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Theory article

Well-posedness and blow-up results for a time-space fractional diffusion-wave equation

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Abstract: In this paper, we demonstrate the local well-posedness and blow up of solutions for a class of time- and space-fractional diffusion wave equation in a fractional power space associated with the Laplace operator. First, we give the definition of the solution operator which is a noteworthy extension of the solution operator of the corresponding time-fractional diffusion wave equation. We have analyzed the properties of the solution operator in the fractional power space and Lebesgue space. Next, based on some estimates of the solution operator and source term, we prove the well-posedness of mild solutions by using the contraction mapping principle. We have also investigated the blow up of solutions by using the test function method. The last result describes the properties of mild solutions when $\alpha \to 1^-$. The main feature of the proof is the reasonable use of continuous embedding between fractional space and Lebesgue space.

Keywords: fractional diffusion wave equation; local well-posedness; blow up

1. Introduction

To date, qualitative theory of fractional differential equations and their applications in computer science [1], physics [2–4], engineering [5, 6], economics, biology and ecology have been extensively discussed and demonstrated [7]. In recent years, research on fractional diffusion equations has attracted the attention of many scholars [8–10]. Moustafa et al. [11] used a potent spectral approach to solve time-fractional diffusion equations. Youssri et al. [12] addressed the time-fractional heat conduction equation in one spatial dimension. Schneider and Wyss [13] pointed out that the following fractional diffusion equation can be used to model some diffusion phenomena in special types of porous media and describe various subdiffusive phenomena:

$$u_t(t,x) = \partial_t(g_\alpha * \Delta u)(t,x) + r(t,x), \tag{1.1}$$

where $\alpha \in (0,1)$, $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and $g_\alpha * \Delta u$ represents convolution, which is defined follows:

$$(f * g)(t) = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau,$$

where $f, g \in L^1(\Omega)$, $0 < t < T$. Subsequently, the well-posedness, blow up and long-time behavior of solutions of time-fractional diffusion equations have been extensively studied in the literature. For example, in [14], the authors considered a class of quasilinear abstract time fractional evolution equations in continuous interpolation spaces. Zacher [15] obtained the results of L_p maximal regularity for abstract parabolic Volterra equations. Wang and Sun [16] investigated the local discontinuous Galerkin finite-element method for the fractional Allen-Cahn equation with the Caputo-Hadamard derivative in the time domain. The global existence and blow up of solutions to time-fractional diffusion equations were also considered [17–19]. We note that the fractional diffusion equations mentioned above mainly have the following form:

$$ {}_0^C D_t^\alpha u = \Delta u + f(t,u), \tag{1.2}$$

where ${}_0^C D_t^\alpha u$ denotes the Caputo fractional derivative of order α.

However, in [20], by using the test function method, Fino and Kirane obtained results for the blow up and global existence of solutions for the following time- and space- fractional equation:

$$\begin{cases} u_t + (-\Delta)^{\frac{\beta}{2}} u = \int_0^t (t-s)^{-\gamma} |u|^{p-1} u ds, \ x \in \mathbb{R}^N, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbb{R}^N, \end{cases}$$

where $0 < \beta \leq 2$.

De Andrade et al. [21–24] discussed a series of results regarding the following equation:

$$u_t(t,x) = \partial_t \int_0^t g_\alpha(s) \Delta u(t-s,x) ds + |u(t,x)|^{\rho-1} u(t,x), \ x \in \mathbb{R}^N, \ t > 0, \tag{1.3}$$

which is a little different from (1.2). In fact, in [22], they studied the global well-posedness and spatiotemporal asymptotic behavior of mild solutions for the following Cauchy problem for fractional reaction-diffusion equations:

$$\begin{cases} u_t(t,x) = \partial_t \int_0^t g_\alpha(s) \Delta u(t-s,x) ds + |u(t,x)|^{\rho-1} u(t,x), \ x \in \mathbb{R}^N, \ t > 0, \\ u(x,0) = u_0(x), \ x \in \mathbb{R}^N. \end{cases}$$

For a nonnegative and nonzero $u_0 \in L^q(\mathbb{R}^N)$, if $\rho > 1 + \frac{2}{\alpha N}$, then there exists $q > 1$ such that the equation has a positive global solution. In [21], under the conditions that $u_0 \in L^q(\Omega)$, $q \geq 1$, $q > \frac{\alpha N}{2}(\rho - 1)$ and $\rho\alpha > 1$, they analyzed the local well-posedness in $L^q(\Omega)$ for the following fractional diffusion equation:

$$\begin{cases} u_t(t,x) = \partial_t \int_0^t g_\alpha(s) \Delta u(t-s,x) ds + |u(t,x)|^{\rho-1} u(t,x), \ x \in \Omega, \ 0 < t < T, \\ u(t,x) = 0, \ x \in \partial\Omega, \ 0 < t < T, \\ u(x,0) = u_0(x), \ x \in \Omega. \end{cases}$$

where $\alpha \in (0,1)$, $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ and Ω is a sufficiently smooth bounded domain in $\mathbb{R}^N$. They also proved the existence of a positive solution and gave sufficient conditions for the blow-up behavior of the solutions.

Inspired by the above results, in this paper, we focus on the following fractional diffusion wave equation:

$$\begin{cases} u_{tt}(t,x) = -\partial_t \int_0^t g_\alpha(s)(-\Delta)^\sigma u(t-s,x)ds + |u(t,x)|^p, \ x \in \Omega, \ 0 < t < T, \\ u(t,x) = 0, \ x \in \mathbb{R}^N \backslash \Omega, \ 0 < t < T, \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \ x \in \Omega, \end{cases} \tag{1.4}$$

where $\alpha \in (0,1)$, $g_\alpha(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$, $p > 1$, $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $u_0, u_1 \in X_q^{\sigma+\beta}$. $(-\Delta)^\sigma$ is the fractional Laplace operator of order σ $(0 < \sigma < 1)$, which may be defined follows:

$$(-\Delta)^\sigma v(x,t) = \mathcal{F}^{-1}\left(|\xi|^{2\sigma}\mathcal{F}(v)(\xi)\right)(x,t),$$

where $\mathcal{F}$ denotes the Fourier transform and $\mathcal{F}^{-1}$ represents the inverse Fourier transform.

Recently, there have been some results obtained in the literature on time-fractional diffusion wave equations. For instance, Kian and Yamamoto [25] investigated a weak solution for the semilinear case of (1.2) in the bounded domain for dimensions of $n = 2, 3$. Alvarez et al. [26] considered the well-posedness for an abstract Cauchy problem in a Hilbert space. Otarola and Salgado [27] studied the time and space regularities of weak solutions for the space- and time-fractional wave equations. Wang et al. [28] considered the existence of local and global solutions to a time-fractional diffusion wave equation with exponential growth. In [29], the authors proved the self-similarity, symmetries, and asymptotic behavior in Morrey spaces for fractional wave equations. Zhang and Li [18] considered the following for the nonlinear time-fractional diffusion wave equation in $\mathbb{R}^N$:

$$\begin{cases} {}_0^C D_t^\alpha u - \Delta u = |u|^p, \ x \in \mathbb{R}^N, \ t > 0, \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \ x \in \mathbb{R}^N. \end{cases} \tag{1.5}$$

where $1 < \alpha < 2$, $p > 1$, $u_0, u_1 \in L^q(\mathbb{R}^N)$ $(q > 1)$. They determined the critical exponents when $u_1 \not\equiv 0$ and $u_1 \equiv 0$, respectively.

In this paper, we first give the definition of the solution operator which is a noteworthy extension of the solution operator of the corresponding time-fractional diffusion wave equation. We analyze the properties of the solution operator in the fractional powers space and Lebesgue space. Next, based on some estimates of the solution operator and source term, we prove the well-posedness of mild solutions by using the contraction mapping principle. We also study the blow up of solutions by test using the function method. Naturally, we want to know the properties of mild solutions when $\alpha \to 1^-$; thus, in the last part of this paper, we show that the mild solutions of (1.4) approach the mild solutions of space-fractional diffusion equations.

