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Research article

Stability of thermoelastic Timoshenko system with variable delay in the internal feedback

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Abstract: Based on the Fourier law of heat conduction, this paper was concerned with the thermoelastic Timoshenko system with memory and variable delay in the internal feedback, which describes the transverse vibration of a beam. By the Lummer-Phillips theorem and the variable norm technique suitable for the nonautonomous operator, the stability of the coupled system has been derived in space $\mathscr{H}$.

Keywords: Timoshenko system; exponential stability; variable delay; memory

1. Introduction

In 1921, S. P. Timoshenko studied the transverse vibration of a beam, and found that the motion could be described by a family of partial differential equations in which the bending moment and shear stress were involved. Based on the constitutive laws in the mathematical elasticity theory, the following system, called the Timoshenko system [1], was derived

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k(\varphi_x + \psi)_x(x,t) = 0, \text{ in } (0,L)\times\mathbb{R}^+, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + k(\varphi_x+\psi)(x,t) = 0, \text{ in } (0,L)\times\mathbb{R}^+, \end{cases} \tag{1.1}$$

where $L > 0$ is the length of beam, φ is the vertical displacement of beam, ψ denotes the rotation angle of the filament of beam, and ρ_1, ρ_2, k, b are positive constants.

So far, many significative results on the dissipative Timoshenko system have been derived, involving the linear and nonlinear cases, dynamical behavior, stability, and so on. Regarding the exponential stability of the dissipative Timoshenko system, the following equal speed condition is usually important:

$$\frac{k}{\rho_1} = \frac{b}{\rho_2}. \tag{1.2}$$

To achieve the desired dissipation, damping terms are needed necessarily to system (1.1), and we can refer to [2–4] and references therein, in which the frictional damping, indefinite damping, and weak damping were involved. In general, the influence of thermal damping was usually considered in research on Timoshenko system [5–12]. The coupled system, called the thermoelastic Timoshenko system, is always combined with other damping terms, in which the heat conduction is ordinarily under some law ($\tau q_t + q + \beta\theta_x = 0$), e.g., the Fourier law ($\tau = 0$) and the Cattaneo law ($\tau \neq 0$).

In 2009, H. D. Fernández Sare and R. Racke studied the thermoelastic Timoshenko system with memory under the Fourier law and obtained the exponential stability, which was not true for the case of Cattaneo law [13]. The stability of thermoelastic system was also obtained in [14], in which the thermal damping was in the shear moment, and it is different from [13]. Many good results could be referred in [15–18]. Furthermore, the influence of delay appears in economics, physics, and other fields, the delay term could be constant delay, continuous delay, or the distributed delay, which is important to the stability of system; We can refer to [19–21] for the results on fluid system with delay, and see [22–24] for the Timoshenko system with delay.

In this paper, under the Fourier law we consider the thermoelastic Timoshenko system subject to memory and variable time delay on $(0, L) \times \mathbb{R}^+$

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k(\varphi_x + \psi)_x(x,t) + a\varphi_t(x, t - \rho(t)) + d\varphi_t(x,t) + \sigma\theta_x = 0, \\ \rho_2\psi_{tt}(x,t) - \tilde{b}\psi_{xx}(x,t) + \int_0^\infty g(s)\psi_{xx}(x, t-s)ds + k(\varphi_x + \psi)(x,t) - \sigma\theta = 0, \\ \rho_3\theta_t(x,t) + q_x(x,t) + \sigma(\varphi_x + \psi)_t(x,t) = 0, \\ q + \beta\theta_x = 0, \end{cases} \tag{1.3}$$

where θ is the temperature, q is the heat flux vector, $a, d, \sigma, \rho_3, \tilde{b}, \beta$ are positive constants satisfying $d > \sqrt{2}a$, $(x,t) \in (0, L) \times \mathbb{R}^+$, the delay function $\rho(\cdot) : \mathbb{R}^+ \to [M_0, h]$ ($h > M_0 > 0$) is C^1 continuous and satisfies that

$$0 < \widetilde{M_0} < \rho'(t) \leq 1 - \frac{2a^2}{d^2} = \widetilde{M_1} < 1,$$

and the positive exponentially decaying kernel function g satisfies for some constant $k_1 > 0$ that

$$b = \tilde{b} - \int_0^\infty g(s)ds = \tilde{b} - C_0 > 0, \; g'(s) \leq -k_1 g(s), \; s > 0. \tag{1.4}$$

The primary characteristics of this article are different from the previous results, and details are as follows:

1) Different from [13], the thermal damping $\sigma\theta_x$ is in the shear moment term of this article, which is similar with the case of [14]. Also, the continuous delay terms $a\varphi_t(x, t - \rho(t))$ and $d\varphi_t(x,t)$ are considered in the shear moment term, which bring us some difficulties in deriving the dissipation and stability.

2) Since the operator $A(t)$ generated in (1.3) is nonautonomous, the autonomous operator method in [25] is invalid. So, the variable norm technique in [26] is used, which is suitable for the case of the nonautonomous operator.

The rest of this article is arranged as follows. In Section 2 we give the equivalent problem of (1.3), and present a useful lemma concerning the perturbation theory and main results in Section 3. The well-posedness of (1.3) is derived by the Lummer-Phillips theorem in Section 4. Four sufficient conditions in Kato's perturbation theory are verified in Section 5, and the exponential stability of (1.3) is proved finally.

2. Equivalent problem

At first, we introduce a new variable for the delay feedback term

$$z(x,\kappa,t) = \varphi_t(x, t-\kappa\rho(t)), \quad (\kappa,t) \in (0,1)\times\mathbb{R}^+, \tag{2.1}$$

and we have

$$\rho(t)z_t(x,\kappa,t) + (1-\kappa\rho'(t))z_\kappa(x,\kappa,t) = 0 \quad \text{in} \quad (0,L)\times(0,1)\times\mathbb{R}^+. \tag{2.2}$$

For the memory term, we present

$$\eta(x,t,s) = \psi(x,t) - \psi(x,t-s),\ t,s \geq 0. \tag{2.3}$$

Using the above transformation, we convert the system (1.3) to the following equivalent form:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k(\varphi_x+\psi)_x(x,t) + az(x,1,t) + d\varphi_t(x,t) + \sigma\theta_x = 0, \\ \rho_2\psi_{tt}(x,t) - b\psi_{xx}(x,t) + k(\varphi_x+\psi)(x,t) - \int_0^\infty g(s)\eta_{xx}(s)ds - \sigma\theta = 0, \\ \rho_3\theta_t(x,t) - \beta\theta_{xx}(x,t) + \sigma(\varphi_x+\psi)_t(x,t) = 0, \\ \eta_t + \eta_s - \psi_t = 0, \\ \rho(t)z_t(x,\kappa,t) + (1-\kappa\rho'(t))z_\kappa(x,\kappa,t) = 0, \end{cases} \tag{2.4}$$

where $(x,\kappa,t) \in (0,L)\times(0,1)\times\mathbb{R}^+$. Also, the new equivalent system (2.4) is equipped with the initial conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x),\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), \\ \theta(x,0) = \theta_0(x),\ z(x,\kappa,0) = \varphi_3(x,-\kappa\rho(0)),\ \eta(x,0,s) = \eta_0(x,s), \eta(x,t,0) = 0, \end{cases} \tag{2.5}$$

and the boundary conditions

$$\begin{cases} \varphi_x(0,t) = \varphi_x(L,t) = \psi(0,t) = \psi(L,t) = \theta(0,t) = \theta(L,t) = 0,\ z(x,0,t) = \varphi_t(x,t), \\ \eta(0,t,s) = \eta(L,t,s) = 0. \end{cases} \tag{2.6}$$

