Y \to X$ is of type D29. We want to show that $a'_5 < 4b+2$. Then

$$a(F,X) = \frac{1}{4b+2} + 2\frac{a'_5}{4b+2} \leq \frac{1}{4b+2} + \frac{8b+2}{4b+2} < 2 = \frac{a}{2}$$

and we can finish the proof. If $z^b \in p(x,z,u)$, then $a'_5 \leq b$. If $a'_2 < 2b+1$, then $a'_5 \leq 4b$. Assume that $z^b \notin p(x,z,u)$ and $a'_2 \geq 2b+1$. Then $a'_1 \leq 2b+1$ and $xz^{\frac{b-1}{2}} \in p(x,z,u)$. Hence,

$$a'_5 \leq 2b + 1 + \frac{b-1}{2} < 4b+2.$$

□

### 4.5. Divisorial contractions to cE points

In this subsection, we assume that $X$ has $cE$ singularities. First, we study $w$-morphisms over $cE$ points. All $w$-morphisms over $cE$ type points are listed in Tables 8 and 9.

**Table 8.** Divisorial contractions to $cE$ points with discrepancy one.

| No. | defining equations | weight | $\frac{\text{type}}{a(X,E)}$ | condition |
|---|---|---|---|---|
| E1 | $x^2+y^3+g_{\geq 4}(y,z,u)$ | $(2,2,1,1)$ | $\frac{cE_6}{1}$ | $\frac{\partial^2}{\partial y^2}g(y,z,u)=0$ |
| E2 | $x^2+xp_2(z,u)+y^3+g_{\geq 5}(y,z,u)$ | $(3,2,1,1)$ | $\frac{cE_{6,7}}{1}$ | |
| E3 | $x^2+y^3+g_{\geq 6}(y,z,u)$ | $(3,2,2,1)$ | $\frac{cE}{1}$ | |
| E4 | $x^2+y^3+y^2p_2(z,u)+g_{\geq 8}(y,z,u)$ | $(4,3,2,1)$ | $\frac{cE}{1}$ | |
| E5 | $x^2+xp_4(y,z,u)+y^3+g_{\geq 9}(y,z,u)$ | $(5,3,2,1)$ | $\frac{cE}{1}$ | |
| E6 | $x^2+y^3+y^2p_3(z,u)+g_{\geq 10}(y,z,u)$ | $(5,4,2,1)$ | $\frac{cE_{7,8}}{1}$ | |
| E7 | $x^2+y^3+g_{\geq 12}(y,z,u)$ | $(6,4,3,1)$ | $\frac{cE}{1}$ | |
| E8 | $x^2+y^3+y^2p_4(z,u)+g_{\geq 14}(y,z,u)$ | $(7,5,3,1)$ | $\frac{cE_{7,8}}{1}$ | |
| E9 | $x^2+xp_7(y,z,u)+y^3+g_{\geq 15}(y,z,u)$ | $(8,5,3,1)$ | $\frac{cE_{7,8}}{1}$ | |
| E10 | $x^2+y^3+g_{\geq 18}(y,z,u)$ | $(9,6,4,1)$ | $\frac{cE_{7,8}}{1}$ | |
| E11 | $x^2+y^3+y^2p_6(z,u)+g_{\geq 20}(y,z,u)$ | $(10,7,4,1)$ | $\frac{cE_8}{1}$ | |
| E12 | $x^2+y^3+g_{\geq 24}(y,z,u)$ | $(12,8,5,1)$ | $\frac{cE_8}{1}$ | |
| E13 | $x^2+y^3+g_{\geq 30}(y,z,u)$ | $(15,10,6,1)$ | $\frac{cE_8}{1}$ | |

**Table 9.** Divisorial contractions to $cE$ points with discrepancy one, continued.

| No. | defining equations | weight | type / $a(X,E)$ | condition |
|---|---|---|---|---|
| E14 | $\begin{cases} x^2+y^3+tz+g_{\geq 6}(y,z,u) \\ p_4(x,y,z,u)+t \end{cases}$ | $(3,2,1,1,5)$ | $\frac{cE_{6,7}}{1}$ | $p(x,y,z,u)$ is irreducible |
| E15 | $x^2+xp_2(z,u)+y^3+g_{\geq 6}(x,y,z,u)$ | $(4,2,1,1)$ | $\frac{cE_6}{1}$ | |
| E16 | $\begin{cases} x^2+y^3+tp_2(z,u)+g_{\geq 6}(y,z,u) \\ q_3(y,z,u)+t \end{cases}$ | $(3,2,1,1,4)$ | $\frac{cE_7}{1}$ | $q(y,z,u)$ is irreducible |
| E17 | $x^2+y^3+yz^3+g_{\geq 6}(y,z,u)$ | $(3,3,1,1)$ | $\frac{cE_7}{1}$ | $y^2u^2 \in g$ |
| E18 | $\begin{cases} x^2+yt+g_{\geq 10}(y,z,u) \\ y^2+p_6(y,z,u)+t \end{cases}$ | $(5,3,2,1,7)$ | $\frac{cE_{7,8}}{1}$ | $y^2+p(y,z,u)$ is irreducible |

We assume that there exist two different $w$-morphisms over $X$, say $Y \to X$ and $Y_1 \to X$. Let $F = exc(Y_1 \to X)$. Let $P = \mathrm{Center}_Y F$. One always has that $a(F,Y) < 1$, so $P$ is a non-Gorenstein point.

**Lemma 4.11.** Assume that both $Y \to X$ and $Y_1 \to X$ are of type E1–E13. Then $Y \to X$ is not of type E1 or E6.

*Proof.* Assume that $Y \to X$ is of type E1. Then the only non-Gorenstein point on $Y$ is the origin of

$$U_y = (x'^2 + y'^2 + g'(z',u') = 0) \subset \mathbb{A}^4_{(x',y',z',u')}/\frac{1}{2}(0,1,1,1).$$

This is a $cAx/2$ point. The exceptional divisor $G$ of discrepancy less than one over this point is given by the weighted blow-up with the weight $w(x',y',z',u') = \frac{1}{2}(2,3,1,1)$. One can compute that $a(G,X) = 2$, hence there is only one $w$-morphism over $X$. Thus, $Y \to X$ is not of type E1.

Assume that $Y \to X$ is of type E6. If $X$ has $cE_8$ singularities, then there is only one non-Gorenstein point on $Y$, namely the origin of $U_y$. If $X$ has $cE_7$ singularities, then the origin of $U_z$ is also a non-Gorenstein point. Assume first that $P$ is the origin of $U_z$. Then $P$ is a cyclic quotient point of index two and there is only one exceptional divisor over $P$ with discrepancy less than one. Hence, $F$ should correspond to this exceptional divisor. One can compute that $v_F(x,y,z,u) = (3,3,1,1)$, so $Y_1 \to X$ should be of type E17. Nevertheless, in this case one can see that $v_F(\sigma) \leq v_E(\sigma)$ for all $\sigma \in \mathcal{O}_X$. This contradicts Corollary 2.17. Hence, $P$ can not be the origin of $U_z$.

We want to show that $P$ is also not the origin of $U_y$. The chart $U_y$ is defined by

$$(x'^2 + y'(y' + p(z',u')) + g'(y',z',u') = 0) \subset \mathbb{A}^4_{(x',y',z',u')}/\frac{1}{4}(1,3,2,1).$$

The origin of $U_y$ is a $cAx/4$ point. Since $(u = 0)$ defines a Du Val section, we know that $v_F(u) = v_F(y') + v_F(u') = 1$. Hence, both $v_F(y')$ and $v_F(u') < 1$. This means that $v_F(y) \leq 3$. Assume that $Y \to X$ and $Y_1 \to X$ correspond to the same embedding $X \hookrightarrow \mathbb{A}^4$. Then, since $v_F(y) \leq 3$, we know that $Y_1 \to X$ is of type E1–E5. However, in those cases one always has that $v_F(\sigma) \leq v_E(\sigma)$ for all $\sigma \in \mathcal{O}_X$. This contradicts Corollary 2.17. Thus, $Y_1 \to X$ corresponds to a different embedding.

Let $Z \to Y$ be a $w$-morphism over the origin of $U_y$. From the classification we know that $Z \to Y$ is a weighted blow-up with the weight $w(x', y', z', u') = \frac{1}{4}(5, k, 2, 1)$ for $k = 3$ or 7. One can compute that non-Gorenstein points on $Z$ over $U_y$ are cyclic quotient points. Let $\bar{Z} \to Z$ be an economic resolution over those cyclic quotient points. Then $F$ appears on $\bar{Z}$ since $a(F, Y) < 1$. Moreover, $\bar{Z} \to X$ can be viewed as a sequence of weighted blow-ups with respect to the embedding $X \hookrightarrow \mathbb{A}^4_{(x,y,z,u)}$. We write $(x_1, ..., x_4) = (x, y, z, u)$ and let $X \hookrightarrow \mathbb{A}^4_{(x'_1,...,x'_4)}$ be the embedding corresponding to the weighted blow-up $Y_1 \to X$. One can always assume that $x'_4 = x_4 = u$ since $v_E(u) = 1$. We write $x'_j = x_j + q_j$. Since $Y \to X$ and $Y_1 \to X$ correspond to different embeddings, there exists $j < 4$ such that $q_j \neq 0$ and $v_F(x'_j) > v_F(x_j) = v_F(q)$. Since $\bar{Z} \to X$ can be viewed as a sequence of weighted blow-ups with respect to the embedding $X \hookrightarrow \mathbb{A}^4_{(x,y,z,u)}$, we know that the defining equation of $\bar{Z}$ is of the form $x_j + q_j + \bar{h}$ such that $v_F(x'_j) = v_F(\bar{h})$. Hence, there is exactly one $j$ such that $q_j \neq 0$, and the defining equation of $X$ is of the form $\xi(x_j + q_j) + h$. One can see that either $x_j = z$, or $x_j = y$ and $q_j = p$.

Now, if $x_j = z$, then $x'_1 = x_1 = x$ and $x'_2 = x_2 = y$. One can see that $v_F(x'_2) = v_F(y) \leq 3$. So, $Y_1 \to X$ is of type E1–E5. In those cases, $v_F(x'_j) \leq 2$, so $v_F(x_j) = v_F(q_j) = 1$ and $v_F(x'_j) = 2$. Hence, $Y_1 \to X$ is of type E3–E5 and $v_F(y) = v_F(x'_2) \geq 2$. Also, since $v_F(q_j) = 1$, $q_j = \lambda u$ for some $\lambda \in \mathbb{C}$. Therefore $v_F(z) = v_F(x'_j - q_j) = 1$. But, then

$$\frac{v_F(z)}{v_E(z)} = \frac{1}{2} \leq \frac{v_F(y)}{v_E(y)}.$$

By Lemma 2.16, $\text{Center}_Y F$ can not be the origin of $U_y$. This leads to a contradiction.

Finally, we assume that $x_j = y$ and $q_j = p$. Notice that $p = \lambda_1 zu + \lambda_2 u^3$, hence $v_F(p) \geq 2$. If $v_F(z) = v_F(x'_3) = 1$, then $Y_1 \to X$ is of type E1 or E2, and so $v_F(x'_j) = 2$. However, we know that $v_F(p) \geq 2$. This contradicts the assumption that $v_F(x'_j) > v_F(q_j) = v_F(p)$. Hence, $v_F(z) \geq 2$ and so $v_F(p) \geq 3$. Since

$$v_F(y) = v_F(x_j) = v_F(q_j) = v_F(p) \geq 3$$

and $v_F(y) \leq 3$ by the previous discussion, we know that $v_F(y) = 3$. Recall that we write

$$U_y = (x'^2 + y'(y' + p(z', u')) + g'(y', z', u') = 0) \subset \mathbb{A}^4_{(x',y',z',u')}/\frac{1}{4}(1, 3, 2, 1).$$

Since $v_F(y) = 3$, $v_F(E) = v_F(y') = \frac{3}{4}$. This means that $a(F, Y) = \frac{1}{4}$, so $F$ corresponds to a $w$-morphism over $U_y$. Nevertheless, as we mentioned before, $w$-morphisms over $U_y$ can be obtained by a weighted blow-up with respect to the above embedding, hence $Y_1 \to X$ and $Y \to X$ correspond to the same four-dimensional embedding, leading to a contradiction. □

**Lemma 4.12.** Assume that both $Y \to X$ and $Y_1 \to X$ are of type E1–E13. If $P$ is the origin of $U_x \subset Y$, then $Y \to X$ is of type E2, E5 or E9, and $Y_1 \to X$ has the same type. One has that $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1 \overset{-}{\underset{X}{\Rightarrow}} Y$.

*Proof.* This assumption implies that the origin of $U_x$ is contained in $Y$, so $Y \to X$ is of type E2, E5 or E9 and $P$ is a cyclic quotient point. If $Y \to X$ is a weight blow-up with the weight $(b, c, d, 1)$, then $b = c + d$ and

$$U_x = (x' + p(z', u') + g'(x', y', z', u') = 0) \subset \mathbb{A}^4_{(x',y',z',u')}/\frac{1}{b}(-1, c, d, 1).$$

Since $a(F, Y) < 1$, $F$ is the valuation described by [4, Proposition 3.1]. Hence, $v_F(y', z', u') = \frac{1}{b}(c', d', a')$ with $c' + d' = b$ and $\frac{a'}{b} = a(F, Y)$. Since $a(F, X) = a(F, Y) + v_F(x') = 1$, we know that $v_F(x') = 1 - \frac{a'}{b}$.

One can compute that

$$v_F(x, y, z, u) = (b - a', c - \frac{(a'c - c')}{b}, d - \frac{a'd - d'}{b}, 1).$$

Since $c' < b$ and $a'c \equiv c'(\text{mod } b)$, we know that $\frac{a'c-c'}{b} \geq 0$, so $v_F(y) \leq c = v_E(y)$. Likewise, we know that $v_F(z) \leq d = v_E(z)$. One also has that $v_F(x) < v_E(x)$ and $v_F(u) = v_E(u)$.

On the other hand, Corollary 2.17 says that there exists $\sigma \in O_X$ such that $v_F(\sigma) > v_E(\sigma)$. This can only happen when

$$v_F(x') = v_F(p(z', u')) < v_F(x' + p(z', u')) = v_F(g'(x', y', z', u'))$$

and in this case one can choose $\sigma = x + p(z, u) \in O_X$. Now, $Y_1 \to X$ can be obtained by a weighted blow-up with respect to the embedding

$$X \hookrightarrow (\sigma^2 - \sigma p(z, u) + y^3 + g(y, z, u) = 0) \subset \mathbb{A}^4_{(\sigma,y,z,u)}$$

and with the weight $w_1(\sigma, y, z, u) = (b_1, c_1, d_1, 1)$ where $c_1 = c - \frac{(a'c-c')}{b}$ and $d_1 = d - \frac{a'd-d'}{b}$. Since $c_1 \leq c$, $d_1 \leq d$ and $b_1 = v_F(v) > v_E(v)$, by Lemma 2.16 we know that $\text{Center}_{Y_1}E$ is the origin of $U_{1,\sigma}$.

Now, if we interchange $Y$ and $Y_1$, then the above argument yields that $c \leq c_1$ and $d \leq d_1$. Hence, $c = c_1$ and $d = d_1$ and so $Y \to X$ and $Y_1 \to X$ are of the same type. One has that $a' = 1$ and $c' = c$, $d' = d$. Thus, $F$ is the exceptional divisor of the $w$-morphism over $P$. Now we know that $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$ by Lemma 3.13 and also $Y_1 \overset{-}{\underset{X}{\Rightarrow}} Y$ by the symmetry. □

**Lemma 4.13.** Assume that both $Y \to X$ and $Y_1 \to X$ are of type E1–E13. Then $P$ is not the origin of $U_y \subset Y$.

*Proof.* By Lemma 4.11, we know that $Y \to X$ is not of type E1 or E6, hence $Y \to X$ is of type E4, E8 or E11 and the origin of $U_y$ is a cyclic quotient point. We assume that $Y \to X$ is a weighted blow-up with the weight $(b, c, d, 1)$. Following the same computation as in the proof of Lemma 4.12, we may write $Y_1 \to X$ as a weighted blow-up with respect to the embedding

$$X \hookrightarrow (x^2 + (\sigma - p(z, u))^2\sigma + g(\sigma, z, u) = 0) \subset \mathbb{A}^4_{(x,\sigma,z,u)}$$

and with the weight $(b_1, c_1, d_1, 1)$, such that $\text{Center}_{Y_1}E$ is the origin of $U_{1,\sigma} \subset Y_1$. Nevertheless, in this case one always has that $b_1 < b$ since $b > c$. The symmetry between $Y$ and $Y_1$ yields that $b > b_1 > b$, which is impossible. □

**Lemma 4.14.** Assume that both $Y \to X$ and $Y_1 \to X$ are of type E1–E13. Let $X \hookrightarrow \mathbb{A}^4_{(x_1,...,x_4)}$ be the embedding that corresponds to $Y \to X$ in Table 8. Then $P$ is the origin of $U_i$ for some $i \leq 4$.

*Proof.* Assume that $P$ is not the origin of $U_i$ for all $i \leq 4$. Then $Y$ has a non-Gorenstein point on $U_i \cap U_j$ for some $i \neq j$. In this case, $Y \to X$ is of type E7 or E10–E13. For simplicity we assume that $i = 1$ and $j = 2$. If $Y \to X$ is a weighted blow-up with the weight $(a_1, ..., a_4)$, then we have the following observation:

(1) $a_1 = dk_1$ and $a_2 = dk_2$ for some integers $k_1$, $k_2$ and $d$. We may assume that $k_2 = 2$ and $k_1$ is odd.