The main difference between (1.4) and (1.3) is the definition of the solution operator. The solution operator of (1.3) was defined by a probability density function and the heat semigroup in Ω under the Dirichlet boundary condition; however, this representation is invalid for the solution operator of (1.4), and we need to estimate the solution operator by using complex integral representations. On the other hand, by performing the basic calculations for the fractional derivative and integral, we can transform (1.5) be as follows:

$$u_{tt}(t,x) = \partial_t \int_0^t g_{\alpha-1}(s)\Delta u(t-s,x)ds + \partial_t \int_0^t g_{\alpha-1}(s)|u(t-s,x)|^p ds,\ \alpha \in (1,2). \tag{1.6}$$

Equation (1.6) shows that both the diffusion process and reaction process in the model are affected by the same memory effect, which is too specific in the physical process. In fact, if we take $\sigma = 1$ in (1.4), this model can be regarded as a modified version of (1.5) (or (1.6)), and the main feature of this model is that we only consider the memory effect on the diffusion term; also, the reaction terms cannot be treated the same as in (1.5). Furthermore, choosing the fractional power space as the workspace can improve the regularity of mild solutions.

This paper is arranged as follows. Section 2 gives some basic notations. In Section 3, we prove the properties of the solution operator in the fractional power space X_q^β. The local well-posedness of the problem (1.4) is analyzed in Section 4. In Section 5, we study the blow-up problem. The behavior of the solutions when α approaches 1 is considered in Section 6.

2. Preliminaries

We first recall the definition of the Riemann-Liouville fractional operators in [6]: for $\gamma \in (0,1)$, we have

$$I_{a^+}^\gamma \varphi(t) = \int_a^t \frac{(t-s)^{\gamma-1}}{\Gamma(\gamma)}\varphi(s)ds,\ \varphi \in L^1(a,b),$$

$$I_{b^-}^\gamma \varphi(t) = \int_t^b \frac{(s-t)^{\gamma-1}}{\Gamma(\gamma)}\varphi(s)ds,\ \varphi \in L^1(a,b),$$

$$D_{a^+}^\gamma \varphi(t) = \frac{d}{dt} I_{a^+}^{1-\gamma}\varphi(t),\ \varphi \in AC[a,b],$$

$$D_{b^-}^\gamma \varphi(t) = -\frac{d}{dt} I_{b^-}^{1-\gamma}\varphi(t),\ \varphi \in AC[a,b].$$

where $AC[a,b]$ denotes the space of absolutely continuous functions defined on $[a,b]$. Moreover, we have the following formula for fractional integration by parts:

$$\int_a^b \varphi(t)(I_{a^+}^\alpha \psi)(t)dt = \int_a^b \psi(t)(I_{b^-}^\alpha \varphi)(t)dt,$$

provided that

$$\varphi(t) \in L^p(a,b),\ \psi(t) \in L^q(a,b),\ \frac{1}{p} + \frac{1}{q} < 1 + \alpha,\ p \geq 1,\ q \geq 1.$$

In order to discuss the well-posedness of mild solutions for problem (1.4), we need to integrate (1.4) with respect to t twice.

$$u_t(t,x) = u_1(x) - \int_0^t g_\alpha(s)(-\Delta)^\sigma u(t-s,x)ds + \int_0^t f(u(s))ds,\ t \geq 0, \tag{2.1}$$

where $f(u) = |u|^p$. Then,

$$u(t,x) = u_0(x) + u_1(x)t - \int_0^t [g_\alpha * (-\Delta)^\sigma u](s)ds + \int_0^t (1 * f(u))(s)ds,\ t \geq 0. \tag{2.2}$$

Taking the Laplace transform about t in (2.2), we have

$$\widehat{u}(\lambda) = \lambda^\alpha(\lambda^{\alpha+1} - (-\Delta)^\sigma)^{-1}u_0(x) + \lambda^{\alpha-1}(\lambda^{\alpha+1} - (-\Delta)^\sigma)^{-1}u_1(x) + \lambda^{\alpha-1}(\lambda^{\alpha+1} - (-\Delta)^\sigma)^{-1}\widehat{f}(\lambda).$$

When $t \geq 0$, because $(-\Delta)^\sigma$ is a sectorial operator in $L^q(\mathbb{R}^N)$ $(1 < q < \infty)$, that is, there exist positive constants C and $\phi \in (\frac{\pi}{2}, \pi)$ such that $\Sigma_\phi := \{\lambda \in \mathbb{C}\backslash\{0\} : \left|\arg(\lambda)\right| < \phi\}$ and

$$\left\|(\lambda - (-\Delta)^\sigma)^{-1}\right\| \leq \frac{C}{|\lambda|},\ \forall \lambda \in \Sigma_\phi.$$

The operator $S_\alpha(t)$ can be defined according to the Cauchy integral properties follows:

$$S_\alpha(t) := \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{H_a} e^{\lambda t}\lambda^\alpha(\lambda^{\alpha+1} - (-\Delta)^\sigma)^{-1}d\lambda,\ t > 0, \\ I, t = 0, \end{cases} \tag{2.3}$$

where $H_a = H_a(t^{-1}, \theta) = \{se^{i\theta} : t^{-1} \leq s < \infty\} \cup \{t^{-1}e^{is} : |s| \leq \theta\} \cup \{se^{-i\theta} : t^{-1} \leq s < \infty\}$ denotes the Hankel path for all $\theta \in (\frac{\pi}{2}, \phi]$, $\phi \in (\frac{\pi}{2}, \pi)$, and I is the identity operator.

According to Theorem 3.2 in [30], for $\lambda > 0$, we have

$$\lambda^{\alpha-1}(\lambda^{\alpha+1} - (-\Delta)^\sigma)^{-1}u_1(x) = \lambda^{-1}\lambda^\alpha(\lambda^{\alpha+1} - (-\Delta)^\sigma)^{-1}u_1(x) = \int_0^\infty e^{-\lambda t}(1 * S_\alpha)(t)dtu_1(x),$$

and

$$\lambda^{\alpha-1}(\lambda^{\alpha+1} - (-\Delta)^\sigma)^{-1}\widehat{f}(\lambda) = \lambda^{-1}\lambda^\alpha(\lambda^{\alpha+1} - (-\Delta)^\sigma)^{-1}\widehat{f}(\lambda) = \int_0^\infty e^{-\lambda t}(1 * S_\alpha * f)(t)dt.$$

Finally, the mild solution of the problem (1.4) can be written as

$$u(t,x) = S_\alpha(t)u_0(x) + (1 * S_\alpha)(t)u_1(x) + (1 * S_\alpha * f)(t), \tag{2.4}$$

where $f(u) = |u|^p$.

Let Ω be a bounded and smooth domain in $\mathbb{R}^N$. Apply $X_q^\beta = D((-\Delta)^\beta)$, $\beta \geqslant 0$, with the norm $\|u\|_{X_q^\beta} = \left\|(-\Delta)^\beta u\right\|_{L^q(\Omega)}$. It follows from [31, 32] that we have the following continuous embeddings.

$$\begin{cases} X_q^\beta \hookrightarrow L^r(\Omega),\ r \leq \dfrac{Nq}{N - 2q\beta},\ 0 < \beta < \dfrac{N}{2q}, \\ X_q^0 = L^q(\Omega), \\ X_q^\beta \hookleftarrow L^r(\Omega),\ r \geq \dfrac{N}{N - 2q\beta},\ -\dfrac{N}{2q'} < \beta \leq 0, \end{cases} \tag{2.5}$$

where $q' = \frac{q}{q-1}$.

Lemma 2.1. *If* $0 < \beta < \frac{N}{2q}$, $q > 1$ *and* $1 < p < \frac{N}{N-2q\beta}$, *there exists a constant* $c > 0$ *such that*

$$\|f(u) - f(v)\|_{L^q(\Omega)} \leq c(\|u\|^{p-1}_{X^\beta_q} + \|v\|^{p-1}_{X^\beta_q})\|u - v\|_{X^\beta_q}, \tag{2.6}$$

and

$$\|f(u)\|_{L^q(\Omega)} \leq c\,\|u\|^p_{X^\beta_q}, \tag{2.7}$$

for all $u, v \in X^\beta_q$.

