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Research article

Existence, uniqueness, and blow-up analysis of a quasi-linear bi-hyperbolic equation with dynamic boundary conditions

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Abstract: For this study, we investigate the existence and uniqueness of local solutions and derive a blow-up solution for a quasi-linear bi-hyperbolic equation under dynamic boundary conditions. We utilize the contraction mapping concept to demonstrate the solution's local well-posedness and employ a concavity approach to establish the blow-up result.

Keywords: wave equation; local existence; blow-up; contraction mapping principle

1. Introduction

In this paper, we study the following quasi-linear bi-hyperbolic equation:

$$w_{tt} + \Delta^2 w - \Delta w = bf(-\Delta w), \quad \text{in} \quad \Omega \times [0, T), \tag{1.1}$$

under the following dynamic boundary conditions,

$$w = 0, \quad \frac{\partial \Delta w}{\partial \eta} = -a\Delta w_t, \qquad \text{in} \quad \Gamma \times [0, T), \tag{1.2}$$

and initial conditions

$$w(x, 0) = w_0(x), \quad w_t(x, 0) = w_1(x), \quad x \in \Omega, \tag{1.3}$$

where $a \geq 0$, $b \geq 0$, $t \geq 0$, $x \in \Omega$, Ω is an open bounded connected region in $\mathbb{R}^n$ $(n \geq 1)$ with a smooth boundary $\Gamma := \partial\Omega$, and $\eta(x)$ represents an outer unit normal vector to the boundary Γ.

The Eq (1.1) represents a mathematical model of a wave process in a physical domain Ω over a time interval $[0, T)$. This wave equation involves w_{tt} as the second-time derivative representing acceleration over time, $\Delta^2 w$ as the second-order spatial Laplacian indicating wave irregularities, Δw as the first-order Laplacian reflecting propagation speed within the wave, and $bf(-\Delta w)$ as a nonlinear term depicting wave interaction with its negative Laplacian. Overall, these equations and conditions provide a framework for understanding wave behavior, interactions at boundaries, and the evolution of waves from specified initial conditions within a physical space. Applications of such equations extend to various areas, including acoustics, electromagnetics, and mechanics, where wave phenomena play a crucial role in modeling and analysis.

First, we mention some known results of higher-order differential equations under dynamic boundary conditions related to the problems (1.1)–(1.3). Dynamic boundary conditions introduce dependencies on both time and space variables, influencing the behavior and evolution of solutions within specific domains. Recent research has highlighted the significance of dynamic boundary conditions in various mathematical contexts, particularly in studying wave propagation, heat transfer, fluid dynamics, and other physical phenomena. Notably, works by Vasconcellos and Teixeira [1] have explored the implications of dynamic boundary conditions on well-posedness. They considered, for $n \leq 3$, the following problem:

$$\begin{cases} u_{tt} + \Delta^2 u - \phi\Big(\int_\Omega |\nabla u|^2 dx\Big)\Delta u + g(u_t) = 0, & \text{on} \quad \Omega \times (0,T), \\ u = \dfrac{\partial u}{\partial \nu} = 0, & \text{on} \quad \Gamma \times (0,T), \end{cases}$$

where ϕ is a non-negative continuous real differentiable function, and g is a continuous non-decreasing real function. They proved the existence and uniqueness of global solutions. Guedda and Labani [2] studied the problem

$$\begin{cases} u_{tt} + \Delta^2 u + \delta u_t - \phi(\int_\Omega |\nabla u|^2 dx)\Delta u = f(u), & \text{on} \quad \Omega \times (0,T), \\ u = 0, \quad \Delta u + p(x)\dfrac{\partial u_t}{\partial \nu} = 0, & \text{on} \quad \Gamma \times (0,T), \end{cases}$$

where $p \geq 0$ is a smooth function defined on the boundary of Ω. They studied the global nonexistence of solutions under certain conditions on f and ϕ. Later, Wu and Tsai [3] considered the initial boundary value problem for a Kirchhoff-type plate equation with a source term in a bounded domain. They established the existence of a global solution using an argument similar to that in [4]. Vitillaro conducted a study in 2017 focusing on dynamic boundary conditions. In this work [5], a wave equation with hyperbolic dynamic boundary conditions, interior and boundary damping effects, and supercritical sources was investigated.

Several authors have extensively studied blow-up phenomena and global nonexistence (see [4, 6–9]). Levine [8] introduced the concavity method and investigated the nonexistence of global solutions with negative initial energy. Subsequently, Georgiev and Todorova [4] expanded upon Levine's work. In 2002, Vitillaro [10] further refined the results of Georgiev and Todorova for systems with positive initial energy. Vitillaro also explored blow-up phenomena for wave equations with dynamic boundary conditions in [9]. Additionally, Can et al. [6, 7] investigated the blow-up properties of (1.1) under various boundary conditions, assuming non-positive initial energy. While their result is achieved by applying the Ladyzhenskaya and Kalantarov lemma [11], along with a generalized concavity method, our approach is based on the blow-up lemma by Korpusov [12], which is another application of the concavity method. In our study of problems (1.1)–(1.3), we obtained both a local existence result and a blow-up result under positive initial energy.