(2) $x_1^{k_2}$ and $x_2^{k_1}$ appear in $f$ where $f$ is the defining equation of $X$. Moreover $\nu_E(x_1^{k_2}) = \nu_E(x_2^{k_1}) = \nu_E(f)$.

(3) $P$ is a cyclic quotient point of index $d$. On $U_1$, the local coordinate system is given by $(x_1', x_3', x_4')$, where $x_l'$ is the strict transform of $x_l$ on $U_1$.

Since $F$ is a valuation of discrepancy less than 1 over $P$, we know that $\nu_F(x_1', x_3', x_4') = \frac{1}{d}(a_1', a_3', a_4')$ with $a_l' < d$ for $l = 1$, 3, and 4. One can compute that

$$\nu_F(x_1, ..., x_4) = (k_1 a_1', k_2 a_1', \frac{1}{d}(a_3' + a_3 a_1'), \frac{1}{d}(a_4' + a_4 a_1')),$$

and $Y_1 \to X$ can be obtained by the weighted blow-up with respect to the same embedding $X \hookrightarrow \mathbb{A}^4_{(x_1,...,x_4)}$ and with the weight $\nu_F$. Nevertheless, one can easily see that $\nu_F(x_l) \le \nu_E(x_l)$ for all $1 \le l \le 4$. This contradicts Corollary 2.17. □

**Proposition 4.15.** Assume that both $Y \to X$ and $Y_1 \to X$ are both of type E1–E13. Then $Y \to X$ is of type E2, E5 or E9, and $Y_1 \to X$ has the same type. One has that $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1 \underset{X}{\overset{-}{\Rightarrow}} Y$.

*Proof.* Let

$$X \hookrightarrow (f(x, y, z, u) = 0) \subset \mathbb{A}^4_{(x,y,z,u)}$$

be the embedding corresponding to $Y \to X$, and

$$X \hookrightarrow (f_1(x_1, y_1, z_1, u_1) = 0) \subset \mathbb{A}^4_{(x_1,y_1,z_1,u_1)}$$

be the embedding corresponding to $Y_1 \to X$. If $\mathrm{Center}_Y F$ is the origin of $U_x \subset Y$ or $\mathrm{Center}_{Y_1} E$ is the origin of $U_{x_1} \subset Y_1$, then the statement follows from Lemma 4.12. We do not consider these cases here. Then, Lemmas 4.13 and 4.14 imply that $\mathrm{Center}_Y F = U_z \subset Y$ and $\mathrm{Center}_{Y_1} E = U_{z_1} \subset Y_1$.

We may assume that $\nu_E(f) \le \nu_F(f_1)$. Since $\nu_E(u) = \nu_F(u_1) = 1$, one can always assume that $u = u_1$ and $(u = 0)$ defines a Du Val section. Lemma 2.16 implies that $\nu_E(z) > \nu_E(z_1)$ and $\nu_F(z) < \nu_F(z_1)$, hence $z \neq z_1$. We may write $z_1 = z + h$. If $\nu_E(h) \ge \nu_E(z)$, then we may replace $z$ by $z + h$, which will lead to a contradiction. Hence, $\nu_E(h) < \nu_E(z)$. Thus, $h = \lambda u^k$ for some $k < \nu_E(z)$. Since $(u = 0)$ defines a Du Val section, we know that $z^4$, $yz^3$ or $z^5 \in f$. It follows that $u^{4k}$, $yu^{3k}$ or $u^{5k}$ appear in either $f$ or $f_1$. This means that $\nu_E(f) \le 4k$, $\nu_E(y) + 3k$ or $5k$ for some $k < \nu_E(z)$. One can easily check that for all the cases in Table 8 this inequality never holds. Thus, we get a contradiction. □

**Proposition 4.16.** Assume that $Y \to X$ is of type E14–E18. Then there exists $Y_1 \to X$ which is of type E3 or E6 such that $Y \underset{X}{\Rightarrow} Y_1$. Moreover, if $Y \to X$ is of type E15 or E17, then $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1$.

*Proof.* Assume that $Y \to X$ is of type E14. Consider the chart

$$U_t = (x'^2 + y'^3 + z' + g'(y', z', u', t') = p(x', y', z', u') + t' = 0) \subset \mathbb{A}^5_{(x',y',z',u',t')} / \frac{1}{5}(3, 2, 1, 1, 4).$$

We choose $(y_1, ..., y_5) = (x', y', u', z', t')$ with $\delta_4 = 1$ and $\delta_5 = 2$ or 4, or $\delta_4 = 2$ and $\delta_5 = 1$ or 4. Then $(\Xi)$ holds. Now, let $F$ be the exceptional divisor that corresponds to the $w$-morphism over the origin of $U_t$. Then

$$\nu_F(y_1, ..., y_5) = \frac{1}{5}(3, 2, 1, 6, 4).$$

One can see that $(\Theta_4)$ holds. Thus, there exists a divisorial contraction $Y_1 \to X$ so that $Y \underset{X}{\Rightarrow} Y_1$ which extracts $F$. One can compute that $v_F(x,y,z,u) = (3,2,2,1)$, so $Y_1 \to X$ is of type E3.

Assume that $Y \to X$ is of type E15. Consider the chart

$$U_x = (x'^2 + p(z',u') + y'^3 + g'(x',y',z',u') = 0) \subset \mathbb{A}^4_{(x',y',z',u')}/\frac{1}{4}(3,2,1,1).$$

We choose $(y_1,...,y_4) = (x',z',u',y')$ with $\delta_4 = 4$. Then $(\Xi')$ holds. Now the origin of $U_x$ is a $cAx/4$ point. After a suitable change of coordinates, we may assume that $u'^2 \notin p(z',u')$. Then the $w$-morphism over this point can be given by a weighted blow-up with the weight $v_F(y_1,...,y_4) = \frac{1}{4}(3,5,1,2)$. One can see that $(\Theta_2)$ and $(\Xi'_-)$ hold. Hence, there exists a divisorial contraction $Y_1 \to X$ such that $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$. One also has $v_F(x,y,z,u) = (3,2,2,1)$, so $Y_1 \to X$ is of type E3.

Assume that $Y \to X$ is of type E16. Consider the chart

$$U_t = (x'^2 + y'^3 + p(z',u') + g'(y',z',u',t') = q(y',z',u') + t' = 0) \subset \mathbb{A}^5_{(x',y',z',u',t')}/\frac{1}{4}(3,2,1,1,3).$$

We choose $(y_1,...,y_5) = (y',z',u',x',t')$ with $\delta_4 = 4$ and $\delta_5 = 1$ or 2. Then $(\Xi)$ holds. Now the origin of $U_t$ is a $cAx/4$ point. After a suitable change of coordinates, we may assume that $u'^2 \notin p(z',u')$. Then the weight $v_F(y_1,...,y_5) = \frac{1}{4}(2,5,1,3,3)$ defines a $w$-morphism over $U_t$. One can see that $(\Theta_2)$ holds. Hence, there exists a divisorial contraction $Y_1 \to X$ such that $Y \underset{X}{\Rightarrow} Y_1$. One can compute that $v_F(x,y,z,u) = (3,2,2,1)$, so again $Y_1 \to X$ is of type E3.

Assume that $Y \to X$ is of type E17. Consider the chart

$$U_y = (x'^2 + y'^3 + z'^3 + g'(y',z',u') = 0) \subset \mathbb{A}^4_{(x',y',z',u')}/\frac{1}{3}(0,2,1,1).$$

We can choose $(y_1,...,y_4) = (y',z',u',x')$ with $\delta_4 = 4$. Then $(\Xi')$ holds. The origin of $U_y$ is a $cD/3$ point. Notice that $y'^2u'^2 \in g'(y',z',u')$, so the $w$-morphism over $U_y$ is given by the weighted blow-up with the weight $v_F(y_1,...,y_4) = \frac{1}{3}(2,4,1,3)$. One can see that $(\Theta_2)$ and $(\Xi'_-)$ hold. One has that $v_F(x,y,z,u) = (3,2,1,1)$, so there exists $Y_1 \to X$ which is of type E3 such that $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$.

Finally, assume that $Y \to X$ is of type E18. Consider the chart

$$U_t = (x'^2 + y' + g'(y',z',u',t') = y'^2 + p(y',z',u') + t = 0) \subset \mathbb{A}^5_{(x',y',z',u',t')}/\frac{1}{7}(5,3,2,1,6).$$

We choose $(y_1,...,y_5) = (y',z',u',x',t')$ with $\delta_4 = 4$ and $\delta_5 = 1$. Then $(\Xi)$ holds. The origin of $U_t$ is a cyclic quotient point. Let $F$ be the exceptional divisor corresponding to the $w$-morphism over this point. Then

$$v_F(y_1,...,y_5) = \frac{1}{7}(10,2,1,5,6)$$

(notice that the irreducibility of $y^2 + p(y,z,u)$ implies that $p(0,z,u) \neq 0$, so $v_F(t) = \frac{6}{7}$). One can see that $(\Theta_1)$ holds. Thus, there exists a divisorial contraction $Y_1 \to X$ which extracts $F$ so that $Y \underset{X}{\Rightarrow} Y_1$. One can compute that $v_F(x,y,z,u) = (5,4,2,1)$, and so $Y_1 \to X$ is of type E6. □

Now we study divisorial contractions over $cE$ points with discrepancy greater than one. Those divisorial contractions are given in Table 10.

**Table 10.** Divisorial contractions to $cE$ points with large disprepancies.

| No. | defining equations | weight | type / $a(X,E)$ | condition |
|---|---|---|---|---|
| E19 | $x^2+(y+p_2(z,u))^3+yu^3+g_{\geq 6}(z,u)$ | $(3,3,2,1)$ | $\frac{cE_6}{2}$ | $z\in p(z,u)$ |
| E20 | $\begin{cases} x^2+yt+g_{\geq 10}(y,z,u) \\ y^2+p_6(z,u)+t \end{cases}$ | $(5,3,2,2,7)$ | $\frac{cE_7}{2}$ | $gcd(p_6,g_{10})=1$ |
| E21 | $x^2+y^3+u^7+g_{\geq 14}(z,u)$ | $(7,5,3,2)$ | $\frac{cE_{7,8}}{2}$ | $yz^3$, $z^5$ or $z^4u\in g(z,u)$ |

**Proposition 4.17.** Assume that $Y\to X$ is a divisorial contraction with discrepancy $a>1$. Then there exists a $w$-morphism $Y_1\to X$ such that $Y\underset{X}{\Rrightarrow} Y_1$.

*Proof.* Assume first that $Y\to X$ is of type E19. The chart $U_y\subset Y$ is defined by

$$(x^2+(y+p(z,u))^3+u^3+g'(y,z,u)=0)\subset\mathbb{A}^4_{(x,y,z,u)}/\frac{1}{3}(0,1,1,2).$$

One can choose $(y_1,...,y_4)=(x,y+p,z,u)$ with $\delta_4=1$. Then $(\Xi')$ holds. The origin of $U_y$ is a $cD/3$ point. The $w$-morphism over this point is given by the weighted blow-up with the weight

$$w(y_1,...,y_4)=(3,4,1,2)\text{ or }(6,4,1,5).$$

One can see that $(\Theta_4)$ holds. Thus, there exists $Y_1\to X$ such that $Y\underset{X}{\Rrightarrow} Y_1$. A direct computation shows that $Y_1\to X$ is a $w$-morphism.

Assume that $Y\to X$ is of type E20. The chart $U_t\subset Y$ is defined by

$$(x^2+y+g'(y,z,u,t)=y^2+p(z,u)+t=0)\subset\mathbb{A}^5_{(x,y,z,u,t)}/\frac{1}{7}(5,3,2,2,6).$$

We take $(y_1,...,y_5)=(y,z,u,x,t)$ with $\delta_4=4$ and $\delta_5=1$. Then $(\Xi)$ holds. The $w$-morphism over the origin of $U_t$ is given by weighted blowing-up the weight $w(y_1,...,y_5)=\frac{1}{7}(5,1,1,6,3)$. One can see that $(\Theta_1)$ holds. Hence, there exists a divisorial contraction $Y_1\to X$ such that $Y\underset{X}{\Rrightarrow} Y_1$. One can compute that $Y_1\to X$ is a $w$-morphism.

Finally assume that $Y\to X$ is of type E21. The chart $U_y\subset Y$ is defined by

$$(x^2+y+u^7+g'(y,z,u)=0)\subset\mathbb{A}^4_{(x,y,z,u)}/\frac{1}{5}(2,4,3,2).$$

One can choose $(y_1,...,y_4)=(y,z,u,x)$. Then $(\Xi')$ holds. The $w$-morphism over $U_y$ is given by the weighted blow-up with the weight $w(y_1,...,y_4)=\frac{1}{5}(2,4,1,1)$. One can see that $(\Theta_2)$ holds. Thus, there exists a divisorial contraction $Y_1\to X$ such that $Y\underset{X}{\Rrightarrow} Y_1$. One can compute that $Y_1\to X$ is a $w$-morphism. □

### *4.6. Divisorial contractions to $cE/2$ points*

Finally, we need to study divisorial contractions over $cE/2$ points. All such divisorial contractions are listed in Table 11.

**Table 11.** Divisorial contractions to $cE/2$ points.

| No. | defining equations | $(r; a_i)$ / weight | type / $a(X,E)$ |
|---|---|---|---|
| E22 | $x^2 + y^3 + g_{\geq 3}(y,z,u)$ | $(2; 1, 0, 1, 1)$ / $\frac{1}{2}(3, 2, 3, 1)$ | $cE/2$ / $1/2$ |
| E23 | $x^2 + y^3 + g_{\geq 5}(y,z,u)$ | $(2; 1, 0, 1, 1)$ / $\frac{1}{2}(5, 4, 3, 1)$ | $cE/2$ / $1/2$ |
| E24 | $x^2 + xp_{\frac{5}{2}}(y,z,u) + y^3 + g_{\geq 6}(y,z,u)$ | $(2; 1, 0, 1, 1)$ / $\frac{1}{2}(7, 4, 3, 1)$ | $cE/2$ / $1/2$ |
| E25 | $x^2 + y^3 + g_{\geq 9}(y,z,u)$ | $(2; 1, 0, 1, 1)$ / $\frac{1}{2}(9, 6, 5, 1)$ | $cE/2$ / $1/2$ |
| E26 | $x^2 + y^3 + z^4 + u^8 + g_{\geq 8}(y,z,u)$ | $(2; 1, 0, 1, 1)$ / $(4, 3, 2, 1)$ | $cE/2$ / $1$ |

**Proposition 4.18.** Let $Y \to X$ be a divisorial contraction.

(1) Assume that there are two $w$-morphisms over $X$. Then:

(1–1) If $Y \to X$ is of type E22 and there exists another $w$-morphism $Y_1 \to X$, then $Y_1 \to X$ is of type E22 or E23 and $Y \underset{X}{\Rightarrow} Y_1$.

(1–2) If $Y \to X$ is of type E23, then there are exactly two $w$-morphisms. The other one, $Y_1 \to X$, is of type E22. Interchanging $Y$ and $Y_1$, we are back to Case (1–1).

(1–3) If $Y \to X$ is of type E24, then there are exactly two $w$-morphisms. They are both of type E24 and are negatively linked to each other.

(1–4) $Y \to X$ is not of type E25.

(2) Assume that $Y \to X$ is of type E26. Then there is a $w$-morphism $Y_1 \to X$ which is of type E22 such that $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1$.

*Proof.* The statement about the number of $w$-morphisms follows from [9, Section 10]. First, assume that there exists two $w$-morphisms over $X$ and $Y \to X$ is of type E22. Let $F = exc(Y_1 \to X)$. The only non-Gorenstein point on $Y$ is the origin of $U_z$, which is a $cD/3$ point defined by

$$(x'^2 + y'^3 + g'(y', z', u') = 0) \subset \mathbb{A}^4_{(x',y',z',u')}/\frac{1}{3}(0, 2, 1, 1).$$

One can see that

$$\frac{1}{2} = a(F, X) = a(F, Y) + \frac{1}{2}v_F(z'),$$

hence $a(F, Y) = v_F(z') = \frac{1}{3}$. Thus, $F$ corresponds to a $w$-morphism over $U_z$. From Table 6 we know that

$$v_F(x', y', z', u' + \lambda z') = \frac{1}{3}(b, c, 1, 4)$$

for some $\lambda \in \mathbb{C}$, where $(b,c) = (3,2)$ or $(6,5)$. Now one can choose $(y_1, ..., y_4) = (y', u' + \lambda z', u' + \xi z', x')$ with $\delta_4 = 4$, where $\xi \in \mathbb{C}$ is a number so that $u + \xi z$ defines a Du Val section on $X$ and $\xi \neq \lambda$. Thus, $(\Xi')$ and $(\Theta_2)$ hold and $Y \underset{X}{\Rightarrow} Y_1$.