To solve the problems (2.4)–(2.6), we write

$$L^2_\star(0,L) = \{w \in L^2(0,L) | \int_0^L w(x)dx = 0\},\ H^1_\star(0,L) = H^1_0(0,L)\cap L^2_\star(0,L),$$

$$L^2_g(\mathbb{R}^+, H^1_0(0,L)) = \{u | \sqrt{g}u \in L^2(\mathbb{R}^+, H^1_0(0,L))\}.$$

For the delay space $L^2((0, L) \times (0, 1))$, we define the following inner product and norm

$$(z_1, z_2)_\xi = \xi \int_0^1 \int_0^L z_1(\kappa) z_2(\kappa) dx d\kappa, \ \|z\|_\xi = \xi \int_0^1 \|z(\kappa)\|_{L^2}^2 d\kappa,$$

where $\xi = \xi(t)$ satisfies

$$d\rho(t) - \frac{\rho(t)}{1 - \rho'(t)} \sqrt{(1 - \rho'(t))d^2 - 2a^2} \le \xi \le d\rho(t) + \frac{\rho(t)}{1 - \rho'(t)} \sqrt{(1 - \rho'(t))d^2 - 2a^2}.$$

We consider the phase (Hilbert) space

$$\begin{aligned}\mathscr{H} =& H_\star^1(0, L) \times L_\star^2(0, L) \times H_0^1(0, L) \times L^2(0, L) \times L^2(0, L) \times L_g^2(\mathbb{R}^+, H_0^1(0, L)) \\ & \times L^2((0, L) \times (0, 1))\end{aligned} \tag{2.7}$$

endowed with the norm

$$\begin{aligned}\|U\|_{\mathscr{H}} =& \rho_1 \|\Phi\|_{L^2}^2 + \rho_2 \|\Psi\|_{L^2}^2 + k\|\varphi_x + \psi\|_{L^2}^2 + b\|\psi_x\|_{L^2}^2 + \rho_3 \|\theta\|_{L^2}^2 \\ & + \int_0^\infty g(s) \|\eta_x(s)\|_{L^2}^2 ds + \xi \int_0^1 \|z(\kappa)\|_{L^2}^2 d\kappa\end{aligned}$$

and corresponding inner product $(\cdot, \cdot)_{\mathscr{H}}$ for all $U = (\varphi, \Phi, \psi, \Psi, \theta, \eta, z)^T \in \mathscr{H}$, where $\Phi = \varphi_t$ and $\Psi = \psi_t$. Basing on the equivalent equations, for any $t \in \mathbb{R}^+$, we define the operator $A(t)$ as the following:

$$A(t)U = \begin{pmatrix} \Phi \\ \frac{k}{\rho_1}(\varphi_x + \psi)_x - \frac{a}{\rho_1} z(1) - \frac{d}{\rho_1}\Phi - \frac{\sigma}{\rho_1}\theta_x \\ \Psi \\ \frac{1}{\rho_2}(b\psi + \int_0^\infty g(s)\eta(s)ds)_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi) + \frac{\sigma}{\rho_2}\theta \\ \frac{\beta}{\rho_3}\theta_{xx} - \frac{\sigma}{\rho_3}(\Phi_x + \Psi) \\ \Psi - \eta_s \\ -\frac{1 - \kappa\rho'(t)}{\rho(t)} z_\kappa \end{pmatrix} \tag{2.8}$$

with domain

$$\begin{aligned}D(A(t)) = \{&U \in \mathscr{H} \mid \varphi \in H^2(0, L),\ \varphi_x \in H_0^1(0, L),\ \Phi \in H_\star^1(0, L),\ \Psi \in H_0^1(0, L),\ b\psi + \\ &\int_0^\infty g(s)\eta(s)ds \in H^2(0, L),\ \theta \in H_0^1(0, L) \cap H^2(0, L),\ \eta(\cdot, 0) = 0,\ z_\kappa \in L^2((0, L) \times (0, 1))\}.\end{aligned}$$

From the definition of $D(A(t))$ we know

$$D(A(t)) = D(A(0)),\ t \in \mathbb{R}^+,$$

and the new system (2.4) can be written as the following abstract form

$$\begin{cases} \frac{dU}{dt} = A(t)U, \quad t > 0, \\ U(0) = U_0 = (\varphi_0, \Phi_0, \psi_0, \Psi_0, \theta_0, \eta_0, z_0)^T. \end{cases} \tag{2.9}$$

3. A useful lemma and main results

To study the nonlinear system (2.9) with nonautonomous operator, we introduce here the perturbation theory (see [26]) suitable for the case of the nonautonomous operator.

Lemma 3.1. *Suppose that for any $t \in \mathbb{R}^+$, there holds*
(P-I) The domain $D(A(t))$ is dense in the phase space $\mathscr{H}$, and $D(A(t)) = D(A(0))$.
(P-II) In space $\mathscr{H}$, the operator $A(t)$ generates a C_0 semigroup $\{S_{A(t)}(s)\}_{s\geq 0}$ for any $t \in [0, T]$, where $T > 0$ is some fixed constant.
(P-III) There exist positive constants K_1, K_2 independent of $t \in [0, T]$ satisfying

$$\|S_{A(t)}(s)U_0\|_{\mathscr{H}} \leq K_1 e^{K_2 s} \|U_0\|_{\mathscr{H}}.$$

(P-IV) The operator $A(t): \ [0, T] \to B(D(A(0)), \mathscr{H})$ is essentially bounded and strongly measurable. Then, for any $U_0 \in \mathscr{H}$, the solution U to system (2.9) is unique and satisfies

$$U \in C([0, T], D(A(0))) \cap C^1([0, T], \mathscr{H}).$$

Theorem 3.1. *For any fixed $t \in [0, T]$ and any $U_0 \in \mathscr{H}$, the solution U to the system (2.9) exists uniquely, generates a C_0 semigroup of contraction $\{S_{A(t)}(s)\}_{s\geq 0}$ in $\mathscr{H}$, whose infinitesimal generator is $A(t)$, and there holds that*

$$U \in C^0([0, T], D(A(0))) \cap C^1([0, T], \mathscr{H}).$$

To prove Theorem 3.1, we apply the Lummer-Phillips theorem, and need to show for any fixed $t \in [0, T]$ that $0 \in \varrho(A(t))$ (spectral set of $A(t)$) and the dissipation of $A(t)$.

Theorem 3.2. *For any $U_0 \in \mathscr{H}$, there exist constants $C, \varsigma > 0$, being independent of U_0, such that for any $s \in \mathbb{R}^+$,*

$$\|U(s)\|_{\mathscr{H}} \leq C \|U_0\|_{\mathscr{H}} e^{-\varsigma s},$$

which means the system (2.9) *is exponentially stable.*

To prove Theorem 3.2, we use the well-known characterization of exponential stability for C_0 semigroup as follows, and we can refer to [25] for a detailed process of proof.

Theorem 3.3. *For any fixed $t \in [0, T]$, let $S_{A(t)}(s) = e^{A(t)s}$ be a C_0 semigroup of contraction in space $\mathscr{H}$. Then, the semigroup is exponentially stable if and only if*

$$i\mathbb{R} \subset \varrho(A(t)), \tag{3.1}$$

$$\limsup_{|\lambda|\to\infty} \|(i\lambda I - A(t))^{-1}\| < \infty. \tag{3.2}$$

4. Proof of Theorem 3.1

Basing on the Lummer-Phillips theorem, we need to derive the following lemmas to get the well-posedness, and we can also see [24] for reference.