The paper is structured as follows. Section 2 provides essential definitions, theorems, and inequalities. In Section 3, we initially employ the Galerkin approximation method to investigate the existence of the corresponding linear problems (3.1)–(3.3). Subsequently, utilizing the contraction mapping principle, we establish the local existence and uniqueness of regular solutions for problems (1.1)–(1.3). Finally, in the last section, we deduce the blow-up solutions for problems (1.1)–(1.3) under the condition of positive initial energy.

2. Preliminaries and notations

The *Sobolev space* is defined by $W^{k,p}(\Omega) := \{u \in L_p(\Omega) : D^\alpha u \in L_p(\Omega), \forall\ 0 \le |\alpha| \le k\}$ for $1 \le p < \infty$, equipped with the following norm:

$$\|u\|_{W^{k,p}(\Omega)} := \Big(\sum_{0\le|\alpha|\le k} \|D^\alpha u\|^p_{L_p(\Omega)} \Big)^{1/p}.$$

We denote by $H^k(\Omega) = W^{k,2}(\Omega)$ the Hilbert-Sobolev space. Throughout this paper, we denote $\|.\|_{L^2(\Omega)} = \|.\|_2$.

Definition 2.1. Let $w(t)$ be a weak solution of the problem defined by Eqs (1.1)–(1.3). We define the maximal existence time T_∞ as follows:

(i) If $w(t)$ exists for $0 \le T < \infty$, then $T_\infty = +\infty$.
(ii) If there exists a $T_0 \in (0, \infty)$ such that $w(t)$ exists for $0 \le T < T_0$,but does not exist at $T = T_0$, then $T_\infty = T_0$.

In order to prove the blow-up result, we will utilize the following lemma due to Korpusov.

Lemma 2.2. *[12] Let $\psi(t) \in C^2(0, T)$ and consider the differential inequality*

$$\psi\psi'' - \alpha(\psi')^2 + \gamma\psi'\psi + \beta\psi \ge 0, \quad \alpha > 1, \beta \ge 0, \gamma \ge 0.$$

Assume that the following conditions

$$\psi'(0) > \frac{\gamma}{\alpha - 1}\psi(0), \quad \text{and} \quad \Big(\psi'(0) - \frac{\gamma}{\alpha - 1}\psi(0)\Big)^2 > \frac{2\beta}{2\alpha - 1}\psi(0),$$

hold with $\psi(t) \ge 0$, and $\psi(0) > 0$. Then the time $T > 0$ can not be arbitrarily large. That is,

$$T < T_\infty = \psi^{1-\alpha}(0)A^{-1},$$

where T_∞ is the maximal existence time interval for $\psi(t)$ and

$$A^2 \equiv (\alpha - 1)^2\psi^{-2\alpha}(0)\Big[\Big(\psi'(0) - \frac{\gamma}{\alpha - 1}\psi(0)\Big)^2 - \frac{2\beta}{2\alpha - 1}\psi(0)\Big],$$

such that $\lim_{t\uparrow T_\infty} \psi(t) = +\infty$.

Now, we state the assumptions on the function f:

(A1) $f : H_0^2(\Omega) \to L^2(\Omega)$ is locally Lipschitz with the Lipschitz constant L_f, that is, for every $x \in H_0^2(\Omega)$, there exists a neighborhood V of x and a positive constant L_f depending on V such that

$$\|f(y) - f(z)\|_2 \leq L_f \|y - z\|_2,$$

for each $y, z \in V$.

(A2) The function f with its primitive $F(u) = \displaystyle\int_0^u f(s)ds$ has the property:

$$f(0) = 0, \qquad uf(u) \geq 2(2\gamma + 1)F(u), \tag{2.1}$$

for all $u \in \mathbb{R}$ and for some positive real number γ.

Example 2.3. *Consider the function $f(u) = u^2$. This function satisfies property (2.1) based on the conditions $f(0) = 0$ and the behavior of its primitive, $F(u) = \int_0^u f(s)ds = \int_0^u s^2 ds = \frac{u^3}{3}$. Specifically, we can establish the inequality $uf(u) = u^3 \geq 2(2\gamma + 1)\frac{u^3}{3}$, which holds true for some $\gamma \leq \frac{1}{4}$.*

3. Local existence

In this section, we delve into the local existence of solutions for the wave Eqs (1.1)–(1.3) employing the contraction mapping principle. Initially, we examine the following linear initial boundary value problem:

$$w_{tt} + \Delta^2 w - \Delta w = h(x,t), \qquad \text{in} \quad \Omega \times [0,T), \tag{3.1}$$

$$w = 0, \quad \frac{\partial \Delta w}{\partial \eta} = -a\Delta w_t, \qquad \text{in} \quad \Gamma \times [0,T), \tag{3.2}$$

$$w(x,0) = w_0(x), \quad w_t(x,0) = w_1(x), \quad x \in \Omega. \tag{3.3}$$

Lemma 3.1. *Suppose that $w_0 \in U$, $w_1 \in H$, and $h \in W^{1,2}(0,T; L^2(\Omega))$. Then, the problems* (3.1)–(3.3) *admit a unique solution w such that*

$$w \in L^\infty(0,T;U), \ w_t \in L^\infty(0,T;H),$$

where $U = \{w \in H_0^2(\Omega) : \frac{\partial \Delta w}{\partial \eta}|_\Gamma = -a\Delta w_t\}$, and $H = H_0^1(\Omega) \cap H^2(\Omega)$.

