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Research article

Hopf algebra structures on generalized quaternion algebras

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Abstract: In this paper, we use elementary linear algebra methods to explore possible Hopf algebra structures within the generalized quaternion algebra. The sufficient and necessary conditions that make the generalized quaternion algebra a Hopf algebra are given. It is proven that not all of the generalized quaternion algebras have Hopf algebraic structures. When the generalized quaternion algebras have Hopf algebraic structures, we describe all the Hopf algebra structures. Finally, we shall prove that all the Hopf algebra structures on the generalized quaternion algebras are isomorphic to Sweedler Hopf algebra, which is consistent with the classification of 4-dimensional Hopf algebras.

Keywords: coalgebra; generalized quaternion; Hopf algebra; Sweedler Hopf algebra

1. Introduction

Representing the natural extension of complex numbers, the quaternion is an effective way of understanding many aspects of physics and kinematics. Nowadays, the quaternion is used especially in the areas of computer vision, computer graphics, animation, and to solve optimization problems involving the estimation of rigid body transformations as well. Brand proved De Moivre's theorem and used it to find the n-th roots of a quaternion [1]. These formulas are also investigated in the cases of dual and split quaternions [2,3]. Generally, a brief introduction to the generalized quaternions is provided in [4], and some algebraic properties of the generalized quaternion have been studied in [5]. Recently, great progress has been made in [6–8].

A Hopf algebra is a bialgebra with an endomorphism satisfying a condition that can be expressed using the algebra and coalgebra structures. The first example of such a structure was observed in algebraic topology by Hopf in 1941. Starting in the late 1960s, Hopf algebras became a subject of study from a strictly algebraic point of view. Perhaps one of the most striking aspects of Hopf algebras is their extraordinary ubiquity in virtually all fields of mathematics: from number theory to algebraic geometry, Lie theory, Galois theory, separable field extension, representation, quantum mechanics, and the list may go on. Refer to [9] and [10] for more knowledge about Hopf algebras.

For the generalized quaternion algebra, its algebraic structure is influenced by two parametric variables. A natural question occurs to us: "what is the Hopf algebra structure on generalized quaternion algebra". The intuition tells us that the changes in two parametric variables may influence the Hopf algebra structures in generalized quaternion algebra. Such influences have aroused our interest. The purpose of this paper is to discuss the Hopf algebra structures in generalized quaternion algebra in order to understand more information on generalized quaternion algebra.

The paper is organized as follows: In Section 2, we recall some basic definitions and results for generalized quaternion algebra and Hopf algebras. The conditions that make the generalized quaternion algebra a Hopf algebra are given in Section 3. In Section 4, we discuss what kinds of generalized quaternion algebras have Hopf algebra structures. When the generalized quaternion algebras have Hopf algebra structures, we describe all the Hopf algebra structures on them. It is proved that all the Hopf algebra structures on the generalized quaternion algebras are isomorphic to Sweedler Hopf algebra.

2. Preliminaries

Throughout the paper, $\mathbb{C}$ denotes the complex number field, $\mathbb{C}^* = \mathbb{C} - \{0\}$. All algebras and coalgebras are over $\mathbb{C}$ and linear means $\mathbb{C}$-linear. Unadorned tensor products $\otimes$ are supposed to be over $\mathbb{C}$. id and $\circ$ mean the identity map and the compound operation of maps. Given a matrix M, M^T means the transpose of M, and

$$M(n) = \begin{bmatrix} M & & 0 \\ & \ddots_{n \text{ times}} & \\ 0 & & M \end{bmatrix},$$

for example, $M(1) = M, M(2) = \begin{pmatrix} M & 0 \\ 0 & M \end{pmatrix}$.

2.1. Coalgebras

A coalgebra (with counit) is a vector space C together with two linear maps, comultiplication $\Delta : C \to C \otimes C$ and counit $\varepsilon : C \to \mathbb{C}$, such that the following conditions hold:

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta, \tag{2.1}$$

$$(id \otimes \varepsilon) \circ \Delta = id = (\varepsilon \otimes id) \circ \Delta. \tag{2.2}$$

For a coalgebra (C, Δ, ε), we use Sweedler-Heynemann notation with the summation sign suppressed for the comultiplication map Δ, namely

$$\Delta(c) = c_1 \otimes c_2,$$

for all $c \in C$.

2.2. *Hopf algebras*

A space H is a bialgebra, if $(H, m, 1_H)$ is an algebra, (H, Δ, ε) is a coalgebra, such that the following conditions hold:

(1) Δ is an algebraic morphism,

(2) ε is an algebraic morphism.

Let $(H, m, 1_H, \Delta, \varepsilon)$ be a bialgebra. Then H is a Hopf algebra, if there exists an element $S : H \to H$ (called an antipode) such that the following conditions hold:

$$S(h_1)h_2 = \varepsilon(h)1_H = h_1S(h_2). \tag{2.3}$$

Let $(H, m, 1_H, \Delta, \varepsilon, S)$ and $(H', m', 1_{H'}, \Delta', \varepsilon', S')$ be two Hopf algebras. A linear map $\sigma : H \to H'$ is called a bialgebra homomorphism, if the following conditions hold:

$$\sigma \circ m = m' \circ (\sigma \otimes \sigma), \sigma(1_H) = 1_{H'}, \Delta' \circ \sigma = (\sigma \otimes \sigma) \circ \Delta, \varepsilon' \circ \sigma = \varepsilon.$$

If σ is a bialgebra homomorphism satisfying $S' \circ \sigma = \sigma \circ S$, then σ is called a Hopf algebra homomorphism. As far as we know, a bialgebra homomorphism is also a Hopf homomorphism. If σ is a bijection, then we call σ a Hopf algebra isomorphism from H to H'. We call H isomorphic to H' if there exists a Hopf algebra isomorphism between H and H'.

2.3. *Sweedler Hopf algebra*

Sweedler Hopf algebra $\mathbb{H}_4$ is generated by two elements g and v, which satisfy

$$g^2 = 1, v^2 = 0, gv + vg = 0.$$

The comultiplication, the antipode, and the counit of $\mathbb{H}_4$ are given by

$$\Delta(g) = g \otimes g, \Delta(v) = g \otimes v + v \otimes 1, \varepsilon(g) = 1, \varepsilon(v) = 0, S(g) = g, S(v) = -gv.$$

Notice that the dimension of $\mathbb{H}_4$ is four, and $1, g, v, gv$ form a basis for $\mathbb{H}_4$.

2.4. *Generalized quaternion algebras*

A generalized quaternion q is an expression of the form

$$q = a_0E_0 + a_1E_1 + a_2E_2 + a_3E_3,$$

where a_0, a_1, a_2, a_3 are complex numbers and E_0, E_1, E_2, E_3 satisfy the following equalities:

$$E_1^2 = -\alpha E_0, E_2^2 = -\beta E_0, E_3^2 = -\alpha\beta E_0, E_1E_2 = E_3 = -E_2E_1,$$

$$E_2E_3 = \beta E_1 = -E_3E_2, E_3E_1 = \alpha E_2 = -E_1E_3, E_0E_i = E_iE_0 = E_i,$$

where $\alpha, \beta \in \mathbb{C}, i = 0, 1, 2, 3$. The set of all generalized quaternions is denoted by $\mathbb{H}_{\alpha\beta}$ [11]. The addition and multiplication of $H_{\alpha\beta}$ are given as follows:

$$(a_0E_0 + a_1E_1 + a_2E_2 + a_3E_3) + (b_0E_0 + b_1E_1 + b_2E_2 + b_3E_3)$$

$$= (a_0 + b_0)E_0 + (a_1 + b_1)E_1 + (a_2 + b_2)E_2 + (a_3 + b_3)E_3,$$

$$\begin{aligned}&(a_0E_0 + a_1E_1 + a_2E_2 + a_3E_3)(b_0E_0 + b_1E_1 + b_2E_2 + b_3E_3)\\ &= (a_0b_0 - \alpha a_1b_1 - \beta a_2b_2 - \alpha\beta a_3b_3)E_0 + (a_1b_0 + a_0b_1 - \beta a_3b_2 + \beta a_2b_3)E_1\\ &\quad +(a_2b_0 + \alpha a_3b_1 + a_0b_2 - \alpha a_1b_3)E_2 + (a_3b_0 - a_2b_1 + a_1b_2 + a_0b_3)E_3.\end{aligned}$$

It is easily checked that $\mathbb{H}_{\alpha\beta}$, with the above addition and multiplication, is an associative algebra. We call $\mathbb{H}_{\alpha\beta}$ an algebra of generalized quaternions (or generalized quaternion algebra). Special cases are considered as follows:

- If $\alpha = \beta = 1$, then $\mathbb{H}_{\alpha\beta}$ is the algebra of real quaternions.
- If $\alpha = 1, \beta = -1$, then $\mathbb{H}_{\alpha\beta}$ is the algebra of split quaternions.
- If $\alpha = 1, \beta = 0$, then $\mathbb{H}_{\alpha\beta}$ is the algebra of semi-quaternions.
- If $\alpha = -1, \beta = 0$, then $\mathbb{H}_{\alpha\beta}$ is the algebra of split semi-quaternions,denoted by $\mathbb{H}_{ss}$.
- If $\alpha = 0, \beta = 0$, then $\mathbb{H}_{\alpha\beta}$ is the algebra of $\frac{1}{4}$-quaternions [12].

3. Hopf algebra structures on $\mathbb{H}_{\alpha\beta}$

In this section, we shall discuss the conditions that make $\mathbb{H}_{\alpha\beta}$ be a Hopf algebra. Let

$$e_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, e_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, e_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, e_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

be the standard basis of $\mathbb{C}^4$.