Lemma 4.1. *For any fixed $t \in [0, T]$, the operator $A(t)$ has the property of dissipation.*

Proof. For any $U = (\varphi, \Phi, \psi, \Psi, \theta, \eta, z)^T \in D(A(t))$, we have

$$\begin{aligned}(A(t)U, U)_{\mathscr{H}} = &\int_0^L [k(\varphi_{xx} + \psi_x)\Phi - az(1)\Phi - d\varphi_t\Phi - \sigma\theta_x\Phi]dx \\ &+ \int_0^L [b\psi_{xx}\Psi - k\varphi_x\Psi - k\psi\Psi + \sigma\theta\Psi]dx \\ &+ \int_0^L [k\Phi_x\varphi_x + k\Phi_x\psi + k\Psi\varphi_x + k\Psi\psi + b\Psi_x\psi_x]dx \\ &+ \int_0^L [\beta\theta_{xx}\theta - \sigma\Phi_x\theta - \sigma\Psi\theta]dx + \int_0^L g(s)(\Psi - \eta_s)_x\eta_x dsdx \\ &+ \xi\int_0^1 (\int_0^L -\frac{1-\kappa\rho'(t)}{\rho(t)} z_k z dx)d\kappa, \end{aligned} \tag{4.1}$$

and it follows that

$$\begin{aligned}(A(t)U, U)_{\mathscr{H}} = &\int_0^L [-\beta\theta_{xx}\theta - az(1)\Phi - d\varphi_t\Phi]dx + \frac{1}{2}\int_0^L\int_0^\infty g'(s)\|\eta_x\|_{L^2}^2 dsdx \\ &+ \xi\int_0^1 [\int_0^L -\frac{1-\kappa\rho'(t)}{\rho(t)} z_k z dx]d\kappa \\ = &-\beta\int_0^L |\theta_x|^2 + \frac{1}{2}\int_0^L\int_0^\infty g'(s)\|\eta_x\|_{L^2}^2 dsdx - a\int_0^L \varphi_t(t-\rho(t))\varphi_t dx - d\int_0^L |\varphi_t|^2 dx \\ &- \frac{\xi}{2\rho(t)}\int_0^L [z^2(x,1,t) - z^2(x,0,t)]dx + \frac{\xi\rho'(t)}{2\rho(t)}\int_0^L [z^2(x,1,t) - \int_0^1 z^2(x,\kappa,t)d\kappa]dx \\ = &-\beta\int_0^L |\theta_x|^2 + \frac{1}{2}\int_0^L\int_0^\infty g'(s)\|\eta_x\|_{L^2}^2 dsdx - a\int_0^L \varphi_t(t-\rho(t))\varphi_t dx \\ &- (d - \frac{\xi}{2\rho(t)})\int_0^L |\varphi_t|^2 dx - (\frac{\xi}{2\rho(t)} - \frac{\xi\rho'(t)}{2\rho(t)})\int_0^L z^2(x,1,t)dx - \frac{\xi\rho'(t)}{2\rho(t)}\int_0^1 z^2(x,\kappa,t)d\kappa dx \\ \le &-\beta\int_0^L |\theta_x|^2 - \frac{k_1}{2}\int_0^L\int_0^\infty g(s)\|\eta_x\|_{L^2}^2 dsdx - (\frac{\xi}{4\rho(t)} - \frac{\xi\rho'(t)}{4\rho(t)})\int_0^L z^2(x,1,t)dx \\ &- \frac{\xi\rho'(t)}{2\rho(t)}\int_0^1 z^2(x,\kappa,t)d\kappa]dx < 0. \end{aligned} \tag{4.2}$$

Lemma 4.2. *For any fixed $t \in [0, T]$, $0 \in \varrho(A(t))$.*

Proof. For any $V = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)^T \in \mathscr{H}$, we study the equations

$$A(t)U = V, \tag{4.3}$$

that is,

$$\Phi = v_1, \tag{4.4}$$

$$\frac{k}{\rho_1}(\varphi_x + \psi)_x - \frac{1}{\rho_1}az(1) - \frac{d}{\rho_1}\varphi_t - \frac{\sigma}{\rho_1}\theta_x = v_2, \tag{4.5}$$

$$\Psi = v_3, \tag{4.6}$$

$$\frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}(\varphi_x + \psi) + \frac{\sigma}{\rho_2}\theta = v_4, \tag{4.7}$$

$$\frac{\beta}{\rho_3}\theta_{xx} - \frac{\sigma}{\rho_3}(\Phi_x + \Psi) = v_5, \tag{4.8}$$

$$\Psi - \eta_s = v_6, \tag{4.9}$$

$$-\frac{1-\kappa\rho'(t)}{\rho(t)} z_\kappa = v_7. \tag{4.10}$$

By (4.4) and (4.6), we get

$$\Phi = v_1 \in H^1_\star(0, L),\ \Psi = v_3 \in H^1_0(0, L).$$

From (4.10), we know

$$z(\kappa) = z(0) + \int_0^\kappa \frac{-\rho(t)}{1-\tau\rho'(t)} v_7 dt = v_1 + \int_0^\kappa \frac{-\rho(t)}{1-\tau\rho'(t)} v_7 dt \in L^2((0,1)\times(0,L)). \tag{4.11}$$

It follows that

$$z(1) = v_1 + \int_0^1 \frac{-\rho(t)}{1-\tau\rho'(t)} v_7 d\tau. \tag{4.12}$$

Here, some equalities could be rewritten as

$$k(\varphi_x + \psi)_x = \sigma\theta_x + \rho_1 v_2 + d v_1 + a z(1), \tag{4.13}$$

$$(b\psi + \int_0^\infty g(s)\eta(s)ds)_{xx} = k(\varphi_x + \psi) - \sigma\theta + \rho_2 v_4, \tag{4.14}$$

$$\beta\theta_{xx} = \rho_3 v_5 + \sigma(v_{1x} + v_3). \tag{4.15}$$

As to (4.15), we use the conclusion that $\rho_3 v_5 + \sigma(v_{1x} + v_3) \in L^2(0, L)$ together with the standard elliptic theory, and we obtain that

$$\theta \in H^1_0(0, L) \cap H^2(0, L).$$

It follows from (4.13) that

$$\varphi_x + \psi = \frac{1}{k}\int_0^x (\sigma\theta_t + \rho_1 v_2 + d v_1 + a z(1))dt \in H^1_0(0, L).$$

In addition, we use the standard elliptic theory again and the relation $k(\varphi_x+\psi)-\sigma\theta+\rho_2 v_4 \in L^2(0, L)$, and we derive that

$$b\psi + \int_0^\infty g(s)\eta(s)ds \in H^1_0(0, L) \cap H^2(0, L).$$

Using the same technique as shown as in [13], we can obtain that

$$\eta \in L^2_g(\mathbb{R}^+; H^1_0),\ \psi \in H^1_0(0, L),\ \varphi_x \in H^1_0(0, L).$$

In summary, the system (4.3) is solved uniquely and $U = (\varphi, \Phi, \psi, \Psi, \theta, \eta, z)^T \in D(A(t))$. From the above derivation process, we can show that there exists a positive constant C such that

$$\|U\|_{\mathscr{H}} \leq C\|V\|_{\mathscr{H}},$$

which means $0 \in \varrho(A(t))$ for any fixed $t \in \mathbb{R}^+$. According to the Lummer-Phillips Theorem, we finish the proof of Theorem 3.1.

5. Verification of conditions in Lemma 3.1

In order to be able to use the autonomous operator theory to the nonautonomous case, we need to verify the conditions in the Kato's perturbation theory, and we can also refer to [27] and draw inspiration.