Now assume that $Y \to X$ is of type E24. The chart $U_x \subset Y$ is defined by

$$(x' + p(y', z', u') + y'^3 + g'(x', y', z', u') = 0) \subset \mathbb{A}^4_{(x',y',z',u')}/\frac{1}{7}(5, 4, 3, 1).$$

The origin is a cyclic quotient point and there is only one $w$-morphism over this point. Let $F$ be the exceptional divisor corresponding to this $w$-morphism. By Lemma 3.13 we know that there exists a divisorial contraction $Y_1 \to X$ which extracts $F$ such that $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1$. One can compute that $a(F, X) = \frac{1}{2}$. Hence, $Y_1 \to X$ is also a $w$-morphism.

Finally, assume that $Y \to X$ is of type E26. Consider the chart $U_y \subset Y$ which is defined by

$$(x'^2 + y' + z'^4 + u'^8 + g'(y', z', u') = 0) \subset \mathbb{A}^4_{(x',y',z',u')}/\frac{1}{6}(1, 2, 5, 1).$$

One can take $(y_1, ..., y_4) = (y', z', u', x')$ with $\delta_4 = 4$. One can see that $(\Xi')$ holds. Now, let $F$ be the exceptional divisor corresponding to the $w$-morphism over $U_t$. Then $v_F(y_1, ..., y_4) = \frac{1}{6}(2, 5, 1, 1)$. Hence, $(\Theta_2)$ and $(\Xi'_-)$ hold. Thus, there exists a divisorial contraction $Y_1 \to X$ which extracts $F$ so that $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1$. One can compute that $v_F(x, y, z, u) = \frac{1}{2}(3, 2, 3, 1)$, so $Y_1 \to X$ is of type E22. □

## 5. Estimating depths

We want to understand the change of singularities after running the minimal model program. The final result is the following proposition.

**Proposition 5.1.**

(1) Assume that $Y \to X$ is a divisorial contraction between terminal and $\mathbb{Q}$-factorial threefolds.

(1–1) If $Y \to X$ is a divisorial contraction to a point, then

$$gdep(X) \leq gdep(Y) + 1 \text{ and } dep(X) \leq dep(Y) + 1.$$

If $Y \to X$ is a divisorial contraction to a curve, then

$$gdep(X) \leq gdep(Y) \text{ and } dep(X) \leq dep(Y).$$

(1–2) $dep_{Gor}(X) \geq dep_{Gor}(Y)$ and the inequality is strict if the non-isomorphic locus on $X$ contains a Gorenstein singular point.

(2) Assume that $X \dashrightarrow X'$ is a flip between terminal and $\mathbb{Q}$-factorial threefolds.

(2–1)

$$gdep(X) > gdep(X') \text{ and } dep(X) > dep(X').$$

(2–2) $dep_{Gor}(X) \leq dep_{Gor}(X')$ and the inequality is strict if the non-isomorphic locus on $X'$ contains a Gorenstein singular point.

**Corollary 5.2.** Assume that

$$X_0 \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_k$$

is a process of the minimal model program. Then:

(1) $\rho(X_0/X_k) \geq gdep(X_k) - gdep(X_0)$ and the equality holds if and only if $X_i \dashrightarrow X_{i+1}$ is a strict $w$-morphism for all $i$.

(2) $dep_{Gor}(X_k) \geq dep_{Gor}(X_0)$.

In particular, if $X$ is a terminal $\mathbb{Q}$-factorial threefold and $W \to X$ is a resolution of singularities, then $\rho(W/X) \geq gdep(X)$ and the equality holds if and only if $W$ is a feasible resolution of $X$.

*Proof.* Statement (2) easily follows from the inequalities in Proposition 5.1. Assume that the sequence contains $m$ flips, then $\rho(X_0/X_k) = k - m$. On the other hand, we know that

$$gdep(X_{i+1}) \leq \begin{cases} gdep(X_i) - 1 & \text{if } X_i \dashrightarrow X_{i+1} \text{ is a flip} \\ gdep(X_i) + 1 & \text{otherwise} \end{cases}.$$

It follows that $gdep(X_k) \leq gdep(X_0) + k - 2m$, hence $gdep(X_k) - gdep(X_0) \leq \rho(X_0/X_k)$. Now $gdep(X_k) - gdep(X_0) = \rho(X_0/X_k)$ if and only if $m = 0$ and $gdep(X_{i+1}) = gdep(X_i) + 1$ for all $i$, which is equivalent to $X_i \dashrightarrow X_{i+1}$ being a strict $w$-morphism for all $i$.

Now assume that $X$ is a terminal $\mathbb{Q}$-factorial threefold and $W \to X$ is a resolution of singularities. We can run $K_W$-MMP over $X$ and the minimal model is $X$ itself. Since $gdep(W) = 0$, one has that $\rho(W/X) \geq gdep(X)$ and the equality holds if and only if $W$ is a feasible resolution of $X$. □

The inequalities for the depth part are exactly Lemma 2.20. We only need to prove the inequalities for the generalized depth and the Gorenstein depth.

**Convention 5.3.** Let $\mathcal{S}$ be a set consisting of birational maps between $\mathbb{Q}$-factorial terminal threefolds. We say that $(*)_{\mathcal{S}}$ holds if, for all $Z \dashrightarrow V$ inside $\mathcal{S}$, one has that:

(1) If $Z \to V$ is a divisorial contraction to a point, then

$$dep_{Gor}(V) \geq dep_{Gor}(Z) \geq dep_{Gor}(V) - (dep(Z) - dep(V) + 1).$$

(2) If $Z \to V$ is a divisorial contraction to a smooth curve, then

$$dep_{Gor}(V) \geq dep_{Gor}(Z) \geq dep_{Gor}(V) - (dep(Z) - dep(V)).$$

(3) If $Z \dashrightarrow V$ is a flip, then

$$dep_{Gor}(V) \geq dep_{Gor}(Z) \geq dep_{Gor}(V) - (dep(Z) - dep(V) - 1).$$

(4) If $Z \dashrightarrow V$ is a flop, then $dep_{Gor}(V) = dep_{Gor}(Z)$.

Moreover, if there exists a Gorenstein singular point $P \in V$ such that $P$ is not contained in the isomorphic locus of $Z \dashrightarrow V$, then $dep_{Gor}(V) > dep_{Gor}(Z)$ unless $V \dashrightarrow Z$ is a flop.

We say that $(*)^{(1)}_{\mathcal{S}}$ holds if statement (1) is true, but statements (2) and (3) are unknown.

If $V \dashrightarrow Z$ is a flip or a divisorial contraction, we denote the condition $(*)_{V \dashrightarrow Z} = (*)_{\mathcal{S}}$ where $\mathcal{S}$ is the set containing only one element $V \dashrightarrow Z$.

It is easy to see that if $Y \to X$ is a divisorial contraction, then $(*)_{Y\to X}$ holds if and only if the inequalities in Proposition 5.1 (1) hold. Likewise, if $X \dashrightarrow X'$ is a flip, then $(*)_{X\dashrightarrow X'}$ holds if and only if the inequalities in Proposition 5.1 (2) hold.

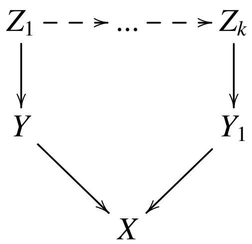
**Remark 5.4.** If $Z \dashrightarrow V$ is a flop, then the singularities on $Z$ and $V$ are the same by [17, Theorem 2.4]. Hence, statement (4) is always true.

**Convention 5.5.** Given $n \in \mathbb{Z}_{\geq 0}$, we denote

$$\mathcal{S}_n = \left\{ \phi : Z \dashrightarrow V \;\middle|\; \begin{array}{c} \phi \text{ is a flip, a flop or a divisorial contraction between} \\ \text{between } \mathbb{Q}\text{-factorial terminal threefolds, } gdep(Z) \leq n \end{array} \right\}.$$

**Lemma 5.6.** Assume that $Y \to X$ and $Y_1 \to X$ are two divisorial contractions between terminal threefolds, such that $Y \underset{X}{\Rightarrow} Y_1$. If $(*)_{\mathcal{S}_{gdep(Y)-1}}$ holds, then $gdep(Y_1) \leq gdep(Y)$. Moreover, $gdep(Y_1) = gdep(Y)$ if and only if $Y_1 \underset{X}{\Rightarrow} Y$.

*Proof.* We have a diagram



such that $Z_1 \to Y$ is a strict $w$-morphism and $Z_i \dashrightarrow Z_{i+1}$ is a flip or a flop for all $1 \leq i \leq k-1$. Since $gdep(Z_1) = gdep(Y) - 1$ and $(*)_{\mathcal{S}_{gdep(Y)-1}}$ holds, we know that $gdep(Z_2) \leq gdep(Y) - 1$. Repeating this argument $k-2$ times, one can say that $gdep(Z_k) \leq gdep(Y) - 1$. Again, since $(*)_{\mathcal{S}_{gdep(Y)-1}}$ holds, we know that $gdep(Y_1) \leq gdep(Z_k) + 1 = gdep(Y)$.

Now, $gdep(Y_1) = gdep(Y)$ if and only if all the inequalities above are equalities. This is equivalent to $Z_k \to Y_1$ being a strict $w$-morphism and $Z_i \dashrightarrow Z_{i+1}$ being a flop for all $i = 1, ..., k-1$, or $k = 1$. In other words, we also have $Y_1 \underset{X}{\Rightarrow} Y$. □

**Corollary 5.7.** Assume that $Y \to X$ is a strict $w$-morphism over $P \in X$ and $Y_1 \to X$ is another divisorial contraction over $P$. If $(*)_{\mathcal{S}_{gdep(Y)}}$ holds, then there exist divisorial contractions $Y_1 \to X, ..., Y_k \to X$ such that

$$Y_1 \underset{X}{\Rightarrow} ... \underset{X}{\Rightarrow} Y_k \underset{X}{\Rightarrow} Y.$$

*Proof.* By Proposition 4.1, we know that there exists $Y_1, ..., Y_l, Y'_1, ..., Y'_{l'}$ such that

$$Y_1 \underset{X}{\Rightarrow} ... \underset{X}{\Rightarrow} Y_l = Y'_{l'} \underset{X}{\Leftarrow} ... \underset{X}{\Leftarrow} Y'_1 \underset{X}{\Leftarrow} Y.$$

One can apply Lemma 5.6 to the sequence $Y \underset{X}{\Rightarrow} Y'_1 \underset{X}{\Rightarrow} ... \underset{X}{\Rightarrow} Y'_{l'}$ and conclude that $gdep(Y'_i) \leq gdep(Y)$ for all $i$. Since $Y'_i \to X \in \mathcal{S}_{gdep(Y)}$ for all $i = 1, ..., l'$, one has $gdep(Y'_i) \geq gdep(X) - 1 = gdep(Y)$. Thus, $gdep(Y'_i) = gdep(Y)$ and Lemma 5.6 says that one has

$$Y'_{l'} \underset{X}{\Rightarrow} ... \underset{X}{\Rightarrow} Y'_1 \underset{X}{\Rightarrow} Y.$$

Now we can take $k = l + l' - 1$ and let $Y_i = Y'_{l'-i+l}$ for $l < i \leq k$. □

**Corollary 5.8.** Assume that $(*)_{\mathcal{S}_{n-2}}$ holds. Assume that $P \in X$ is a $cA/r$ point or a $cAx/r$ point such that $gdep(X) = n$. Then every $w$-morphism over $P$ is a strict $w$-morphism. In particular, one can always assume that the morphism $Y_1 \to X$ in Theorem 2.18 is a strict $w$-morphism.

*Proof.* Propositions 4.2 and 4.3 say that if $Y \to X$ and $Y_1 \to X$ are two different $w$-morphisms over $P$, then there exists $Y_2 \to X$, ..., $Y_k \to X$ such that

$$Y_1 \underset{X}{\Leftrightarrow} Y_2 \underset{X}{\Leftrightarrow} ... \underset{X}{\Leftrightarrow} Y_k = Y.$$

We can assume that $Y \to X$ is a strict $w$-morphism, so $gdep(Y) = n - 1$. Lemma 5.6 implies that $Y_1$ is also a strict $w$-morphism. Hence, every $w$-morphism over $P$ is a strict $w$-morphism.

Now assume that $X$ is in the diagram in Theorem 2.18 and $P$ is a non-Gorenstein point in the exceptional set of $X \to W$. If $P$ is a $cA/r$ or a $cAx/r$ point, then we already know that every $w$-morphism over $P$ is a strict $w$-morphism. Otherwise, by Remark 2.19 (2) we know that any $w$-morphism $Y_1 \to X$ over $P$ induces a diagram in Theorem 2.18. Hence, we can choose $Y_1 \to X$ to be a strict $w$-morphism. □

**Convention 5.9.** Let $DV$ be the set of symbols

$$DV = \{A_i, D_j, E_k\}_{i\in\mathbb{N}, j\in\mathbb{N}_{\geq 4}, k=6,7,8}.$$

One can define an ordering on $DV$ by

$$A_i < A_{i'} < D_j < D_{j'} < E_k < E_{k'} \text{ for all } i < i', j < j', k < k'.$$

Given $\Box \in DV$, define

$$\mathcal{T}_\Box = \left\{ \begin{array}{c} X \text{ is a terminal} \\ \mathbb{Q}\text{-factorial threefold} \end{array} \middle| \begin{array}{c} GE(P \in X) \leq \Box \text{ for all} \\ \text{non-Gorenstein point } P \in X \end{array} \right\},$$

$$\mathcal{T}_{\Box,n} = \{X \in \mathcal{T}_\Box \,|\, gdep(X) \leq n\}$$

and

$$\mathcal{T}_n = \bigcup_{\Box \in DV} \mathcal{T}_{\Box,n}.$$

Here, $GE(P \in X)$ denotes the type of the general elephant near $P$. That is, the type of a general Du Val section $H \in |-K_X|$ near an analytic neighborhood of $P \in X$.

**Convention 5.10.** Let $\mathcal{T}$ be a set of terminal threefolds. We say that the condition $(\Pi)_{\mathcal{T}}$ holds if for all $X \in \mathcal{T}$ and for all strict $w$-morphisms $Y \to X$ over non-Gorenstein points of $X$, one has that $dep(Y) = dep(X) - 1$.

**Remark 5.11.**

(1) Assume that $Y \to X$ is a $w$-morphism over a non-Gorenstein point $P$. Then the general elephant of $Y$ over $X$ is better than the general elephant of $X$ near $P$. This is because if $H \in |-K_X|$ near $P$, then $H_Y \in |-K_Y|$ and $H_Y \to H$ is a partial resolution by [3, Lemma 2.7].

(2) One has that $(\Pi)_{\mathcal{T}_{A_1}}$ always holds since if $GE(P \in X) = A_1$ for some non-Gorenstein point $P$, then $P$ is a cyclic quotient point of index two (cf. [7, (6.4)]). In this case, there is only one $w$-morphism $Y \to X$ over $P$ and $Y$ is smooth over $X$. Hence,

$$gdep(P \in X) = dep(P \in X) = 1$$

and $gdep(Y) = dep(Y) = 0$ over $X$. Thus, $Y \to X$ is a strict $w$-morphism and also $dep(Y) = dep(X) - 1$.

**Lemma 5.12.** Assume that $Y \to X$ is a strict $w$-morphism over a non-Gorenstein point $P$. If $(*)_{\mathcal{S}_{gdep(Y)}}$ holds and $(\Pi)_{\mathcal{T}_\square}$ holds for all $\square < GE(P \in X)$, then $dep(Y) = dep(X) - 1$.

*Proof.* By the definition we know that $dep(Y) \geq dep(X) - 1$. Assume that $dep(Y) > dep(X) - 1$. Then there exists $Y_1 \to X$ such that $dep(Y_1) = dep(X) - 1 < dep(Y)$. Corollary 5.7 says that there exist divisorial contractions $Y_2 \to X, ..., Y_k \to X$ such that $Y_1 \underset{X}{\Rightarrow} Y_2 \underset{X}{\Rightarrow} ... \underset{X}{\Rightarrow} Y_k \underset{X}{\Rightarrow} Y$. From Remark 5.11 (1) we know that $Y_i \in \mathcal{T}_\square$ for some $\square < GE(P \in X)$ for all $i$, hence, if

$$\begin{array}{ccc} Z_{i,1} & \dashrightarrow \ldots \dashrightarrow & Z_{i,k_i} \\ \downarrow & & \downarrow \\ Y_i & & Y_{i+1} \\ & \searrow \qquad \swarrow & \\ & X & \end{array}$$

is the induced diagram of $Y_i \underset{X}{\Rightarrow} Y_{i+1}$, then $dep(Z_{i,1}) = dep(Y_i) - 1$. By Lemma 2.20 we know that

$$dep(Y_{i+1}) \leq dep(Z_{i,k_i}) + 1 \leq dep(Z_{i,1}) + 1 = dep(Y_i)$$

for all $i$. Hence, $dep(Y_1) \leq dep(Y)$. This leads to a contradiction as we assume that $dep(Y) > dep(Y_1)$. □

**Lemma 5.13.** Fix an integer $n$ and assume that $(*)_{\mathcal{S}_{n-1}}$ holds. Then $(\Pi)_{\mathcal{T}_n}$ holds.

*Proof.* We need to show that for all $X \in \mathcal{T}_n$ and for all strict $w$-morphism $Y \to X$ over a non-Gorenstein point, one has that $dep(Y) = dep(X) - 1$. By Remark 5.11 (2) we know that $(\Pi)_{\mathcal{T}_{A_1}}$ holds, hence $(\Pi)_{\mathcal{T}_{A_m,n}}$ and $(\Pi)_{\mathcal{T}_{D_4,n}}$ hold for all $m \in \mathbb{N}$ by Lemma 5.12 and by induction on $m$. Then, one can prove that $(\Pi)_{\mathcal{T}_{D_m,n}}$ and $(\Pi)_{\mathcal{T}_{E_6,n}}$ hold by again applying Lemma 5.12 and by induction on $m$. Statements $(\Pi)_{\mathcal{T}_{E_7},n}$ and $(\Pi)_{\mathcal{T}_{E_8},n}$ can be proved in the same way. □

**Lemma 5.14.** Fix an integer $n$ and assume that $(*)_{\mathcal{S}_{n-1}}$ holds. Assume that we have a diagram

$$\begin{array}{ccc} Y_1 & \dashrightarrow \cdots \dashrightarrow & Y_k \\ \downarrow & & \downarrow \\ X & & X' \\ & \searrow \qquad \swarrow & \\ & W & \end{array}$$

such that $gdep(X) = n$, $Y_1 \to X$ is a strict $w$-morphism, $Y_i \dashrightarrow Y_{i+1}$ is a flip or a flop for $i = 1, ..., k-1$, and $Y_k \to X'$ is a divisorial contraction. Then,

(1) $dep_{Gor}(X) = dep_{Gor}(Y_1) \le dep_{Gor}(X')$ and $gdep(X') \le gdep(X)$.