Proof. We initially employed the Galerkin approximation method to investigate the existence of solutions to this linear problem. Let $(\phi_n)_{n\in\mathbb{N}}$ be a basis in U, and V_n denote the subspace generated by $\phi_1, ..., \phi_n$ ($n = 1, 2, ...$). Consider $w_n(t) = \sum_{i=1}^n r_{in}(t)\phi_i$ as the solution of the approximation problem corresponding to (3.1)–(3.3) for $\phi \in V_n$. Then, we have:

$$\int_\Omega w_n'' \phi dx + \int_\Omega \Delta\phi \Delta w_n dx + \int_\Omega \nabla w_n \nabla \phi dx = \int_\Omega h(x,t)\phi dx, \tag{3.4}$$

with initial conditions satisfying

$$w_n(0) \equiv \sum_{i=1}^n \Big(\int_\Omega w_0 \phi_i dx\Big)\phi_i \to w_0 \quad \text{in} \quad U, \tag{3.5}$$

$$w_n'(0) \equiv \sum_{i=1}^{n} (\int_\Omega w_1\phi_i dx)\phi_i \to w_1 \quad \text{in} \quad H. \tag{3.6}$$

First, we verify the existence of solutions to (3.4)–(3.6) on some interval $[0, t_n)$, $0 < t_n < T$, and then use standard differential equations techniques [13] to extend the solution across the entire interval $[0, T]$. To achieve this, we need to establish the following a priori estimates.

Setting $\phi = 2w_n'(t)$ in (3.4), integrating over $(0, t)$, and utilizing boundary conditions yield:

$$\begin{aligned} \|w_n'(t)\|_2^2 + \|\Delta w_n(t)\|_2^2 + \|\nabla w_n(t)\|_2^2 \;&\leq\; \|w_n'(0)\|_2^2 + \|\Delta w_n(0)\|_2^2 + \|\nabla w_n(0)\|_2^2 \\ &+\; 2\int_0^t \int_\Omega h(x,t) w_n'(t) dx. \end{aligned}$$

From this, we obtain:

$$\|w_n'(t)\|_2^2 + \|\Delta w_n(t)\|_2^2 + \|\nabla w_n(t)\|_2^2 \leq C_0 + \int_0^t \Big(\|w_n'(s)\|_2^2 + \|\Delta w_n(s)\|_2^2\Big) dt, \tag{3.7}$$

where $C_0 = \|w_n'(0)\|_2^2 + \|\Delta w_n(0)\|_2^2 + \|\nabla w_n(0)\|_2^2 + \int_0^T \|h\|_2^2 dt$, and utilizing the estimate:

$$2|\int_\Omega h(x,t) w_n'(t) dx| \leq \|h\|_2^2 + \|w_n'(t)\|_2^2. \tag{3.8}$$

The conditions (3.5) and (3.6), and the property of h imply that C_0 is bounded. Now, for all $0 \leq t \leq T$, applying Gronwall's inequality in (3.7), we obtain

$$\|w_n'(t)\|_2^2 + \|\Delta w_n(t)\|_2^2 + \|\nabla w_n(t)\|_2^2 \leq M_1, \tag{3.9}$$

where M_1 is a positive constant.

To estimate $w_n''(0)$ in L^2-norm, we set $t = 0$ in (3.4) and $\phi = 2w_n''(0)$:

$$\|w_n''(0)\|_2^2 \leq \|w_n''(0)\|_2 \Big[\|\Delta^2 w_n(0)\|_2 + \|\Delta w_n(0)\|_2 + \|h\|_2\Big]. \tag{3.10}$$

By employing (3.5) and (3.6), we find a positive constant M_2 such that:

$$\|w_n''(0)\|_2 \leq M_2. \tag{3.11}$$

Next, we aim to establish an upper bound for $\|w_n''(t)\|_2$. Replacing $\phi = 2w_n''(t)$ in (3.4) after differentiating it with respect to t gives

$$\frac{d}{dt}\Big[\|w_n''(t)\|_2^2 + \|\Delta w_n'(t)\|_2^2 + \|\nabla w_n'(t)\|_2^2\Big] \leq 2\int_\Omega h'(x,t) w_n''(t) dx. \tag{3.12}$$

Hence, by integrating (3.12) over $(0, t)$ and using the inequalities (3.8), (3.9) and (3.11), we obtain

$$\underbrace{\|w_n''(t)\|_2^2 + \|\Delta w_n'(t)\|_2^2 + \|\nabla w_n'(t)\|_2^2}_{=:Y(t)} \leq \underbrace{Y(0) + \int_0^T \|h'\|_2^2}_{=:C_1} + \int_0^t \Big(\|w_n''(s)\|_2^2 + \|\Delta w_n'(s)\|_2^2\Big) dt. \tag{3.13}$$

Using Gronwall's inequality for the inequality

$$Y(t) \leq C_1 + \int_0^t \Big(\|w_n''(s)\|_2^2 + \|\Delta w_n'(s)\|_2^2\Big) dt,$$

and (3.5) and (3.6), we can derive

$$\|w_n''(t)\|_2^2 + \|\Delta w_n'(t)\|_2^2 + \|\nabla w_n'(t)\|_2^2 \leq M_3, \tag{3.14}$$

for any $t \in [0, T]$ with a positive M_3, which is independent of $n \in \mathbb{N}$. Using (3.9) and (3.14), we may conclude that

$$w_i \to w \quad \text{weak}^* \quad \text{in} \quad L^\infty(0, T; H_0^2(\Omega)), \tag{3.15}$$

$$w_i' \to w' \quad \text{weak}^* \quad \text{in} \quad L^\infty(0, T; H), \tag{3.16}$$

$$w_i' \to w' \quad \text{and} \quad w_i'' \to w'' \quad \text{weak}^* \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)). \tag{3.17}$$