Lemma 5.1. $\overline{D(A(0))} = \mathscr{H}$.

Proof. Suppose that $U_1 = (\varphi_1, \Phi_1, \psi_1, \Psi_1, \theta_1, \eta_1, z_1)^T \perp D(A(0))$, then there holds for any $U = (\varphi, \Phi, \psi, \Psi, \theta, \eta, z)^T \in D(A(0))$ that

$$\begin{aligned}(U_1, U)_{\mathscr{H}} &= \rho_1 \int_0^L \Phi_1 \Phi dx + \rho_2 \int_0^L \Psi_1 \Psi dx + k \int_0^L (\varphi_{1x} + \psi_1)(\varphi_x + \psi) dx \\ &+ b \int_0^L \psi_{1x} \psi_x dx + \rho_3 \int_0^L \theta_1 \theta dx + \int_0^L \int_0^\infty g(s) \eta_1 \eta ds dx + \xi \int_0^1 \int_0^L z_1(\kappa) z(\kappa) dx d\kappa = 0. \qquad (5.1)\end{aligned}$$

When $U = (0, 0, 0, 0, 0, 0, z)^T$ and $z \in C_0^\infty((0, L) \times (0, 1))$, $U \in D(A(0))$, and from (5.1) we have

$$\xi \int_0^1 \int_0^L z_1(\kappa) z(\kappa) dx d\kappa = 0.$$

The fact that $C_0^\infty((0, L) \times (0, 1))$ is dense in $L^2((0, L) \times (0, 1))$ leads to $z_1(x, \kappa, t) = 0$ in $(0, L) \times (0, 1) \times \mathbb{R}^+$. Using the same technique, we can show that

$$\varphi_1 = 0, \Phi_1 = 0, \psi_1 = 0, \Psi_1 = 0, \theta_1 = 0, \eta_1 = 0.$$

It follows that $\overline{D(A(0))} = \mathscr{H}$, and the condition (P-I) is verified.

Lemma 5.2. *For any $t \in [0, T]$, the operator family $\{A(t)\}$ is stable in $\mathscr{H}$.*

Proof. For any $U \in \mathscr{H}$, we write $||U||_t^2 = ||(\varphi, \Phi, \psi, \Psi, \theta, \eta, \sqrt{\rho(t)} z)^T||_{\mathscr{H}}^2$. From assumptions on $\rho(t)$, we know that for any $t \in [0, T]$ there holds

$$M_0 \xi \int_0^1 \int_0^L z^2(\kappa) d\kappa dx < \rho(t) \xi \int_0^1 \int_0^L z^2(\kappa) d\kappa dx < h \xi \int_0^1 \int_0^L z^2(\kappa) d\kappa dx. \qquad (5.2)$$

If we choose $C' = min\{M_0, 1\}, \ C'' = max\{h, 1\}$, then

$$C' ||(\varphi, \Phi, \psi, \Psi, \theta, \eta, z)^T||_{\mathscr{H}}^2 \leq ||(\varphi, \Phi, \psi, \Psi, \theta, \eta, \sqrt{\rho(t)} z)^T||_{\mathscr{H}}^2 \leq C'' ||(\varphi, \Phi, \psi, \Psi, \theta, \eta, z)^T||_{\mathscr{H}}^2.$$

It follows that $||U||_t^2 = ||(\varphi, \Phi, \psi, \Psi, \theta, \eta, \sqrt{\rho(t)} z)^T||_{\mathscr{H}}^2$ is an equivalent norm of space $\mathscr{H}$, and we claim here that

$$||U||_t \leq ||U||_{t_0} e^{\frac{\widetilde{M_1}}{2M_0}|t - t_0|}. \qquad (5.3)$$

In fact, for any $0 \leq t_0 \leq t \leq T$, there holds

$$||U||_t^2 \leq ||U||_{t_0}^2 e^{\frac{\widetilde{M_1}}{2M_0}(t - t_0)} = (1 - e^{\frac{\widetilde{M_1}}{2M_0}(t - t_0)}).$$

$$(\rho_1\|\Phi\|_{L^2}^2+\rho_2\|\Psi\|_{L^2}^2+k\|\varphi_x+\psi\|_{L^2}^2+b\|\psi_x\|_{L^2}^2+\rho_3\|\theta\|_{L^2}^2+\int_0^\infty g(s)\|\eta_x(s)\|_{L^2}^2ds)$$
$$+(\rho(t)-\rho(t_0)e^{\frac{\widetilde{M_1}}{2M_0}(t-t_0)})\xi\int_0^1\|z(\kappa)\|_{L^2}^2d\kappa.$$

From the properties of $\rho(t)$, we know for any $\vartheta\in(t_0,t)$ there holds

$$\rho(t)=\rho(t_0)+\rho'(\vartheta)(t-t_0)\le\rho(t_0)+\widetilde{M_1}(t-t_0),$$

which means

$$\frac{\rho(t)}{\rho(t_0)}\le 1+\frac{\widetilde{M_1}}{M_0}(t-t_0)\le e^{\frac{\widetilde{M_1}}{M_0}(t-t_0)}.$$

From the fact that $1-e^{\frac{\widetilde{M_1}}{2M_0}(t-t_0)}\le 0$, we derive the conclusion, which means the condition (P-III) is verified.

Lemma 5.3. *For any $t\in[0,T]$, the operator $A(t)\in L_\star^\infty([0,T],B(D(L(0)),\mathscr{H}))$.*

Proof. For any $U=(\varphi,\Phi,\psi,\Psi,\theta,\eta,z)^T\in\mathscr{H}$, from (2.8), we know

$$A(t)=\begin{pmatrix}0&1&0&0&0&0&0\\ \frac{k}{\rho_1}\frac{\partial^2}{\partial x^2}&-\frac{d}{\rho_1}&\frac{k}{\rho_1}\frac{\partial}{\partial x}&0&-\frac{\sigma}{\rho_1}\frac{\partial}{\partial x}&0&-\frac{a}{\rho_1}|_{\kappa=1}\\ 0&0&1&0&0&0&0\\ -\frac{k}{\rho_2}\frac{\partial}{\partial x}&0&\frac{b}{\rho_2}\frac{\partial^2}{\partial x^2}&0&\frac{\sigma}{\rho_2}&\frac{1}{\rho_2}\int_0^\infty g(s)\frac{\partial^2}{\partial x^2}ds&0\\ 0&-\frac{\sigma}{\rho_3}\frac{\partial}{\partial x}&0&-\frac{\sigma}{\rho_3}&\frac{\beta}{\rho_3}\frac{\partial^2}{\partial x^2}&0&0\\ 0&0&0&1&0&-\frac{\partial}{\partial s}&0\\ 0&0&0&0&0&0&-\frac{1-\kappa\rho'(t)}{\rho(t)}\partial\kappa\end{pmatrix}. \tag{5.4}$$

It follows that

$$A_t(t)U=\begin{pmatrix}0\\0\\0\\0\\0\\0\\ \frac{\kappa(\rho(t)\rho''(t)-\rho'^2(t))+\rho'(t)}{\rho^2(t)}z_\kappa\end{pmatrix}, \tag{5.5}$$

which is bounded for $t\in[0,T]$, and the conclusion is finished, which means the condition (P-IV) is verified.

To sum up, basing on the result in Theorem 3.1, we confirm the four conditions in Lemma 3.1.