Proof. For $0 < \beta < \frac{N}{2q}$, $q > 1$ and $1 < p \leq \frac{N}{N-2q\beta}$, it follows that $X^\beta_q \hookrightarrow L^{pq}$; then, $f : X^\beta_q \to L^q$ is well defined. Thus

$$\begin{aligned}\|f(u) - f(v)\|_{L^q(\Omega)} &\leq \||u|^p - |v|^p\|_{L^q(\Omega)} \\ &\leq (\|u\|^{p-1}_{L^q(\Omega)} + \|v\|^{p-1}_{L^q(\Omega)})\|u - v\|_{L^q(\Omega)} \\ &\leq c(\|u\|^{p-1}_{X^\beta_q} + \|v\|^{p-1}_{X^\beta_q})\|u - v\|_{X^\beta_q},\end{aligned}$$

and

$$\|f(u)\|_{L^q(\Omega)} \leq \|u\|^p_{L^q(\Omega)} \leq c\,\|u\|^p_{X^\beta_q}.$$

3. Properties of solution operator $S_\alpha(t)$

Lemma 3.1. *Suppose that* $\alpha \in (0, 1)$, $q > 1$ *and* $1 < p \leq \frac{N}{N-2q\beta}$. *Given* $0 < \beta < \min\{\frac{\sigma}{\alpha+1}, \frac{N}{2q}\}$, *there exists a constant* $M \geq 0$ *such that, for any* $\varphi \in L^q(\Omega)$,

$$\|S_\alpha(t)\varphi\|_{X^\beta_q} \leq Mt^{-\frac{\beta}{\sigma}(\alpha+1)}\|\varphi\|_{L^q(\Omega)}, \tag{3.1}$$

$$\|(1 * S_\alpha)(t)\varphi\|_{X^\beta_q} \leq Mt^{1-\frac{\beta}{\sigma}(\alpha+1)}\|\varphi\|_{L^q(\Omega)}, \tag{3.2}$$

for all $t > 0$.

Proof. Denote $H_a = H_{a_1} \cup H_{a_2} \cup H_{a_3}$, where

$$\begin{aligned}H_{a_1} &= \{se^{i\theta} : t^{-1} \leq s < \infty\}, \\ H_{a_2} &= \{t^{-1}e^{is} : |s| \leq \theta\}, \\ H_{a_3} &= \{se^{-i\theta} : t^{-1} \leq s < \infty\},\end{aligned}$$

for any $t > 0$. Consider that $\theta \in (\frac{\pi}{2}, \phi]$ and $\phi \in (\frac{\pi}{2}, \pi)$ and let $\sum_\phi$ be the sector associated with the sectorial operator $(-\Delta)^\sigma$. Then, we can estimate the integral for all $\varphi \in L^q(\Omega)$:

$$\begin{aligned}\|S_\alpha(t)\varphi\|_{X^\beta_q} &= \left\|\frac{1}{2\pi i}\int_{H_a} e^{\lambda t}\lambda^\alpha(-\Delta)^\beta\left(\lambda^{\alpha+1} - (-\Delta)^\sigma\right)^{-1}\varphi d\lambda\right\|_{L^q(\Omega)} \\ &\leq \frac{C}{2\pi}\int_{H_a}\left|e^{\lambda t}\right|\left|\lambda^{\frac{\beta}{\sigma}(\alpha+1)-1}\right||d\lambda|\,\|\varphi\|_{L^q(\Omega)}.\end{aligned}$$

For H_{a_1}, we have

$$\left\|\frac{C}{2\pi}\int_{H_{a_1}} e^{\lambda t}\lambda^{\frac{\beta}{\sigma}(\alpha+1)-1}\varphi d\lambda\right\|_{L^q(\Omega)} \leq \frac{C}{2\pi}\int_{\frac{1}{t}}^{\infty}\left|e^{tse^{i\theta}}\right|\left|se^{i\theta}\right|^{\frac{\beta}{\sigma}(\alpha+1)-1}\left|e^{i\theta}\right| ds\|\varphi\|_{L^q(\Omega)}$$

$$\leq \frac{C}{2\pi}t^{1-\frac{\beta}{\sigma}(\alpha+1)}\int_{\frac{1}{t}}^{\infty} e^{ts\cos\theta}ds\|\varphi\|_{L^q(\Omega)}$$

$$= -\frac{C}{2\pi}t^{-\frac{\beta}{\sigma}(\alpha+1)}\frac{e^{\cos\theta}}{\cos\theta}\|\varphi\|_{L^q(\Omega)}.$$

For H_{a_2}, we have

$$\left\|\frac{C}{2\pi}\int_{H_{a_2}} e^{\lambda t}\lambda^{\frac{\beta}{\sigma}(\alpha+1)-1}\varphi d\lambda\right\|_{L^q(\Omega)} \leq \frac{C}{2\pi}\int_{-\theta}^{\theta}\left|e^{e^{is}}\right|\left|t^{-1}e^{is}\right|^{\frac{\beta}{\sigma}(\alpha+1)-1}t^{-1}ds\|\varphi\|_{L^q(\Omega)}$$

$$\leq \frac{C}{2\pi}\int_{-\theta}^{\theta} e^{|\cos s|}t^{-\frac{\beta}{\sigma}(\alpha+1)}ds\|\varphi\|_{L^q(\Omega)}$$

$$\leq \frac{C}{2\pi}t^{-\frac{\beta}{\sigma}(\alpha+1)}\int_{-\theta}^{\theta} eds\|\varphi\|_{L^q(\Omega)}$$

$$\leq \frac{Ce\theta}{\pi}t^{-\frac{\beta}{\sigma}(\alpha+1)}\|\varphi\|_{L^q(\Omega)}.$$

For H_{a_3} we proceed in the same way as for H_{a_1}. From the above estimate for each part of H_a we have

$$M \geq \frac{C}{\pi}\left(-\frac{e^{\cos\theta}}{\cos\theta} + e\theta\right) > 0,$$

such that

$$\|S_\alpha(t)\varphi\|_{X_q^\beta} \leq Mt^{-\frac{\beta}{\sigma}(\alpha+1)}\|\varphi\|_{L^q(\Omega)}.$$

Using the same approach as that used in the proof above we can prove the estimate (3.2). For all $\varphi \in L^q(\Omega)$

$$\|(1 * S_\alpha)(t)\varphi\|_{X_q^\beta} = \left\|\frac{1}{2\pi i}\int_{H_a} e^{\lambda t}\lambda^{\alpha-1}(-\Delta)^\beta(\lambda^{\alpha+1} - (-\Delta)^\sigma)^{-1}\varphi d\lambda\right\|_{L^q(\Omega)}$$

$$\leq \frac{C}{2\pi}\int_{H_a}\left|e^{\lambda t}\right|\left|\lambda^{\frac{\beta}{\sigma}(\alpha+1)-2}\right||d\lambda|\,\|\varphi\|_{L^q(\Omega)}.$$

Next, we estimate each part of H_a. Ultimately, we can obtain

$$\|(1 * S_\alpha)(t)\varphi\|_{X_q^\beta} \leq Mt^{1-\frac{\beta}{\sigma}(\alpha+1)}\|\varphi\|_{L^q(\Omega)}.$$

Remark 3.1. *(1) When $\beta = 0$, Theorem 3.1 implies that*

$$\|S_\alpha(t)\varphi\|_{L^q(\Omega)} \leq M\|\varphi\|_{L^q(\Omega)},\ t \geq 0,$$

$$\|(1 * S_\alpha)(t)\varphi\|_{L^q(\Omega)} \leq Mt\|\varphi\|_{L^q(\Omega)},\ t > 0.$$

(2) When $\alpha = 1$ and $\sigma = 1$, Lemma 3.1 becomes the estimation of the solution operator for space-fractional wave equations.