(2) $(*)_{Y_i \dashrightarrow Y_{i+1}}$ holds for $i = 1, ..., k-1$.

(3) $(*)_{Y_k \to X'}$ holds.

Moreover, if the non-isomorphic locus of $X \dashrightarrow X'$ on $X'$ contains a Gorenstein singular point, then $dep_{Gor}(X) < dep_{Gor}(X')$.

*Proof.* Since $(*)_{\mathcal{S}_{n-1}}$ holds, we know that $(\Pi)_{\mathcal{T}_n}$ holds by Lemma 5.13. Hence, $dep_{Gor}(Y_1) = dep_{Gor}(X)$. We know that $gdep(Y) = n-1$. Since $(*)_{\mathcal{S}_{n-1}}$ holds, one can prove that $gdep(Y_i) \le gdep(Y_1) \le n-1$ for all $i$ and hence $(*)_{Y_i \dashrightarrow Y_{i+1}}$ and $(*)_{Y_k \to X'}$ hold. Thus,

$$gdep(X') \le gdep(Y_k) - 1 \le gdep(Y_1) - 1 \le gdep(X)$$

and

$$dep_{Gor}(X) = dep_{Gor}(Y_1) \le ... \le dep_{Gor}(Y_k) \le dep_{Gor}(X').$$

Now, if the non-isomorphic locus of $X \dashrightarrow X'$ on $X'$ contains a Gorenstein singular point, then either the non-isomorphic locus of $Y_i \dashrightarrow Y_{i+1}$ on $Y_{i+1}$ contains a Gorenstein singular point or the non-isomorphic locus of $Y_k \to X'$ on $X'$ contains a Gorenstein singular point. Hence, at least one of the above inequalities is strict. Thus, one has $dep_{Gor}(X) < dep_{Gor}(X')$. □

**Lemma 5.15.** Assume that $(*)_{\mathcal{S}_{n-1}}$ holds. Then $(*)^{(1)}_{\mathcal{S}_n}$ holds.

*Proof.* Let $Y \to X$ be a divisorial contraction to a point which belongs to $\mathcal{S}_n$. We know that $gdep(Y) = n$. Assume first that $Y \to X$ is a strict $w$-morphism over a point $P \in X$. Notice that $dep(Y) \ge dep(X) - 1$ by Lemma 2.20. If $P$ is a non-Gorenstein point, then $dep_{Gor}(Y) = dep_{Gor}(X)$ by Lemma 5.13, hence $(*)_{Y \to X}$ holds. If $P$ is a Gorenstein point, then

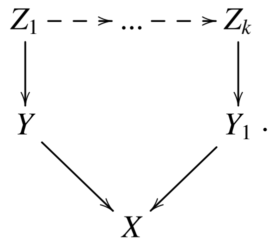
$$dep_{Gor}(X) = gdep(X) - dep(X) = gdep(Y) + 1 - dep(X) = dep_{Gor}(Y) + dep(Y) - dep(X) + 1.$$

Moreover, since $dep(P \in X) = 0$, $dep(Y) - dep(X) \ge 0$, hence

$$dep_{Gor}(X) > dep_{Gor}(Y) = dep_{Gor}(X) - (dep(Y) - dep(X) + 1).$$

Thus, $(*)_{Y \to X}$ holds.

In general, by Corollary 5.7 there exist $Y_1 \to X, ..., Y_k \to X$ such that $Y_k \to X$ is a strict $w$-morphism and one has $Y \underset{X}{\Rightarrow} Y_1 \underset{X}{\Rightarrow} ... \underset{X}{\Rightarrow} Y_k$. By induction on $k$ we may assume that $(*)_{Y_i \to X}$ holds for all $i$ (notice that $gdep(Y_i) \le gdep(Y) = n$ for all $i$ by Lemma 5.6). Now we have a diagram



By Lemma 5.14 we know that $dep_{Gor}(Y) \le dep_{Gor}(Y_1)$ and $(*)_{Z_i \dashrightarrow Z_{i+1}}$, $(*)_{Z_k \to Y_1}$ hold. Since $(*)_{Y_1 \to X}$ holds, we know that $dep_{Gor}(X) \ge dep_{Gor}(Y_1) \ge dep_{Gor}(Y)$ and

$$
\begin{aligned}
dep_{Gor}(Y) + (dep(Y) - dep(X) + 1) &= dep_{Gor}(Z_1) + (dep(Z_1) - dep(X) + 2) \\
&\geq gdep(Z_1) - dep(X) + 2 \\
&\geq gdep(Z_k) - dep(X) + 2 \\
&\geq gdep(Y_1) - dep(X) + 1 \\
&\geq dep_{Gor}(Y_1) + (dep(Y_1) - dep(X) + 1) \\
&\geq dep_{Gor}(X).
\end{aligned}
$$

Moreover, if $Y \to X$ is a divisorial contraction to a Gorenstein point, then $Y_1 \to X$ is also a divisorial contraction to a Gorenstein point. Hence, $dep_{Gor}(Y_1) < dep_{Gor}(X)$ and we also have $dep_{Gor}(Y) < dep_{Gor}(X)$. □

*Proof of Proposition 5.1.* We need to say that $(*)_{\mathcal{S}_n}$ holds for all $n$ and we will prove this by induction on $n$. If $n = 0$, then $\mathcal{S}_0$ consists only smooth blow-downs and smooth flops. One can see that $(*)_{\mathcal{S}_0}$ holds. In general, assume that $(*)_{\mathcal{S}_{n-1}}$ holds. By Lemma 5.15 we know that $(*)^{(1)}_{\mathcal{S}_n}$ holds. Hence, it is enough to show that, given a flip $X \dashrightarrow X'$ or a divisorial contraction to a curve $X \to V$ such that $gdep(X) = n$, $(*)_{X \dashrightarrow X'}$ or $(*)_{X \to V}$ holds.

If $X \to V$ is a smooth blow-down, then $dep_{Gor}(X) = dep_{Gor}(V)$, and so there is nothing to prove. In general, we have a diagram as in Theorem 2.18:

$$
\begin{array}{ccc}
Y_1 & \dashrightarrow \cdots \dashrightarrow & Y_k \\
\downarrow & & \downarrow \\
X & & X' \\
& \searrow \quad \swarrow & \\
& V &
\end{array}
$$

By Lemma 5.14 we know that $dep_{Gor}(X) \leq dep_{Gor}(X')$ and $(*)_{Y_i \dashrightarrow Y_{i+1}}$, $(*)_{Y_k \to X'}$ hold. One has that

$$
\begin{aligned}
dep_{Gor}(X) + (dep(X) - dep(X')) &= gdep(X) - dep(X') \\
&= gdep(Y_1) - dep(X') + 1 \\
&\geq gdep(Y_k) - dep(X') + 1 \\
&\geq gdep(X') - dep(X') = dep_{Gor}(X').
\end{aligned}
$$

Moreover, if $X \dashrightarrow X'$ is a flip, then either one of $Y_i \dashrightarrow Y_{i+1}$ is a flip or $Y_k \to X'$ is a divisorial contraction to a curve by [3, Remark 3.4]. This implies that either $gdep(Y_1) > gdep(Y_k)$ or $gdep(Y_k) \geq gdep(X')$. If $X \to V$ is a divisorial contraction to a curve, then $Y_k \to X'$ is a divisorial contraction to a curve, hence one always has that $gdep(Y_k) \geq gdep(X')$. In conclusion, we have

$$dep_{Gor}(X) \geq dep_{Gor}(X') - (dep(X) - dep(X') - 1).$$

If $X \dashrightarrow X'$ is a flip, then one can see that $(*)_{X \dashrightarrow X'}$ holds. Now assume that $X \to V$ is a divisorial contraction to a curve. Then $X' \to V$ is a divisorial contraction to a point. We know that $(*)_{X' \to V}$ holds since $(*)^{(1)}_{\mathcal{S}_n}$ holds and $gdep(X') \leq n$ by Lemma 5.14. One can see that $dep_{Gor}(X) \leq dep_{Gor}(X') \leq dep_{Gor}(V)$ and

$$
\begin{aligned}
dep_{Gor}(X) &\geq dep_{Gor}(X') - (dep(X) - dep(X') - 1) \\
&\geq dep_{Gor}(V) - (dep(X') - dep(V) + 1) - (dep(X) - dep(X') - 1) \\
&= dep_{Gor}(V) - (dep(X) - dep(V)).
\end{aligned}
$$

Thus, $(*)_{X \to V}$ holds. □

## 6. Comparing different feasible resolutions

In this section, we describe the difference between two different feasible resolutions of a terminal $\mathbb{Q}$-factorial threefold. The final result is the diagrams in Theorem 1.1.

### *6.1. General discussion*

**Lemma 6.1.** Assume that $X$ is a terminal threefold and $Y \to X$, $Y_1 \to X$ are two different strict $w$-morphisms over $P \in X$ such that $Y \underset{X}{\Rightarrow} Y_1$. Let

$$
\begin{array}{ccccc}
Z_1 & \dashrightarrow & \dots & \dashrightarrow & Z_k \\
\downarrow & & & & \downarrow \\
Y & & & & Y_1 \\
& \searrow & & \swarrow & \\
& & X & &
\end{array}
$$

be the corresponding diagram. Then one of the following holds:

(1) $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1 \underset{X}{\overset{-}{\Rightarrow}} Y$ and $k = 1$.

(2) $k = 2$ and $Z_1 \dashrightarrow Z_2$ is a smooth flop.

(3) $k = 2$, $Z_1 \dashrightarrow Z_2$ is a singular flop, $P \in X$ is of type $cA/r$ and both $Y \to X$ and $Y_1 \to X$ are of type A1 in Table 2.

*Proof.* Since $Y \to X$ and $Y_1 \to X$ are both strict $w$-morphisms, we know that $gdep(Y) = gdep(Y_1)$. Hence, $Z_i \dashrightarrow Z_{i+1}$ can not be a flip. Thus, if $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1$, then $k = 1$.

Now assume that $Y \underset{X}{\overset{-}{\not\Rightarrow}} Y_1$. According to the result in Section 4, one has that:

(i) If $P$ is of type $cA/r$, then by Proposition 4.2 we know that $Y \to X$ and $Y_1 \to X$ are both of type A1. Since $Y \underset{X}{\overset{-}{\not\Rightarrow}} Y_1$, we know that $f_4^{\circ} = \eta_4 = y$ is irreducible. Hence, there is only one $K_{Z_1}$-trivial curve on $Z_1$ and so $k = 2$.

(ii) $P$ is not of type $cAx/r$ since one always has that $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1$ if $Y \to X$ and $Y_1 \to X$ are two different $w$-morphisms over $P$ by Proposition 4.3.

(iii) $P$ is not of type $cD$ by Proposition 4.5.

(iv) $P$ is not of type $cD/3$ by Proposition 4.7.

(v) If $P$ is of type $cD/2$, then by Propositions 4.8 and 4.9 we know that $Y \to X$ is of type D17 in Table 6. We know that $Z_1 \to Y$ is a $w$-morphism over the origin of

$$U_t = (x'^2 + y' + g'(z', u', t') = z'u' + y'^3 + t' = 0) \subset \mathbb{A}^5_{(x',y',z',u',t')}/\frac{1}{5}(3, 1, 1, 2, 3)$$

and we take $(y_1, ..., y_5) = (y', u', y' + z', x', t')$. One can see that there are two curves contained in $y_3 = y_5 = 0$, namely $\Gamma_1 = (x'^2 + g'(0, u', 0) = y' = y' + z' = u' = 0)$ and $\Gamma_2 = (x'^2 + g'(-y', -y'^2, 0) = u' + y'^2 = y' + z' = t' = 0)$. We know that $(\Theta_1)$ holds. When computing the intersection number for $\Gamma_2$ one can take $\delta_4 = 4$ and $\delta_5 = 2$, hence $(\Xi_-)$ holds in this case. This implies that the proper transform $\Gamma_{2,Z_1}$ of $\Gamma_2$ on $Z_1$ is a $K_{Z_1}$-trivial curve and is not contained in $exc(Z_1 \to Y)$. Thus, $Z_i \dashrightarrow Z_{i+1}$ is a flip along $\Gamma_{2,Z_i}$ for some $i$, and hence $gdep(Y_1) < gdep(Y)$. This is a contradiction. Thus, this case will not happen.

(vi) If $P$ is of type $cE$, then by Proposition 4.15 and Proposition 4.16 we know that $Y \to X$ is of type E14, E16, or E18 in Table 9. In those cases, $Y \to X$ are five-dimensional weighted blow-ups. Let $\Gamma \subset Y$ be a curve contained in $exc(Y \to X)$ such that the proper transform $\Gamma_{Z_1}$ of $\Gamma$ on $Z_1$ is a possibly $K_{Z_1}$-trivial curve. From the proof of Proposition 4.16, one can see that:

(vi–i) $Y \to X$ is of type E14. In this case, $Z_1 \to Y$ is a $w$-morphism over the origin of

$$U_t = (x'^2 + y'^3 + z' + g'(y', z', u', t') = p(x', y', z', u') + t' = 0)$$
$$\subset \mathbb{A}^5_{(x',y',z',u',t')}/\frac{1}{5}(3, 2, 1, 1, 4).$$

We choose $(y_1, ..., y_5) = (x', y', u', z', t')$ with $\delta_4 = 1$ and $\delta_5 = 2$ or 4, or $\delta_4 = 2$ and $\delta_5 = 1$ or 4. We also know that $(\Theta_4)$ holds. If both $\delta_4$ and $\delta_5 \neq 4$, then $(\Xi_-)$ holds, which implies that $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$. This contradicts our assumption. Hence, $\delta_5 = 4$. Now, $\Gamma = (x'^2 + y'^3 = z' = u' = t' = 0)$.

(vi–ii) $Y \to X$ is of type E16. In this case $Z_1 \to Y$ is a $w$-morphism over the origin of

$$U_t = (x'^2 + y'^3 + p(z', u') + g'(y', z', u', t') = q(y', z', u') + t' = 0)$$
$$\subset \mathbb{A}^5_{(x',y',z',u',t')}/\frac{1}{4}(3, 2, 1, 1, 3).$$

As in the proof of Proposition 4.16, we choose $(y_1, ..., y_5) = (y', z', u', x', t')$ with $\delta_4 = 4$ and $\delta_5 = 1$ or 2. We know that $(\Theta_2)$ holds. Hence, if $\delta_5 = 1$, then $(\Xi_-)$ holds, and so $Y \overset{-}{\underset{X}{\Rightarrow}} Y_1$. This leads to a contradiction. Thus, $\delta_5 = 2$ and one has $\Gamma = (x'^2 + y'^3 = z' = u' = t' = 0)$.

(vi–iii) $Y \to X$ is of type E18. In this case, $Z_1 \to Y$ is a $w$-morphism over the origin of

$$U_t = (x'^2 + y' + g'(y', z', u', t') = y'^2 + p(y', z', u') + t = 0)$$
$$\subset \mathbb{A}^5_{(x',y',z',u',t')}/\frac{1}{7}(5, 3, 2, 1, 6).$$

We choose $(y_1, ..., y_5) = (y', z', u', x', t')$ with $\delta_4 = 4$ and $\delta_5 = 1$. We know that $(\Theta_1)$ holds. If $z'^3 \in p$, then we can choose $\delta_5 = 2$. Then, since $(\Theta_1)$ holds, we know that $(\Xi_-)$ holds, and so $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1$. This contradicts our assumption. Hence, $z'^3 \notin p$. In this case, $u$ divides $p$ and $X$ has $cE_8$ singularities, so $z'^5 \in g'$. One can see that $\Gamma = (x'^2 + z'^5 = y' = u' = t' = 0)$.

Now the origin of $U_t$ is a cyclic quotient point and the $w$-morphism over this point is a weighted blow-up with weights $v_F(x', y', z', u', t') = \frac{1}{5}(3, 2, 6, 1, 4)$, $\frac{1}{4}(3, 2, 5, 1, 3)$ and $\frac{1}{7}(5, 10, 2, 1, 6)$, respectively. An easy computation shows that $\Gamma_{Z_1}$ does not pass through any singular point of $Z_1$, hence $Z_1 \dashrightarrow Z_2$ is a smooth flop. Since there is only one $K_{Z_1}$-trivial curve, we know that $k = 2$.