6. Proof of Theorem 3.2

6.1. *Proof of* (3.1)

Assume that $i\mathbb{R}\subset\varrho(A(t))$ is not true. Therefore, there is a constant C_*, a sequence $\{\beta_n\}$ satisfying $0<\beta_n\to C_*$ and $i\beta_n\in\varrho(A(t))$, and a sequence of functions

$$U_n=(\varphi_n,\Phi_n,\psi_n,\Psi_n,\theta_n,\eta_n,z_n)^T\in D(A(t)),\ \|U_n\|_{\mathscr{H}}=1, \tag{6.1}$$

such that

$$i\beta_n U_n - A(t)U_n \to 0 \text{ in } \mathscr{H}. \tag{6.2}$$

That is,

$$i\beta_n\varphi_n - \Phi_n \to 0 \text{ in } H^1_\star(0,L), \tag{6.3}$$

$$i\beta_n\Phi_n - \frac{k}{\rho_1}(\varphi_{nx} + \psi_n)_x + \frac{1}{\rho_1}az_n(1) + \frac{d}{\rho_1}\Phi_n + \frac{\sigma}{\rho_1}\theta_{nx} \to 0 \text{ in } L^2_\star(0,L), \tag{6.4}$$

$$i\beta_n\psi_n - \Psi_n \to 0 \text{ in } H^1_0(0,L), \tag{6.5}$$

$$i\beta_n\Psi_n - \frac{1}{\rho_2}(b\psi_n + \int_0^\infty g(s)\eta_n(s)ds)_{xx} + \frac{k}{\rho_2}(\varphi_{nx} + \psi_n) - \frac{\sigma}{\rho_2}\theta_n \to 0 \text{ in } L^2(0,L), \tag{6.6}$$

$$i\beta_n\theta_n - \frac{\beta}{\rho_3}\theta_{nxx} + \frac{\sigma}{\rho_3}(\Phi_{nx} + \Psi_n) \to 0 \text{ in } L^2(0,L), \tag{6.7}$$

$$i\beta_n\eta_n - \Psi + \eta_s \to 0 \text{ in } L^2_g(\mathbb{R}^+; H^1_0(0,L)), \tag{6.8}$$

$$i\beta_n z_n + \frac{1-\kappa\rho'(t)}{\rho(t)}z_{n\kappa} \to 0 \text{ in } L^2(0,1; H^1_0(0,L)). \tag{6.9}$$

Lemma 6.1. *For any $t \in \mathbb{R}^+$, we have, as $n \to \infty$,*

$$\theta_n \to 0 \textit{ in } H^1_0(0,L), \tag{6.10}$$

$$\int_0^L \int_0^\infty g(s)\|\eta_x\|^2_{L^2}dsdx \to 0 \textit{ in } L^2_g(\mathbb{R}^+; H^1_0(0,L)), \tag{6.11}$$

$$z_n(1) \to 0 \textit{ in } L^2(0,L), \tag{6.12}$$

$$z_n(x,\kappa,t) \to 0 \textit{ in } L^2(0,1; L^2(0,L)). \tag{6.13}$$

Proof. Multiplying (6.2) by U_n in $\mathscr{H}$ yields that

$$-Re(A(t)U_n, U_n)_{\mathscr{H}} = \beta\int_0^L |\theta_{nx}|^2 + Rea\int_0^L \varphi_{nt}(t-\rho(t))\varphi_{nt}dx + (d - \frac{\xi}{2\rho(t)})\int_0^L |\varphi_{nt}|^2dx$$
$$+ (\frac{\xi}{2\rho(t)} - \frac{\xi\rho'(t)}{2\rho(t)})\int_0^L z_n^2(x,1,t)dx + \frac{\xi\rho'(t)}{2\rho(t)}\int_0^1 z_n^2(x,\kappa,t)d\kappa dx \to 0, \tag{6.14}$$

which means

$$\beta\int_0^L |\theta_x|^2 + \frac{k_1}{2}\int_0^L\int_0^\infty g(s)\|\eta_x\|^2_{L^2}dsdx + (\frac{\xi}{4\rho(t)} - \frac{\xi\rho'(t)}{4\rho(t)})\int_0^L z^2(x,1,t)dx$$
$$+ \frac{\xi\rho'(t)}{2\rho(t)}\int_0^1 z^2(x,\kappa,t)d\kappa dx \to 0, \tag{6.15}$$

and the conclusion holds finally.

Thus, the system (6.2) could be reduced into the following simplified form:

$$i\beta_n\varphi_n - \Phi_n \to 0 \text{ in } H^1_\star(0,L), \tag{6.16}$$

$$i\beta_n\rho_1\Phi_n - k(\varphi_{nx} + \psi_n)_x + az_n(1) + d\Phi_n \to 0 \text{ in } L^2_\star(0,L), \tag{6.17}$$

$$i\beta_n\psi_n - \Psi_n \to 0 \text{ in } H^1_0(0,L), \tag{6.18}$$

$$i\beta_n\rho_2\Psi_n - (b\psi_n + \int_0^\infty g(s)\eta_n(s)ds)_{xx} + k(\varphi_{nx} + \psi_n) \to 0 \text{ in } L^2(0,L), \tag{6.19}$$

$$i\beta_n\rho_3\theta_n - \beta\theta_{nxx} + \sigma(\Phi_{nx} + \Psi_n) \to 0 \text{ in } L^2(0,L), \tag{6.20}$$

$$i\beta_n\eta_n - \Psi + \eta_s \to 0 \text{ in } L^2_g(\mathbb{R}^+; H^1_0(0,L)), \tag{6.21}$$

$$i\beta_n z_n + \frac{1-\kappa\rho'(t)}{\rho(t)} z_{n\kappa} \to 0 \text{ in } L^2(0,1; H^1_0(0,L)). \tag{6.22}$$

Lemma 6.2. *For any $t \in \mathbb{R}^+$, we have, as $n \to \infty$,*

$$\Phi_n \to 0 \text{ in } L^2(0,L). \tag{6.23}$$

Proof. Multiplying (6.22) by $\overline{z_n}$ and considering the real part, we obtain

$$\int_0^1 \frac{1-\kappa\rho'(t)}{2\rho(t)} \frac{d}{d\kappa}\|z_n(\kappa)\|^2_{L^2} d\kappa \to 0, \tag{6.24}$$

which means

$$\begin{aligned} &\frac{1}{2\rho(t)}\int_0^1 \frac{d}{d\kappa}\|z_n(\kappa)\|^2_{L^2} d\kappa - \frac{\rho'(t)}{2\rho(t)}\int_0^1 \kappa\frac{d}{d\kappa}\|z_n(\kappa)\|^2_{L^2} d\kappa \\ =& \frac{1}{2\rho(t)}(\|z_n(1)\|^2_{L^2} - \|z_n(0)\|^2_{L^2}) - \frac{\rho'(t)}{2\rho(t)}\int_0^1 \kappa d(\|z_n(\kappa)\|^2_{L^2}) \to 0, \end{aligned}$$

and

$$(1-\rho'(t))(\|z_n(1)\|^2_{L^2} - \|z_n(0)\|^2_{L^2}) \to 0.$$

It follows from (6.12) that the conclusion is finished.