Lemma 3.2. *Let* $0 < \alpha < 1$, $0 < \sigma + \beta < \frac{N}{2q}$, $0 \le t_0 < t < \infty$, *for* $\psi \in X_q^{\sigma+\beta}$,

$$\|S_\alpha(t)\psi - S_\alpha(t_0)\psi\|_{X_q^\beta} \to 0.$$

Proof. For $\psi \in X_q^{\sigma+\beta}$,

$$\begin{aligned}
\|S_\alpha(t)\psi - \psi\|_{X_q^\beta} &= \frac{1}{2\pi i}\int_{H_a} e^{\lambda t}(-\Delta)^\beta \lambda^\alpha (\lambda^{\alpha+1} - (-\Delta)^\sigma)^{-1}\psi d\lambda - (-\Delta)^\beta\psi \\
&= \frac{1}{2\pi i}\int_{H_a} e^{\lambda t}\lambda^\alpha(-\Delta)^\beta(\lambda^{\alpha+1} - (-\Delta)^\sigma)^{-1}\psi - \lambda^{-1}(-\Delta)^\beta\psi d\lambda \\
&= \frac{1}{2\pi i}\int_{H_a} e^{\lambda t}(\lambda^\alpha(\lambda^{\alpha+1} - (-\Delta)^\sigma)^{-1} - \lambda^{-1})d\lambda(-\Delta)^\beta\psi \\
&= \frac{1}{2\pi i}\int_{H_a} e^{\lambda t}\lambda^{-1}(-\Delta)^\sigma(\lambda^{\alpha+1} - (-\Delta)^\sigma)^{-1}d\lambda(-\Delta)^\beta\psi \\
&= \frac{1}{2\pi i}\int_{H_a} e^{\lambda t}\lambda^{-1}(\lambda^{\alpha+1} - (-\Delta)^{\sigma-1})^{-1}d\lambda(-\Delta)^{\sigma+\beta}\psi,
\end{aligned}$$

Let $\lambda t = \mu$; then,

$$\begin{aligned}
\|S_\alpha(t)\psi - \psi\|_{X_q^\beta} &\le \frac{1}{2\pi}\int_{H_a} |e^\mu|\,|\mu|^{-1}[(\frac{\mu}{t})^{\alpha+1} - (-\Delta)^\sigma]^{-1}|d\mu|\,\|\psi\|_{X_q^{\sigma+\beta}} \\
&\le C\int_{H_a} \frac{|e^\mu|}{|\mu|^{\alpha+2}}|d\mu|\,\|\psi\|_{X_q^{\sigma+\beta}} t^{\alpha+1},
\end{aligned}$$

when $t \to 0^+$ and $\|S_\alpha(t)\psi - \psi\|_{X_q^\beta} \to 0$. For $t > 0$ and $\psi \in X_q^{\sigma+\beta}$, we have

$$S_\alpha(t)\psi - S_\alpha(t_0)\psi = \frac{1}{2\pi i}\int_{H_a} (e^{\lambda t} - e^{\lambda t_0})\lambda^\alpha(\lambda^{\alpha+1} - (-\Delta)^\sigma)^{-1}\psi d\lambda,$$

By the dominated convergence theorem, it follows that

$$\|S_\alpha(t)\psi - S_\alpha(t_0)\psi\|_{X_q^\beta} \to 0, \text{ as } t \to t_0.$$

Therefore,

$$\|S_\alpha(t)\psi - S_\alpha(t_0)\psi\|_{X_q^\beta} \to 0.$$

4. The well-posedness of mild solutions

Definition 4.1. *Let* $T > 0$, $\alpha \in (0,1)$, $0 < \sigma < 1$, $q > 1$ *and* $u_0, u_1 \in X_q^{\sigma+\beta}$. *We say that u is a mild solution of problem* (1.4) *if* $u \in C([0,T]; X_q^\beta)$ *and*

$$u(t) = S_\alpha(t)u_0 + (1 * S_\alpha)(t)u_1 + (1 * S_\alpha * f)(t).$$

Theorem 4.1. *Suppose that* $u_0, u_1 \in X_q^{\sigma+\beta}$, $\alpha \in (0,1)$, $0 < \sigma < 1$, $0 < \sigma + \beta < \frac{N}{2q}$, $q > 1$, $1 < p \leq \frac{N}{N-2q\beta}$ *and* $0 < \beta < \min\{\frac{\sigma}{p(\alpha+1)}, \frac{N}{2q}\}$*. Then there exists* $T > 0$ *such that* (1.4) *has a unique mild solution* $u \in C([0,T]; X_q^{\beta})$*. Furthermore,*

$$t^{\frac{\beta}{\sigma}(\alpha+1)}\|u(t)\|_{X_q^\beta} \to 0, \text{ as } t \to 0^+.$$

Moreover, the solution of problem (1.4) *depends continuously on the initial data.*

Proof. We will use the contraction principle. Let $\alpha \in (0,1)$, $0 < \sigma < 1$, $p > 1$ and $0 < \beta < \min\{\frac{\sigma}{p(\alpha+1)}, \frac{N}{2q}\}$. Take $T > 0$ and $R > 0$ such that

$$0 < T^{2-\frac{p\beta}{\sigma}(\alpha+1)} < \frac{R^{1-p}}{3McB(2-\frac{\beta}{\sigma}(\alpha+1), 1-\frac{\beta}{\sigma}(\alpha+1))},$$

and

$$M\|u_0\|_{L^q(\Omega)} \leq \frac{R}{3},\ MT\|u_1\|_{L^q(\Omega)} \leq \frac{R}{3}.$$

Let $E := C([0,T]; X_q^\beta)$ and $\|u\|_E = \sup\limits_{0<t<T} t^{\frac{\beta}{\sigma}(\alpha+1)}\|u(t)\|_{X_q^\beta}$; then, E is a Banach space. Denote

$$\mathcal{B} = \{u \in E : \|u\|_E \leq R\},$$

and $F : E \to E$ by

$$Fu(t) = S_\alpha(t)u_0 + (1 * S_\alpha)(t)u_1 + (1 * S_\alpha * f)(t). \tag{4.1}$$

It is easy to see that $F : E \to E$ is well defined. Let $0 < t_2 < t_1 < T$; then, we have

$$\begin{aligned}\|Fu(t_1) - Fu(t_2)\|_{X_q^\beta} \leq& \|S_\alpha(t_1)u_0 - S_\alpha(t_2)u_0\|_{X_q^\beta} + \|(1 * S_\alpha(t_1))u_1 - (1 * S_\alpha(t_2))u_1\|_{X_q^\beta}\\ &+ \|(1 * S_\alpha * f)(t_1) - (1 * S_\alpha * f)(t_2)\|_{X_q^\beta}\\ =& I_1 + I_2 + I_3.\end{aligned}$$

Given Lemma 3.2, we have

$$I_1 = \|S_\alpha(t_1)u_0 - S_\alpha(t_2)u_0\|_{X_q^\beta} \to 0,\ \text{as } t_2 \to t_1.$$

From Lemma 3.1, we have

$$I_2 \leq \int_{t_2}^{t_1} \|S_\alpha(t)u_1\|_{X_q^\beta} dt \leq \int_{t_2}^{t_1} Mt^{-\frac{\beta}{\sigma}(\alpha+1)} dt \|u_1\|_{L^q(\Omega)} \to 0,\ \text{as } t_2 \to t_1.$$

Using the mean value theorem, we deduce that

$$\begin{aligned}
I_3 \leq& \int_0^{t_2} (t_1 - t_2)\left\|S_\alpha(t)|_{t=t_2-s+\theta(t_1-t_2)} f(u(s))\right\|_{X_q^\beta} ds \\
&+ \int_{t_2}^{t_1} \|(1 * S_\alpha)(t_2 - s) f(u(s))\|_{X_q^\beta} ds \\
\leq& M \int_0^{t_2} (t_1 - t_2)(t_2 - s + \theta(t_1 - t_2))^{-\frac{\beta}{\sigma}(\alpha+1)} \|f(u(s))\|_{L^q(\Omega)} ds \\
&+ M \int_{t_2}^{t_1} (t_2 - s)^{1-\frac{\beta}{\sigma}(\alpha+1)} \|f(u(s))\|_{L^q(\Omega)} ds \\
\leq& Mc \int_0^{t_2} (t_1 - t_2)(t_2 - s + \theta(t_1 - t_2))^{-\frac{\beta}{\sigma}(\alpha+1)} \|u(s)\|^p_{X_q^\beta} ds \\
&+ Mc \int_{t_2}^{t_1} (t_2 - s)^{1-\frac{\beta}{\sigma}(\alpha+1)} \|u(s)\|^p_{X_q^\beta} ds \\
\leq& McR^p (t_1 - t_2) \int_0^{t_2} (t_2 - s + \theta(t_1 - t_2))^{-\frac{\beta}{\sigma}(\alpha+1)} s^{-\frac{p\beta}{\sigma}(\alpha+1)} ds \\
&+ McR^p \int_{t_2}^{t_1} (t_2 - s)^{1-\frac{\beta}{\sigma}(\alpha+1)} s^{-\frac{p\beta}{\sigma}(\alpha+1)} ds \\
\to& 0,
\end{aligned}$$

as $t_2 \to t_1$, where $\theta \in [0, 1]$ is a constant.