For the sake of convenience, we adopt $E_{ij} = E_i \otimes E_j, i, j = 0, 1, 2, 3$ and

$$\mathbf{E_{0,1,2,3}} = (E_{00}, E_{10}, E_{20}, E_{30}, E_{01}, E_{11}, E_{21}, E_{31}, E_{02}, E_{12}, E_{22}, E_{32}, E_{03}, E_{13}, E_{23}, E_{33}).$$

Define a map

$$\Delta : \mathbb{H}_{\alpha\beta} \to \mathbb{H}_{\alpha\beta} \otimes \mathbb{H}_{\alpha\beta}$$

as follows:

$$\Delta(E_0) = \mathbf{E_{0,1,2,3}}X_0,\ \Delta(E_1) = \mathbf{E_{0,1,2,3}}X_1,$$

$$\Delta(E_2) = \mathbf{E_{0,1,2,3}}X_2,\ \Delta(E_3) = \mathbf{E_{0,1,2,3}}X_3,$$

where

$$X_0 = \begin{pmatrix} a_{00} \\ a_{10} \\ a_{20} \\ a_{30} \\ a_{01} \\ a_{11} \\ a_{21} \\ a_{31} \\ a_{02} \\ a_{12} \\ a_{22} \\ a_{32} \\ a_{03} \\ a_{13} \\ a_{23} \\ a_{33} \end{pmatrix} = \begin{pmatrix} \gamma_0 \\ \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{pmatrix}, \gamma_i = \begin{pmatrix} a_{0i} \\ a_{1i} \\ a_{2i} \\ a_{3i} \end{pmatrix}, A = (\gamma_0, \gamma_1, \gamma_2, \gamma_3), X_1 = \begin{pmatrix} b_{00} \\ b_{10} \\ b_{20} \\ b_{30} \\ b_{01} \\ b_{11} \\ b_{21} \\ b_{31} \\ b_{02} \\ b_{12} \\ b_{22} \\ b_{32} \\ b_{03} \\ b_{13} \\ b_{23} \\ b_{33} \end{pmatrix} = \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}, \eta_i = \begin{pmatrix} b_{0i} \\ b_{1i} \\ b_{2i} \\ b_{3i} \end{pmatrix}, B = (\eta_0, \eta_1, \eta_2, \eta_3),$$

$$X_2 = \begin{pmatrix} c_{00} \\ c_{10} \\ c_{20} \\ c_{30} \\ c_{01} \\ c_{11} \\ c_{21} \\ c_{31} \\ c_{02} \\ c_{12} \\ c_{22} \\ c_{32} \\ c_{03} \\ c_{13} \\ c_{23} \\ c_{33} \end{pmatrix} = \begin{pmatrix} \zeta_0 \\ \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{pmatrix}, \zeta_i = \begin{pmatrix} c_{0i} \\ c_{1i} \\ c_{2i} \\ c_{3i} \end{pmatrix}, C = (\zeta_0, \zeta_1, \zeta_2, \zeta_3), X_3 = \begin{pmatrix} d_{00} \\ d_{10} \\ d_{20} \\ d_{30} \\ d_{01} \\ d_{11} \\ d_{21} \\ d_{31} \\ d_{02} \\ d_{12} \\ d_{22} \\ d_{32} \\ d_{03} \\ d_{13} \\ d_{23} \\ d_{33} \end{pmatrix} = \begin{pmatrix} \xi_0 \\ \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}, \xi_i = \begin{pmatrix} d_{0i} \\ d_{1i} \\ d_{2i} \\ d_{3i} \end{pmatrix}, D = (\xi_0, \xi_1, \xi_2, \xi_3).$$

Let $\mathbf{X} = (X_0, X_1, X_2, X_3)$, which is useful in the following discussions. Notice that $\Delta(E_0) = E_0 \otimes E_0$ if and only if $X_0 = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)^T$. In the definition of Hopf algebra, Δ is an algebra homomorphism, so it should preserve the unit element. From now on, we will assume that $\Delta(E_0) = E_0 \otimes E_0$. So $A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Lemma 3.1. Δ *satisfies the condition (2.1) if and only if the following equalities hold:*

$$\mathbf{X}B = B(4)(e_1(4), e_2(4), e_3(4), e_4(4))\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}, \tag{3.1}$$

$$\mathbf{X}C = C(4)(e_1(4), e_2(4), e_3(4), e_4(4))\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}, \tag{3.2}$$

$$\mathbf{X}D = D(4)(e_1(4), e_2(4), e_3(4), e_4(4))\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}. \tag{3.3}$$

Proof. For E_1, since

$$\begin{aligned}
& (\Delta \otimes id) \circ \Delta(E_1) \\
= \; & (\Delta \otimes id)(\sum_{j=0}^{3} b_{0j}E_{0j} + \sum_{j=0}^{3} b_{1j}E_{1j} + \sum_{j=0}^{3} b_{2j}E_{2j} + \sum_{j=0}^{3} b_{3j}E_{3j}) \\
= \; & \sum_{j=0}^{3}\sum_{k,l=0}^{3} b_{0j}a_{kl}E_k \otimes E_l \otimes E_j + \sum_{j=0}^{3}\sum_{k,l=0}^{3} b_{1j}b_{kl}E_k \otimes E_l \otimes E_j \\
& + \sum_{j=0}^{3}\sum_{k,l=0}^{3} b_{2j}c_{kl}E_k \otimes E_l \otimes E_j + \sum_{j=0}^{3}\sum_{k,l=0}^{3} b_{3j}d_{kl}E_k \otimes E_l \otimes E_j \\
= \; & \sum_{k,l=0}^{3} b_{00}a_{kl}E_k \otimes E_l \otimes E_0 + \sum_{k,l=0}^{3} b_{10}b_{kl}E_k \otimes E_l \otimes E_0 \\
& + \sum_{k,l=0}^{3} b_{20}c_{kl}E_k \otimes E_l \otimes E_0 + \sum_{k,l=0}^{3} b_{30}d_{kl}E_k \otimes E_l \otimes E_0 \\
& + \sum_{k,l=0}^{3} b_{01}a_{kl}E_k \otimes E_l \otimes E_1 + \sum_{k,l=0}^{3} b_{11}b_{kl}E_k \otimes E_l \otimes E_1 \\
& + \sum_{k,l=0}^{3} b_{21}c_{kl}E_k \otimes E_l \otimes E_1 + \sum_{k,l=0}^{3} b_{31}d_{kl}E_k \otimes E_l \otimes E_1 \\
& + \sum_{k,l=0}^{3} b_{02}a_{kl}E_k \otimes E_l \otimes E_2 + \sum_{k,l=0}^{3} b_{12}b_{kl}E_k \otimes E_l \otimes E_2 \\
& + \sum_{k,l=0}^{3} b_{22}c_{kl}E_k \otimes E_l \otimes E_2 + \sum_{k,l=0}^{3} b_{32}d_{kl}E_k \otimes E_l \otimes E_2
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{k,l=0}^{3} b_{03}a_{kl}E_k\otimes E_l\otimes E_3+\sum_{k,l=0}^{3} b_{13}b_{kl}E_k\otimes E_l\otimes E_3\\
&+\sum_{k,l=0}^{3} b_{23}c_{kl}E_k\otimes E_l\otimes E_3+\sum_{k,l=0}^{3} b_{33}d_{kl}E_k\otimes E_l\otimes E_3\\
=&\sum_{k,l=0}^{3}(b_{00}a_{kl}+b_{10}b_{kl}+b_{20}c_{kl}+b_{30}d_{kl})E_k\otimes E_l\otimes E_0\\
&+\sum_{k,l=0}^{3}(b_{01}a_{kl}+b_{11}b_{kl}+b_{21}c_{kl}+b_{31}d_{kl})E_k\otimes E_l\otimes E_1
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{k,l=0}^{3}(b_{02}a_{kl}+b_{12}b_{kl}+b_{22}c_{kl}+b_{32}d_{kl})E_k\otimes E_l\otimes E_2\\
&+\sum_{k,l=0}^{3}(b_{03}a_{kl}+b_{13}b_{kl}+b_{23}c_{kl}+b_{33}d_{kl})E_k\otimes E_l\otimes E_3
\end{aligned}
$$

and

$$
\begin{aligned}
&(id\otimes\Delta)\circ\Delta(E_1)\\
=&(id\otimes\Delta)(\sum_{j=0}^{3} b_{j0}E_{j0}+\sum_{j=0}^{3} b_{j1}E_{j1}+\sum_{j=0}^{3} b_{j2}E_{j2}+\sum_{j=0}^{3} b_{j3}E_{j3})\\
=&\sum_{j=0}^{3}\sum_{k,l=0}^{3} b_{j0}a_{kl}E_j\otimes E_k\otimes E_l+\sum_{j=0}^{3}\sum_{k,l=0}^{3} b_{j1}b_{kl}E_j\otimes E_k\otimes E_l\\
&+\sum_{j=0}^{3}\sum_{k,l=0}^{3} b_{j2}c_{kl}E_j\otimes E_k\otimes E_l+\sum_{j=0}^{3}\sum_{k,l=0}^{3} b_{j3}d_{kl}E_j\otimes E_k\otimes E_l\\
=&\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j0}a_{k0}E_j\otimes E_k\otimes E_0+\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j1}b_{k0}E_j\otimes E_k\otimes E_0\\
&+\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j2}c_{k0}E_j\otimes E_k\otimes E_0+\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j3}d_{k0}E_j\otimes E_k\otimes E_0\\
&+\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j0}a_{k1}E_j\otimes E_k\otimes E_1+\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j1}b_{k1}E_j\otimes E_k\otimes E_1\\
&+\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j2}c_{k1}E_j\otimes E_k\otimes E_1+\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j3}d_{k1}E_j\otimes E_k\otimes E_1\\
&+\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j0}a_{k2}E_j\otimes E_k\otimes E_2+\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j1}b_{k2}E_j\otimes E_k\otimes E_2
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j2}c_{k2}E_j\otimes E_k\otimes E_2+\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j3}d_{k2}E_j\otimes E_k\otimes E_2\\
&+\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j0}a_{k3}E_j\otimes E_k\otimes E_3+\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j1}b_{k3}E_j\otimes E_k\otimes E_3\\
&+\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j2}c_{k3}E_j\otimes E_k\otimes E_3+\sum_{j=0}^{3}\sum_{k=0}^{3} b_{j3}d_{k3}E_j\otimes E_k\otimes E_3\\
=\ &\sum_{j=0}^{3}\sum_{k=0}^{3}(b_{j0}a_{k0}+b_{j1}b_{k0}+b_{j2}c_{k0}+b_{j3}d_{k0})E_j\otimes E_k\otimes E_0\\
&+\sum_{j=0}^{3}\sum_{k=0}^{3}(b_{j0}a_{k1}+b_{j1}b_{k1}+b_{j2}c_{k1}+b_{j3}d_{k1})E_j\otimes E_k\otimes E_1
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{j=0}^{3}\sum_{k=0}^{3}(b_{j0}a_{k2}+b_{j1}b_{k2}+b_{j2}c_{k2}+b_{j3}d_{k2})E_j\otimes E_k\otimes E_2\\
&+\sum_{j=0}^{3}\sum_{k=0}^{3}(b_{j0}a_{k3}+b_{j1}b_{k3}+b_{j2}c_{k3}+b_{j3}d_{k3})E_j\otimes E_k\otimes E_3,
\end{aligned}
$$

it follows that, for all $k, l = 0, 1, 2, 3$,

$$b_{00}a_{kl}+b_{10}b_{kl}+b_{20}c_{kl}+b_{30}d_{kl}=b_{k0}a_{l0}+b_{k1}b_{l0}+b_{k2}c_{l0}+b_{k3}d_{l0},$$

$$b_{01}a_{kl}+b_{11}b_{kl}+b_{21}c_{kl}+b_{31}d_{kl}=b_{k0}a_{l1}+b_{k1}b_{l1}+b_{k2}c_{l1}+b_{k3}d_{l1},$$

$$b_{02}a_{kl}+b_{12}b_{kl}+b_{22}c_{kl}+b_{32}d_{kl}=b_{k0}a_{l2}+b_{k1}b_{l2}+b_{k2}c_{l2}+b_{k3}d_{l2},$$

$$b_{03}a_{kl}+b_{13}b_{kl}+b_{23}c_{kl}+b_{33}d_{kl}=b_{k0}a_{l3}+b_{k1}b_{l3}+b_{k2}c_{l3}+b_{k3}d_{l3},$$

which is equivalent to (3.1). For E_2, E_3, as what we do for E_1, it follows that the other equations hold. □

Next, we define a map $\varepsilon : \mathbb{H}_{\alpha\beta} \to \mathbb{C}, \varepsilon(E_0) = 1, \varepsilon(E_i) = x_i, i = 1, 2, 3$. Set $\mathbf{x} = \begin{pmatrix} 1 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$.