Lemma 6.3. *For any $t \in \mathbb{R}^+$, we have, as $n \to \infty$,*

$$-\rho_1\|\Phi_n\|^2_{L^2} - \rho_2\|\Psi_n\|^2_{L^2} + k\|\varphi_{nx} + \psi_n\|^2_{L^2} + b\|\psi_{nx}\|^2_{L^2} \to 0. \tag{6.25}$$

Proof. Multiplying (6.16) and (6.17) by $\rho_1\overline{\Phi_n}$ and $\overline{\varphi_n}$, respectively, we get

$$i\beta_n\rho_1(\varphi_n, \Phi_n) - \rho_1\|\Phi_n\|^2_{L^2} \to 0 \tag{6.26}$$

and

$$i\beta_n\rho_1(\Phi_n, \varphi_n) - k((\varphi_{nx} + \varphi_n)_x, \varphi_n) + a(z_n(1), \varphi_n) + d(\Phi_n, \varphi_n) \to 0. \tag{6.27}$$

Adding (6.26) and (6.27), and taking the real part, we have

$$-\rho_1\|\Phi_n\|^2_{L^2} - Rek((\varphi_{nx} + \varphi_n)_x, \varphi_n) + Rea(z_n(1), \varphi_n) + Red(\Phi_n, \varphi_n) \to 0, \tag{6.28}$$

and from Lemmas (6.1) and (6.2), we derive that

$$-\rho_1\|\Phi_n\|_{L^2}^2 - Rek((\varphi_{nx}+\varphi_n)_x,\varphi_n) \to 0. \tag{6.29}$$

Multiplying (6.18) and (6.19) by $\rho_2\overline{\Psi_n}$ and $\overline{\psi_n}$, respectively, we get

$$i\beta_n\rho_2(\psi_n,\Psi_n) - \rho_2\|\Psi_n\|_{L^2}^2 \to 0 \tag{6.30}$$

and

$$i\beta_n\rho_2(\Psi_n,\psi_n) - ((b\psi_n + \int_0^\infty g(s)\eta_n(s)ds)_{xx},\psi_n) + k(\varphi_{nx}+\psi_n,\psi_n) \to 0. \tag{6.31}$$

Adding (6.30) and (6.31), we have

$$-\rho_2\|\Psi_n\|_{L^2}^2 - (b\psi_{nxx},\psi_n) + (\int_0^\infty g(s)\eta_n(s)ds)_{xx},\psi_n) + Rek(\varphi_{nx}+\psi_n,\psi_n) \to 0, \tag{6.32}$$

and

$$-\rho_2\|\Psi_n\|_{L^2}^2 + b\|\psi_{nx}\|_{L^2}^2 + Rek(\varphi_{nx}+\psi_n,\psi_n) \to 0. \tag{6.33}$$

Using (6.29) and (6.33), we derive that

$$-\rho_1\|\Phi_n\|_{L^2}^2 - \rho_2\|\Psi_n\|_{L^2}^2 + k\|\varphi_{nx}+\psi_n\|_{L^2}^2 + b\|\psi_{nx}\|_{L^2}^2 \to 0, \tag{6.34}$$

and the result is obtained.

From the Lemma 2.5 in [14], we know

$$\|\Psi_{nx}\|_{L^2}^2 + \|\psi_{nx}\|_{L^2}^2 \to 0. \tag{6.35}$$

Combining (6.2), (6.34), and (6.35), from the fact that $\|U_n\|_{\mathscr{H}}^2 = \rho_1\|\Phi_n\|_{L^2}^2 + \rho_2\|\Psi_n\|_{L^2}^2 + k\|\varphi_{nx} + \psi_n\|_{L^2}^2 + b\|\psi_{nx}\|_{L^2}^2 = 1$, we can show that

$$k\|\varphi_{nx}+\psi_n\|_{L^2}^2 \to 1. \tag{6.36}$$

Multiplying (6.17) and (6.19) by $\frac{b}{\kappa}\overline{\psi_{nx}}$ and $\overline{\varphi_{nx}+\psi_n}$, respectively, we get

$$i\beta_n\frac{\rho_1 b}{\kappa}(\psi_{nx},\Phi_n) - b(\psi_{nx},(\varphi_{nx}+\psi_n)_x) + \frac{ab}{\kappa}(\psi_{nx},z_n(1)) + \frac{db}{\kappa}(\psi_{nx},\Phi_n) \to 0 \tag{6.37}$$

and

$$i\beta_n\rho_2(\Psi_n,\varphi_{nx}+\psi_n) + ((b\psi_n + \int_0^\infty g(s)\eta_n(s)ds)_x,(\varphi_{nx}+\psi_n)_x) + k\|\varphi_{nx}+\psi_n\|_{L^2}^2 \to 0. \tag{6.38}$$

Adding (6.37) and (6.38), we have

$$i\beta_n\rho_2(\Psi_n,\varphi_{nx}+\psi_n)+i\beta_n\frac{\rho_1 b}{\kappa}(\psi_{nx},\Phi_n)+k\|\varphi_{nx}+\psi_n\|_{L^2}^2 + \frac{ab}{\kappa}(\psi_{nx},z_n(1))+\frac{db}{\kappa}(\psi_{nx},\Phi_n)\to 0. \tag{6.39}$$

Multiplying (6.16) and (6.18) by $\overline{\Psi_n}$ and $i\beta_n\rho_2\overline{\psi_n}$, respectively, we get

$$i\beta_n\rho_2(\Psi_n,\varphi_{nx})-\rho_2(\Psi_n,\Phi_{nx})\to 0 \tag{6.40}$$

and

$$\beta_n^2\rho_2\|\psi_n\|_{L^2}^2+i\beta_n\rho_2(\Psi_n,\psi_n)\to 0. \tag{6.41}$$

Adding (6.40) and (6.41), we have

$$i\beta_n\rho_2(\Psi_n,\varphi_{nx}+\psi_n)-\rho_2(\Psi_n,\Phi_{nx})+\beta_n^2\rho_2\|\psi_n\|_{L^2}^2\to 0. \tag{6.42}$$

Adding (6.39) and (6.42), we have

$$i\beta_n\frac{\rho_1 b}{\kappa}(\psi_{nx},\Phi_n)+k\|\varphi_{nx}+\psi_n\|_{L^2}^2+\frac{ab}{\kappa}(\psi_{nx},z_n(1))+\frac{db}{\kappa}(\psi_{nx},\Phi_n) - \rho_2(\Psi_n,\Phi_{nx})+\beta_n^2\rho_2\|\psi_n\|_{L^2}^2\to 0. \tag{6.43}$$

Multiplying (6.18) by $\rho_2\overline{\Phi_{nx}}$, we get

$$i\beta_n\rho_2(\psi_n,\Phi_{nx})-\rho_2(\Psi_n,\Phi_{nx})\to 0. \tag{6.44}$$

Adding (6.43) and (6.44), we get

$$i\beta_n\rho_2(\psi_n,\Phi_{nx})+i\beta_n\frac{\rho_1 b}{\kappa}(\psi_{nx},\Phi_n)+k\|\varphi_{nx}+\psi_n\|_{L^2}^2 + \frac{ab}{\kappa}(\psi_{nx},z_n(1))+\frac{db}{\kappa}(\psi_{nx},\Phi_n)-\beta_n^2\rho_2\|\psi_n\|_{L^2}^2\to 0. \tag{6.45}$$

From the equal speed condition, we have

$$k\|\varphi_{nx}+\psi_n\|_{L^2}^2+\frac{ab}{\kappa}(\psi_{nx},z_n(1))+\frac{db}{\kappa}(\psi_{nx},\Phi_n)-\beta_n^2\rho_2\|\psi_n\|_{L^2}^2\to 0, \tag{6.46}$$

and there holds

$$k\|\varphi_{nx}+\psi_n\|_{L^2}^2\to 0. \tag{6.47}$$

This is a contradiction with (6.36), and we prove that $i\mathbb{R}\subset\varrho(A(t))$.