(vii) If $P$ is of type $cE/2$, then by Proposition 4.18 we know that $Y \to X$ is of type E22 or E23 in Table 11. Moreover, if $Y \to X$ is of type E23, then $Y_1 \to X$ is of type E22. Thus, interchanging $Y$ and $Y_1$ if necessary, we can always assume that $Y \to X$ is of type E22. As in the proof of Proposition 4.18, we know that $Z_1 \to Y$ is a $w$-morphism over the origin of $U_z \subset Y$, which is a $cD/3$ point defined by

$$(x'^2 + y'^3 + g'(y', z', u') = 0) \subset \mathbb{A}^4_{(x',y',z',u')}/\frac{1}{3}(0, 2, 1, 1).$$

The only possible $K_{Z_1}$-trivial curve $\Gamma_{Z_1}$ is the lifting of the curve $\Gamma = (x'^2 + y'^3 = z' = u' = 0)$ on $Z_1$. Moreover, $Z_1 \to Y$ is defined by a weighted blow-up with the weight

$$v_F(x', y', z', u' + \lambda z') = \frac{1}{3}(b, c, 1, 4)$$

for some $\lambda \in \mathbb{C}$, where $(b, c) = (3, 2)$ or $(6, 5)$. If $(b, c) = (6, 5)$, then one can see that $(\Xi_-)$ holds by considering the function $y$, hence $Y \underset{X}{\overset{-}{\Rightarrow}} Y_1$, which contradicts our assumption. Hence, $(b, c) = (3, 2)$. In this case, an easy computation shows that $\Gamma_{Z_1}$ does not pass through any singular point of $Z_1$, so $Z_1 \dashrightarrow Z_2$ is a smooth flop. Since there is only one $K_{Z_1}$-trivial curve, we know that $k = 2$.

□

We need to construct a factorization of the flop in Lemma 6.1 (3). Assume that

$$X = (xy + f(z, u) = 0) \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{r}(\beta, -\beta, 1, 0)$$

is a $cA/r$ singularity with $f(z, u) = z^{rk} + g(z, u)$ and $Y \to X$, $Y_1 \to X$ are two different strict $w$-morphisms with the factorization

$$\begin{array}{ccc} Z_1 & \dashrightarrow & Z_2 \\ \downarrow & & \downarrow \\ Y & & Y_1 \\ & \searrow \quad \swarrow & \\ & X & \end{array}$$

such that $Z_1 \dashrightarrow Z_2$ is a flop.

**Remark 6.2.** One always has that $k > 1$ since, if $k = 1$, then there is only one $w$-morphism over $X$ by classification Table 2.

**Lemma 6.3.** $Z_1 \dashrightarrow Z_2$ is a flop over

$$V = (\bar{x}\bar{y} + \bar{u}f_1(\bar{z},\bar{u}) = 0) \subset \mathbb{A}^4_{(\bar{x},\bar{y},\bar{z},\bar{u})}/\frac{1}{r}(\beta, -\beta, 1, 0),$$

where $f_1 = f(zu^{\frac{1}{r}}, u)/u^k$. Moreover, $Z_1 = Bl_{(\bar{x},\bar{u})}V$ and $Z_2 = Bl_{(\bar{y},\bar{u})}V$.

*Proof.* We may assume that $Y \to X$ is a weighted blow-up with the weight $w(x, y, z, u) = \frac{1}{r}(b, c, 1, r)$ with $b > r$. The chart $U_u \subset Y$ is defined by

$$(x_1y_1 + f_1(z_1, u_1) = 0) \subset \mathbb{A}^4_{(x_1,y_1,z_1,u_1)}/\frac{1}{r}(b, c, 1, r)$$

and the chart $U_x \subset Y$ is defined by

$$(y_2 + f_2(x_2, z_2, u_2) = 0) \subset \mathbb{A}^4_{(x_2,y_2,z_2,u_2)}/\frac{1}{b}(b - r, c, 1, r)$$

for some $f_2$. As in the proof of Proposition 4.2, we know that $Z_1 \to Y$ is a weighted blow-up over the origin of $U_x$ with the weight $w'(x_2, y_2, z_2, u_2) = \frac{1}{b}(b - r, rk, 1, r)$. The flopping curve $\Gamma$ of $Z_1 \to Z_2$ is the strict transform of the curve $\Gamma_Y \subset Y$ such that $\Gamma_Y|_{U_u} = (y_1 = z_1 = u_1 = 0)$ and $\Gamma_Y|_{U_x} = (x_2 = y_2 = z_2 = 0)$. One can see that $\Gamma$ intersects $exc(Z_1 \to Y)$ at the origin of $U'_u \subset Z_1$, which is defined by

$$(y' + f'(x', z', u') = 0) \subset \mathbb{A}^4_{(x',y',z',u')}/\frac{1}{r}(b, 0, 1, -b).$$

It is easy to see that $\Gamma$ is contained in $U'_u \cup U_u$, and on $U'_u$ we know that $\Gamma$ is defined by $(x' = z' = 0)$.

We have the following change of coordinates formula:

$$x = x_1u_1^{\frac{b}{r}}, \quad y = y_1u_1^{\frac{c}{r}}, \quad z = z_1u_1^{\frac{1}{r}}, \quad u = u_1, \text{ and}$$

$$x = x_2^{\frac{b}{r}}, \quad y = y_2x_2^{\frac{c}{r}}, \quad z = z_2x_2^{\frac{1}{r}}, \quad u = u_2x_2.$$

Also,

$$x_2 = x'u'^{\frac{b-r}{b}}, y_2 = y'u'^{\frac{rk}{b}}, \quad z_2 = z'u'^{\frac{1}{b}} \quad , u_2 = u'^{\frac{r}{b}}.$$

One can see that

$$x = x'^{\frac{b}{r}}u'^{\frac{b-r}{r}}, y = y'x'^{\frac{c}{r}}u'^{\frac{rk}{b}+\frac{c}{r}-\frac{c}{b}} = y'x'^{\frac{c}{r}}u'^{\frac{c}{r}+1}, \quad z = z'x'^{\frac{1}{r}}u'^{\frac{1}{r}} \text{ and } u = x'u'.$$

It follows that

$$u_1 = x'u', \quad z_1 = z', \quad y_1 = y'u' \text{ and } x_1 = u'^{-1}.$$

If we choose an isomorphism

$$U'_u \cong (y_1 + u'f'(x', z', u') = 0) \subset \mathbb{A}^4_{(x',y_1,z',u')}/\frac{1}{r}(b, -b, 1, -b),$$

then $U_u \cup U'_u = Bl_{(x',u_1)}V$, where

$$V = (x'y_1 + u_1f_1(z_1, u_1) = 0) \subset \mathbb{A}^4_{(x',y_1,z_1,u_1)}/\frac{1}{r}(b, -b, 1, 0)$$

by noticing that

$$\begin{aligned} f'(x',z',u') &= f_2(x'u'^{\frac{b-r}{b}}, z'u'^{\frac{1}{b}}, u'^{\frac{r}{b}})/u'^{\frac{rk}{b}} \\ &= f(z_2 x_2^{\frac{1}{r}}, u_2 x_2)/x_2^k u'^{\frac{rk}{b}} \\ &= f(z' x'^{\frac{1}{r}} u'^{\frac{1}{r}}, x'u')/x'^k u'^k \\ &= f(z_1 u_1^{\frac{1}{r}}, u_1)/u_1{}^k \\ &= f_1(z_1, u_1). \end{aligned}$$

Now we can choose $(\bar{x}, \bar{y}, \bar{z}, \bar{u}) = (x', y_1, z_1, u_1)$.

Finally, we know that $Y_1 \to X$ is a weighted blow-up with the weight $\frac{1}{r}(b-r, c+r, 1, r)$ and $Z_2 \to Y_1$ is a $w$-morphism over the origin of $U_{1,y} \subset Y_1$. Hence, the local picture of $Z_2 \to V$ can be obtained by a similar computation, but interchanging the role of $x$ and $y$. One then has that $Z_2 = Bl_{(\bar{y},\bar{u})}V$. □

### 6.2. *Explicit factorization of flops*

In this subsection, we assume that

$$V = (xy + uf(z,u) = 0) \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{r}(\beta, -\beta, 1, 0)$$

with $f(z,u) = z^{rk} + g(z,u)$ for some $k > 1$. Let $Z_1 = Bl_{(x,u)}V$ and $Z_2 = Bl_{(y,u)}V$ such that $Z_1 \dashrightarrow Z_2$ is a $\mathbb{Q}$-factorial terminal flop. Let $w$ be the weight $w(z,u) = \frac{1}{r}(1,r)$ and $m = w(f(z,u))$. Then $m \leq k$. Let $U_{1,x} \subset Z_1$ be the chart

$$(y + u_1 f_1(z,u_1) = 0) \subset \mathbb{A}^4_{(x,y,z,u_1)}/\frac{1}{r}(\beta, -\beta, 1, -\beta)$$

and $U_{1,u} \subset Z_1$ be the chart

$$(x_1 y + f(z,u) = 0) \subset \mathbb{A}^4_{(x_1,y,z,u)}/\frac{1}{r}(\beta, -\beta, 1, 0)$$

with the relations $u = u_1 x$, $x = x_1 u$ and $f_1 = f(z, xu_1)$. Similarly, let $U_{2,y} \subset Z_2$ be the chart

$$(x + u_2 f_2(z,u_2) = 0) \subset \mathbb{A}^4_{(x,y,z,u_2)}/\frac{1}{r}(\beta, -\beta, 1, \beta)$$

and $U_{2,u} \subset Z_2$ be the chart

$$(xy_2 + f(z,u) = 0) \subset \mathbb{A}^4_{(x,y_2,z,u)}/\frac{1}{r}(\beta, -\beta, 1, 0)$$

with the relations $u = u_2 y$, $y = y_2 u$ and $f_2 = f(z, yu_2)$.

**Lemma 6.4.** Let $\phi : V' \to V$ be a strict $w$-morphism. Then:

(1) The chart $U'_u \subset V'$ is $\mathbb{Q}$-factorial.

(2) If $m = k$ then $U'_z \subset V'$ is smooth. Otherwise, $U'_z$ contains exactly one non-$\mathbb{Q}$-factorial $cA$ point which is defined by $xy + uf''(z,u) = 0$ where $f'' = f(z^{\frac{1}{r}}, zu)/z^m$. One then has that $w(f'') < m$.

(3) All other singular points on $V'$ are cyclic quotient points.

*Proof.* From Table 2, we know that $V' \to V$ is a weighted blow-up with the weight $\frac{1}{r}(b,c,1,r)$ with $b + c = r(m+1)$. Statement (3) follows from direct computations. One can compute that

$$U'_u \cong (xy + f'(z,u) = 0) \subset \mathbb{A}^4_{(x,y,z,u)} / \frac{1}{r}(\beta, -\beta, 1, 0)$$

with $f' = f(zu^{\frac{1}{r}}, u)/u^m$. By [18, 2.2.7] we know that $U'_u$ is $\mathbb{Q}$-factorial if and only if $f'$ is irreducible as a $\mathbb{Z}/r\mathbb{Z}$-invariant function. If $U'_u$ is not $\mathbb{Q}$-factorial, then $f' = f'_1 f'_2$ for some non-unit $\mathbb{Z}/r\mathbb{Z}$-invariant functions $f'_1$ and $f'_2$. Write

$$f'_1 = \sum_{(i,j)\in\mathbb{Z}^2_{\geq 0}} \xi_{i,j} z^{ri} u^j \text{ and } f'_2 = \sum_{(i,j)\in\mathbb{Z}^2_{\geq 0}} \zeta_{i,j} z^{ri} u^j.$$

Let

$$m_1 = -\min_{\xi_{i,j}\neq 0}\{j - i\}, \quad m_2 = -\min_{\zeta_{i,j}\neq 0}\{j - i\}.$$

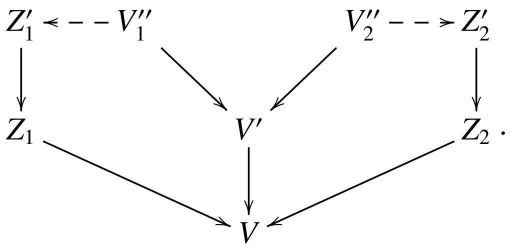
Notice that if $f' = \sum_{(i,j)} \sigma_{i,j} z^{ri} u^j$, then $j + m - i \geq 0$ if $\sigma_{i,j} \neq 0$ because $f' = f(zu^{\frac{1}{r}}, u)/u^m$. Hence, we have the relation $m_1 + m_2 \leq m$. Now let

$$f_1 = \sum_{(i,j)} \xi_{ij} z^{ir} u^{j-i+m_1} \text{ and } f_2 = \sum_{(i,j)} \zeta_{ij} z^{ir} u^{j-i+m_2}.$$

Then $f_1$ and $f_2$ can be viewed as $\mathbb{Z}/r\mathbb{Z}$-invariant functions on $V$ such that $\phi^* f_i = u^{m_i} f'_i$ for $i = 1, 2$. This means that $f = u^{m-m_1-m_2} f_1 f_2$ is not irreducible. Nevertheless, the chart $U_{1,u} \subset Z_1$ is defined by $x_1 y + f(z,u) = 0$ and has $\mathbb{Q}$-factorial singularities. This leads to a contradiction. Thus, $U'_u$ is $\mathbb{Q}$-factorial.

The chart $U'_z$ is defined by $xy + uf''(z,u) = 0$. When $m = k$, we know that $f''$ is a unit along $u = 0$, hence $U'_z$ is smooth. If $m < k$, then $f''(z,u)$ is a non-unit. In this case, the origin of $U'_z$ is a non-$\mathbb{Q}$-factorial $cA$ singularity. □

From the construction in [19], for any strict $w$-morphism $V' \to V$ we have the following diagram




where $V''_i \to V'$ is a $\mathbb{Q}$-factorization and $V''_i \dashrightarrow Z'_i$ is a composition of flips for $i = 1, 2$. By Lemma 6.4, we know that if $m = k$, then $V''_1 = V' = V''_2$. Otherwise, $V''_1 \dashrightarrow V''_2$ is a flop over the singularity $xy + uf''(z,u) = 0$.

First we discuss the factorization of $V''_1 \dashrightarrow Z'_1$. If $m = k$, then $V''_1 = V'$ is covered by four affine charts $U'_x$, $U'_y$, $U'_z$ and $U'_u$. The origin of $U'_x$ and $U'_y$ are cyclic quotient points and all other singular points are contained in $U'_u$. When $m < k$, the chart $U'_z$ has a non-$\mathbb{Q}$-factorial point and there are two charts $U''_{1,x}$ and $U''_{1,u}$ over this point. We fix the following notation for the latter discussion.

- $\bar{U}_1'' = U_x' = (y_1'' + u_1'' f_1''(x_1'', z_1'', u_1'') = 0) \subset \mathbb{A}^4_{(x_1'', y_1'', z_1'', u_1'')}/\frac{1}{b}(b-r, c, 1, r)$
  with $x = x_1''^{\frac{b}{r}}$, $y = y_1'' x_1''^{\frac{c}{r}}$, $z = z_1'' x_1''^{\frac{1}{r}}$, $u = u_1'' x_1''$ and $f_1'' = f(z_1'' x_1''^{\frac{1}{r}}, u_1'' x_1'')/x_1''^m$.
- $\bar{U}_2'' = U_y' = (x_2'' + u_2'' f_2''(y_2'', z_2'', u_2'') = 0) \subset \mathbb{A}^4_{(x_2'', y_2'', z_2'', u_2'')}/\frac{1}{c}(b, c-r, 1, r)$
  with $x = x_2'' y_2''^{\frac{b}{r}}$, $y = y_2''^{\frac{c}{r}}$, $z = z_2'' y_2''^{\frac{1}{r}}$, $u = u_2'' y_2''$ and $f_2'' = f(z_2'' y_2''^{\frac{1}{r}}, u_2'' y_2'')/y_2''^m$.
- $\bar{U}_3'' = U_u' = (x_3'' y_3'' + f_3''(z_3'', u_3'') = 0) \subset \mathbb{A}^4_{(x_3'', y_3'', z_3'', u_3'')}/\frac{1}{r}(b, c, 1, r)$
  with $x = x_3'' u_3''^{\frac{b}{r}}$, $y = y_3'' u_3''^{\frac{c}{r}}$, $z = z_3'' u_3''^{\frac{1}{r}}$, $u = u_3''$ and $f_3'' = f(z_3'' u_3''^{\frac{1}{r}}, u_3'')/u_3''^m$.

If $m = k$, define $\bar{U}_4'' = U_z'$. In this case this chart is smooth. When $m < k$ we define

- $\bar{U}_4'' = U_{1,x}'' = (y_4'' + u_4'' f_4''(x_4'', z_4'', u_4'') = 0) \subset \mathbb{A}^4_{(x_4'', y_4'', z_4'', u_4'')}$
  with $x = x_4'' z_4''^{\frac{b}{r}}$, $y = y_4'' z_4''^{\frac{c}{r}}$, $z = z_4''^{\frac{1}{r}}$, $u = x_4'' u_4'' z_4''$ and $f_4'' = f(z_4''^{\frac{1}{r}}, x_4'' u_4'' z_4'')/z_4''^m$.
- $\bar{U}_5'' = U_{1,u}'' = (x_5'' y_5'' + f_5''(z_5'', u_5'') = 0) \subset \mathbb{A}^4_{(x_5'', y_5'', z_5'', u_5'')}$
  with $x = x_5'' u_5'' z_5''^{\frac{b}{r}}$, $y = y_5'' z_5''^{\frac{c}{r}}$, $z = z_5''^{\frac{1}{r}}$, $u = u_5'' z_5''$ and $f_5'' = f(z_5''^{\frac{1}{r}}, u_5'' z_5'')/z_5''^m$.