Lemma 3.2. *$\varepsilon : \mathbb{H}_{\alpha\beta} \to \mathbb{C}$ satisfies the condition (2.2) if and only if the following equalities hold:*

$$(A, B, C, D)\mathbf{x}(4) = E = (A^T, B^T, C^T, D^T)\mathbf{x}(4) \tag{3.4}$$

Proof. Straightforward. □

Next, we discuss the conditions which make Δ and ε be algebra homomorphisms. Set

$$\mathbb{U}_0=\begin{pmatrix}1&0&0&0\\0&-\alpha&0&0\\0&0&-\beta&0\\0&0&0&-\alpha\beta\end{pmatrix},\mathbb{U}_1=\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&0&\beta\\0&0&-\beta&0\end{pmatrix},$$

$$\mathbb{U}_2=\begin{pmatrix}0&0&1&0\\0&0&0&-\alpha\\1&0&0&0\\0&\alpha&0&0\end{pmatrix},\mathbb{U}_3=\begin{pmatrix}0&0&0&1\\0&0&1&0\\0&-1&0&0\\1&0&0&0\end{pmatrix}.$$

Firstly, we compute

$$\begin{aligned}
&\mathbf{E_{0,1,2,3}}^T\mathbf{E_{0,1,2,3}}\\
=&\begin{pmatrix}\mathbb{U}_0&0&0&0\\0&(-\alpha)\mathbb{U}_0&0&0\\0&0&(-\beta)\mathbb{U}_0&0\\0&0&0&(-\alpha\beta)\mathbb{U}_0\end{pmatrix}E_{00}\\
&+\begin{pmatrix}\mathbb{U}_1&0&0&0\\0&(-\alpha)\mathbb{U}_1&0&0\\0&0&(-\beta)\mathbb{U}_1&0\\0&0&0&(-\alpha\beta)\mathbb{U}_1\end{pmatrix}E_{10}+\begin{pmatrix}\mathbb{U}_2&0&0&0\\0&(-\alpha)\mathbb{U}_2&0&0\\0&0&(-\beta)\mathbb{U}_2&0\\0&0&0&(-\alpha\beta)\mathbb{U}_2\end{pmatrix}E_{20}\\
&+\begin{pmatrix}\mathbb{U}_3&0&0&0\\0&(-\alpha)\mathbb{U}_3&0&0\\0&0&(-\beta)\mathbb{U}_3&0\\0&0&0&(-\alpha\beta)\mathbb{U}_3\end{pmatrix}E_{30}+\begin{pmatrix}0&\mathbb{U}_0&0&0\\\mathbb{U}_0&0&0&0\\0&0&0&\beta\mathbb{U}_0\\0&0&(-\beta)\mathbb{U}_0&0\end{pmatrix}E_{01}
\end{aligned}$$

$$\begin{aligned}
&+\begin{pmatrix}0&\mathbb{U}_1&0&0\\\mathbb{U}_1&0&0&0\\0&0&0&\beta\mathbb{U}_1\\0&0&(-\beta)\mathbb{U}_1&0\end{pmatrix}E_{11}+\begin{pmatrix}0&\mathbb{U}_2&0&0\\\mathbb{U}_2&0&0&0\\0&0&0&\beta\mathbb{U}_2\\0&0&(-\beta)\mathbb{U}_2&0\end{pmatrix}E_{21}\\
&+\begin{pmatrix}0&\mathbb{U}_3&0&0\\\mathbb{U}_3&0&0&0\\0&0&0&\beta\mathbb{U}_3\\0&0&(-\beta)\mathbb{U}_3&0\end{pmatrix}E_{31}+\begin{pmatrix}0&0&\mathbb{U}_0&0\\0&0&0&(-\alpha)\mathbb{U}_0\\\mathbb{U}_0&0&0&0\\0&\alpha\mathbb{U}_0&0&0\end{pmatrix}E_{02}\\
&+\begin{pmatrix}0&0&\mathbb{U}_1&0\\0&0&0&(-\alpha)\mathbb{U}_1\\\mathbb{U}_1&0&0&0\\0&\alpha\mathbb{U}_1&0&0\end{pmatrix}E_{12}+\begin{pmatrix}0&0&\mathbb{U}_2&0\\0&0&0&(-\alpha)\mathbb{U}_2\\\mathbb{U}_2&0&0&0\\0&\alpha\mathbb{U}_2&0&0\end{pmatrix}E_{22}\\
&+\begin{pmatrix}0&0&\mathbb{U}_3&0\\0&0&0&(-\alpha)\mathbb{U}_3\\\mathbb{U}_3&0&0&0\\0&\alpha\mathbb{U}_3&0&0\end{pmatrix}E_{32}+\begin{pmatrix}0&0&0&\mathbb{U}_0\\0&0&\mathbb{U}_0&0\\0&-\mathbb{U}_0&0&0\\\mathbb{U}_0&0&0&0\end{pmatrix}E_{03}
\end{aligned}$$

$$+\begin{pmatrix} 0 & 0 & 0 & \mathbb{U}_1 \\ 0 & 0 & \mathbb{U}_1 & 0 \\ 0 & -\mathbb{U}_1 & 0 & 0 \\ \mathbb{U}_1 & 0 & 0 & 0 \end{pmatrix} E_{13} + \begin{pmatrix} 0 & 0 & 0 & \mathbb{U}_2 \\ 0 & 0 & \mathbb{U}_2 & 0 \\ 0 & -\mathbb{U}_2 & 0 & 0 \\ \mathbb{U}_2 & 0 & 0 & 0 \end{pmatrix} E_{23}$$

$$+\begin{pmatrix} 0 & 0 & 0 & \mathbb{U}_3 \\ 0 & 0 & \mathbb{U}_3 & 0 \\ 0 & -\mathbb{U}_3 & 0 & 0 \\ \mathbb{U}_3 & 0 & 0 & 0 \end{pmatrix} E_{33}.$$

Let N_{ij} be the coefficient matrices of E_{ij}, $i, j = 0, 1, 2, 3$. We can construct the following matrix

$$\mathbf{W} = (N_{00}, N_{10}, N_{20}, N_{30}, N_{01}, N_{11}, N_{21}, N_{31}, N_{02}, N_{12}, N_{22}, N_{32}, N_{03}, N_{13}, N_{23}, N_{33}).$$

Notice that $\mathbf{W}$ is a 16×256 matrix.

Lemma 3.3. *With* $\mathbf{W}$ *defined above,* Δ *is an algebra homomorphism if and only if the following equalities hold:*

$$\mathbf{X}^T\mathbf{W}\mathbf{X}(16)\mathbf{V} = \mathbf{L}_1\mathbf{X}(4)^T, \tag{3.5}$$

where

$$\mathbf{V} = (e_1(16), e_2(16), e_3(16), e_4(16)),$$

$$\mathbf{L}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -\alpha & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\alpha & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -\beta & 0 & 0 & 0 & 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \alpha & 0 & 0 & -\beta & 0 & 0 & -\alpha\beta & 0 & 0 & 0 \end{pmatrix}.$$

Proof. For all $i, j = 0, 1, 2, 3$, since

$$\begin{aligned} & \Delta(E_i)\Delta(E_j) \\ = \;& X_i^T \mathbf{E_{0,1,2,3}}^T \mathbf{E_{0,1,2,3}} X_j \\ = \;& X_i^T N_{00} X_j E_{00} + X_i^T N_{10} X_j E_{10} + \cdots + X_i^T N_{33} X_j \\ = \;& \mathbf{E_{0,1,2,3}} \mathbf{X_j}(16)^T \mathbf{W}^T X_i. \end{aligned}$$

Now, we compute $\Delta(E_iE_j)$, for $i, j = 0, 1, 2, 3$. When $j = 1$, we have

$$\Delta(E_0E_1) = \mathbf{E_{0,1,2,3}} X_1, \Delta(E_1E_1) = \mathbf{E_{0,1,2,3}}(-\alpha X_0),$$

$$\Delta(E_2E_1) = \mathbf{E_{0,1,2,3}}(-X_3), \Delta(E_3E_1) = \mathbf{E_{0,1,2,3}}(\alpha X_2).$$

Moreover,

$$\Delta(E_0E_1, E_1E_1, E_2E_1, E_3E_1) = \mathbf{E_{0,1,2,3}}(X_0, X_1, X_2, X_3) \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \alpha & 0 \end{pmatrix}^T.$$

Similar to what we do for $j = 1$, we can discuss the cases $j = 0, 2, 3$, and get the following equalities:

$$\Delta(E_0E_0, E_1E_0, E_2E_0, E_3E_0) = \mathbf{E_{0,1,2,3}}(X_0, X_1, X_2, X_3)\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^T,$$

$$\Delta(E_0E_2, E_1E_2, E_2E_2, E_3E_2) = \mathbf{E_{0,1,2,3}}(X_0, X_1, X_2, X_3)\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}^T,$$

$$\Delta(E_0E_1, E_1E_1, E_2E_1, E_3E_1) = \mathbf{E_{0,1,2,3}}(X_0, X_1, X_2, X_3)\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\alpha & 0 \\ 0 & \beta & 0 & 0 \\ -\alpha\beta & 0 & 0 & 0 \end{pmatrix}^T.$$

Since

$$\begin{aligned} \Delta(E_0E_1, E_1E_1, E_2E_1, E_3E_1) &= (\Delta(E_0)\Delta(E_1), \Delta(E_1)\Delta(E_1), \Delta(E_2)\Delta(E_1), \Delta(E_3)\Delta(E_1)) \\ &= \mathbf{E_{0,1,2,3}X_1}(16)^T\mathbf{W}^T\mathbf{X}, \end{aligned}$$

it follows that

$$\mathbf{X_1}(16)^T\mathbf{W}^T\mathbf{X} = \mathbf{X}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \alpha & 0 \end{pmatrix}^T.$$

Similarly, we have

$$\mathbf{X_0}(16)^T\mathbf{W}^T\mathbf{X} = \mathbf{X}E,\ \mathbf{X_2}(16)^T\mathbf{W}^T\mathbf{X} = \mathbf{X}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \end{pmatrix}^T,$$

$$\mathbf{X_3}(16)^T\mathbf{W}^T\mathbf{X} = \mathbf{X}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\alpha & 0 \\ 0 & \beta & 0 & 0 \\ -\alpha\beta & 0 & 0 & 0 \end{pmatrix}^T.$$

Using the equalities above, we can get the Eq (3.5).