For $t_2 = 0$, we have

$$\begin{aligned}
\|Fu(t_1) - Fu(0)\|_{X_q^\beta} \leq& \|S_\alpha(t_1)u_0 - u_0\|_{X_q^\beta} + \|(1 * S_\alpha(t_1))u_1\|_{X_q^\beta} \\
&+ \int_0^{t_1} \|(1 * S_\alpha)(t_1 - s) f(u(s))\|_{X_q^\beta} ds,
\end{aligned}$$

where

$$\begin{aligned}
&\int_0^{t_1} \|(1 * S_\alpha)(t_1 - s) f(u(s))\|_{X_q^\beta} ds \\
&\leq M \int_0^{t_1} (t_1 - s)^{1-\frac{\beta}{\sigma}(\alpha+1)} \|u(s)\|^p_{X_q^\beta} ds \\
&\leq McR^p \int_0^{t_1} (t_1 - s)^{1-\frac{\beta}{\sigma}(\alpha+1)} s^{-\frac{p\beta}{\sigma}(\alpha+1)} ds \\
&= McR^p t_1^{2-\frac{\beta}{\sigma}(\alpha+1)-\frac{p\beta}{\sigma}(\alpha+1)} B\left(2 - \frac{\beta}{\sigma}(\alpha + 1), 1 - \frac{p\beta}{\sigma}(\alpha + 1)\right),
\end{aligned}$$

and $B(a, b) = \int_0^1 (1 - s)^{a-1} s^{b-1} ds$ is the beta function. Thus, $\|Fu(t_1) - Fu(0)\|_{X_q^\beta} \to 0$ as $t_1 \to 0$.

Now, we shall show that $Fu \in \mathcal{B}$ for $u \in \mathcal{B}$ when $t \in [0, T]$. Considering Lemmas 3.1 and 2.1, we obtain

$$\begin{aligned}
&t^{\frac{\beta}{\sigma}(\alpha+1)}\|Fu(t)\|_{X_q^\beta}\\
&\le t^{\frac{\beta}{\sigma}(\alpha+1)}\|S_\alpha(t)u_0\|_{X_q^\beta}+t^{\frac{\beta}{\sigma}(\alpha+1)}\|(1*S_\alpha)(t)u_1\|_{X_q^\beta}+t^{\frac{\beta}{\sigma}(\alpha+1)}\|(1*S_\alpha*f)(t)\|_{X_q^\beta}\\
&\le M\|u_0\|_{L^q(\Omega)}+MT\|u_1\|_{L^q(\Omega)}+Mt^{\frac{\beta}{\sigma}(\alpha+1)}\int_0^t(t-s)^{1-\frac{\beta}{\sigma}(\alpha+1)}\|f(u(s))\|_{L^q(\Omega)}ds\\
&\le \frac{2R}{3}+Mct^{\frac{\beta}{\sigma}(\alpha+1)}\int_0^t(t-s)^{1-\frac{\beta}{\sigma}(\alpha+1)}\|u(s)\|_{X_q^\beta}^p ds\\
&\le \frac{2R}{3}+McR^pT^{2-\frac{p\beta}{\sigma}(\alpha+1)}B(2-\frac{\beta}{\sigma}(\alpha+1),1-\frac{p\beta}{\sigma}(\alpha+1))\\
&\le R.
\end{aligned}$$

If $u, v \in \mathcal{B}$, by using Lemmas 3.1 and 2.1, we have

$$\begin{aligned}
&t^{\frac{\beta}{\sigma}(\alpha+1)}\|Fu(t)-Fv(t)\|_{X_q^\beta}\\
&\le t^{\frac{\beta}{\sigma}(\alpha+1)}\int_0^t\|(1*S_\alpha)(t-s)[f(u(s))-f(v(s))\|_{X_q^\beta}ds\\
&\le Mt^{\frac{\beta}{\sigma}(\alpha+1)}\int_0^t(t-s)^{1-\frac{\beta}{\sigma}(\alpha+1)}\|f(u(s))-f(v(s))\|_{L^q(\Omega)}ds\\
&\le Mct^{\frac{\beta}{\sigma}(\alpha+1)}\int_0^t(t-s)^{1-\frac{\beta}{\sigma}(\alpha+1)}\|u(s)-v(s)\|_{X_q^\beta}(\|u(s)\|_{X_q^\beta}^{p-1}+\|v(s)\|_{X_q^\beta}^{p-1})ds\\
&\le 2McR^{p-1}t^{\frac{\beta}{\sigma}(\alpha+1)}\int_0^t(t-s)^{1-\frac{\beta}{\sigma}(\alpha+1)}s^{-\frac{p\beta}{\sigma}(\alpha+1)}ds\|u-v\|_E\\
&\le 2McR^{p-1}T^{2-\frac{p\beta}{\sigma}(\alpha+1)}B(2-\frac{\beta}{\sigma}(\alpha+1),1-\frac{p\beta}{\sigma}(\alpha+1))\|u-v\|_E\\
&< \frac{1}{3}\|u-v\|_E.
\end{aligned}$$

By using the contraction principle, we obtain a unique fixed point u in $\mathcal{B}$.

On the other hand,

$$\begin{aligned}
t^{\frac{\beta}{\sigma}(\alpha+1)}\|u(t)\|_{X_q^\beta}\le&t^{\frac{\beta}{\sigma}(\alpha+1)}\|S_\alpha(t)u_0\|_{X_q^\beta}+t^{\frac{\beta}{\sigma}(\alpha+1)}\|(1*S_\alpha)(t)u_1\|_{X_q^\beta}\\
&+Mc\|u\|_E^p t^{\frac{\beta}{\sigma}(\alpha+1)}\int_0^t(t-s)^{1-\frac{\beta}{\sigma}(\alpha+1)}s^{-\frac{p\beta}{\sigma}(\alpha+1)}ds\\
<&t^{\frac{\beta}{\sigma}(\alpha+1)}\|S_\alpha(t)u_0\|_{X_q^\beta}+t^{\frac{\beta}{\sigma}(\alpha+1)}\|(1*S_\alpha)(t)u_1\|_{X_q^\beta}\\
&+Mc\|u\|_E^p B(2-\frac{\beta}{\sigma}(\alpha+1),1-\frac{p\beta}{\sigma}(\alpha+1))t^{2-\frac{\beta}{\sigma}(\alpha+1)}\\
\to&0,
\end{aligned}$$

as $t \to 0^+$.

Next, we prove that u is unique in E. Let $v \in E$ be another solution of the problem (1.4); then, take

$0 < \tilde{T} \le T$ such that $\sup_{t\in(0,T)} t^{\frac{\beta}{\sigma}(\alpha+1)}\|v(t)\|_{X_q^\beta} \le \tilde{R}$. For $t \in [0,T]$, we have

$$\begin{aligned}
\|u-v\|_E &= t^{\frac{\beta}{\sigma}(\alpha+1)}\|u(t)-v(t)\|_{X_q^\beta} \\
&\le Mc(R^{p-1}+\tilde{R}^{p-1})t^{\frac{\beta}{\sigma}(\alpha+1)}\int_0^t (t-s)^{1-\frac{\beta}{\sigma}(\alpha+1)}s^{-\frac{p\beta}{\sigma}(\alpha+1)}(s^{\frac{\beta}{\sigma}(\alpha+1)}\|u(s)-v(s)\|_{X_q^\beta})ds \\
&\le Mc(R^{p-1}+\tilde{R}^{p-1})t^{\frac{\beta}{\sigma}(\alpha+1)}(\tilde{T})^{-\frac{p\beta}{\sigma}(\alpha+1)}\int_{\tilde{T}}^t (t-s)^{1-\frac{\beta}{\sigma}(\alpha+1)}(s^{\frac{\beta}{\sigma}(\alpha+1)}\|u(s)-v(s)\|_{X_q^\beta})ds.
\end{aligned}$$

Then,

$$\xi(t) \le Mc(R^{p-1}+\tilde{R}^{p-1})t^{\frac{\beta}{\sigma}(\alpha+1)}\tilde{T}^{-\frac{p\beta}{\sigma}(\alpha+1)}\int_0^t (t-s)^{1-\frac{\beta}{\sigma}(\alpha+1)}\xi(s)ds,$$

where $\xi : [0,T] \to [0,\infty)$ and $\xi(t) = t^{\frac{\beta}{\sigma}(\alpha+1)}\|u(t)-v(t)\|_{X_q^\beta}$. According to Lemma 7.1.1 in [32], we can derive the uniqueness of the mild solution by using the singular Gronwall inequality.