On the other hand, we will see later that it is enough to assume that $Z_1' \to Z_1$ is a divisorial contraction over the origin of $U_{1,u} \subset Z_1$. This means that $Z_1'$ is covered by five affine charts $U_{1,x}$, $U_{1,x}'$, $U_{1,y}'$, $U_{1,z}'$ and $U_{1,u}'$ where the latter four charts correspond to the weighted blow-up $Z_1' \to Z_1$. Notice that $exc(Z_1' \to Z_1)$ and $exc(V' \to V)$ are the same divisor since $V_1''$ and $Z_1'$ are isomorphic in codimension one. One can see that $Z_1' \to Z_1$ is a weighted blow-up with the weight $w'(x_1, y, z, u) = \frac{1}{r}(b-r, c, 1, r)$. Again, we use the following notation:

- $\bar{U}_1' = U_{1,x}' = (y_1' + f_1'(x_1', z_1', u_1') = 0) \subset \mathbb{A}^4_{(x_1', y_1', z_1', u_1')}/\frac{1}{b-r}(b-2r, c, 1, r)$
  with $x = x_1'^{\frac{b}{r}} u_1'$, $y = y_1' x_1'^{\frac{c}{r}}$, $z = z_1' x_1'^{\frac{1}{r}}$, $u = u_1' x_1'$ and $f_1' = f(z_1' x_1'^{\frac{1}{r}}, u_1' x_1')/x_1'^m$.
- $\bar{U}_2' = U_{1,y}' = (x_2' + f_2'(y_2', z_2', u_2') = 0) \subset \mathbb{A}^4_{(x_2', y_2', z_2', u_2')}/\frac{1}{c}(b-r, c-r, 1, r)$
  with $x = x_2' y_2'^{\frac{b}{r}} u_2'$, $y = y_2'^{\frac{c}{r}}$, $z = z_2' y_2'^{\frac{1}{r}}$, $u = u_2' y_2'$ and $f_2' = f(z_2' y_2'^{\frac{1}{r}}, u_2' y_2')/y_2'^m$.
- $\bar{U}_3' = U_{1,u}' = (x_3' y_3' + f_3'(z_3', u_3') = 0) \subset \mathbb{A}^4_{(x_3', y_3', z_3', u_3')}/\frac{1}{r}(b-r, c, 1, r)$
  with $x = x_3' u_3'^{\frac{b}{r}}$, $y = y_3' u_3'^{\frac{c}{r}}$, $z = z_3' u_3'^{\frac{1}{r}}$, $u = u_3'$ and $f_3' = f(z_3' u_3'^{\frac{1}{r}}, u_3')/u_3'^m$.
- $\bar{U}_4' = U_{1,x} = (y_4' + u_4' f_4'(z_4', u_4') = 0) \subset \mathbb{A}^4_{(x_4', y_4', z_4', u_4')}/\frac{1}{r}(\beta, -\beta, 1, -\beta)$
  with $x = x_4'$, $y = y_4'$, $z = z_4'$, $u = u_4' x_4'$ and $f_4' = f(z_4', u_4' x_4')$.
- $\bar{U}_5' = U_{1,z}' = (x_5' y_5' + f_5'(z_5', u_5') = 0) \subset \mathbb{A}^4_{(x_5', y_5', z_5', u_5')}$
  with $x = x_5' z_5'^{\frac{b}{r}} u_5'$, $y = y_5' z_5'^{\frac{c}{r}}$, $z = z_5'^{\frac{1}{r}}$, $u = u_5' z_5'$ and $f_5' = f(z_5'^{\frac{1}{r}}, u_5' z_5')/z_5'^m$.

**Lemma 6.5.** Assume that $Z_1' \to Z_1$ is a divisorial contraction over the origin of $U_{1,u}$. Then:

(1) $gdep(Z_1') = gdep(V_1'') - 1$.

(2) All the singular points on the non-isomorphic loci of $V_1'' \dashrightarrow Z_1'$ on both $V_1''$ and $Z_1'$ are cyclic quotient points.

(3) The flip $V_1'' \dashrightarrow Z_1'$ is of type IA in the convention of [20, Theorem 2.2].

*Proof.* It is easy to see that $\bar{U}_i' \cong \bar{U}_i''$ for $i = 2, 3$ and $i = 5$ if $m < k$. Since $\bar{U}_4'$ is smooth, the only singular point contained in the non-isomorphic locus of $V_1'' \dashrightarrow Z_1'$ on $V_1''$ is the origin of $\bar{U}_1''$. This point is a cyclic quotient point of index $b$, so it has generalized depth $b - 1$.

On the other hand, singular points on the non-isomorphic locus of $V_1'' \to Z_1'$ on $Z_1'$ are origins of $\bar{U}_1'$ and $\bar{U}_4'$. They are cyclic quotient points of indices $b - r$ and $r$, respectively. One can then see that

$$gdep(V_1'') - gdep(Z_1') = b - 1 - (b - r - 1 + r - 1) = 1.$$

Now we know that that the flipping curve contains only one singular point which is a cyclic quotient point. Also, the general elephant of the flip is of $A$-type since it comes from the factorization of a flop over a $cA/r$ point. Thus, the flip is of type IA by the classification from [20, Theorem 2.2]. □

**Lemma 6.6.** Assume that $T \searrow V \swarrow T'$ is a flip of type IA. Assume that

$$\begin{array}{ccc} S_1 & \dashrightarrow \dots \dashrightarrow & S_k \\ \downarrow & & \downarrow \\ T & \dashrightarrow & T' \end{array}$$

is the factorization in Theorem 2.18. Then:

(1) If $S_1 \dashrightarrow S_2$ is a flop, then it is a Gorenstein flop.

(2) If $S_1 \dashrightarrow S_2$ is a flip, or $S_1 \dashrightarrow S_2$ is a flop and $S_2 \dashrightarrow S_3$ is a flip, then the flip is of type IA.

*Proof.* Let $C \subset T$ be the flipping curve. Since $T \dashrightarrow T'$ is a type IA flip, there is exactly one non-Gorenstein point which is contained in $C$, and this point is a $cA/r$ point. From the construction, we know that $S_1 \to T$ is a $w$-morphism over this $cA/r$ point. Also, there exists a Du Val section $H \in |-K_T|$ such that $C \not\subset H$. We know that $H_{S_1} \in |-K_{S_1}|$ by [3, Lemma 2.7 (2)]. Hence, all non-Gorenstein point of $S_1$ is contained in $H_{S_1}$. Now assume that $C_{S_1}$ contains a non-Gorenstein point. Then $H_{S_1}$ intersects $C_{S_1}$ non-trivially. Since $C_{S_1} \not\subset H_{S_1}$, we know that $H_{S_1}.C_{S_1} > 0$, hence $C_{S_1}$ is a $K_{S_1}$-negative curve and so $S_1 \dashrightarrow S_2$ is a flip. Thus, if $S_1 \dashrightarrow S_2$ is a flop, then it is a Gorenstein flop.

If $S_1 \dashrightarrow S_2$ is a flip, then $C_{S_1}$ passes through a non-Gorenstein point since $0 > K_{S_1}.C_{S_1} > -1$ by [21, Theorem 0]. Since $C \subset T$ passes through exactly one non-Gorenstein point, $C_{S_1}$ passes through exactly one non-Gorenstein point and this point is contained in $E = exc(S_1 \to T)$. Since $S_1 \to T$ is a $w$-morphism over a $cA/r$ point, an easy computation shows that $E$ contains only $cA/r$ singularities. Also, we know that $H_{S_1}$ is a Du Val section which does not contain $C_{S_1}$. Thus, $S_1 \dashrightarrow S_2$ is also a flip of type IA by the classification [20, Theorem 2.2].

Now assume that $S_1 \dashrightarrow S_2$ is a flop and $S_2 \dashrightarrow S_3$ is a flip. Then the flipping curve $\Gamma$ of $S_2 \dashrightarrow S_3$ is contained in $E_{S_2}$ since all $K_{S_2}$-negative curves over $V$ are contained in $E_{S_2}$. Since $S_1 \dashrightarrow S_2$ is a flop, we know that $E_{S_2}$ contains only $cA/r$ singularities [17, Theorem 2.4] and $H_{S_2} \in |-K_{S_2}|$ is also a Du Val section. If $\Gamma \not\subset H_{S_2}$, then $S_2 \dashrightarrow S_3$ is a flip of type IA by the classification from [20, Theorem 2.2].

Thus, we only need to prove that $\Gamma \not\subset H_{S_2}$. Assume that $\Gamma \subset H_{S_2} \cap E_{S_2}$. Then, since $H_{S_1}$ does not intersect $C_{S_1}$ (otherwise $C_{S_1}$ is a $K_{S_1}$-negative curve and then $S_1 \dashrightarrow S_2$ is not a flop), we know that $\Gamma$ does not intersect the flopping curve $C'_{S_2} \subset S_2$. Let $B \subset E_{S_2}$ be a curve which intersects $C'_{S_2}$ non-trivially. Then $\Gamma_{S_1} \equiv \lambda B_{S_1}$ for some $\lambda \in \mathbb{Q}$ since the both curves are contracted by $S_1 \to T$. Hence, for all divisors $D \subset S_2$ such that $D.C'_{S_2} = 0$, we know that $D.\Gamma = \lambda D.B$. On the other hand, $S_1 \dashrightarrow S_2$ is a $K_{S_1} + E$-anti-flip since $C_{S_1}$ is not contained in $E$. By Corollary 2.22 we know that $(K_{S_2} + E_{S_2}).B_{S_2} > (K_{S_1} + E).B_{S_1}$, hence $E_{S_2}.B_{S_2} > E.B_{S_1}$. Now we know that $\rho(S_2/V) = 2$ and $B$ is not numerically equivalent to a multiple of $C'_{S_2}$, hence we may write $\Gamma \equiv \lambda B + \mu C'_{S_2}$ for some $\mu \in \mathbb{Q}$. Since

$$E_{S_2}.\Gamma = E_{S_1}.\Gamma_{S_1} = \lambda E.B_{S_1} < \lambda E_{S_2}.B_{S_2}$$

and $E_{S_2}.C'_{S_2} < 0$, we know that $\mu > 0$. Hence, $\Gamma$ is not contained in the boundary of the relative effective cone $NE(S_2/V)$. Thus, $\Gamma$ cannot be the flipping curve of $S_2 \dashrightarrow S_3$. This leads to a contradiction. □

**Lemma 6.7.** Assume that $T \dashrightarrow T'$ is a three-dimensional terminal $\mathbb{Q}$-factorial flip which satisfies conditions (1)–(3) of Lemma 6.5. Then, the factorization in Theorem 2.18 for $T \dashrightarrow T'$ is one of the following diagrams:

(1)

$$\begin{array}{ccc} S_1 & \dashrightarrow & S_2 \\ \downarrow & & \downarrow \\ T & \dashrightarrow & T' \end{array}$$

where $S_1 \dashrightarrow S_2$ is a flip which also satisfies conditions (1)–(3) of Lemma 6.5 and $S_2 \to T'$ is a strict $w$-morphism.

(2)

$$\begin{array}{ccccc} S_1 & \dashrightarrow & S_2 & \dashrightarrow & S_3 \\ \downarrow & & & & \downarrow \\ T & & \dashrightarrow & & T' \end{array}$$

where $S_1 \dashrightarrow S_2$ is a smooth flop and $S_2 \dashrightarrow S_3$ is a flip which also satisfies conditions (1)–(3) of Lemma 6.5 and $S_3 \to T'$ is a strict $w$-morphism.

(3)

$$\begin{array}{ccc} S_1 & \dashrightarrow & S_2 \\ \downarrow & & \downarrow \\ T & \dashrightarrow & T' \end{array}$$

where $S_1 \dashrightarrow S_2$ is a smooth flop and $S_2 = Bl_{C'}T'$ where $C'$ is a smooth curve contained in the smooth locus of $T'$.

*Proof.* We have the factorization

$$\begin{array}{ccccc} S_1 & \dashrightarrow & \dots & \dashrightarrow & S_k \\ \downarrow & & & & \downarrow \\ T & & \dashrightarrow & & T' \end{array}$$

such that $S_1 \dashrightarrow S_2$ is a flip or a flop and $S_i \dashrightarrow S_{i+1}$ is a flip for all $2 \leq i \leq k-1$. One has that

$$gdep(S_k) \leq gdep(S_1) = gdep(T) - 1 = gdep(T') \leq gdep(S_k) + 1.$$

If $gdep(S_k) = gdep(S_1)$, then $k = 2$ and $S_1 \dashrightarrow S_2$ is a flop. By [3, Remark 3.4] we know that $S_2 \to T'$ is a divisorial contraction to a curve. Since singular points on the non-isomorphic locus of $T \dashrightarrow T'$ are all cyclic quotient points and there is no divisorial contraction to a curve which passes through a cyclic quotient point [8, Theorem 5], we know that $S_2 \to T'$ is a divisorial contraction to a curve $C'$ contained in the smooth locus. Now, we also have that $gdep(S_2) = gdep(T')$, hence $S_2$ is smooth over $T'$ and so $C'$ is also a smooth curve.

Now assume that $gdep(S_k) < gdep(S_1)$. Then $gdep(S_k) = gdep(S_1) - 1 = gdep(T') - 1$, hence either $k = 2$ or $k = 3$ and $S_1 \dashrightarrow S_2$ is a flop. Also, $S_k \to T'$ is a $w$-morphism. Since singular points on the non-isomorphic locus of $T \dashrightarrow T'$ are all cyclic quotient points, singular points on the exceptional divisor of $S_k \to T'$ are all cyclic quotient points. Since flops do not change singularities [17, Theorem 2.4], we know that the singular points on the non-isomorphic locus of $S_i \dashrightarrow S_{i+1}$ are all cyclic quotient points for $i = 1, ..., k-1$. If $S_1 \dashrightarrow S_2$ is a flop, then by Lemma 6.6 we know that it is a Gorenstein flop. Since cyclic quotient points are not Gorenstein, we know that the flop is in fact a smooth flop. Now assume that $S_i \dashrightarrow S_{i+1}$ is a flip for $i = 1$ or 2. Then again, by Lemma 6.6 we know that it is a flip of type IA. Hence, conditions (1)–(3) of Lemma 6.5 are satisfied for this flip. □

**Corollary 6.8.** Assume that $T \dashrightarrow T'$ is a three-dimensional terminal $\mathbb{Q}$-factorial flip which satisfies conditions (1)–(3) of Lemma 6.5. Then we have a factorization

$$\begin{array}{ccc} \tilde{T} & \dashrightarrow & \bar{T}' \\ \Big| & & \downarrow \\ \Big| & & \tilde{T}' \\ \downarrow & & \downarrow \\ T & \dashrightarrow & T' \end{array}$$

where $\tilde{T} \to T$ and $\tilde{T}' \to T$ are feasible resolutions, $\bar{T}' = Bl_{C'}\tilde{T}'$ where $C' \subset \tilde{T}'$ is a smooth curve, and $\tilde{T} \dashrightarrow \bar{T}'$ is a sequence of smooth flops.

*Proof.* For convenience we denote the diagram $\begin{array}{ccc} \tilde{T} & \dashrightarrow & \bar{T}' \\ \Big| & & \downarrow \\ \Big| & & \tilde{T}' \\ \downarrow & & \downarrow \\ T & \dashrightarrow & T' \end{array}$ by $(A)_{T \dashrightarrow T'}$.

We know that the factorization of $T \dashrightarrow T'$ is of the form (1)–(3) in Lemma 6.7. Notice that, since the only singular points on the non-isomorphic locus of $T \dashrightarrow T'$ are cyclic quotient points, the feasible resolutions $\tilde{T}$ and $\tilde{T}'$ are uniquely determined. Hence, $\tilde{T}$ is also a feasible resolution of $S_1$. If the factorization of $T \dashrightarrow T'$ is of type (3) in Lemma 6.7, then the non-isomorphic locus of $S_1 \dashrightarrow T'$ contains no singular points. This means that $S_2 \to T'$ induces a smooth blow-up $\bar{T}' \to \tilde{T}'$ on $\tilde{T}'$, and $S_1 \dashrightarrow S_2$ induces a smooth flop $\tilde{T} \dashrightarrow \bar{T}'$. Thus, $(A)_{T \dashrightarrow T'}$ exists.