□

Lemma 3.4. *ε is an algebra homomorphism if and only if the following equality holds:*

$$\mathbf{x}\mathbf{x}^T = \mathbf{L}_1\mathbf{x}(4). \tag{3.6}$$

Proof. Straightforward. □

Let $S : \mathbb{H}_{\alpha\beta} \to \mathbb{H}_{\alpha\beta}$ be a transformation,

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14} \\ k_{21} & k_{22} & k_{23} & k_{24} \\ k_{31} & k_{32} & k_{33} & k_{34} \\ k_{41} & k_{42} & k_{43} & k_{44} \end{pmatrix} = (\omega_1, \omega_2, \omega_3, \omega_4)$$

the matrix of S with respect to the basis E_0, E_1, E_2, E_3. Thus we have

$$S(E_0, E_1, E_2, E_3) = (E_0, E_1, E_2, E_3)K.$$

If S is an antipode, then $S(E_0) = E_0$, which yields $k_{11} = 1, k_{21} = k_{31} = k_{41} = 0$.

Next, we shall discuss when S is an antipode. Firstly, we prove a useful result: for

$$\mu = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix}, \nu = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \in \mathbb{C}^4,$$

we have

$$\mu^T \begin{pmatrix} E_0 \\ E_1 \\ E_2 \\ E_3 \end{pmatrix} (E_0, E_1, E_2, E_3)\nu = \mu^T \mathbb{U} \begin{pmatrix} \nu & & & \\ & \nu & & \\ & & \nu & \\ & & & \nu \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \\ E_2 \\ E_3 \end{pmatrix}, \tag{3.7}$$

where

$$\mathbb{U} = (\mathbb{U}_0, \mathbb{U}_1, \mathbb{U}_2, \mathbb{U}_3).$$

In fact,

$$\begin{aligned} & \mu^T \begin{pmatrix} E_0 \\ E_1 \\ E_2 \\ E_3 \end{pmatrix} (E_0, E_1, E_2, E_3)\nu \\ = \ & \mu^T \begin{pmatrix} E_0 & E_1 & E_2 & E_3 \\ E_0 & -\alpha E_0 & E_3 & -\alpha E_2 \\ E_2 & -E_3 & -\beta E_0 & \beta E_1 \\ E_3 & \alpha E_2 & -\beta E_1 & -\alpha\beta E_0 \end{pmatrix} \nu \\ = \ & \mu^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\alpha & 0 & 0 \\ 0 & 0 & -\beta & 0 \\ 0 & 0 & 0 & -\alpha\beta \end{pmatrix} \nu E_0 + \mu^T \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{pmatrix} \nu E_1 \\ & + \mu^T \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\alpha \\ 1 & 0 & 0 & 0 \\ 0 & \alpha & 0 & 0 \end{pmatrix} \nu E_2 + \mu^T \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \nu E_3 \end{aligned}$$

$$= \mu^T \mathbb{U} \begin{pmatrix} \nu & & & \\ & \nu & & \\ & & \nu & \\ & & & \nu \end{pmatrix} \begin{pmatrix} E_0 \\ E_1 \\ E_2 \\ E_3 \end{pmatrix}.$$

Lemma 3.5. *S satisfies the condition (2.3) if and only if the following equalities hold:*

$$\begin{pmatrix} \omega_1(4) \\ \omega_2(4) \\ \omega_3(4) \\ \omega_4(4) \end{pmatrix}^T \mathbb{U}(4)^T \mathbf{X} = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.8}$$

$$\mathbf{L}_2^T K(4) \mathbf{X} = \begin{pmatrix} 1 & x_1 & x_2 & x_3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.9}$$

where

$$\mathbf{L}_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\beta & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\alpha & 0 \\ 0 & \beta & 0 & 0 \\ -\alpha\beta & 0 & 0 & 0 \end{pmatrix}.$$

Proof. For E_1, on one hand, we have

$$\begin{aligned} x_1 E_0 &= \varepsilon(E_1)E_0 = m(id \otimes S)\Delta(E_1) \\ &= (E_0S(E_0), E_1S(E_0), E_2S(E_0), E_3S(E_0), E_0S(E_1), E_1S(E_1), E_2S(E_1), E_3S(E_1), \\ &\quad E_0S(E_2), E_1S(E_2), E_2S(E_2), E_3S(E_2), E_0S(E_3), E_1S(E_3), E_2S(E_3), E_3S(E_3)) \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \end{pmatrix} \\ &= (E_0S(E_0), E_1S(E_0), E_2S(E_0), E_3S(E_0))\eta_1 + (E_0S(E_1), E_1S(E_1), E_2S(E_1), E_3S(E_1))\eta_2 \\ &\quad + (E_0S(E_2), E_1S(E_2), E_2S(E_2), E_3S(E_2))\eta_3 + (E_0S(E_3), E_1S(E_3), E_2S(E_3), E_3S(E_3))\eta_4 \end{aligned}$$

$$
\begin{aligned}
&= \eta_1^T\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix}S(E_0)+\eta_2^T\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix}S(E_1)+\eta_3^T\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix}S(E_2)+\eta_4^T\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix}S(E_3)\\
&= \eta_1^T\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix}(E_0,E_1,E_2,E_3)\omega_1+\eta_2^T\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix}(E_0,E_1,E_2,E_3)\omega_2\\
&\quad +\eta_3^T\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix}(E_0,E_1,E_2,E_3)\omega_3+\eta_4^T\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix}(E_0,E_1,E_2,E_3)\omega_4\\
&= \eta_1^T\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix}S(E_0)+\eta_2^T\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix}S(E_1)+\eta_3^T\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix}S(E_2)+\eta_4^T\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix}S(E_3)\\
&= \eta_1^T\mathbb{U}\begin{pmatrix}\omega_1&&&\\&\omega_1&&\\&&\omega_1&\\&&&\omega_1\end{pmatrix}\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix}+\eta_2^T\mathbb{U}\begin{pmatrix}\omega_2&&&\\&\omega_2&&\\&&\omega_2&\\&&&\omega_2\end{pmatrix}\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix}\\
&\quad +\eta_3^T\mathbb{U}\begin{pmatrix}\omega_3&&&\\&\omega_3&&\\&&\omega_3&\\&&&\omega_3\end{pmatrix}\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix}+\eta_4^T\mathbb{U}\begin{pmatrix}\omega_4&&&\\&\omega_4&&\\&&\omega_4&\\&&&\omega_4\end{pmatrix}\begin{pmatrix}E_0\\E_1\\E_2\\E_3\end{pmatrix},
\end{aligned}
$$

which follows that

$$\eta_1^T\mathbb{U}\omega_1(4)+\eta_2^T\mathbb{U}\omega_2(4)+\eta_3^T\mathbb{U}\omega_3(4)+\eta_4^T\mathbb{U}\omega_4(4)=(x_1,0,0,0),$$

For E_0,E_2 and E_3, as we do for E_1, it follows that

$$\gamma_1^T\mathbb{U}\omega_1(4)+\gamma_2^T\mathbb{U}\omega_2(4)+\gamma_3^T\mathbb{U}\omega_3(4)+\gamma_4^T\mathbb{U}\omega_4(4)=(1,0,0,0).$$

$$\zeta_1^T\mathbb{U}\omega_1(4)+\zeta_2^T\mathbb{U}\omega_2(4)+\zeta_3^T\mathbb{U}\omega_3(4)+\zeta_4^T\mathbb{U}\omega_4(4)=(x_2,0,0,0),$$

$$\xi_1^T\mathbb{U}\omega_1(4)+\xi_2^T\mathbb{U}\omega_2(4)+\xi_3^T\mathbb{U}\omega_3(4)+\xi_4^T\mathbb{U}\omega_4(4)=(x_3,0,0,0).$$

Thus, we have

$$\begin{pmatrix}\omega_1(4)\\\omega_2(4)\\\omega_3(4)\\\omega_4(4)\end{pmatrix}^T\mathbb{U}(4)^T\mathbf{X}=\begin{pmatrix}1&x_1&x_2&x_3\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}.$$

Using the two equations above, we can get the desired equation (3.8). On the other hand,

$$x_1E_0 \;=\; \varepsilon(E_1)E_0=m(S\otimes id)\Delta(E_1)$$

$$
\begin{aligned}
&= (S(E_0)E_0, S(E_1)E_0, S(E_2)E_0, S(E_3)E_0, S(E_0)E_1, S(E_1)E_1, S(E_2)E_1, S(E_3)E_1,\\
&\quad S(E_0)E_2, S(E_1)E_2, S(E_2)E_2, S(E_3)E_2, S(E_0)E_3, S(E_1)E_3, S(E_2)E_3, S(E_3)E_3)\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{pmatrix}\\
&= \eta_1^T\begin{pmatrix}S(E_0)\\ S(E_1)\\ S(E_2)\\ S(E_3)\end{pmatrix}E_0 + \eta_2^T\begin{pmatrix}S(E_0)\\ S(E_1)\\ S(E_2)\\ S(E_3)\end{pmatrix}E_1 + \eta_3^T\begin{pmatrix}S(E_0)\\ S(E_1)\\ S(E_2)\\ S(E_3)\end{pmatrix}E_2 + \eta_4^T\begin{pmatrix}S(E_0)\\ S(E_1)\\ S(E_2)\\ S(E_3)\end{pmatrix}E_3\\
&= \eta_1^T S(\begin{pmatrix}E_0\\ E_1\\ E_2\\ E_3\end{pmatrix})E_0 + \eta_2^T S(\begin{pmatrix}E_0\\ E_1\\ E_2\\ E_3\end{pmatrix})E_1 + \eta_3^T S(\begin{pmatrix}E_0\\ E_1\\ E_2\\ E_3\end{pmatrix})E_2 + \eta_4^T S(\begin{pmatrix}E_0\\ E_1\\ E_2\\ E_3\end{pmatrix})E_3\\
&= \eta_1^T K^T E_0\begin{pmatrix}E_0\\ E_1\\ E_2\\ E_3\end{pmatrix} + \eta_2^T K^T\begin{pmatrix}0 & 1 & 0 & 0\\ -\alpha & 0 & 0 & 0\\ 0 & 0 & 0 & -1\\ 0 & 0 & \alpha & 0\end{pmatrix}\begin{pmatrix}E_0\\ E_1\\ E_2\\ E_3\end{pmatrix}\\
&\quad +\eta_3^T K^T\begin{pmatrix}0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -\beta & 0 & 0 & 0\\ 0 & -\beta & 0 & 0\end{pmatrix}\begin{pmatrix}E_0\\ E_1\\ E_2\\ E_3\end{pmatrix} + \eta_4^T K^T\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & -\alpha & 0\\ 0 & \beta & 0 & 0\\ -\alpha\beta & 0 & 0 & 0\end{pmatrix}\begin{pmatrix}E_0\\ E_1\\ E_2\\ E_3\end{pmatrix},
\end{aligned}
$$

which yield

$$
\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\\ \eta_4\end{pmatrix}^T K(4)^T\mathbf{L}_2 = (x_1, 0, 0, 0).
$$