Finally we shall prove the continuous dependence. If u and v are solutions of the problem (1.4) starting from u_0, u_1 and v_0, v_1, respectively, and if they belong to E, we have

$$\begin{aligned}
t^{\frac{\beta}{\sigma}(\alpha+1)}\|u(t)-v(t)\|_{X_q^\beta} \le& M\|u_0-v_0\|_{L^q(\Omega)} + Mt\|u_1-v_1\|_{L^q(\Omega)} \\
&+ Mt^{\frac{\beta}{\sigma}(\alpha+1)}\int_0^t (t-s)^{1-\frac{\beta}{\sigma}(\alpha+1)}\||u(s)|^p-|v(s)|^p\|_{L^q(\Omega)}ds \\
\le& M\|u_0-v_0\|_{L^q(\Omega)} + Mt\|u_1-v_1\|_{L^q(\Omega)} \\
&+ 2McR^{p-1}T^{2-\frac{p\beta}{\sigma}(\alpha+1)}B(2-\frac{\beta}{\sigma}(\alpha+1), 1-\frac{p\beta}{\sigma}(\alpha+1))\|u-v\|_E \\
\le& M\|u_0-v_0\|_{L^q(\Omega)} + Mt\|u_1-v_1\|_{L^q(\Omega)} + \frac{1}{3}\|u-v\|_E.
\end{aligned}$$

Thus, it follows that

$$\|u-v\|_E \le \frac{3}{2}M\|u_0-v_0\|_{L^q(\Omega)} + \frac{3}{2}MT\|u_1-v_1\|_{L^q(\Omega)},$$

which implies continuous dependencies on initial values.

Definition 4.2. *Let $u \in C([0,T]; X_q^\beta)$ be a mild solution of the problem (1.4); if $\bar{u} : [0,T] \to X_q^\beta$ is a mild solution of the problem (1.4) for $T' > T$ and $\bar{u} = u$ when $t \in [0,T]$, then we say that $\bar{u}$ is a continuation of u on $[0,T']$.*

Theorem 4.2. *Under the assumptions of Theorem 4.1, we let u be a mild solution of (1.4). Then, there exists a maximal time T_{max} such that u has a unique continuation on $[0,T_{max}]$.*

Proof. Let $u \in E$ be the solution given by Theorem 4.1. Let K be the set of all $v \in C((0,T']; X_q^\beta)$ such that $u \equiv v$ on $t \in [0,T]$ and

$$\sup_{T\le t\le T'} \|v(t)-u(T)\|_{X_q^\beta} < R'.$$

Define the operator Λ on K by

$$\Lambda v(t) = S_\alpha(t)u_0 + (1*S_\alpha)(t)u_1 + \int_0^t (1*S_\alpha)(t-s)f(v(s))ds.$$

Similar to the proof of Theorem 4.1, we can get the continuity of $\Lambda v(t) : (0, T'] \to X_q^\beta$ for given $v \in K$. And it is easy to see that $\Lambda v(t) = u(t)$ for all $t \in [0, T]$.

Now, for all $T < t < T'$, we have

$$\begin{aligned}\|\Lambda v(t) - u(T)\|_{X_q^\beta} \leq& \|[S_\alpha(t) - S_\alpha(T)]u_0\|_{X_q^\beta} + \left\|\int_0^T [S_\alpha(t-s) - S_\alpha(T-s)]u_1 ds\right\|_{X_q^\beta} \\ &+ \left\|\int_0^T [(1 * S_\alpha)(t-s) - (1 * S_\alpha)(T-s)]f(u(s))ds\right\|_{X_q^\beta} \\ &+ \left\|\int_T^t S_\alpha(t-s)u_1 ds\right\|_{X_q^\beta} + \left\|\int_T^t (1 * S_\alpha)(t-s)f(v(s))ds\right\|_{X_q^\beta}.\end{aligned}$$

By Lebesgue's dominated convergence theorem, the above first three terms can be proved to approach zero as $t \to T^+$. Moreover

$$\|\int_T^t S_\alpha(t-s)u_1 ds\|_{X_q^\beta} \leq M(t-T)^{-\frac{\beta}{\sigma}(\alpha+1)}\|u_1\|_{L^q(\Omega)},$$

and

$$\begin{aligned}&\left\|\int_T^t (1 * S_\alpha)(t-s)f(v(s))ds\right\|_{X_q^\beta} \\ &\leq Mc\int_T^t (t-s)^{1-\frac{\beta}{\sigma}(\alpha+1)} \|f(v(s))\|_{L^q(\Omega)} ds \\ &\leq Mc(R' + \|u(T)\|_{X_q^\beta})^p t^{2-\frac{\beta}{\sigma}(\alpha+1)-\frac{p\beta}{\sigma}(\alpha+1)p} \int_{\frac{T}{t}}^1 (1-s)^{1-\frac{\beta}{\sigma}(\alpha+1)} s^{-\frac{p\beta}{\sigma}(\alpha+1)} ds.\end{aligned}$$

So, the last two terms go to zero as $t \to T^+$. Therefore, if we choose T small enough, for $t \in [T, T']$

$$\|[S_\alpha(t) - S_\alpha(T)]u_0\|_{X_q^\beta} \leq \frac{R'}{5},\ M(t-T)\|u_1\|_{X_q^\beta} \leq \frac{R'}{5},$$

$$\left\|\int_0^T [S_\alpha(t-s) - S_\alpha(T-s)]u_1 ds\right\|_{X_q^\beta} \leq \frac{R'}{5},$$

$$\left\|\int_0^T [(1 * S_\alpha)(t-s) - (1 * S_\alpha)(T-s)]f(u(s))ds\right\|_{X_q^\beta} \leq \frac{R'}{5},$$

and

$$Mc(R' + \|u(T)\|_{X_q^\beta})^p t^{2-\frac{\beta}{\sigma}(\alpha+1)-\frac{p\beta}{\sigma}(\alpha+1)p} \int_{\frac{T}{t}}^1 (1-s)^{1-\frac{\beta}{\sigma}(\alpha+1)} s^{-\frac{p\beta}{\sigma}(\alpha+1)} ds \leq \frac{R'}{5},$$

then

$$\sup_{T \leq t \leq T'} \|\Lambda v(t) - u(T)\|_{X_q^\beta} \leq R'.$$

The proof of Λ is a contraction and the uniqueness is similar to that described by Theorem 4.1.

Theorem 4.3. *If the assumptions of Theorem 4.1 hold and we let u be the mild solution of the problem* (1.4) *with a maximal time of existence* $T_{\max} < \infty$ *then*

$$\lim_{t \to T_{\max}^-} \sup \|u(t)\|_{X_q^\beta} = \infty.$$

Proof. If $T_{max} < \infty$ and there exists a constant $\mu > 0$ such that $\|u(t)\|_{X_q^\beta} < \mu$ for all $t \in [0, T_{\max})$, we can choose a sequence $\{t_n\}_{n\in N} \subset [0, T_{\max})$ such that $t_n \to T_{\max}^-$ as $n \to \infty$. We will show that $\{u(t_n)\}_{n\in N}$ is a Cauchy sequence in X_q^β. Indeed, for $0 < t_m < t_n < T_{\max}$, we have

$$\begin{aligned} u(t_n) - u(t_m) = &[S_\alpha(t_n) - S_\alpha(t_m)]u_0 \\ &+ \int_0^{t_m} [S_\alpha(t_n - s) - S_\alpha(t_m - s)]u_1 ds \\ &+ \int_0^{t_m} [(1 * S_\alpha)(t_n - s) - (1 * S_\alpha)(t_m - s)]f(u(s))ds \\ &+ \int_{t_m}^{t_n} S_\alpha(t_n - s)u_1 ds + \int_{t_m}^{t_n} (1 * S_\alpha)(t_n - s)f(u(s))ds, \end{aligned}$$

The estimation is similar to that in Theorem 4.2. And the last term is more direct, that is,

$$\|\int_{t_m}^{t_n} (1 * S_\alpha)(t_n - s)f(u(s))ds\|_{X_q^\beta} \le Mc\mu^p \int_{t_m}^{t_n} (t_n - s)^{1-\frac{\beta}{\sigma}(\alpha+1)}ds.$$

Then, for $m, n \to \infty$, we have

$$\|u(t_n) - u(t_m)\|_{X_q^\beta} \to 0.$$

Hence, there is a $u(T_{max}) \in X_q^\beta$ such that

$$\lim_{n\to\infty} u(t_n) = u(T_{\max}).$$

Therefore, by Theorem 4.2 we can extend the solution to some larger interval and this contradicts the maximality of $T_{\max}$.