6.2. Proof of (3.2)

To achieve the goal, for any $V = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)^T \in \mathscr{H}$, we consider the resolvent equation

$$(i\lambda I_d - A(t))U = V, \tag{6.48}$$

that is,

$$i\lambda\varphi - \Phi = v_1, \tag{6.49}$$
$$i\lambda\Phi - \frac{k}{\rho_1}(\varphi_x + \psi)_x + \frac{1}{\rho_1}az(1) + \frac{d}{\rho_1}\Phi + \frac{\sigma}{\rho_1}\theta_x = v_2, \tag{6.50}$$
$$i\lambda\psi - \Psi = v_3, \tag{6.51}$$
$$i\lambda\Psi - \frac{1}{\rho_2}(b\psi + \int_0^\infty g(s)\eta(s)ds)_{xx} + \frac{k}{\rho_2}(\varphi_x + \psi) - \frac{\sigma}{\rho_2}\theta = v_4, \tag{6.52}$$
$$i\lambda\theta - \frac{\beta}{\rho_3}\theta_{xx} + \frac{\sigma}{\rho_3}(\Phi_x + \Psi) = v_5, \tag{6.53}$$
$$i\lambda\eta - \Psi + \eta_s = v_6, \tag{6.54}$$
$$i\lambda z + \frac{1 - \kappa\rho'(t)}{\rho(t)}z_\kappa = v_7. \tag{6.55}$$

Lemma 6.4. *For any $t \in \mathbb{R}^+$, there exists a constant C_0 independent of V such that*

$$\|\theta_x\|_{L^2}^2 + \int_0^L \int_0^\infty g(s)\|\eta_x\|_{L^2}^2 ds dx + \xi \int_0^1 \|z(x, \kappa, t)\|_{L^2}^2 \leq C_0\|U\|_{\mathscr{H}}\|V\|_{\mathscr{H}}. \tag{6.56}$$

Proof. Multiplying (6.48) by U in $\mathscr{H}$, combining Lemma 6.1, and taking the real part, we derive that

$$\begin{aligned}&\|\theta_{nx}\|_{L^2}^2 + \int_0^L \int_0^\infty g(s)\|\eta_x\|_{L^2}^2 ds dx + \xi \int_0^1 \|z_n(x, \kappa, t)\|_{L^2}^2\\ &\leq CRe(V, U)_{\mathscr{H}} \leq C_0\|U\|_{\mathscr{H}}\|V\|_{\mathscr{H}}.\end{aligned}$$

Lemma 6.5. *For λ large enough and any $t \in \mathbb{R}^+$, there holds that*

$$k\|\varphi_x + \psi\|_{L^2}^2 \leq \varepsilon\|U\|_{\mathscr{H}}^2 + C_\varepsilon\|V\|_{\mathscr{H}}^2. \tag{6.57}$$

Proof. Combining (6.49), (6.51), and (6.53), we have

$$i\lambda\rho_3\theta - \beta\theta_{xx} + i\lambda\sigma(\varphi_x + \psi) = \rho_3 v_5 + \sigma(v_{1x} + v_3). \tag{6.58}$$

Multiplying (6.58) by $\overline{k(\varphi_x + \psi)}$ in $L^2(0, L)$, we have

$$\begin{aligned}i\lambda\sigma k \int_0^L |\varphi_x + \psi|^2 dx = &-\beta \int_0^L \theta_x \overline{k(\varphi_x + \psi)_x} dx + \rho_3 k \int_0^L \theta \overline{i\lambda(\varphi_x + \psi)} dx\\ &+ k \int_0^L (\rho_3 v_5 + \sigma(v_{1x} + v_3))\overline{(\varphi_x + \psi)} dx.\end{aligned} \tag{6.59}$$

In the following way, we estimate each term of (6.59). From (6.50), we derive that

$$-\beta\int_0^L \theta_x\overline{k(\varphi_x+\psi)_x}dx = i\lambda\beta\rho_1\int_0^L \theta_x\overline{\Phi}dx - \beta\sigma\int_0^L |\theta_x|^2dx - \beta a\int_0^L \theta_x z(1)dx - \beta d\int_0^L \theta_x\Phi dx + \beta\rho_1\int_0^L \theta_x\overline{v_2}dx. \tag{6.60}$$

Next, from (6.49) and (6.51), we derive

$$\rho_3 k\int_0^L \theta\overline{i\lambda(\varphi_x+\psi)}dx = -\rho_3 k\int_0^L \theta_x\overline{\Phi}dx + \rho_3 k\int_0^L \theta\overline{\Psi}dx + \rho_3 k\int_0^L \theta\overline{v_{1x}+v_3}dx, \tag{6.61}$$

and it follows that

$$\begin{aligned} i\lambda\sigma k\|\varphi_x+\psi\|_{L^2}^2dx =& i\lambda\beta\rho_1\int_0^L \theta_x\overline{\Phi}dx - \beta\sigma\int_0^L |\theta_x|^2dx - \beta a\int_0^L \theta_x z(1)dx - \beta d\int_0^L \theta_x\Phi dx \\ & - \rho_3 k\int_0^L \theta_x\overline{\Phi}dx + \rho_3 k\int_0^L \theta\overline{\Psi}dx + \rho_3 k\int_0^L \theta\overline{v_{1x}+v_3}dx \\ & + k\int_0^L (\rho_3 v_5 + \sigma(v_{1x}+v_3))\overline{(\varphi_x+\psi)}dx + \beta\rho_1\int_0^L \theta_x\overline{v_2}dx. \end{aligned} \tag{6.62}$$

We see that

$$\lambda\sigma k\|\varphi_x+\psi\|_{L^2}^2dx \le \lambda\beta\rho_1\|\theta_x\|_{L^2}\|\Phi\|_{L^2} + C\|\theta_x\|_{L^2}\|U\|_{\mathscr{H}} + C\|U\|_{\mathscr{H}}\|V\|_{\mathscr{H}} + C\|\theta_x\|_{L^2}\|V\|_{\mathscr{H}}, \tag{6.63}$$

that is,

$$k\|\varphi_x+\psi\|_{L^2}^2dx \le \frac{\beta\rho_1}{\sigma}\|\theta_x\|_{L^2}\|\Phi\|_{L^2} + \frac{C}{\lambda}\|\theta_x\|_{L^2}\|U\|_{\mathscr{H}} + \frac{C}{\lambda}\|U\|_{\mathscr{H}}\|V\|_{\mathscr{H}} + \frac{C}{\lambda}\|\theta_x\|_{L^2}\|V\|_{\mathscr{H}}, \tag{6.64}$$

and Young's inequality leads to the conclusion finally.

Lemma 6.6. *For λ large enough, any $t \in \mathbb{R}^+$, and any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ satisfying*

$$\rho_1\|\Phi\|_{L^2}^2 \le \varepsilon\|U\|_{\mathscr{H}}^2 + C_\varepsilon\|V\|_{\mathscr{H}}^2. \tag{6.65}$$

Proof. Multiplying (6.50) by $-\overline{\varphi}$ in $L^2(0, L)$ and using (6.49), we have

$$\begin{aligned} \rho_1\int_0^L |\Phi|^2dx =& k\int_0^L |\varphi_x+\psi|^2dx - k\int_0^L (\varphi_x+\psi)\overline{\psi}dx + \frac{ia}{\lambda}\int_0^L z(1)\overline{\Phi+v_1}dx \\ & + \frac{id}{\lambda}\int_0^L \Phi\overline{\Phi+v_1}dx + \frac{i\sigma}{\lambda}\int_0^L \theta_x\overline{\Phi+v_1}dx - \rho_1\int_0^L (\Phi\overline{v_1} + v_2\overline{\varphi})dx. \end{aligned} \tag{6.66}$$

From Lemma 6.4, we can derive that

$$Re\left(\frac{ia}{\lambda}\int_0^L z(1)\overline{\Phi+v_1}dx + \frac{id}{\lambda}\int_0^L \Phi\overline{\Phi+v_1}dx\right) \le \frac{C}{\lambda}\|U\|_{\mathscr{H}}^{1/2}\|V\|_{\mathscr{H}}^{1/2}(\|U\|_{\mathscr{H}} + \|V\|_{\mathscr{H}}), \tag{6.67}$$

$$Re\frac{i\sigma}{\lambda}\int_0^L \theta_x\overline{\Phi + v_1}dx \le \frac{C}{\lambda}\|U\|_{\mathscr{H}}^{1/2}\|V\|_{\mathscr{H}}^{1/2}(\|U\|_{\mathscr{H}} + \|V\|_{\mathscr{H}}), \tag{6.68}$$

and

$$Re\rho_1\int_0^L(\Phi\overline{v_1} + v_2\overline{\varphi})dx \le C\|U\|_{\mathscr{H}}\|V\|_{\mathscr{H}}. \tag{6.69}$$

It follows from Lemma 6.5 and Young's inequality that we can derive the conclusion.