For E_0, E_2 and E_3, as we do for E_1, it follows that

$$
\begin{pmatrix}\gamma_1\\ \gamma_2\\ \gamma_3\\ \gamma_4\end{pmatrix}^T K(4)^T\mathbf{L}_2 = (1, 0, 0, 0), \begin{pmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\\ \zeta_4\end{pmatrix}^T K(4)^T\mathbf{L}_2 = (x_2, 0, 0, 0),
$$

$$
\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_3\\ \xi_4\end{pmatrix}^T K(4)^T\mathbf{L}_2 = (x_3, 0, 0, 0).
$$

Thus, we have the desired equation (3.9).

□

From the lemmas above, we can get the main result in this section.

Theorem 3.6. *With Δ, ε, S defined above. Then $\mathbb{H}_{\alpha\beta}$ is a Hopf algebra if and only if there exist the matrices B, C, D, K and $x_1, x_2, x_3 \in \mathbb{C}$ satisfying the conditions (3.1), (3.2), (3.3), (3.4), (3.6), (3.7), (3.9).*

4. Description of Hopf algebra structures in generalized quaternion algebra

In this section, we shall discuss the value of α, β and whether there exist Hopf algebra structures on $\mathbb{H}_{\alpha\beta}$. Using (3.6), we have

$$\alpha + x_1^2 = 0, x_1x_2 - x_3 = 0, \alpha x_2 + x_1x_3 = 0, x_1x_2 + x_3 = 0, \beta + x_2^2 = 0,$$

$$x_2x_3 - \beta x_1 = 0, x_1x_3 - \alpha x_2 = 0, \beta x_1 + x_2x_3 = 0, \alpha\beta + x_3^2 = 0.$$

Thus, it follows that $x_3 = 0$ and $\alpha\beta = 0$. If $\alpha = 0$ and $\beta = 0$, then $x_1 = x_2 = x_3 = 0$. Using (3.4), we have

$$b_{22}x_1 + b_{32}x_2 + b_{42}x_3 + b_{12} = 1, b_{12}x_1 + b_{13}x_2 + b_{14}x_3 + b_{11} = 0, b_{22}x_1 + b_{23}x_2 + b_{24}x_3 + b_{21} = 1.$$

Thus, it follows that $b_{12} = 1, b_{11} = 0, b_{21} = 1$. Using (3.1), we get $b_{12}b_{21} + b_{11}b_{22} = 0$, which is in contradiction with $b_{12}b_{21} + b_{11}b_{22} = 1$. Thus, there are no Hopf algebra structures on $\frac{1}{4}$-quaternion algebra. From $\alpha\beta = 0$, it follows that at least one of α, β is a zero.

By Theorem 3.6, we have the following result:

Theorem 4.1. *If $\alpha = 0, \beta = 0$ or $\alpha\beta \neq 0$, then there are no Hopf algebra structures in the generalized quaternion algebra.*

By computing and analyzing with the help of the scientific computing software–Mathematica, we have the following result:

Theorem 4.2. *If $\alpha \neq 0, \beta = 0$, then there exists a bijection between all Hopf algebra structures on $\mathbb{H}_{\alpha\beta}$ and the $(B, C, D, K, x_1, x_2, x_3)$ consists of*

(I)

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{i}{\sqrt{\alpha}} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, C = \begin{pmatrix} 0 & 0 & a & \frac{i\sqrt{(a-1)a}}{\sqrt{\alpha}} \\ 0 & 0 & \frac{i(a-1)}{\sqrt{\alpha}} & -\frac{\sqrt{(a-1)a}}{\alpha} \\ 1-a & -\frac{ia}{\sqrt{\alpha}} & 0 & 0 \\ -\frac{i\sqrt{(a-1)a}}{\sqrt{\alpha}} & \frac{\sqrt{(a-1)a}}{\alpha} & 0 & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 0 & i\sqrt{(a-1)a}\sqrt{\alpha} & 1-a \\ 0 & 0 & -\sqrt{(a-1)a} & -\frac{ia}{\sqrt{\alpha}} \\ -i\sqrt{(a-1)a}\sqrt{\alpha} & \sqrt{(a-1)a} & 0 & 0 \\ a & \frac{i(a-1)}{\sqrt{\alpha}} & 0 & 0 \end{pmatrix},$$

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2\sqrt{(a-1)a} & i(1-2a)\sqrt{\alpha} \\ 0 & 0 & -\frac{i(2a-1)}{\sqrt{\alpha}} & 2\sqrt{(a-1)a} \end{pmatrix},$$

$$(x_1, x_2, x_3) = \begin{pmatrix} i\sqrt{\alpha} & 0 & 0 \end{pmatrix}.$$

(II)

$$B=\begin{pmatrix}0&0&0&0\\0&\frac{i}{\sqrt{\alpha}}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},C=\begin{pmatrix}0&0&a&-\frac{i\sqrt{(a-1)a}}{\sqrt{\alpha}}\\0&0&-\frac{i(a-1)}{\sqrt{\alpha}}&-\frac{\sqrt{(a-1)a}}{\alpha}\\1-a&\frac{ia}{\sqrt{\alpha}}&0&0\\\frac{i\sqrt{(a-1)a}}{\sqrt{\alpha}}&\frac{\sqrt{(a-1)a}}{\alpha}&0&0\end{pmatrix},$$

$$D=\begin{pmatrix}0&0&-i\sqrt{(a-1)a}\sqrt{\alpha}&1-a\\0&0&-\sqrt{(a-1)a}&\frac{ia}{\sqrt{\alpha}}\\i\sqrt{(a-1)a}\sqrt{\alpha}&\sqrt{(a-1)a}&0&0\\a&-\frac{i(a-1)}{\sqrt{\alpha}}&0&0\end{pmatrix},$$

$$K=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-2\sqrt{(a-1)a}&i(2a-1)\sqrt{\alpha}\\0&0&\frac{i(2a-1)}{\sqrt{\alpha}}&2\sqrt{(a-1)a}\end{pmatrix},$$

$$(x_1,x_2,x_3)=\begin{pmatrix}-i\sqrt{\alpha}&0&0\end{pmatrix}.$$

(III)

$$B=\begin{pmatrix}0&0&0&0\\0&-\frac{i}{\sqrt{\alpha}}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},C=\begin{pmatrix}0&0&a&-\frac{i\sqrt{(a-1)a}}{\sqrt{\alpha}}\\0&0&\frac{i(a-1)}{\sqrt{\alpha}}&\frac{\sqrt{(a-1)a}}{\alpha}\\1-a&-\frac{ia}{\sqrt{\alpha}}&0&0\\\frac{i\sqrt{(a-1)a}}{\sqrt{\alpha}}&-\frac{\sqrt{(a-1)a}}{\alpha}&0&0\end{pmatrix},$$

$$D=\begin{pmatrix}0&0&-i\sqrt{(a-1)a}\sqrt{\alpha}&1-a\\0&0&\sqrt{(a-1)a}&-\frac{ia}{\sqrt{\alpha}}\\i\sqrt{(a-1)a}\sqrt{\alpha}&-\sqrt{(a-1)a}&0&0\\a&\frac{i(a-1)}{\sqrt{\alpha}}&0&0\end{pmatrix},$$

$$K=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&2\sqrt{(a-1)a}&i(1-2a)\sqrt{\alpha}\\0&0&-\frac{i(2a-1)}{\sqrt{\alpha}}&-2\sqrt{(a-1)a}\end{pmatrix},$$

$$(x_1,x_2,x_3)=\begin{pmatrix}i\sqrt{\alpha}&0&0\end{pmatrix}.$$

(IV)

$$B=\begin{pmatrix}0&0&0&0\\0&\frac{i}{\sqrt{\alpha}}&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix},C=\begin{pmatrix}0&0&a&\frac{i\sqrt{(a-1)a}}{\sqrt{\alpha}}\\0&0&-\frac{i(a-1)}{\sqrt{\alpha}}&\frac{\sqrt{(a-1)a}}{\alpha}\\1-a&\frac{ia}{\sqrt{\alpha}}&0&0\\-\frac{i\sqrt{(a-1)a}}{\sqrt{\alpha}}&-\frac{\sqrt{(a-1)a}}{\alpha}&0&0\end{pmatrix},$$

$$D=\begin{pmatrix} 0 & 0 & i\sqrt{(a-1)a}\sqrt{\alpha} & 1-a \\ 0 & 0 & \sqrt{(a-1)a} & \frac{ia}{\sqrt{\alpha}} \\ -i\sqrt{(a-1)a}\sqrt{\alpha} & -\sqrt{(a-1)a} & 0 & 0 \\ a & -\frac{i(a-1)}{\sqrt{\alpha}} & 0 & 0 \end{pmatrix},$$

$$K=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2\sqrt{(a-1)a} & i(2a-1)\sqrt{\alpha} \\ 0 & 0 & \frac{i(2a-1)}{\sqrt{\alpha}} & -2\sqrt{(a-1)a} \end{pmatrix},$$

$$(x_1,x_2,x_3)=\begin{pmatrix} -i\sqrt{\alpha} & 0 & 0 \end{pmatrix}.$$

(V)