Thus, by taking the limit in (3.4) and utilizing the above convergences, we obtain:

$$\int_0^T \int_\Omega (w_{tt} + \Delta^2 w - \Delta w) u\sigma dx dt = \int_0^T \int_\Omega h(x, t) u\sigma dx dt,$$

for all $\sigma \in D(0, T)$ and for all $u \in U$. From the above identity, we have

$$w_{tt} + \Delta^2 w - \Delta w = h(x, t) \quad \text{in} \quad L^\infty(0, T; L^2(\Omega)), \tag{3.18}$$

since $w'', \Delta w$ and $h \in L^\infty(0, T; L^2(\Omega))$ and we deduce $\Delta^2 w \in L^\infty(0, T; L^2(\Omega))$, so $w \in L^\infty(0, T; U)$.

To prove the uniqueness of the solution, let w_1 and w_2 be two solutions of (3.1)–(3.3). Then $v = w_1 - w_2$ satisfies

$$\int_\Omega v''(t)\phi dx + \int_\Omega \Delta v \Delta\phi dx + \int_\Omega \nabla v \nabla\phi dx = 0, \tag{3.19}$$

for $\phi \in U$. Also, we have

$$v(x, 0) = 0,\ v'(x, 0) = 0 \quad \text{in} \quad \Omega, \text{ and } v(x, t) = 0,\ \frac{\partial \Delta v}{\partial \eta} = -a\Delta v_t \quad \text{on} \quad \Gamma.$$

Now, if we set $\phi = 2v'(t)$ in (3.19), then we have

$$\|v'(t)\|_2^2 + \|\nabla v(t)\|_2^2 + \|\Delta v(t)\|_2^2 \leq \int_0^t \|v'(s)\|_2^2 + \|\nabla v(s)\|_2^2.$$

By Gronwall's inequality, we conclude that

$$\|v'(t)\|_2 = \|\Delta v(t)\|_2 = \|\nabla v(t)\|_2 = 0, \quad \forall t \in [0, T].$$

Therefore, we have uniqueness. Now, we establish the local existence of the problems (1.1)–(1.3).

Theorem 3.2. *Suppose that $f : H_0^2(\Omega) \to L^2(\Omega)$, and that $w_0 \in U$, and $w_1 \in H$, then there exists a unique solution w with $w \in L^\infty(0, T; U)$ and $w_t \in L^\infty(0, T; H)$.*

Proof. Define the following space for $T > 0$ and $R_0 > 0$:

$$X_{T,R_0} = \Big\{ v \in L^\infty(0,T;U), v_t \in L^\infty(0,T;H) : e(v(t)) \equiv \|v_t(t)\|_2^2 + \|\Delta v(t)\|_2^2 \le R_0^2, t \in [0,T] \Big\}.$$

Then X_{T,R_0} is a complete metric space with the distance

$$d(x,y) = \sup_{0\le t\le T} \Big[\|\Delta(x-y)\|^2 + \|(x-y)_t\|^2 \Big]^{\frac{1}{2}}, \tag{3.20}$$

where $x, y \in X_{T,R_0}$.

By Lemma 3.1, for any $u \in X_{T,R_0}$, the problem

$$w_{tt} + \Delta^2 w - \Delta w = bf(-\Delta u) \tag{3.21}$$

has a unique solution w of (3.21). We define the nonlinear mapping $Bu = w$, and then, we shall show that there exists $T > 0$ and $R_0 > 0$ such that

(i) $B : X_{T,R_0} \to X_{T,R_0}$,
(ii) In the space X_{T,R_0}, the mapping B is a contraction according to the metric given in (3.20).

After multiplication by $2w_t$ in Eq (3.21), and integration over Ω, we find

$$e_1(w(t)) := \int_0^t [\|w_t\|_2^2 + \|\Delta w\|_2^2 + \|\nabla w\|_2^2] = \underbrace{2b\int_0^t \int_\Omega f(-\Delta u) w_t dx}_{I_1}. \tag{3.22}$$

Taking into account the assumption (A1) on f, we obtain

$$\begin{aligned} |I_1| &= 2b\int_0^t \int_\Omega bf(-\Delta u)w_t(t) d\Omega dt \le b\int_0^t \|f(-\Delta u)\|_2.\|w_t(t)\|_2 dt \\ &\le 2bL_f \int_0^T \|-\Delta u(t)\|_2.\|w_t(t)\|_2 + 2b\int_0^t \underbrace{\|f(0)\|_2}_{=0} \|w_t(t)\|_2 dt \\ &\le (4b^2L_f^2 + 1)\int_0^t \underbrace{\Big(\|\Delta u(t)\|_2^2 + \|w_t(t)\|_2^2\Big) dt}_{\le e_1(w(s))}. \end{aligned}$$

Then, by integrating (3.22) over $(0,t)$ and using the above inequality, we deduce

$$e_1(w(t)) \le e_1(w_0) + (4b^2L_f^2 + 1)\int_0^t e_1(w(s))ds.$$

Thus, by Gronwall's inequality, we have

$$e_1(w(t)) \le e_1(w_0) e^{\int_0^t 4b^2L_f^2+1}. \tag{3.23}$$

Therefore, if the parameters T and R_0 satisfy $e_1(w_0)e^{\int_0^t 4b^2L_f^2+1} \le R_0^2$, we obtain

$$e(w(t)) \le (e_1(w_0)) e^{\int_0^t 4b^2L_f^2+1} \le R_0^2. \tag{3.24}$$

Hence, it implies that B maps X_{T,R_0} into itself.