Now, if the factorization of $T \dashrightarrow T'$ is of type (1), then $\tilde{T}'$ is a feasible resolution of $S_2$. Since $gdep(S_1) = gdep(T) - 1$, we may utilize induction on $gdep(T)$ and assume that $(A)_{S_1 \dashrightarrow S_2}$ exists, and then $(A)_{T \dashrightarrow T'}$ can be induced by $(A)_{S_1 \dashrightarrow S_2}$. If the factorization of $T \dashrightarrow T'$ is of type (2), then again by induction we may assume that

$$(A)_{S_2 \dashrightarrow S_3} = \begin{array}{ccc} \tilde{S}_2 & \dashrightarrow & \bar{S}_3 \\ \Big| & & \downarrow \\ \Big| & & \tilde{S}_3 \\ \downarrow & & \downarrow \\ S_2 & \dashrightarrow & S_3 \end{array}$$

exists. Since $S_3 \to T'$ is a strict $w$-morphism, we know that $\tilde{S}_3 = \tilde{T}'$. Also, since $S_1 \dashrightarrow S_2$ is a smooth flop, it induces a smooth flop $\tilde{T} = \tilde{S}_1 \dashrightarrow \tilde{S}_2$. If we let $\bar{T}' = \bar{S}_3$ and let $\tilde{T} \dashrightarrow \bar{T}'$ be the composition $\tilde{T} \dashrightarrow \tilde{S}_2 \dashrightarrow \bar{T}'$, then we get the diagram $(A)_{T \dashrightarrow T'}$. □

**Definition 6.9.** Let $W_1 \dashrightarrow W_2$ be a birational map between smooth threefolds. We say that $W_1 \dashrightarrow W_2$ is of type $\Omega_0$ if it is a composition of smooth flops. We say that $W_1 \dashrightarrow W_2$ is of type $\Omega_n$ if there exists the diagram

$$\begin{array}{ccccccc} \bar{W}_1 & \dashleftarrow & \bar{W}'_1 & \dashrightarrow & \bar{W}'_2 & \dashrightarrow & \bar{W}_2 \\ \downarrow & & & & & & \downarrow \\ W_1 & & & & & & W_2 \end{array}$$

such that:

(1) $\bar{W}_i = Bl_{C_i} W_i$ for some smooth curve $C_i \subset W_i$ for $i = 1, 2$.

(2) $\bar{W}'_i \dashrightarrow \bar{W}_i$ is a composition of smooth flops for $i = 1, 2$.

(3) $\bar{W}'_1 \dashrightarrow \bar{W}'_2$ has the factorization

$$\bar{W}'_1 = \bar{W}'_{1,1} \dashrightarrow \bar{W}'_{1,2} \dashrightarrow ... \dashrightarrow \bar{W}'_{1,m} = \bar{W}'_2$$

such that $\bar{W}'_{1,j} \dashrightarrow \bar{W}'_{1,j+1}$ is a birational map of type $\Omega_{m_j}$ for some $m_j < n$.

**Example 6.10.** In the following diagrams, dashmaps stand for smooth flops and all other maps are blowing-down smooth curves.

(1)

$$\begin{array}{ccc} \bar{W}_1 & \dashrightarrow & \bar{W}_2 \\ \downarrow & & \downarrow \\ W_1 & & W_2 \end{array} .$$

(2)

$$\begin{array}{ccccccc} & & \bar{W}'_1 & \dashrightarrow & \bar{W}'_2 & & \\ & & \downarrow & & \downarrow & & \\ \bar{W}_1 & \dashrightarrow & W'_1 & & W'_2 & \dashleftarrow & \bar{W}_2 \\ \downarrow & & & & & & \downarrow \\ W_1 & & & & & & W_2 \end{array} .$$

(3)

$$\begin{array}{ccccccccccc} & & \bar{W}'_1 & \dashleftarrow & \bar{W}''_1 & & \bar{W}''_2 & \dashrightarrow & \bar{W}'_2 & & \\ & & \downarrow & & & \searrow \quad \swarrow & & & \downarrow & & \\ \bar{W}_1 & \dashrightarrow & W'_1 & & & W'' & & & W'_2 & \dashleftarrow & \bar{W}_2 \\ \downarrow & & & & & & & & & & \downarrow \\ W_1 & & & & & & & & & & W_2 \end{array} .$$

In diagram (1), $W_1 \dashrightarrow W_2$ is of type $\Omega_1$. In both diagram (2) and (3), $W_1 \dashrightarrow W_2$ is of type $\Omega_2$.

**Definition 6.11.** Let $W \dashrightarrow W'$ be a birational map between smooth threefolds. We say that $W \dashrightarrow W'$ has an $\Omega$-type factorization if there exists birational maps between smooth threefolds

$$W = W_1 \dashrightarrow W_2 \dashrightarrow ... \dashrightarrow W_k = W'$$

such that $W_i \dashrightarrow W_{i+1}$ is of type $\Omega_{n_i}$ for some $n_i \in \mathbb{Z}_{\geq 0}$.

**Proposition 6.12.** Assume that $X$ is a $\mathbb{Q}$-factorial terminal threefold and $W \to X$, $W' \to X$ are two different feasible resolutions. Then the birational map $W \dashrightarrow W'$ has an $\Omega$-type factorization.

*Proof.* First, notice that if $gdep(X) = 1$, then $X$ has either a cyclic quotient point of index 2, or a $cA_1$ point defined by $xy + z^2 + u^n$ for $n = 2$ or 3 by [22, Corollary 3.4]. In those cases, there is exactly one feasible resolution (which is obtained by blowing-up the singular point). Hence, one may assume that $gdep(X) > 1$. Let $Y \to X$ (resp. $Y' \to X$) be the strict $w$-morphism which is the first factor of $W \to X$ (resp. $W' \to X$). If $Y = Y'$, then $W$ and $W'$ are two different feasible resolutions of $Y$. In this case, the statement can be proved by induction on $gdep(X)$. Thus, we may assume that $Y \neq Y'$.

Since both $Y$ and $Y'$ are strict $w$-morphisms, by Corollary 5.7 there exists a sequence of strict $w$-morphisms $Y_1 = Y \to X$, $Y_2 \to X$, ..., $Y_k = Y' \to X$ such that $Y_i \underset{X}{\Rightarrow} Y_{i+1}$ for $i = 1, ..., k-1$. For each $2 \leq i \leq k-1$, let $W_i \to Y_i$ be a feasible resolution. Then $W_i$ is also a feasible resolution of $X$ and it is enough to prove that our statement holds for $W_i$ and $W_{i+1}$, for all $i = 1, ..., k-1$. Thus, we may assume that $Y \underset{X}{\Rightarrow} Y'$.

We have the diagram

$$\begin{array}{ccc} Z_1 & \dashrightarrow \cdots \dashrightarrow & Z_k \\ \downarrow & & \downarrow \\ Y & & Y' \\ & \searrow \qquad \swarrow & \\ & X & \end{array}$$

Lemma 6.1 says that there are three possibilities. If $k = 1$, then $W$ and $W'$ are two different feasible resolutions of $Z_1$. Since $gdep(Z_1) = gdep(X) - 2$, again by induction on $gdep(X)$ we know that $W \dashrightarrow W'$ has an $\Omega$-type factorization. Assume that $k = 2$ and $Z_1 \dashrightarrow Z_2$ is a smooth flop. Then it induces a smooth flop $\tilde{Z}_1 \dashrightarrow \tilde{Z}_2$ where $\tilde{Z}_i \to Z_i$ is a feasible resolution of $Z_i$ for $i = 1, 2$. Also, we know that $W$ and $\tilde{Z}_1$ are two feasible resolutions of $Y$, and $W'$ and $\tilde{Z}_2$ are two feasible resolutions of $Y'$. Again, by induction on $gdep(X)$, we know that both $W \dashrightarrow \tilde{Z}_1$ and $\tilde{Z}_2 \dashrightarrow W'$ have $\Omega$-type factorizations, hence $W \dashrightarrow W'$ does as well.

Finally, assume that we are in the case of Lemma 6.1 (3), namely that $X$ has a $cA/r$ singularity and $Z_1 \dashrightarrow Z_2$ is a singular flop. By Lemma 6.3 we know that $Z_1 \dashrightarrow Z_2$ is a flop over

$$V = (xy + uf(z,u) = 0) \subset \mathbb{A}^4_{(x,y,z,u)}/\frac{1}{r}(\beta, -\beta, 1, 0).$$

We have the factorization of the flop $Z_1 \dashrightarrow Z_2$

$$\begin{array}{ccccc} Z'_1 \dashleftarrow V''_1 & & & & V''_2 \dashrightarrow Z'_2 \\ \downarrow & \searrow & & \swarrow & \downarrow \\ Z_1 & & V' & & Z_2\,. \\ & \searrow & \downarrow & \swarrow & \\ & & V & & \end{array}$$

We use the notation at the beginning of this subsection. Assume that $m > 1$. Then we can choose $V' \to V$ to be the weighted blow-up with the weight $w'(x,y,z,u) = \frac{1}{r}(\beta + r, mr - \beta, 1, r)$. In this case, $Z'_i \to Z_i$ is a divisorial contraction over the origin of $U_{i,u}$ for $i = 1, 2$ since both $w'(x_1)$ and $w'(y_2) > 0$.

When $m = 1$, let $V' \to V$ be the weighted blow-up with the weight $w'(x, y, z, u) = \frac{1}{r}(r + \beta, r - \beta, 1, r)$. Then, $Z'_1 \to Z_1$ is a divisorial contraction over the origin of $U_{1,u}$, but $Z'_2 \to Z_2$ is a divisorial contraction over the origin of $U_{2,y}$. Since $m = 1$ and $k > 1$ by Remark 6.2, we know that $f(z, u) = \lambda u + z^{rk}$ for some unit $\lambda$. If we let $\bar{u} = \lambda u + z^{rk}$, then we can write the defining equation of $V$ as $xy + \bar{u}\bar{f}(z, \bar{u})$ where $\bar{f} = \frac{1}{\lambda}(\bar{u} - z^{rk})$. One has that $Z_2 \cong Bl_{(x,\bar{u})}V$, and under this notation one also has that $Z'_2 \to Z_2$ is a divisorial contraction over the origin of $U_{2,\bar{u}}$. In conclusion, Corollary 6.8 holds for both $V''_1 \dashrightarrow Z'_1$ and $V''_2 \dashrightarrow Z'_2$.

Let $\tilde{Z}'_i \to Z'_i$ be a feasible resolution. If $V''_1 = V''_2$, then one can see that $\tilde{Z}'_1 \dashrightarrow \tilde{Z}'_2$ is of type $\Omega_1$. Assume that $V''_1 \dashrightarrow V''_2$ is a flop. Notice that $Z'_i \to Z_i$ is a $w$-morphism since $a(E, Z_i) = a(E, V)$ where $E = exc(Z'_i \to Z_i) = exc(V' \to V)$. One has that

$$gdep(V''_i) = gdep(Z'_i) + 1 = gdep(Z_i)$$

where the second equality follows from Corollary 5.8. Now we know that $V''_1 \dashrightarrow V''_2$ is a flop over $V'$ with $gdep(V') < gdep(V)$. By induction on $gdep(V)$, we may assume that $\tilde{V}''_1 \dashrightarrow \tilde{V}''_2$ has an $\Omega$-type factorization where $\tilde{V}''_i \to V''_i$ is a feasible resolution corresponding to the diagram in Corollary 6.8. One can see that $\tilde{Z}'_2 \dashrightarrow \tilde{Z}'_2$ can be connected by a diagram of the form $\Omega_n$ for some $n \in \mathbb{N}$. Finally, we know that $W$ and $\tilde{Z}'_1$ (resp. $W'$ and $\tilde{Z}'_2$) are feasible resolutions of $Y$ (resp. $Y'$). Again, by induction on $gdep(X)$, we may assume that $W \dashrightarrow \tilde{Z}'_1$ and $W' \dashrightarrow \tilde{Z}'_2$ have $\Omega$-type factorizations. Hence, $W \dashrightarrow W'$ has an $\Omega$-type factorization. □

**Remark 6.13.** Assume that $X = (xy + z^m + u^k = 0) \subset \mathbb{A}^4$ is a $cA$ singularity. Then there exists feasible resolutions $W$ and $W'$ such that the birational map $W \dashrightarrow W'$ is connected by $\Omega_n$ for $n = \lceil \frac{m}{k-m} \rceil$.

## 7. Minimal resolutions of threefolds

In this section, we prove our main theorems. First, we recall some definitions which are defined in the introduction section.

**Definition 7.1.** Let $X$ be a projective variety. We say that a resolution of singularities $W \to X$ is a P-minimal resolution if for any smooth model $W' \to X$ one has that $\rho(W) \leq \rho(W')$.

**Definition 7.2.** Let $W \dashrightarrow W'$ be a birational map between smooth varieties. We say that this birational map is a P-desingularization of a flop if there exists a flop $X \dashrightarrow X'$ such that $W \to X$ and $W' \to X'$ are P-minimal resolutions.

**Proposition 7.3.** Assume that $X$ is a threefold. Then, $W \to X$ is a P-minimal resolution if and only if $W$ is a feasible resolution of a terminalization of $X$. In particular, if $X$ is a terminal and $\mathbb{Q}$-factorial threefold, then P-minimal resolutions of $X$ coincide with feasible resolutions.

*Proof.* Let $W \to X$ be a resolution of singularities and let $W \dashrightarrow X_W$ be the $K_W$-MMP over $X$. Then $X_W$ is a terminalization of $X$. We know that $\rho(W/X_W) \geq gdep(X_W)$ by Corollary 5.2. Assume first that $W \to X$ is $P$-minimal. Let $W_1 \to X_W$ be a feasible resolution of $X_W$, then $\rho(W_1/X) = gdep(X_W) \leq \rho(W/X_W)$. Since $W_1$ is also a smooth resolution of $X$, the inequality is an equality. Therefore, $\rho(W/X_W) = gdep(X_W)$, which implies that $W \dashrightarrow X_W$ is a sequence of strict $w$-morphisms by Corollary 5.2, or, equivalently, $W \to X_W$ is a feasible resolution.

Conversely, assume that $W \to X$ is not $P$-minimal, but it is a feasible resolution of some $X_W$ which is a terminalization of $X$. There exists a $P$-minimal resolution $W' \to X$ such that $\rho(W/X) > \rho(W'/X)$. From the above argument, there exists a terminalization $X_{W'}$ of $X$ such that $W' \to X_{W'}$ is a feasible resolution. Hence, $\rho(W'/X) = gdep(X')$. However, since terminalizations are connected by flops [23, Theorem 1] and flops do not change singularities by [17, Theorem 2.4], we know that $gdep(X_W) = gdep(X_{W'})$. This means that $\rho(W/X_W) > gdep(X)$, so $W$ can not be a feasible resolution of $X_W$. This is a contradiction. □

*Proof of Theorem 1.1.* Let $X$ be a threefold and $W \to X$, $W' \to X$ be two P-minimal resolutions. By Proposition 7.3, we know that $W$ (resp. $W'$) is a feasible resolution of a terminalization $X_W \to X$ (resp. $X_{W'} \to X$). If $X_W \not\cong X_{W'}$, then $X_W$ and $X_{W'}$ are connected by flops [23, Theorem 1], hence $W \dashrightarrow W'$ is connected by P-desingularizations of terminal $\mathbb{Q}$-factorial flops.

Now assume that $X_W = X_{W'}$. Then $W$ and $W'$ are two different feasible resolutions of $X_W$. The first two paragraphs in the proof of Proposition 6.12 and Lemma 6.7 imply that $W \dashrightarrow W'$ can be also connected by P-desingularizations of terminal $\mathbb{Q}$-factorial flops. Moreover, Proposition 6.12 says that those P-desingularizations of flops can be factorized into compositions of diagrams of the form $\Omega_i$. This finishes the proof. □

**Remark 7.4.** Assume that $X$ is a terminal $\mathbb{Q}$-factorial threefold and $W \to X$, $W' \to X$ are two different P-minimal resolutions. We know that $W$ and $W'$ can be connected by P-desingularizations of flops. Let $W_i \dashrightarrow W_{i+1}$ be a P-desingularization of a flop $X_i \dashrightarrow X_{i+1}$ which appears in the factorization of $W \dashrightarrow W'$. Then, from the construction we know that $gdep(X_i) < gdep(X)$.

Now we compare an arbitrary resolution of singularities to a P-minimal resolution.

**Definition 7.5.** Let $W \dashrightarrow X$ be a birational map where $W$ is a smooth threefold and $X$ is a terminal threefold. We say that the birational map has a *bfw-factorization* if $W \dashrightarrow X$ can be factorized into a composition of smooth blow-downs, P-desingularizations of flops, and strict $w$-morphisms.

**Remark 7.6.** If $X_2 \to X_1$ is a strict $w$-morphism and $X_1 \dashrightarrow X$ is a smooth blow-down or a P-desingularization of a flop, then on $X_1$ the indeterminacy locus of $X_1 \dashrightarrow X$ is disjoint to the indeterminacy locus of $X_1 \dashrightarrow X_2$ since the former one lies on the smooth locus of $X_1$ and the latter one is a singular point. Hence, there exists $X_2 \dashrightarrow X_1' \to X$ where $X_2 \dashrightarrow X_1'$ is a smooth blow-down or a P-desingularization of a flop, and $X_1' \to X$ is a strict $w$-morphism. In other words, $W \dashrightarrow X$ has a bfw-factorization if and only if there exists a birational map $W \dashrightarrow \bar{X}$ which is a composition of smooth blow-downs and P-desingularization of flops, where $\bar{X}$ is a feasible resolution of $X$.

**Proposition 7.7.** Assume that a birational map $W \dashrightarrow X$ has a bfw-factorization where $W$ is a smooth threefold and $X$ is a terminal threefold.

(1) If $X \dashrightarrow X'$ is a flop, then there is a birational map $W \dashrightarrow X'$ which has a bfw-factorization.