$$B=\begin{pmatrix} 0 & 0 & a & 0 \\ 0 & -\frac{i}{\sqrt{\alpha}} & \frac{ia}{\sqrt{\alpha}} & 0 \\ 0 & 0 & 0 & b \\ -\frac{i\sqrt{\alpha}b}{a} & \frac{b}{a} & -b & 0 \end{pmatrix}, C=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{i}{\sqrt{\alpha}} & \frac{ia}{\sqrt{\alpha}} & \frac{b}{a} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{i}{\sqrt{\alpha}} \\ 0 & 0 & 0 & \frac{ia}{\sqrt{\alpha}} \\ 1 & 0 & 0 & \frac{b}{a} \end{pmatrix}, K=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{a^2+\alpha b}{a} & 0 & -i\sqrt{\alpha} \\ 0 & -\frac{i\left(a^2-\alpha b\right)}{a\sqrt{\alpha}} & -\frac{i}{\sqrt{\alpha}} & 0 \end{pmatrix},$$

$$(x_1,x_2,x_3)=\begin{pmatrix} i\sqrt{\alpha} & 0 & 0 \end{pmatrix}.$$

(VI)

$$B=\begin{pmatrix} 0 & 0 & a & 0 \\ 0 & \frac{i}{\sqrt{\alpha}} & -\frac{ia}{\sqrt{\alpha}} & 0 \\ 0 & 0 & 0 & b \\ \frac{i\sqrt{\alpha}b}{a} & \frac{b}{a} & -b & 0 \end{pmatrix}, C=\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{i}{\sqrt{\alpha}} & -\frac{ia}{\sqrt{\alpha}} & \frac{b}{a} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$D=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{i}{\sqrt{\alpha}} \\ 0 & 0 & 0 & -\frac{ia}{\sqrt{\alpha}} \\ 1 & 0 & 0 & \frac{b}{a} \end{pmatrix}, K=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{a^2+\alpha b}{a} & 0 & i\sqrt{\alpha} \\ 0 & \frac{i\left(a^2-\alpha b\right)}{a\sqrt{\alpha}} & \frac{i}{\sqrt{\alpha}} & 0 \end{pmatrix},$$

$$(x_1,x_2,x_3)=\begin{pmatrix} -i\sqrt{\alpha} & 0 & 0 \end{pmatrix}.$$

(VII)

$$B=\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -\frac{i}{\sqrt{\alpha}} & 0 & 0 \\ a & \frac{ia}{\sqrt{\alpha}} & 0 & 0 \\ b & \frac{ib}{\sqrt{\alpha}} & 0 & 0 \end{pmatrix},$$

$$C=\begin{pmatrix} 0 & 0 & \frac{\alpha b^2}{a^2+\alpha b^2} & -\frac{ab}{a^2+\alpha b^2}\\ 0 & 0 & -\frac{ia^2}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} & -\frac{iab}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)}\\ \frac{a^2}{a^2+\alpha b^2} & -\frac{i\sqrt{\alpha}b^2}{a^2+\alpha b^2} & \frac{ia}{\sqrt{\alpha}} & \frac{ib}{\sqrt{\alpha}}\\ \frac{ab}{a^2+\alpha b^2} & \frac{iab}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} & 0 & 0 \end{pmatrix},$$

$$D=\begin{pmatrix} 0 & 0 & -\frac{a\alpha b}{a^2+\alpha b^2} & \frac{a^2}{a^2+\alpha b^2}\\ 0 & 0 & -\frac{ia\sqrt{\alpha}b}{a^2+\alpha b^2} & -\frac{i\sqrt{\alpha}b^2}{a^2+\alpha b^2}\\ \frac{a\alpha b}{a^2+\alpha b^2} & \frac{ia\sqrt{\alpha}b}{a^2+\alpha b^2} & 0 & 0\\ \frac{\alpha b^2}{a^2+\alpha b^2} & -\frac{ia^2}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} & \frac{ia}{\sqrt{\alpha}} & \frac{ib}{\sqrt{\alpha}} \end{pmatrix},$$

$$K=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -a-i\sqrt{\alpha}b & -\frac{2ia\sqrt{\alpha}b}{a^2+\alpha b^2} & -i\sqrt{\alpha}\left(1-\frac{2a^2}{a^2+\alpha b^2}\right)\\ 0 & -b+\frac{ia}{\sqrt{\alpha}} & \frac{i\left(a^2-\alpha b^2\right)}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} & \frac{2ia\sqrt{\alpha}b}{a^2+\alpha b^2} \end{pmatrix},$$

$$(x_1,x_2,x_3)=\begin{pmatrix} i\sqrt{\alpha} & 0 & 0 \end{pmatrix}.$$

(VIII)

$$B=\begin{pmatrix} 0 & 0 & 0 & 0\\ 0 & \frac{i}{\sqrt{\alpha}} & 0 & 0\\ a & -\frac{ia}{\sqrt{\alpha}} & 0 & 0\\ b & -\frac{ib}{\sqrt{\alpha}} & 0 & 0 \end{pmatrix},$$

$$C=\begin{pmatrix} 0 & 0 & \frac{\alpha b^2}{a^2+\alpha b^2} & -\frac{ab}{a^2+\alpha b^2}\\ 0 & 0 & \frac{ia^2}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} & \frac{iab}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)}\\ \frac{a^2}{a^2+\alpha b^2} & \frac{i\sqrt{\alpha}b^2}{a^2+\alpha b^2} & -\frac{ia}{\sqrt{\alpha}} & -\frac{ib}{\sqrt{\alpha}}\\ \frac{ab}{a^2+\alpha b^2} & -\frac{iab}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} & 0 & 0 \end{pmatrix},$$

$$D=\begin{pmatrix} 0 & 0 & -\frac{a\alpha b}{a^2+\alpha b^2} & \frac{a^2}{a^2+\alpha b^2}\\ 0 & 0 & \frac{ia\sqrt{\alpha}b}{a^2+\alpha b^2} & \frac{i\sqrt{\alpha}b^2}{a^2+\alpha b^2}\\ \frac{a\alpha b}{a^2+\alpha b^2} & -\frac{ia\sqrt{\alpha}b}{a^2+\alpha b^2} & 0 & 0\\ \frac{\alpha b^2}{a^2+\alpha b^2} & \frac{ia^2}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} & -\frac{ia}{\sqrt{\alpha}} & -\frac{ib}{\sqrt{\alpha}} \end{pmatrix},$$

$$K=\begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & -a+i\sqrt{\alpha}b & \frac{2ia\sqrt{\alpha}b}{a^2+\alpha b^2} & i\sqrt{\alpha}\left(1-\frac{2a^2}{a^2+\alpha b^2}\right)\\ 0 & -b-\frac{ia}{\sqrt{\alpha}} & \frac{i\left(1-\frac{2a^2}{a^2+\alpha b^2}\right)}{\sqrt{\alpha}} & -\frac{2ia\sqrt{\alpha}b}{a^2+\alpha b^2} \end{pmatrix},$$

$$(x_1,x_2,x_3)=\begin{pmatrix} -i\sqrt{\alpha} & 0 & 0 \end{pmatrix}.$$

(IX)

$$B=\begin{pmatrix} 0 & 0 & a & b \\ 0 & -\frac{i}{\sqrt{\alpha}} & \frac{ia}{\sqrt{\alpha}} & \frac{ib}{\sqrt{\alpha}} \\ c & \frac{ic}{\sqrt{\alpha}} & 0 & -\frac{ic\left(a^2+\alpha b^2\right)}{\alpha^{3/2}b} \\ -\frac{ac}{\alpha b} & -\frac{iac}{\alpha^{3/2}b} & \frac{ic\left(a^2+\alpha b^2\right)}{\alpha^{3/2}b} & 0 \end{pmatrix},$$

$$C=\begin{pmatrix} 0 & 0 & \frac{a^2}{a^2+\alpha b^2} & \frac{ab}{a^2+\alpha b^2} \\ 0 & 0 & -\frac{i\sqrt{\alpha}b^2}{a^2+\alpha b^2} & \frac{iab}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} \\ \frac{\alpha b^2}{a^2+\alpha b^2} & -\frac{ia^2}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} & \frac{i(a+c)}{\sqrt{\alpha}} & -\frac{iac}{\alpha^{3/2}b} \\ -\frac{ab}{a^2+\alpha b^2} & -\frac{iab}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} & \frac{ib}{\sqrt{\alpha}} & 0 \end{pmatrix},$$

$$D=\begin{pmatrix} 0 & 0 & \frac{a\alpha b}{a^2+\alpha b^2} & \frac{\alpha b^2}{a^2+\alpha b^2} \\ 0 & 0 & \frac{ia\sqrt{\alpha}b}{a^2+\alpha b^2} & -\frac{ia^2}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} \\ -\frac{a\alpha b}{a^2+\alpha b^2} & -\frac{ia\sqrt{\alpha}b}{a^2+\alpha b^2} & 0 & \frac{ia}{\sqrt{\alpha}} \\ \frac{a^2}{a^2+\alpha b^2} & -\frac{i\sqrt{\alpha}b^2}{a^2+\alpha b^2} & \frac{ic}{\sqrt{\alpha}} & \frac{i\left(\alpha b^2-ac\right)}{\alpha^{3/2}b} \end{pmatrix},$$

$$K=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a\left(-1+\frac{ic}{\sqrt{\alpha}b}\right)+i\sqrt{\alpha}b-c & \frac{2ia\sqrt{\alpha}b}{a^2+\alpha b^2} & \frac{i\sqrt{\alpha}\left(\alpha b^2-a^2\right)}{a^2+\alpha b^2} \\ 0 & \frac{ac}{\alpha b}-\frac{i(a-c)}{\sqrt{\alpha}}-b & \frac{i\left(\alpha b^2-a^2\right)}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} & -\frac{2ia\sqrt{\alpha}b}{a^2+\alpha b^2} \end{pmatrix},$$

$$(x_1,x_2,x_3)=\begin{pmatrix} i\sqrt{\alpha} & 0 & 0 \end{pmatrix}.$$

(X)

$$B=\begin{pmatrix} 0 & 0 & a & b \\ 0 & \frac{i}{\sqrt{\alpha}} & -\frac{ia}{\sqrt{\alpha}} & -\frac{ib}{\sqrt{\alpha}} \\ c & -\frac{ic}{\sqrt{\alpha}} & 0 & \frac{ic\left(a^2+\alpha b^2\right)}{\alpha^{3/2}b} \\ -\frac{ac}{\alpha b} & \frac{iac}{\alpha^{3/2}b} & -\frac{ic\left(a^2+\alpha b^2\right)}{\alpha^{3/2}b} & 0 \end{pmatrix},$$