Let us now prove (ii). To demonstrate that B is a contraction mapping with respect to the metric $d(.,.)$ given above, we consider $u_i \in X_{T,R_0}$ and $w_i \in X_{T,R_0}$, where $i = 1, 2$ are the corresponding solutions to (3.21). Let $v(t) = (w_1 - w_2)(t)$, then v satisfies the following system:

$$v_{tt} + \Delta^2 v - \Delta v = f(-\Delta u_1) - f(-\Delta u_2), \tag{3.25}$$

with initial conditions

$$v(0) = 0, v_t(0) = 0,$$

and boundary conditions

$$v = 0, \quad \frac{\partial \Delta v}{\partial \eta} = -a\Delta v_t.$$

Multiplying (3.25) by $2v_t$, and integrating it over Ω, we find

$$\frac{d}{dt}\Big[\|v_t\|_2^2 + \|\nabla v\|_2^2 + \|\Delta v\|_2^2\Big] \leq I_2 + I_3, \tag{3.26}$$

where

$$I_2 = 2b \int_\Omega (f(-\Delta u_1) - f(-\Delta u_2))v_t dx,$$

and

$$I_3 = 2\int_\Omega \Delta w_2 v_t dx.$$

To proceed the estimates of I_i, $i = 2, 3$, we observe that

$$\begin{aligned} |I_2| \leq 2b\|f(-\Delta u_1) - f(-\Delta u_2)\|_2.\|v_t\|_2 &\leq 2bL_f\|\Delta u_1 - \Delta u_2\|_2.\|v_t\|_2 \\ &\leq 2bL_f e(u_1 - u_2)^{1/2} e(v(t))^{1/2}, \end{aligned} \tag{3.27}$$

and

$$|I_3| \leq \|\Delta w_2\|_2.\|v_t\|_2 \leq R_0^2 e(v(t))^{1/2}. \tag{3.28}$$

Thus, by using (3.27) and (3.28) in (3.26), we get

$$e(v(t)) \leq \int_0^t [2bL_f e(u_1 - u_2)^{1/2} e(v(s))^{1/2} + R_0^2 e(v(s))^{1/2}]ds.$$

So, from Gronwall's inequality, it follows that

$$e(v(t)) \leq 4b^2 L_f^2 T^2 e^{R_0^2 T} \sup_{0\leq t\leq T} e(u_1 - u_2).$$

By (3.20), we have

$$d(w_1, w_2) \leq C(T, R_0)^{1/2} d(u_1, u_2), \tag{3.29}$$

where $C(T, R_0) = 4b^2 L_f^2 T^2 e^{R_0^2 T}$. Hence, under inequality (3.24), B is a contraction mapping if $C(T, R_0) < 1$. Indeed, we choose R_0 to be sufficiently large and T to be sufficiently small so that (3.24) and (3.29) are simultaneously satisfied. By applying the contraction mapping theorem, we obtain the local existence result.

Remark 3.3. *The application of the contraction mapping theorem in Theorem 3.2 guarantees the existence of a unique local solution $w(t)$ defined in the ball $B(0, R_0) \subset H_0^2(\Omega)$. Since $U \times (H_0^1(\Omega) \cap H^2(\Omega))$ is dense in $H_0^2(\Omega) \times L^2(\Omega)$, we can obtain the similar priori estimates in Theorem 3.2 for $\|w(t)\|_{H_0^2(\Omega)}$ and this norm remains bounded as $t \to T_\infty$. So, we can conclude that the solution can be extended to the whole space $H_0^2(\Omega)$.*

Next, we define a weak solution for the initial and boundary value problem, as follows:

Definition 3.4. A weak solution to the problems (1.1)–(1.3) on $(0, T)$ is any function $w \in C(0, T; H_0^2(\Omega)) \cap C(0, T; L^2(\Omega))$, with $w_0 \in H_0^2(\Omega)$ and $w_1 \in L^2(\Omega)$ verifying

$$\begin{aligned} &\int_0^T \int_\Omega (-w_t\phi_t + \Delta w \Delta\phi + \nabla w \nabla\phi) d\Omega dt + \int_0^T \int_\Gamma (a\Delta w_t \phi) d\Gamma dt \\ = \ & -\int_\Omega (w_t\phi)|_0^T + b\int_0^T \int_\Omega f(-\Delta w)\phi d\Omega dt, \end{aligned}$$

for all test functions ϕ in $C(0, T; U) \cap C(0, T; L^2(\Omega))$.

4. Blow-up

In this section, we study the existence of blow-up solutions for the initial and boundary value problems (1.1)–(1.3). We recall the definition for blow-up of the solutions to the problems (1.1)–(1.3).

Definition 4.1. Suppose w is a solution to (1.1)–(1.3) in the maximal existence time interval $[0, T_\infty)$, $0 < T_\infty \leq \infty$. Then w blows up at T_∞ if $\limsup_{t \to T_\infty, t < T_\infty} \|w\|_2 = +\infty$.