Lemma 6.7. *For λ large enough, any $t \in \mathbb{R}^+$, and any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ satisfying*

$$\rho_2\|\Psi\|_{L^2}^2 \le \varepsilon\|U\|_{\mathscr{H}}^2 + C_\varepsilon\|V\|_{\mathscr{H}}^2. \tag{6.70}$$

Proof. Multiplying (6.52) by $\int_0^\infty g(s)\overline{\eta(s)}ds$ in $L^2(0, L)$, we have

$$\begin{aligned}&-\rho_2\int_0^L\int_0^\infty g(s)\Psi\overline{i\lambda\eta(s)}dsdx - \sigma\int_0^L\int_0^\infty g(s)\theta\overline{\eta(s)}dsdx + b\int_0^L\int_0^\infty g(s)\overline{\eta_x(s)}\psi_x dsdx\\&+\int_0^L|\int_0^\infty g(s)\eta_x(s)ds|^2dx + k\int_0^L\int_0^\infty g(s)(\varphi_x + \psi)\overline{\eta(s)}dsdx\\&= \rho_2\int_0^L\int_0^\infty g(s)v_4\overline{\eta(s)}dsdx.\end{aligned} \tag{6.71}$$

By (6.54), we obtain

$$\begin{aligned}&-\rho_2\int_0^L\int_0^\infty g(s)\Psi\overline{i\lambda\eta(s)}dsdx\\&= -\rho_2\int_0^\infty g(s)ds\int_0^L|\Psi|^2dx - \rho_2\int_0^L\int_0^\infty g(s)\Psi\overline{v_6}dsdx + \rho_2\int_0^L\int_0^\infty g(s)\Psi\overline{\eta_s}dsdx,\end{aligned} \tag{6.72}$$

and integration by parts leads to

$$|\rho_2\int_0^L\int_0^\infty g(s)\Psi\overline{\eta_s}dsdx| \le \rho_2 C_0^{1/2}\|\Psi\|_{L^2}(\int_0^\infty(-g'(s))\|\eta_s\|^2ds)^{1/2}. \tag{6.73}$$

Using (6.49) and (6.51), we show that

$$\begin{aligned}&|k\int_0^L\int_0^\infty g(s)(\varphi_x + \psi)\overline{\eta(s)}dsdx|\\&= |-\frac{ik}{\lambda}\int_0^L\int_0^\infty g(s)(v_{1x} + v_3)\overline{\eta(s)}dsdx - \frac{ik}{\lambda}\int_0^L\int_0^\infty g(s)\Psi\overline{\eta(s)}dsdx\\&+\frac{ik}{\lambda}\int_0^L\int_0^\infty g(s)\Phi\overline{\eta_x(s)}dsdx| \le \frac{C}{\lambda}\|\eta\|_{L_g^2}(\|\Phi\| + \|\Psi\| + \|v_{1x} + v_3\|),\end{aligned} \tag{6.74}$$

and from the Hölder inequality we get

$$\int_0^L|\int_0^\infty g(s)\eta_x(s)ds|^2 \le \int_0^L(\int_0^\infty g(s)ds)\int_0^\infty g(s)|\eta_x(s)|^2dsdx \le C_0\|\eta\|_{L_g^2}^2. \tag{6.75}$$

Combining Lemmas 6.4 and 6.6, and (6.71)–(6.75), and using the Hölder inequality and the Young inequality, we can derive the conclusion. We can also refer to [14] for excellent details.

Lemma 6.8. *For λ large enough, any $t \in \mathbb{R}^+$, and any $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ satisfying*

$$b\|\psi_x\|_{L^2}^2 \le \varepsilon\|U\|_{\mathscr{H}}^2 + C_\varepsilon\|V\|_{\mathscr{H}}^2. \tag{6.76}$$

Proof. Multiplying (6.19) by $\overline{\psi}$ in $L^2(0, L)$, we have

$$\begin{aligned}&-\rho_2\int_0^L \Psi\overline{i\lambda\psi}dx + b\int_0^L |\psi_x|^2dx + \int_0^L\int_0^\infty g(s)\eta_x(s)\overline{\psi_x}dsdx\\&+ k\int_0^L(\varphi_x+\psi)\overline{\psi}dx - \sigma\int_0^L \theta\overline{\psi}dx = \rho_2\int_0^L v_4\overline{\psi}dx.\end{aligned} \tag{6.77}$$

From (6.51), we have

$$\begin{aligned}b\int_0^L |\psi_x|^2dx =& -\int_0^L\int_0^\infty g(s)\eta_x(s)\overline{\psi_x}dsdx - k\int_0^L(\varphi_x+\psi)\overline{\psi}dx\\&+\rho_2\int_0^L \Psi\overline{i\lambda\psi}dx + \sigma\int_0^L\theta\overline{\psi}dx + \rho_2\int_0^L v_4\overline{\psi}dx\\ \le& -\int_0^L\int_0^\infty g(s)\eta_x(s)\overline{\psi_x}dsdx - \frac{ik}{\lambda}\int_0^L(\varphi_x+\psi)\overline{\Psi}dx - \frac{ik}{\lambda}\int_0^L(\varphi_x+\psi)\overline{v_3}dx\\&+\rho_2\int_0^L|\Psi|^2dx + \rho_2\int_0^L\Psi\overline{v_3}dx + \sigma\int_0^L\theta\overline{\psi}dx + \rho_2\int_0^L v_4\overline{\psi}dx,\end{aligned} \tag{6.78}$$

and using Lemmas 6.6 and 6.7 leads to

$$b\|\psi_x\|^2 \le C\|U\|_{\mathscr{H}}\|V\|_{\mathscr{H}} + C\|U\|_{\mathscr{H}}\|\eta\|_{L_g^2} + C\|U\|_{\mathscr{H}}\|\theta_x\|_{\mathscr{H}} + C\|\theta_x\|_{\mathscr{H}}\|V\|_{\mathscr{H}}. \tag{6.79}$$

It follows from Lemma 6.4 and the Young inequality that the conclusion is derived.

In summary, combination of Lemmas 6.4–6.8 help us to obtain

$$\|U\|_{\mathscr{H}} \le C\|V\|_{\mathscr{H}},$$

and (3.2) is finished finally.

7. Further research

In this article, we study the exponential stability of thermoelastic Timoshenko with variable delay in the internal feedback. If the variable delay is replaced by the distributed delay, the relating problem is still open, which is our next objective.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Author contribution

Conceptualization, methodology, writing-original draft preparation, K. Su; software, writing-review and editing, X. Ge. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare there is no conflict of interest.

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