$$C=\begin{pmatrix} 0 & 0 & \frac{a^2}{a^2+\alpha b^2} & \frac{ab}{a^2+\alpha b^2} \\ 0 & 0 & \frac{i\sqrt{\alpha}b^2}{a^2+\alpha b^2} & -\frac{iab}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} \\ \frac{\alpha b^2}{a^2+\alpha b^2} & \frac{ia^2}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} & -\frac{i(a+c)}{\sqrt{\alpha}} & \frac{iac}{\alpha^{3/2}b} \\ -\frac{ab}{a^2+\alpha b^2} & \frac{iab}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} & -\frac{ib}{\sqrt{\alpha}} & 0 \end{pmatrix},$$

$$D=\begin{pmatrix} 0 & 0 & \frac{a\alpha b}{a^2+\alpha b^2} & \frac{\alpha b^2}{a^2+\alpha b^2} \\ 0 & 0 & -\frac{ia\sqrt{\alpha}b}{a^2+\alpha b^2} & \frac{ia^2}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)} \\ -\frac{a\alpha b}{a^2+\alpha b^2} & \frac{ia\sqrt{\alpha}b}{a^2+\alpha b^2} & 0 & -\frac{ia}{\sqrt{\alpha}} \\ \frac{a^2}{a^2+\alpha b^2} & \frac{i\sqrt{\alpha}b^2}{a^2+\alpha b^2} & -\frac{ic}{\sqrt{\alpha}} & \frac{i\left(ac-\alpha b^2\right)}{\alpha^{3/2}b} \end{pmatrix},$$

$$K=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&a\left(-1-\frac{ic}{\sqrt{\alpha}b}\right)-i\sqrt{\alpha}b-c&-\frac{2ia\sqrt{\alpha}b}{a^2+\alpha b^2}&\frac{i\sqrt{\alpha}\left(a^2-\alpha b^2\right)}{a^2+\alpha b^2}\\0&\frac{ac}{\alpha b}+\frac{i(a-c)}{\sqrt{\alpha}}-b&\frac{i\left(a^2-\alpha b^2\right)}{\sqrt{\alpha}\left(a^2+\alpha b^2\right)}&\frac{2ia\sqrt{\alpha}b}{a^2+\alpha b^2}\end{pmatrix},$$

$$(x_1,x_2,x_3)=\begin{pmatrix}-i\sqrt{\alpha}&0&0\end{pmatrix}.$$

Theorem 4.3. *If $\alpha=0,\beta\neq 0$, then there exists a bijection between all Hopf algebra structures on $\mathbb{H}_{\alpha\beta}$ and the (B,C,D,K,x_1,x_2,x_3) consists of*

(I')

$$B=\begin{pmatrix}0&0&0&0\\1&a&0&0\\0&-\frac{i}{\sqrt{\beta}}&0&0\\0&b&0&0\end{pmatrix},C=\begin{pmatrix}0&0&0&-ib\sqrt{\beta}\\-ia\sqrt{\beta}&0&a&iab\sqrt{\beta}\\0&0&-\frac{i}{\sqrt{\beta}}&b\\0&-iab\sqrt{\beta}&0&0\end{pmatrix},$$

$$D=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&a&-\frac{i}{\sqrt{\beta}}&b\end{pmatrix},K=\begin{pmatrix}1&0&0&0\\0&0&-b\beta+ia\sqrt{\beta}&-i\sqrt{\beta}\\0&0&1&0\\0&-\frac{i}{\sqrt{\beta}}&-a+ib\sqrt{\beta}&0\end{pmatrix},$$

$$(x_1,x_2,x_3)=\begin{pmatrix}0&i\sqrt{\beta}&0\end{pmatrix}.$$

(II')

$$B=\begin{pmatrix}0&0&0&0\\1&a&0&0\\0&\frac{i}{\sqrt{\beta}}&0&0\\0&b&0&0\end{pmatrix},C=\begin{pmatrix}0&0&0&ib\sqrt{\beta}\\ia\sqrt{\beta}&0&a&-iab\sqrt{\beta}\\0&0&\frac{i}{\sqrt{\beta}}&b\\0&iab\sqrt{\beta}&0&0\end{pmatrix},$$

$$D=\begin{pmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&a&\frac{i}{\sqrt{\beta}}&b\end{pmatrix},K=\begin{pmatrix}1&0&0&0\\0&0&-b\beta+(-i)a\sqrt{\beta}&i\sqrt{\beta}\\0&0&1&0\\0&\frac{i}{\sqrt{\beta}}&-a-ib\sqrt{\beta}&0\end{pmatrix},$$

$$(x_1,x_2,x_3)=\begin{pmatrix}0&-i\sqrt{\beta}&0\end{pmatrix}.$$

(III')

$$B=\begin{pmatrix}0&1&0&0\\0&a&-\frac{i}{\sqrt{\beta}}&b\\0&0&0&0\\0&0&0&0\end{pmatrix},$$

$$C=\begin{pmatrix}0&-ia\sqrt{\beta}&0&0\\0&0&0&-iab\sqrt{\beta}\\0&a&-\frac{i}{\sqrt{\beta}}&0\\-ib\sqrt{\beta}&iab\sqrt{\beta}&b&0\end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & -\frac{i}{\sqrt{\beta}} \\ 1 & 0 & 0 & b \end{pmatrix},$$

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & b\beta + ia\sqrt{\beta} & i\sqrt{\beta} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{i}{\sqrt{\beta}} & a + ib\sqrt{\beta} & 0 \end{pmatrix},$$

$$(x_1, x_2, x_3) = \begin{pmatrix} 0 & i\sqrt{\beta} & 0 \end{pmatrix}.$$

(IV')

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & a & \frac{i}{\sqrt{\beta}} & b \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & ia\sqrt{\beta} & 0 & 0 \\ 0 & 0 & 0 & iab\sqrt{\beta} \\ 0 & a & \frac{i}{\sqrt{\beta}} & 0 \\ ib\sqrt{\beta} & -iab\sqrt{\beta} & b & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a \\ 0 & 0 & 0 & \frac{i}{\sqrt{\beta}} \\ 1 & 0 & 0 & b \end{pmatrix},$$

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & b\beta - ia\sqrt{\beta} & -i\sqrt{\beta} \\ 0 & 0 & 1 & 0 \\ 0 & -\frac{i}{\sqrt{\beta}} & a - ib\sqrt{\beta} & 0 \end{pmatrix},$$

$$(x_1, x_2, x_3) = \begin{pmatrix} 0 & -i\sqrt{\beta} & 0 \end{pmatrix}.$$

(V')

$$B = \begin{pmatrix} 0 & a & 0 & -\frac{\sqrt{-((a-1)a)}}{\sqrt{\beta}} \\ 1-a & b & \frac{\sqrt{1-a}a}{\sqrt{a-1}\sqrt{\beta}} & c \\ 0 & -\frac{\sqrt{-(a-1)^2}}{\sqrt{\beta}} & 0 & -\frac{\sqrt{a-1}\sqrt{a}}{\beta} \\ \frac{\sqrt{-((a-1)a)}}{\sqrt{\beta}} & \frac{-\frac{\sqrt{-((a-1)a)}b}{\sqrt{\beta}} - ac + c}{a} & \frac{\sqrt{a-1}\sqrt{a}}{\beta} & 0 \end{pmatrix},$$

$$C=\begin{pmatrix} 0 & \frac{\sqrt{-((a-1)a)}b\sqrt{\beta}+(a-1)\beta c}{\sqrt{a-1}\sqrt{a}} & 0 & \frac{\sqrt{a-1}\left(ab-\sqrt{-((a-1)a)}\sqrt{\beta}c\right)}{a^{3/2}} \\ -\sqrt{\frac{a-1}{a}}\beta c & 0 & \frac{\sqrt{1-a}\sqrt{\beta}c}{\sqrt{a}} & \frac{\sqrt{-((a-1)a)}b\sqrt{\beta}c+(a-1)\beta c^2}{\sqrt{a-1}a^{3/2}} \\ 0 & \frac{(a-1)\sqrt{\beta}c}{\sqrt{-((a-1)a)}}+b & \frac{\sqrt{1-a}}{\sqrt{a-1}\sqrt{\beta}} & \frac{-\frac{\sqrt{-((a-1)a)}b}{\sqrt{\beta}}-ac+c}{a} \\ \frac{\sqrt{1-a}\sqrt{\beta}c}{\sqrt{a-1}} & \frac{\left(-\sqrt{-((a-1)a)}\right)b\sqrt{\beta}c-\left((a-1)\beta c^2\right)}{\sqrt{a-1}a^{3/2}} & c & 0 \end{pmatrix},$$

$$D=\begin{pmatrix} 0 & -\sqrt{-((a-1)a)}\sqrt{\beta} & 0 & 1-a \\ \sqrt{-((a-1)a)}\sqrt{\beta} & 0 & \sqrt{a-1}\sqrt{a} & \frac{(a-1)\sqrt{\beta}c}{\sqrt{-((a-1)a)}}+b \\ 0 & -\sqrt{a-1}\sqrt{a} & 0 & \frac{\sqrt{1-a}a}{\sqrt{a-1}\sqrt{\beta}} \\ a & \frac{\sqrt{1-a}\sqrt{\beta}c}{\sqrt{a}} & -\frac{\sqrt{-(a-1)^2}}{\sqrt{\beta}} & \frac{c-\frac{\sqrt{-((a-1)a)}b}{\sqrt{\beta}}}{a} \end{pmatrix},$$

$$K=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\sqrt{a-1}\sqrt{a} & \frac{\left(-a+\sqrt{a-1}\sqrt{a}+1\right)b\sqrt{\beta}}{\sqrt{-((a-1)a)}}+\frac{(2a-1)\beta c}{a} & \frac{(1-2a)\sqrt{1-a}\sqrt{\beta}}{\sqrt{a-1}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{(1-2a)\sqrt{1-a}}{\sqrt{a-1}\sqrt{\beta}} & \frac{a\left(-ab+\sqrt{a-1}\sqrt{a}b-2\sqrt{-(a-1)^2}\sqrt{\beta}c+b\right)-\sqrt{-((a-1)a)}\sqrt{\beta}c}{\sqrt{a-1}a^{3/2}} & -2\sqrt{a-1}\sqrt{a} \end{pmatrix},$$

$$(x_1,x_2,x_3)=\begin{pmatrix} 0 & \frac{\sqrt{a-1}\sqrt{\beta}}{\sqrt{1-a}} & 0 \end{pmatrix}.$$