We introduce the energy functional $E(t)$ as:

$$E(t) := \|\nabla w_t\|_2^2 + \|\Delta w\|_2^2 + \|\nabla \Delta w\|_2^2 - 2b\langle F(-\Delta w), 1\rangle. \tag{4.1}$$

Furthermore, we define the function $\psi(t)$ as follows:

$$\psi(t) = \|\nabla w\|_2^2 + \int_0^t \int_\Gamma a(\Delta w)^2 d\sigma ds + \int_\Gamma a(\Delta w_0)^2 d\sigma. \tag{4.2}$$

The subsequent lemma demonstrates that our energy functional $E(t)$ defined in (4.1) is a non-increasing function.

Lemma 4.2. *Under the assumption* (2.1) *for the energy function $E(t), t > 0$, the inequality $E(t) \leq E(0)$ holds.*

Proof. Multiplying Eq (1.1) by $-2\Delta w_t$ in $L^2(\Omega)$ yields the equality:

$$-2\int_\Omega w_{tt}\Delta w_t dx + 2\int_\Omega \Delta w \Delta w_t dx - 2\int_\Omega \Delta^2 w \Delta w_t dx = -2b\int_\Omega f(-\Delta w)\Delta w_t dx. \tag{4.3}$$

By using Green's Formula and the boundary conditions (1.2), we obtain

$$\frac{d}{dt}[\|\nabla w_t\|_2^2 + \|\Delta w\|_2^2 + \|\nabla \Delta w\|_2^2 - 2b\langle F(-\Delta w), 1\rangle] = -2\int_\Gamma a(\Delta w_t)^2 d\sigma.$$

Then, we have,

$$\frac{d}{dt}E(t) = -2\int_{\Gamma} a(\Delta w_t)^2 d\sigma. \tag{4.4}$$

It is obvious from (4.4) that $E(t) \le E(0)$ for all $t \ge 0$.

Theorem 4.3. *Under the assumptions on the parameter of our problem, the functional $\psi(t)$ given by* (4.2) *satisfies the following inequality:*

$$\psi''(t)\psi(t) - (\gamma+1)[\psi'(t)]^2 + d_0\psi(t) \ge 0,$$

where

$$d_0 := 2(2\gamma+1)E(0) + 2(\gamma+1)\int_{\Gamma} a(\Delta w_0)^2 d\sigma.$$

Proof. Differentiating the function ψ defined in Eq (4.2) for t, we obtain

$$\psi'(t) = 2\langle \nabla w, \nabla w_t\rangle + 2\int_0^t\int_{\Gamma} a\Delta w \Delta w_t d\sigma ds + \int_{\Gamma} a(\Delta w_0)^2 d\sigma. \tag{4.5}$$

Taking one more derivative with respect to t and utilizing Green's formula gives:

$$\begin{aligned}
\psi''(t) &= 2\|\nabla w_t\|_2^2 + 2\langle \nabla w, \nabla w_{tt}\rangle + 2a\int_{\Gamma} \Delta w\Delta w_t d\sigma \\
&= 2\|\nabla w_t\|_2^2 - 2\int_{\Omega} w_{tt}\Delta w + 2\int_{\Gamma}\frac{\partial w}{\partial \eta} w_{tt} d\sigma + 2a\int_{\Gamma}\Delta w\Delta w_t d\sigma \\
&= 2\|\nabla w_t\|_2^2 - 2\int_{\Omega}(\Delta w - \Delta^2 w + bf(-\Delta w))\Delta w dx + 2a\int_{\Gamma}\Delta w\Delta w_t d\sigma.
\end{aligned}$$

Since

$$2\int_{\Omega}\Delta w\Delta^2 w dx = 2\int_{\Gamma}\frac{\partial \Delta w}{\partial \eta}\Delta w d\sigma - 2\int_{\Omega}\nabla(\Delta w)\nabla(\Delta w)dx,$$

we obtain,

$$\begin{aligned}
\psi''(t) &= 2\|\nabla w_t\|_2^2 - 2\|\Delta w\|_2^2 - 2\|\nabla\Delta w\|_2^2 + 2b\langle f(-\Delta w), -\Delta w\rangle \\
&+ \underbrace{2\int_{\Gamma}\frac{\partial \Delta w}{\partial \eta}\Delta w d\sigma + 2a\Delta w\Delta w_t d\sigma}_{=0}.
\end{aligned}$$

By using the inequality (2.1) we have,

$$\begin{aligned}
\psi''(t) &\ge 2\|\nabla w_t\|_2^2 - 2\|\Delta w\|_2^2 - 2\|\nabla\Delta w\|_2^2 + 4b(2\gamma+1)\langle F(-\Delta w), 1\rangle \\
&= -2(2\gamma+1)E(t) + 4(\gamma+1)\|\nabla w_t\|_2^2 + 4\gamma\|\Delta w\|_2^2 + 4\gamma\|\nabla\Delta w\|_2^2.
\end{aligned} \tag{4.6}$$