(2) If $Y \to X$ is a strict $w$-morphism, then there exists a birational map $W \dashrightarrow Y$ which also has a bfw-factorization.

(3) If $X \dashrightarrow X'$ is a flip or a divisorial contraction, then the induced birational map $W \dashrightarrow X'$ has a bfw-factorization.

*Proof.* Assume first that $X \dashrightarrow X'$ is a flop. By Remark 7.6 we know that there exists a bfw-map $W \dashrightarrow \bar{X}$, where $\bar{X}$ is a feasible resolution of $X$. Let $\bar{X}' \to X'$ be a feasible resolution of $X'$. Then $\bar{X}' \to X'$ is a composition of strict $w$-morphisms and the induced birational map $\bar{X} \dashrightarrow \bar{X}'$ is a P-desingularization of the flop $X \dashrightarrow X'$. It follows that the composition

$$W \dashrightarrow \bar{X} \dashrightarrow \bar{X}' \to X'$$

is a bfw-map. This proves (1).

We will prove (2) and (3) by induction on $gdep(X)$. If $gdep(X) = 0$, then $X$ is smooth. In this case, there is no strict $w$-morphism $Y \to X$ or flip $X \dashrightarrow X'$. Assume that $X \to X'$ is a divisorial contraction. If $X'$ is smooth, then it is a smooth blow-down by [22, Theorem 3.3, Corollary 3.4], and if $X'$ is singular, then $X \to X'$ should be a strict $w$-morphism since in this case $X'$ is terminal $\mathbb{Q}$-factorial and $X$ is a P-minimal resolution of $X'$. Now we may assume that $gdep(X) > 0$ and statements (2) and (3) hold for threefolds with generalized depth less then $gdep(X)$.

Let

$$W = X_k \dashrightarrow X_{k-1} \dashrightarrow ... \dashrightarrow X_1 \dashrightarrow X_0 = X$$

be a sequence of birational maps so that $X_{i+1} \dashrightarrow X_i$ is a smooth blow-down, a P-desingularization of a flop, or a strict $w$-morphism for all $1 \le i \le k-1$. By Remark 7.6 we can assume that $X_1 \to X$ is a strict $w$-morphism. Now, given a strict $w$-morphism $Y \to X$, if $Y \cong X_1$, then there is nothing to prove. Otherwise, by Corollary 5.7 there exists a sequence of strict $w$-morphisms $Y_2 \to X$, ..., $Y_{m-1} \to X$ such that

$$X_1 = Y_1 \underset{X}{\Rightarrow} Y_2 \underset{X}{\Rightarrow} ... \underset{X}{\Rightarrow} Y_{m-1} \underset{X}{\Rightarrow} Y_m = Y.$$

For each $1 \le i \le m-1$, one has the factorization

$$\begin{array}{ccccc} Z_{i,1} & \dashrightarrow & ... & \dashrightarrow & Z_{i,k_i} \\ \downarrow & & & & \downarrow \\ Y_i & & & & Y_{i+1} \\ & \searrow & & \swarrow & \\ & & X & & \end{array}$$

such that $Z_{i,1} \dashrightarrow Z_{i,k_i}$ is a composition of flops, $Z_{i,1} \to Y_i$ is a strict $w$-morphism, and

$$gdep(Z_{i,k_i}) = gdep(Z_{i,1}) < gdep(Y_i) < gdep(X).$$

By the induction hypothesis, we know that if there exists a bfw-map $W \dashrightarrow Y_i$, then there exists a bfw-map $W \dashrightarrow Y_{i+1}$. Now one can prove statement (2) by induction on $m$.

Assume that $X \dashrightarrow X'$ is a flip. Then we have a factorization

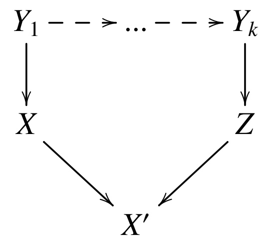
$$\begin{array}{ccccc} Y_1 & \dashrightarrow & ... & \dashrightarrow & Y_k \\ \downarrow & & & & \downarrow \\ X & & & & X' \end{array}$$

as in Theorem 2.18. Since $Y_1 \to X$ is a strict $w$-morphism by Corollary 5.8, there exists a bfw-map $W \dashrightarrow Y_1$. Since

$$gdep(Y_k) \le ... \le gdep(Y_1) < gdep(X),$$

the induction hypothesis implies that there exists a bfw-map $W \dashrightarrow X'$.

Finally, assume that $X \to X'$ is a divisorial contraction. If it is a smooth blow-down or a strict $w$-morphism, then there is nothing to prove. Otherwise, there exists a diagram



such that $Y_1 \to X$ is a strict $w$-morphism and $Y_i \dashrightarrow Y_{i+1}$ is a flip or a flop for all $1 \le i \le k-1$. One has that

$$gdep(Y_k) \le ... \le gdep(Y_1) < gdep(X),$$

hence there exists a bfw-map $W \dashrightarrow Z$. If $X \to X'$ is a divisorial contraction to a curve, then $Y_k \to Z$ is a divisorial contraction to a curve as in Theorem 2.18. In this case, we also have $gdep(Z) \le gdep(Y_k) < gdep(X)$, so there exists a bfw-map $W \dashrightarrow X'$. If $X \to X'$ is a divisorial contraction to a point, then the discrepancy of $Z \to X'$ is less than the discrepancy of $X \to X'$ unless $X \to X'$ is a $w$-morphism. Also, when $X \to X'$ is a $w$-morphism, we know that $gdep(Z) < gdep(X)$ by Lemma 5.6. Thus, we can prove statement (3) by induction on the generalized depth and the discrepancy of $X$ over $X'$. □

One can easily see the following corollary:

**Corollary 7.8.** Assume that $W$ is a smooth threefold and $W \dashrightarrow X$ is a birational map which is a composition of steps of MMP. Then, this birational map can be factorized into a composition of smooth blow-downs, P-desingularizations of flops, and strict $w$-morphisms.

*Proof of Theorem 1.2.* By Corollary 7.8 and Remark 7.6 we know that there exists a feasible resolution $\tilde{X}_W \to X_W$ such that $W \dashrightarrow \tilde{X}_W$ is a composition of smooth blow-downs and P-desingularizations of flops, where $X_W$ is a minimal model of $W$ over $X$. By Proposition 7.3 we know that $\tilde{X}_W$ is also a P-minimal resolution of $X$, hence the birational map $\tilde{X}_W \dashrightarrow \tilde{X}$ is connected by P-desingularizations of flops. Thus, the composition $W \dashrightarrow \tilde{X}_W \dashrightarrow \tilde{X}$ is connected by smooth blow-downs and P-desingularizations of flops. □

*Proof of Corollary 1.3.* Let

$$W = \tilde{X}_k \dashrightarrow ... \dashrightarrow \tilde{X}_1 \dashrightarrow \tilde{X}_0 = \tilde{X}$$

be a sequence of smooth blow-downs and P-desingularization of flops as in Theorem 1.2. We only need to show that if $\tilde{X}_{i+1} \dashrightarrow \tilde{X}_i$ is a P-desingularization of a flop $X_{i+1} \dashrightarrow X_i$, then $b_j(\tilde{X}_{i+1}) = b_j(\tilde{X}_i)$ for all $j = 0, ..., 6$.

By [24, Lemma 2.12] we know that $b_j(X_{i+1}) = b_j(X_j)$ for all $j$. Since $X_i$ and $X_{i+1}$ have the same analytic singularities [17, Theorem 2.4], there exists a feasible resolution $\tilde{X}'_{i+1} \to X_{i+1}$ such that $b_j(\tilde{X}_i) = b_j(\tilde{X}'_{i+1})$ for all $j$. Now, $\tilde{X}'_{i+1}$ and $\tilde{X}_{i+1}$ are two different P-minimal resolutions of $X_{i+1}$, so they can be connected by P-desingularizations of flops with smaller generalized depth by Remark 7.4. By induction on the generalized depth, one can see that $b_j(\tilde{X}'_{i+1}) = b_j(\tilde{X}_{i+1})$. Hence, $b_j(\tilde{X}_{i+1}) = b_j(\tilde{X}_i)$ for all $j = 0, ..., 6$. □

## 8. Further discussion

This section is dedicated to exploring minimal resolutions for singularities in higher dimensions and the potential applications of our main theorems.

### 8.1. Higher-dimensional minimal resolutions

In three dimensions, $P$-minimal resolutions appear to be a viable generalization of minimal resolutions for surfaces. However, in higher dimensions, $P$-minimal resolutions are not good enough. For example, let $X \dashrightarrow X'$ be a smooth flip (eg. a standard flip [1, Section 11.3]). Then, $X$ and $X'$ are both $P$-minimal resolutions of the underlying space, but $X'$ is better than $X$. It is reasonable to assume that $X'$ is a minimal resolution, while $X$ is not. Inspired by Corollary 1.3, we define a new kind of minimal resolution:

**Definition 8.1.** Let $X$ be a projective variety over complex numbers. We say that a resolution of singularities $W \to X$ is a $B$-minimal resolution if for any smooth model $W' \to X$ one has that $b_i(W) \leq b_i(W')$ for all $0 \leq i \leq 2\dim X$.

As stated in Corollary 1.3, B-minimal resolutions coincide with P-minimal resolutions in dimension three. Our main theorems say that B-minimal resolutions of threefolds satisfy certain nice properties. It is logical to anticipate that B-minimal resolutions of higher-dimensional varieties share similar properties.

**Conjecture.** For any projective variety $X$ over the complex numbers, one has that:

(1) B-minimal resolutions of $X$ exist.

(2) Two different B-minimal resolutions are connected by desingularizations of $\mathbb{Q}$-factorial terminal flops.

(3) If $\tilde{X} \to X$ is a B-minimal resolution and $W \to X$ is an arbitrary resolution of singularities, then $W \dashrightarrow \tilde{X}$ can be connected by smooth blow-downs, smooth flips, and desingularizations of $\mathbb{Q}$-factorial terminal flops.

### 8.2. The strong factorization theorem

Let

$$\mathcal{X}_3 = \{\text{ smooth threefolds }\}/\sim,$$

where $W_1 \sim W_2$ if $W_1 \dashrightarrow W_2$ is connected by P-desingularizations of $Q$-factorial terminal flops. For $\eta_1, \eta_2 \in \mathcal{X}_3$ we say that $\eta_1 > \eta_2$ if there exist $W_1$ and $W_2$ so that $\eta_i = [W_i]$ and $W_1 \to W_2$ is a smooth blow-down. Then, Theorems 1.1 and 1.2 imply the following.

**Corollary 8.2.** Given a threefold $X$, let

$$\mathcal{X}_{3,X} = \{\, [W] \in \mathcal{X}_3 \mid \text{There exists a birational morphism } W \to X\}.$$

Then $\mathcal{X}_{3,X}$ has a unique minimal element.

In other words, if we consider the resolution of singularities inside $\mathcal{X}_3$, then there is a unique minimal resolution, which behaves similarly to the minimal resolution of a surface.

As a consequence, inside the space $\mathcal{X}_3$ the following strong factorization theorem holds.

**Theorem 8.3** (Strong factorization theorem for $\mathcal{X}_3$)**.** Assume that $W_1$ and $W_2$ are smooth threefolds which are birational to each other. Then there exists a smooth threefold $\bar{W}$ such that inside $\mathcal{X}_3$ one has $[\bar{W}] \geq [W_i]$ for $i = 1, 2$.

*Proof.* Let $W_1 \leftarrow \bar{W} \rightarrow W_2$ be a common resolution. Then $[\bar{W}] \in \mathcal{X}_{3,W_i}$ for $i = 1, 2$. Since the minimal element of $\mathcal{X}_{3,W_i}$ is $[W_i]$ itself, one has that $[\bar{W}] \geq [W_i]$ for $i = 1, 2$. □

## 8.3. Essential valuations

One can characterize a surface singularity by the information of exceptional curves on the minimal resolution. One may ask, does a similar phenomenon happen for higher-dimensional singularities? Since for higher-dimensional singularities there is no unique minimal resolution, what we really want to study is the following object.

**Definition 8.4.** Let $X$ be a projective threefold over the complex numbers. We say that a divisorial valuation $v_E$ over $X$ is an almost essential valuation if for any P-minimal resolution $\tilde{X} \rightarrow X$ one has that $\text{Center}_{\tilde{X}} E$ is an irreducible component of the exceptional locus of $\tilde{X} \rightarrow X$.

This name comes from the "essential valuation" in the theories of arc spaces.

**Definition 8.5.** Let $X$ be a variety. We say that a divisorial valuation $v_E$ over $X$ is an essential valuation if for any resolution of singularities $W \rightarrow X$ one has that $\text{Center}_W E$ is an irreducible component of the exceptional locus of $W \rightarrow X$.

From the definition, one can see that essential valuations are almost essential, but an almost essential valuation may not be essential.

**Example 8.6.** Let $X = (xy + z^2 + u^{2n+1}) \subset \mathbb{A}^4$ for some $n > 2$. There is exactly one $w$-morphism $X_1 \rightarrow X$ over the singular point, which is obtained by blowing-up the origin. There is only one singular point on $X_1$, which is defined by $xy + z^2 + u^{2(n-1)+1}$. Blowing-up the singular point $n - 1$ more times, we get a resolution of singularities $\tilde{X} \rightarrow X$. From the construction we know that $\tilde{X}$ is a unique feasible resolution of $X$. Since $X$ is terminal and $\mathbb{Q}$-factorial, $\tilde{X}$ is the unique P-minimal resolution of $X$. Hence, almost essential valuations of $X$ are those divisorial valuations which appear on $\tilde{X}$. One can compute that $exc(\tilde{X} \rightarrow X) = E_1 \cup ... \cup E_n$ such that $v_{E_i}(x, y, z, u) = (i, i, i, 1)$. On the other hand, by [25, Lemma 15] we know that essential valuations of $X$ are $v_{E_1}$ and $v_{E_2}$. Hence, $v_{E_3}, ..., v_{E_n}$ are almost essential valuations which are not essential.

Notice that the set of essential valuations does not really characterize the singularity since it is independent of $n$. The set of almost essential valuations carries more information of the singularity.

## 8.4. Derived categories

Let $X$ be a smooth variety. The bounded derived category of coherent sheaves of $X$, denoted by $D^b(X)$, is an interesting subject of investigation. One possible method to study $D^b(X)$ is to construct a semi-orthogonal decomposition of $D^b(X)$ (refer to [26] for more information). Orlov [27] proved that a smooth blow-down yields a semi-orthogonal decomposition. In particular, if $X$ is a smooth surface, then the $K_X$-MMP is a series of smooth blow-downs, thereby resulting in a semi-orthogonal decomposition of $D^b(X)$.

Now assume that $X$ is a smooth threefold and let

$$X = X_0 \dashrightarrow X_1 \dashrightarrow \ldots \dashrightarrow X_k$$

be the process of $K_X$-minimal model program. According to Corollary 7.8 and Remark 7.6, $\tilde{X}_i \dashrightarrow \tilde{X}_{i+1}$ can be factored into a composition of smooth blow-downs and P-desingularizations of flops, where $\tilde{X}_i \to X_i$ is a P-minimal resolution of $X_i$. If every P-desingularizations of flops that appears in the factorization is a smooth flop, then the sequence induces a semi-orthogonal decomposition of $D^b(X)$ since smooth flops are derived equivalent [28].

**Example 8.7.** Let $X_1 \dashrightarrow X_2$ be the flip which is a quotient of an Atiyah flop by an $\mathbb{Z}/2\mathbb{Z}$-action [16, Example 2.7]. Then $X_2$ is smooth and $X_1$ has a $\frac{1}{2}(1,1,1)$ singular point. Let $X \to X_1$ be the smooth resolution obtained by blowing-up the singular point. Then $X \to X_1 \dashrightarrow X_2$ is a sequence of MMP.

The factorization of the flip is exactly diagram (3) in Lemma 6.7, namely the diagram

$$\begin{array}{ccc} X & \dashrightarrow & X' \\ \downarrow & & \downarrow \\ X_1 & \dashrightarrow & X_2 \end{array}$$

where $X \dashrightarrow X'$ is a smooth flop and $X' \to X_2$ is a blow-down of a smooth curve. We know that there exists an equivalence of category $\Phi : D^b(X') \to D^b(X)$ and a semi-orthogonal decomposition $D^b(X') = \langle D_{-1}, D^b(X_2)\rangle$. Hence, $D^b(X) = \langle \Phi(D_{-1}), \Phi(D^b(X_2))\rangle$ is a semi-orthogonal decomposition.

In general, a P-desingularization of a flop $\tilde{X}_i \dashrightarrow \tilde{X}_{i+1}$ may not be derived equivalent since $\tilde{X}_i$ and $\tilde{X}_{i+1}$ may not be isomorphic in codimension one. Nevertheless, due to the symmetry between $\tilde{X}_i$ and $\tilde{X}_{i+1}$, one might expect that a semi-orthogonal decomposition on $D^b(\tilde{X}_i)$ will result in a semi-orthogonal decomposition on $D^b(\tilde{X}_{i+1})$. It still hopeful that our approach will be effective for all smooth threefolds.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

## Acknowledgments

The author thanks Jungkai Alfred Chen for his helpful comments. The author thanks the referees for carefully reading the paper and for their useful comments. The author is supported by KIAS individual Grant MG088901.

## Conflict of interest

The authors declare there is no conflicts of interest.

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