Remark 4.1. *When the nonlinear term $f = |u(t, x)|^{p-1}u(t, x)$, the above theorem also holds.*

5. The blow-up result

Definition 5.1. *Let u be a solution of the problem (1.4) for $T < \infty$and$1 < p < \infty$. If $\lim_{t\to T} \|u\|_{L^p} = \infty$, we say that u blows up in finite time.*

Theorem 5.1. *Under the assumptions of Theorem 4.1, let $\alpha \in (0, 1)$ and $1 < p \le \frac{Nq}{N-2q\beta}$ be such that $p\alpha < 1$. Suppose that the solution u given by Theorem 4.1 is a classical solution starting at u_0, u_1. Then $T_{\max} < \infty$ and $\|u\|_{L^p} \to \infty$.*

Proof. From the continuous embedding $X_q^\beta \hookrightarrow L^p(\Omega)$ for $p \le \frac{Nq}{N-2q\beta}$, we know that the solution u obtained in Section 4 is in $C([0, T_{\max}); L^p(\Omega))$ $(p \le \frac{Nq}{N-2q\beta})$. To prove that $T_{\max} < \infty$ and $\|u\|_{L^p} \to \infty$, we assume that $T_{\max} = \infty$, $\phi(x) = (\phi(T^{-\frac{\alpha}{2}}x))^{\frac{2p\sigma}{p-1}}$,

$$\phi(x) = \begin{cases} 1, |x| \le 1, \\ 0, |x| \ge 2, \end{cases}$$

and $0 \le |\phi(x)| \le 1$. Then,

$$\int_\Omega u\phi dx = \int_\Omega u_0\phi dx + \int_\Omega u_1 t\phi dx - \int_\Omega \int_0^t (g_\alpha * (-\Delta)^\sigma u)(s)\phi ds dx + \int_\Omega \int_0^t (1 * f(u))(s)\phi ds dx.$$

We have

$$\frac{d^2}{dt^2}\left(\int_\Omega u\phi dx\right) = \int_\Omega |u|^p \phi dx - D_{0^+}^{1-\alpha} \int_\Omega u(s)(-\Delta)^\sigma \phi dx, \tag{5.1}$$

where $D_{0^+}^{1-\alpha}$ denotes the Riemann-Liouville fractional derivative.

For a given $T > 0$ and $l > \frac{p+1}{p-1}$, define

$$\psi(t) = \begin{cases} (1 - \frac{t}{T})^l, 0 \le t \le T, \\ 0, t > T. \end{cases}$$

It is easy to see that ψ is a nonnegative function and $\psi(T) = 0$. We can use fractional integration by parts, as in [6], to obtain

$$\begin{aligned} \int_0^T D_{0^+}^{1-\alpha} \int_\Omega u\phi dx \psi dt &= \int_0^T \left(I_{0^+}^\alpha \int_\Omega u\phi\right)' \psi dt \\ &= I_{0^+}^\alpha\left(\int_\Omega u\phi\right)(T)\psi(T) - I_{0^+}^\alpha\left(\int_\Omega u\phi\right)(0)\psi(0) - \int_0^T I_{0^+}^\alpha\left(\int_\Omega u\phi\right)\psi' dt \\ &= -\int_0^T \int_\Omega u\phi dx I_{T^-}^\alpha \psi' dt. \end{aligned} \tag{5.2}$$

Let us multiply both sides of (5.1) by ψ and integrate it over $(0, T)$; then, by applying (5.2) and $|(-\Delta)^\sigma \phi(x)| \le T^{-\alpha} \phi^{\frac{1}{p}}$, we have

$$\begin{aligned} \int_0^T \int_\Omega |u|^p \phi dx \psi dt \le& \int_0^T \int_\Omega |u\phi\psi''| dx dt + \int_0^T \int_\Omega \left|u(-\Delta)^\sigma \phi(-I_{T^-}^\alpha \psi')\right| dx dt \\ \le& \int_0^T \int_\Omega u\phi^{\frac{1}{p}} \phi^{1-\frac{1}{p}} \psi^{\frac{1}{p}} \psi^{-\frac{1}{p}} \psi'' dx dt \\ &+ T^{-\alpha} \int_0^T \int_\Omega u\phi^{\frac{1}{p}} \psi^{\frac{1}{p}} \psi^{-\frac{1}{p}} (-I_{T^-}^\alpha \psi') dx dt \\ \le& \frac{1}{2} \int_0^T \int_\Omega |u|^p \phi dx \psi dt + c_1(\varepsilon) \int_0^T \int_\Omega \left(\phi^{1-\frac{1}{p}} \psi^{-\frac{1}{p}} \psi''\right)^{\frac{p}{p-1}} dx dt \\ &+ c_2(\varepsilon) T^{\frac{-\alpha p}{p-1}} \int_0^T \int_\Omega \left(\psi^{-\frac{1}{p}}(-I_{T^-}^\alpha \psi')\right)^{\frac{p}{p-1}} dx dt. \end{aligned}$$

Then

$$(\psi^{-\frac{1}{p}} \psi'')^{\frac{p}{p-1}} = l^{\frac{p}{p-1}} (l-1)^{\frac{p}{p-1}} (T-t)^{(-\frac{l}{p}+l-2)\frac{p}{p-1}} T^{(\frac{l}{p}-l)\frac{p}{p-1}}.$$

Since $l > \frac{p+1}{p-1}$, we obtain the following integrable equation:

$$\int_0^T \left(\psi^{-\frac{1}{p}} \psi''\right)^{\frac{p}{p-1}} dt = \frac{l^{\frac{p}{p-1}} (l-1)^{\frac{p}{p-1}} (p-1)}{(-l + pl - p - 1)} T^{-\frac{2p}{p-1}+1}. \tag{5.3}$$

Then,

$$\begin{aligned} -I_{T^-}^\alpha \psi' &= -\int_t^T \frac{(s-t)^{\alpha-1}}{\Gamma(\alpha)} \psi'(s) ds \\ &= \frac{lT^{-l}}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} (T-s)^{l-1} ds \\ &= \frac{lT^{-l}}{\Gamma(\alpha)} B(\alpha, l)(T-t)^{\alpha+l-1}. \end{aligned}$$

So,

$$(\psi^{-\frac{1}{p}}(-I_{T^-}^{\alpha}\psi'))^{\frac{p}{p-1}} = (\frac{lB(\alpha,l)}{\Gamma(\alpha)})^{\frac{p}{p-1}}(T-t)^{(\alpha+l-\frac{l}{p}-1)\frac{p}{p-1}}T^{(-l+\frac{l}{p})\frac{p}{p-1}},$$

which is integrable since $l > \frac{p+1}{p-1}$. Thus,

$$T^{\frac{-\alpha p}{p-1}}\int_0^T (\psi^{-\frac{1}{p}}(-I_{T^-}^{\alpha}\psi'))^{\frac{p}{p-1}}dt = \frac{p-1}{p\alpha+pl-l-1}(\frac{lB(\alpha,l)}{\Gamma(\alpha)})^{\frac{p}{p-1}}T^{-\frac{1}{p-1}}. \tag{5.4}$$

By assumption, since $p > 1$ and (5.3) and (5.4) approach zero as $T \to \infty$, we can get a contradiction and see that $\|u\|_{L^p} \to \infty$.