Thus, we obtain from the inequalities (4.6) and (4.4) that

$$
\begin{aligned}
\psi''(t) &\geq -2(2\gamma+1)E(0) + 4(2\gamma+1)\int_0^t \int_\Gamma a(\Delta w_t)^2 d\sigma ds \\
&+ 4(\gamma+1)\|\nabla w_t\|_2^2 + 4\gamma\|\Delta w\|_2^2 + 4\gamma\|\nabla\Delta w\|_2^2 \\
&\geq 4(\gamma+1)\left[\|\nabla w_t\|_2^2 + \int_0^t \int_\Gamma a(\Delta w_t)^2 d\sigma ds + \frac{1}{2}\int_\Gamma a(\Delta w_0)^2 d\sigma\right] - d_0.
\end{aligned}
$$

Multiplying both sides of the following inequality by $\psi(t)$:

$$
\psi''(t) \geq 4(\gamma+1)\underbrace{\left[\|\nabla w_t\|_2^2 + \int_0^t \int_\Gamma a(\Delta w_t)^2 d\sigma ds + \frac{1}{2}\int_\Gamma a(\Delta w_0)^2 d\sigma\right]}_{A} - d_0,
$$

we get

$$
\psi''(t)\psi(t) \geq 4(\gamma+1)A\psi(t) - d_0\psi(t). \tag{4.7}
$$

From (4.5), we obtain:

$$
(1+\gamma)[\psi'(t)]^2 = 4(1+\gamma)\Big[\langle\nabla w, \nabla w_t\rangle + \int_0^t \int_\Gamma a\Delta w \Delta w_t d\sigma ds + \frac{1}{2}\int_\Gamma a(\Delta w_0)^2 d\sigma\Big]^2. \tag{4.8}
$$

Applying Schwartz's and Hölder's inequalities, we obtain:

$$
\begin{aligned}
(1+\gamma)[\psi'(t)]^2 &\leq 4(1+\gamma)\Big[\|\nabla w\|_2.\|\nabla w_t\|_2 + \Big\{\int_0^t \Big[\int_\Gamma a(\Delta w)^2 d\sigma\Big] ds\Big\}^{\frac{1}{2}}. \\
&\qquad \Big\{\int_0^t \Big[\int_\Gamma a(\Delta w_t)^2 d\sigma\Big] ds\Big\}^{\frac{1}{2}} + \frac{1}{2}\int_\Gamma a(\Delta w_0)^2 d\sigma\Big]^2.
\end{aligned} \tag{4.9}
$$

Now, we introduce the following notations:

$$
X := \|\nabla w\|_2, \qquad X' := \Big\{\int_0^t \Big[\int_\Gamma a(\Delta w)^2 d\sigma\Big] ds\Big\}^{\frac{1}{2}},
$$

$$
Y := \|\nabla w_t\|_2, \qquad Y' := \Big\{\int_0^t \Big[\int_\Gamma a(\Delta w_t)^2 d\sigma\Big] ds\Big\}^{\frac{1}{2}}, \qquad Z := \int_\Gamma a(\Delta w_0)^2 d\sigma.
$$

Hence, from (4.9), we have

$$
\begin{aligned}
&4(1+\gamma)\Big[XY + X'Y' + \frac{Z}{2}\Big]^2 \\
=\ &4(1+\gamma)\Big[\Big(X^2Y^2 + (X')^2(Y')^2 + \frac{Z^2}{4}\Big) + 2\Big(XYX'Y' + XY\frac{Z}{2} + X'Y'\frac{Z}{2}\Big)\Big].
\end{aligned}
$$

By Cauchy's inequality, we obtain

$$
XYZ \leq \Big(\frac{X^2}{2} + \frac{Y^2}{2}\Big)Z \qquad \text{and} \qquad X'Y'Z \leq \Big(\frac{(X')^2}{2} + \frac{(Y')^2}{2}\Big)Z.
$$

On the other hand,

$$4(1+\gamma)A\psi(t) = 4(1+\gamma)[Y^2 + (Y')^2 + \frac{Z}{2}][X^2 + (X')^2 + \frac{Z}{2}]$$
$$= 4(1+\gamma)[X^2Y^2 + (X')^2Y^2 + X^2(Y')^2 + Y^2C + (X')^2Y^2 + (Y')^2C + X^2\frac{Z}{2} + (X')^2\frac{Z}{2} + \frac{Z^2}{2}],$$

and we also have

$$X^2(Y')^2 + (X')^2Y^2 = (XY' - X'Y)^2 + 2XX'YY',$$

so, we get

$$(\gamma+1)[\psi'(t)]^2 \leq 4(\gamma+1)A\psi(t). \tag{4.10}$$

Consequently, by subtracting (4.10) from (4.7), we obtain,

$$\psi''(t)\psi(t) - (\gamma+1)[\psi'(t)]^2 + d_0\psi(t) \geq 0,$$

as desired.

Theorem 4.4. *For each fixed $w_0 \in W_0^{1,p}(\Omega)$, there exists $w_1 \in L^2(\Omega)$ satisfying the conditions*

$$(\psi'(0))^2 > \frac{2\beta}{2\alpha-1}\psi(0), \qquad E(0) > 0. \tag{4.11}$$

Hence, by Lemma 2.2 we have the following upper bound for the existence time $T_0 = T_0(u_0, u_1) > 0$ of the solution:

$$T_0 \leq \psi^{1-\alpha}(0)A^{-1}, \quad \lim_{t\uparrow T_\infty} \psi(t) = +\infty \ \text{ for } \ T_\infty \geq T_0,$$

where

$$\alpha = 1+\gamma, \quad \beta = 2(2\gamma+1)E(0) + 2(\gamma+1)\int_\Gamma a(\Delta w_0)^2 d\sigma,$$

and

$$E(0) = \|\nabla w_1\|_2^2 + \|\Delta w_0\|_2^2 + \|\nabla\Delta w_0\|_2^2 - 2b\int_\Gamma F(\Delta w_0)dx, \tag{4.12}$$

with

$$\psi(0) = \|\nabla w_0\|_2^2 + \int_\Gamma a(\Delta w_0)^2 d\sigma, \quad \psi'(0) = 2\langle\nabla w_0, \nabla w_1\rangle + \int_\Gamma a(\Delta w_0)^2 d\sigma.$$

Proof. It is sufficient to prove the resulting conditions in (4.11) are compatible. Firstly, we choose a non-trivial initial function $w_0(x) \in W_0^{1,p}(\Omega)$ in such a way that

$$\int_\Gamma F(\Delta w_0)dx + \frac{4a^{1/2}\|\nabla w_0\|_2^2 \int_\Gamma (\Delta w_0)^2 d\sigma + a^2 \int_\Gamma (\Delta w_0)^4}{8b\|\nabla w_0\|_2^2 + 8ba\int_\Gamma (\Delta w_0)^2 d\sigma} > \frac{\|\nabla w_0\|_2^2 \int_\Gamma (\Delta w_0)^2 d\sigma}{2b(\|\nabla w_0\|_2^2 + \int_\Gamma a(\Delta w_0)^2}$$
$$+ \frac{a(1+\gamma)}{2b(1+2\gamma)}\int_\Gamma \Delta w_0^2 d\sigma + \frac{\|\Delta w_0\|_2^2}{2b} + \frac{\|\nabla\Delta w_0\|_2^2}{2b}. \tag{4.13}$$

Fix $w_0(x)$ and put $w_1(x) = \lambda w_0(x)$ with $\lambda > 0$ so large that the initial energy is guaranteed to be positive:

$$E(0) = \lambda^2\|\nabla w_0\|_2^2 + \|\Delta w_0\|_2^2 + \|\nabla\Delta w_0\|_2^2 - 2b\int_\Gamma F(\Delta w_0)dx > 0.$$

Note that $\psi'(0) = 2\lambda\|\nabla w_0\|_2^2 + \int_\Gamma a(\Delta w_0)^2 > 0$. Then the condition (4.11) takes the form,

$$\begin{aligned}
& 4\lambda^2\|\nabla w_0\|_2^4 + 4\lambda\|\nabla w_0\|_2^2\int_\Gamma a\Delta w_0^2 d\sigma + \Big(\int_\Gamma a\Delta w_0^2\Big)^2 \\
> \;& \frac{1}{1+2\gamma}\Big(4(1+2\gamma)E(0) + 4(1+\gamma)\int_\Gamma a\Delta w_0^2 d\sigma\Big).\Big(\|\nabla w_0\|_2^2 + \int_\Gamma a\Delta w_0^2 d\sigma\Big) \\
= \;& \Big(4E(0) + 4\Big(\frac{1+\gamma}{1+2\gamma}\Big)\int_\Gamma a\Delta w_0^2 d\sigma)\Big).\Big(\|\nabla w_0\|_2^2 + \int_\Gamma a\Delta w_0^2 d\sigma\Big) \\
= \;& \Big(4\lambda^2\|\nabla w_0\|_2^2 + 4\|\Delta w_0\|_2^2 + 4\|\nabla\Delta w_0\|_2^2 - 8b\int_\Gamma F(-\Delta w_0)dx + 4\Big(\frac{1+\gamma}{1+2\gamma}\Big)\int_\Gamma a\Delta w_0^2 d\sigma\Big) \\
. \;& \Big(\|\nabla w_0\|_2^2 + \int_\Gamma a\Delta w_0^2 d\sigma\Big) \\
= \;& 4\lambda^2\|\nabla w_0\|_2^4 + 4\lambda\|\nabla w_0\|_2^2\int_\Gamma a\Delta w_0^2 d\sigma + \Big(4\Big(\frac{1+\gamma}{1+2\gamma}\Big)\int_\Gamma a\Delta w_0^2 d\sigma + 4\|\Delta w_0\|_2^2 \\
+ \;& 4\|\nabla\Delta w_0\|_2^2 - 8b\int_\Gamma F(-\Delta w_0)dx\Big).\Big(\|\nabla w_0\|_2^2 + \int_\Gamma a\Delta w_0^2 d\sigma\Big).
\end{aligned} \tag{4.14}$$

Write $\lambda = 1/a^{1/2}$, where $a > 0$. Then a series of the transformations in (4.14) yields the inequality that coincides with (4.13). This proves that the conditions (4.11) are compatible for sufficiently small $a > 0$.

Remark 4.5. *Consider the function f from Assumption (A2) and the functions w_0 and w_1 that satisfy the following conditions:*
(i) By Theorem 4.4, the bounded function ψ defined in Eq (4.2) *and its derivative ψ' satisfy Lemma 2.2.*
(ii) Additionally, the initial energy functional $E(0)$ defined in Eq (4.12) *is positive.*
Therefore, a positive number exists $T > 0$ such as $T < T_\infty$, where $\psi(t) \to +\infty$ as $t \to T_\infty$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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