6. The behavior of the solutions when $\alpha \to 1^-$

To prove the continuity of the solution as $\alpha \to 1^-$, we denote by u_α the solution of problem (1.4) and by u the solution of the following problem:

$$\begin{cases} u_{tt}(t,x) + (-\Delta)^{\sigma}u(t-s,x) = |u(t,x)|^p, \ x \in \Omega, \ 0 < t < T, \\ u(t,x) = 0, \ x \in \mathbb{R}^N \backslash \Omega, \ 0 < t < T, \\ u(0,x) = u_0(x), \ u_t(0,x) = u_1(x), \ x \in \Omega. \end{cases} \tag{6.1}$$

By applying similar operations for the problem (1.4) to the problem (6.1), we have

$$u(t,x) = S(t)u_0(x) + (1*S)(t)u_1(x) + (1*S*f)(t), \tag{6.2}$$

where

$$S(t) = \frac{1}{2\pi i}\int_{Ha} e^{\lambda t}\lambda(\lambda^2 - (-\Delta)^{\sigma})^{-1}d\lambda,$$

and

$$(1*S)(t) = \frac{1}{2\pi i}\int_{Ha} e^{\lambda t}(\lambda^2 - (-\Delta)^{\sigma})^{-1}d\lambda.$$

Lemma 6.1. *Let $\alpha \in (0,1)$, $\varphi \in X_q^\beta$, and $t \geq 0$. Then,*

$$\|S_\alpha(t)\varphi - S(t)\varphi\|_{X_q^\beta} \to 0, as\ \alpha \to 1^-.$$

And, the convergence is uniform for t on bounded subintervals and φ in bounded subsets of X_q^β.

Proof. For all $\alpha \in (0,1)$ and $\varphi \in X_q^\beta$, we have

$$S_\alpha(t)\varphi - S(t)\varphi = \frac{1}{2\pi i}\int_{H_a} e^{\lambda t}[\lambda^\alpha(\lambda^{\alpha+1} - (-\Delta)^{\sigma})^{-1}\varphi - \lambda(\lambda^2 - (-\Delta)^{\sigma})^{-1}\varphi]d\lambda,$$

Since $(-\Delta)^\sigma$ is a sectorial operator in $L^q(\Omega)$, we have

$$\begin{aligned} &\left\|\lambda^\alpha(\lambda^{\alpha+1} - (-\Delta)^{\sigma})^{-1}\varphi - \lambda(\lambda^2 - (-\Delta)^{\sigma})^{-1}\varphi\right\|_{X_q^\beta} \\ \leq &\left\|\lambda^\alpha(\lambda^{\alpha+1} - (-\Delta)^{\sigma})^{-1}\varphi\right\|_{X_q^\beta} + \left\|\lambda(\lambda^2 - (-\Delta)^{\sigma})^{-1}\varphi\right\|_{X_q^\beta} \\ \leq &\frac{2c}{|\lambda|}\|\varphi\|_{X_q^\beta}. \end{aligned}$$

For each $\lambda \in \rho(A)$,

$$\left|e^{\lambda t}\right|\left\|\lambda^{\alpha}(\lambda^{\alpha+1}-(-\Delta)^{\sigma})^{-1}\varphi-\lambda(\lambda^{2}-(-\Delta)^{\sigma})^{-1}\varphi\right\|_{X_q^{\beta}}\to 0,$$

as $\alpha \to 1^-$. By applying Lebesgue's dominated convergence theorem, we can conclude the proof.

Theorem 6.1. *Let* $\alpha \in (0,1)$. *Suppose that* u_α *and* u *are the mild solutions of the problems* (1.4) *and* (6.1)*, respectively. Then,*

$$\|u_\alpha(t)-u(t)\|_{X_q^{\beta}}\to 0,\ \text{as } \alpha\to 1^-, \tag{6.3}$$

where $t \in [0,\tilde{T}]$ *and* $\tilde{T} > 0$ *is any existence time for* u_α.

Proof. For $t \in [0,\tilde{T}]$, we estimate taht

$$\begin{aligned}\|u_\alpha(t)-u(t)\|_{X_q^{\beta}}\le&\|[S_\alpha(t)-S(t)]u_0\|_{X_q^{\beta}}+\|[(1*S_\alpha)(t)-(1*S)(t)]u_1\|_{X_q^{\beta}}\\&+\int_0^t\|[(1*S_\alpha)(t)-(1*S)(t)]|u_\alpha|^p\|_{X_q^{\beta}}ds\\&+\int_0^t\|(1*S)(t)[|u_\alpha|^p-|u|^p]\|_{X_q^{\beta}}ds.\end{aligned}$$

According to Lemma 6.1, we can see that

$$\begin{aligned}&\|[S_\alpha(t)-S(t)]u_0\|_{X_q^{\beta}}\to 0,\\&\|[(1*S_\alpha)(t)-(1*S)(t)]u_1\|_{X_q^{\beta}}\to 0,\\&\int_0^t\|[(1*S_\alpha)(t)-(1*S)(t)]|u_\alpha|^p\|_{X_q^{\beta}}ds\to 0,\end{aligned}\tag{6.4}$$

as $\alpha \to 1^-$. Define

$$\begin{aligned}a(t)=&\|[S_\alpha(t)-S(t)]u_0\|_{X_q^{\beta}}+\|[(1*S_\alpha)(t)-(1*S)(t)]u_1\|_{X_q^{\beta}}\\&+\int_0^t\|[(1*S_\alpha)(t)-(1*S)(t)]|u_\alpha|^p\|_{X_q^{\beta}}ds.\end{aligned}$$

Therefore, it is easy to see that $a(t) \in C[0,\tilde{T}]$. For the last part of the estimate, it follows that

$$\begin{aligned}&\int_0^t\|(1*S)(t)(|u_\alpha|^p-|u|^p)\|_{X_q^{\beta}}ds\\&\le\Big(\sup_{t\in[0,\tilde{T}]}\|u_\alpha(t)\|_{X_q^{\beta}}^{p-1}+\sup_{t\in[0,\tilde{T}]}\|u(t)\|_{X_q^{\beta}}^{p-1}\Big)\int_0^t(t-s)^{1-\frac{\beta}{\sigma}(\alpha+1)}\|u_\alpha(s)-u(s)\|_{X_q^{\beta}}ds,\end{aligned}$$

when α is sufficiently close to 1. Thus, the map $s \mapsto s^{1-\frac{\beta}{\sigma}(\alpha+1)}$ is decreasing and belongs to $L^1((0,\widetilde{T});R^+)$. Therefore, the generalized Gronwall inequality (Lemma 3.3 in [31]) implies that

$$\|u_\alpha(t)-u(t)\|_{X_q^{\beta}}\le 2\max_{t\in[0,\tilde{T}]}a(t)\exp(\varepsilon t),\ \text{for } 0\le t\le\tilde{T},$$

where $a(t) \to 0$ as $\alpha \to 1^-$ given (6.4). Thus, we obtain (6.3).

7. Conclusions

In this work, we first derived the definition of the solution operator through the use of complex integral representations; this definition is a noteworthy extension of the solution operator of the corresponding time-fractional diffusion wave equation. We have analyzed the properties of the solution operator in the fractional power space and Lebesgue space. Next, by applying some estimates of the solution operator and source term, we have proved the well-posedness of mild solutions by using the contraction mapping principle. We have also studied the blow up of solutions by using the test function method. Naturally, we want to know the properties of mild solutions when $\alpha \to 1^-$; thus, in the last part of this paper, we demonstrate that the mild solutions of (1.4) approach the mild solutions of space-fractional diffusion equations. Furthermore, choosing the fractional power space as the workspace can improve the regularity of mild solutions. The model proposed here is different from those considered in other studies, because both the diffusion process and reaction process in other models are subject to the same memory effect, which is too specific in physical masses. The main feature of this model is that we only consider the memory effect on the diffusion term, and the reaction terms cannot be treated as if they are subject to the same memory effect. In addition, regarding this spatiotemporal fractional diffusion equation, we can also consider the properties of the solution to problem (1.4) in the entire, as well as and the well-posedness of the solution in Besov space, when the initial values are u_0 and u_1 as in the current work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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