(VI')

$$B=\begin{pmatrix} 0 & a & 0 & \frac{\sqrt{-((a-1)a)}}{\sqrt{\beta}} \\ 1-a & b & -\frac{\sqrt{1-a}a}{\sqrt{a-1}\sqrt{\beta}} & c \\ 0 & \frac{\sqrt{-(a-1)^2}}{\sqrt{\beta}} & 0 & -\frac{\sqrt{a-1}\sqrt{a}}{\beta} \\ -\frac{\sqrt{-((a-1)a)}}{\sqrt{\beta}} & \frac{\frac{\sqrt{-((a-1)a)}b}{\sqrt{\beta}}-ac+c}{a} & \frac{\sqrt{a-1}\sqrt{a}}{\beta} & 0 \end{pmatrix},$$

$$C=\begin{pmatrix} 0 & \frac{(a-1)\beta c-\sqrt{-((a-1)a)}b\sqrt{\beta}}{\sqrt{a-1}\sqrt{a}} & 0 & \frac{\sqrt{a-1}\left(ab+\sqrt{-((a-1)a)}\sqrt{\beta}c\right)}{a^{3/2}} \\ -\sqrt{\frac{a-1}{a}}\beta c & 0 & \frac{(a-1)\sqrt{\beta}c}{\sqrt{-((a-1)a)}} & \frac{(a-1)\beta c^2-\sqrt{-((a-1)a)}b\sqrt{\beta}c}{\sqrt{a-1}a^{3/2}} \\ 0 & \frac{\sqrt{1-a}\sqrt{\beta}c}{\sqrt{a}}+b & \frac{\sqrt{a-1}}{\sqrt{1-a}\sqrt{\beta}} & \frac{\frac{\sqrt{-((a-1)a)}b}{\sqrt{\beta}}-ac+c}{a} \\ \frac{\sqrt{a-1}\sqrt{\beta}c}{\sqrt{1-a}} & \frac{\sqrt{-((a-1)a)}b\sqrt{\beta}c-(a-1)\beta c^2}{\sqrt{a-1}a^{3/2}} & c & 0 \end{pmatrix},$$

$$D=\begin{pmatrix} 0 & \sqrt{-((a-1)a)}\sqrt{\beta} & 0 & 1-a \\ -\sqrt{-((a-1)a)}\sqrt{\beta} & 0 & \sqrt{a-1}\sqrt{a} & \frac{\sqrt{1-a}\sqrt{\beta}c}{\sqrt{a}}+b \\ 0 & -\sqrt{a-1}\sqrt{a} & 0 & -\frac{\sqrt{1-a}a}{\sqrt{a-1}\sqrt{\beta}} \\ a & \frac{(a-1)\sqrt{\beta}c}{\sqrt{-((a-1)a)}} & \frac{\sqrt{-(a-1)^2}}{\sqrt{\beta}} & \frac{\frac{\sqrt{-((a-1)a)}b}{\sqrt{\beta}}+c}{a} \end{pmatrix},$$

$$K=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2\sqrt{a-1}\sqrt{a} & \frac{2a^2\beta c+\sqrt{a}(1-a)^{3/2}b\sqrt{\beta}+a\sqrt{-(a-1)^2}b\sqrt{\beta}-3a\beta c+\beta c}{(a-1)a} & \frac{\sqrt{1-a}(2a-1)\sqrt{\beta}}{\sqrt{a-1}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{\sqrt{1-a}(2a-1)}{\sqrt{a-1}\sqrt{\beta}} & \frac{a\left(-ab+\sqrt{a-1}\sqrt{a}b+2\sqrt{-(a-1)^2}\sqrt{\beta}c+b\right)+\sqrt{-((a-1)a)}\sqrt{\beta}c}{\sqrt{a-1}a^{3/2}} & -2\sqrt{a-1}\sqrt{a} \end{pmatrix},$$

$$(x_1, x_2, x_3) = \begin{pmatrix} 0 & \frac{\sqrt{1-a}\sqrt{\beta}}{\sqrt{a-1}} & 0 \end{pmatrix}.$$

(VII')

$$B = \begin{pmatrix} 0 & a & 0 & -\frac{\sqrt{-((a-1)a)}}{\sqrt{\beta}} \\ 1-a & b & -\frac{\sqrt{1-a}a}{\sqrt{a-1}\sqrt{\beta}} & c \\ 0 & \frac{\sqrt{-(a-1)^2}}{\sqrt{\beta}} & 0 & \frac{\sqrt{a-1}\sqrt{a}}{\beta} \\ \frac{\sqrt{-((a-1)a)}}{\sqrt{\beta}} & \frac{-\frac{\sqrt{-((a-1)a)}b}{\sqrt{\beta}}-ac+c}{a} & -\frac{\sqrt{a-1}\sqrt{a}}{\beta} & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & \frac{\left(-\sqrt{-((a-1)a)}\right)b\sqrt{\beta}-((a-1)\beta c)}{\sqrt{a-1}\sqrt{a}} & 0 & \frac{\sqrt{a-1}\left(\sqrt{-((a-1)a)}\sqrt{\beta}c-ab\right)}{a^{3/2}} \\ \sqrt{\frac{a-1}{a}\beta c} & 0 & \frac{\sqrt{1-a}\sqrt{\beta}c}{\sqrt{a}} & \frac{\left(-\sqrt{-((a-1)a)}\right)b\sqrt{\beta}c-\left((a-1)\beta c^2\right)}{\sqrt{a-1}a^{3/2}} \\ 0 & \frac{(a-1)\sqrt{\beta}c}{\sqrt{-((a-1)a)}}+b & \frac{\sqrt{a-1}}{\sqrt{1-a}\sqrt{\beta}} & \frac{-\frac{\sqrt{-((a-1)a)}b}{\sqrt{\beta}}-ac+c}{a} \\ \frac{\sqrt{a-1}\sqrt{\beta}c}{\sqrt{1-a}} & \frac{\sqrt{-((a-1)a)}b\sqrt{\beta}c+(a-1)\beta c^2}{\sqrt{a-1}a^{3/2}} & c & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & -\sqrt{-((a-1)a)}\sqrt{\beta} & 0 & 1-a \\ \sqrt{-((a-1)a)}\sqrt{\beta} & 0 & -\sqrt{a-1}\sqrt{a} & \frac{(a-1)\sqrt{\beta}c}{\sqrt{-((a-1)a)}}+b \\ 0 & \sqrt{a-1}\sqrt{a} & 0 & -\frac{\sqrt{1-a}a}{\sqrt{a-1}\sqrt{\beta}} \\ a & \frac{\sqrt{1-a}\sqrt{\beta}c}{\sqrt{a}} & \frac{\sqrt{-(a-1)^2}}{\sqrt{\beta}} & \frac{c-\frac{\sqrt{-((a-1)a)}b}{\sqrt{\beta}}}{a} \end{pmatrix},$$

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2\sqrt{a-1}\sqrt{a} & \frac{(2a-1)\beta c}{a} - \frac{\left(a+\sqrt{a-1}\sqrt{a}-1\right)b\sqrt{\beta}}{\sqrt{-((a-1)a)}} & \frac{\sqrt{1-a}(2a-1)\sqrt{\beta}}{\sqrt{a-1}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{\sqrt{1-a}(2a-1)}{\sqrt{a-1}\sqrt{\beta}} & \frac{\sqrt{a-1}a^{3/2}b+(a-1)ab-2\sqrt{-(a-1)^2}a\sqrt{\beta}c+\sqrt{-((a-1)a)}\sqrt{\beta}c}{\sqrt{a-1}a^{3/2}} & 2\sqrt{a-1}\sqrt{a} \end{pmatrix},$$

$$(x_1, x_2, x_3) = \begin{pmatrix} 0 & \frac{\sqrt{1-a}\sqrt{\beta}}{\sqrt{a-1}} & 0 \end{pmatrix}.$$

(VIII')

$$B = \begin{pmatrix} 0 & a & 0 & \frac{\sqrt{-((a-1)a)}}{\sqrt{\beta}} \\ 1-a & b & \frac{\sqrt{1-a}a}{\sqrt{a-1}\sqrt{\beta}} & c \\ 0 & -\frac{\sqrt{-(a-1)^2}}{\sqrt{\beta}} & 0 & \frac{\sqrt{a-1}\sqrt{a}}{\beta} \\ -\frac{\sqrt{-((a-1)a)}}{\sqrt{\beta}} & \frac{\frac{\sqrt{-((a-1)a)}b}{\sqrt{\beta}}-ac+c}{a} & -\frac{\sqrt{a-1}\sqrt{a}}{\beta} & 0 \end{pmatrix},$$

$$C = \begin{pmatrix} 0 & \frac{\sqrt{-((a-1)a)}b\sqrt{\beta}-(a-1)\beta c}{\sqrt{a-1}\sqrt{a}} & 0 & -\frac{\sqrt{a-1}\left(ab+\sqrt{-((a-1)a)}\sqrt{\beta}c\right)}{a^{3/2}} \\ \sqrt{\frac{a-1}{a}\beta c} & 0 & \frac{(a-1)\sqrt{\beta}c}{\sqrt{-((a-1)a)}} & \frac{\sqrt{-((a-1)a)}b\sqrt{\beta}c-(a-1)\beta c^2}{\sqrt{a-1}a^{3/2}} \\ 0 & \frac{\sqrt{1-a}\sqrt{\beta}c}{\sqrt{a}}+b & \frac{\sqrt{1-a}}{\sqrt{a-1}\sqrt{\beta}} & \frac{\frac{\sqrt{-((a-1)a)}b}{\sqrt{\beta}}-ac+c}{a} \\ \frac{\sqrt{1-a}\sqrt{\beta}c}{\sqrt{a-1}} & \frac{(a-1)\beta c^2-\sqrt{-((a-1)a)}b\sqrt{\beta}c}{\sqrt{a-1}a^{3/2}} & c & 0 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & \sqrt{-((a-1)a)}\sqrt{\beta} & 0 & 1-a \\ -\sqrt{-((a-1)a)}\sqrt{\beta} & 0 & -\sqrt{a-1}\sqrt{a} & \frac{\sqrt{1-a}\sqrt{\beta}c}{\sqrt{a}} + b \\ 0 & \sqrt{a-1}\sqrt{a} & 0 & \frac{\sqrt{1-a}a}{\sqrt{a-1}\sqrt{\beta}} \\ a & \frac{(a-1)\sqrt{\beta}c}{\sqrt{-((a-1)a)}} & -\frac{\sqrt{-(a-1)^2}}{\sqrt{\beta}} & \frac{\frac{\sqrt{-((a-1)a)}b}{\sqrt{\beta}}+c}{a} \end{pmatrix},$$

$$K = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -2\sqrt{a-1}\sqrt{a} & \frac{2a^2\beta c+\sqrt{a}(1-a)^{3/2}b\sqrt{\beta}-\sqrt{-(a-1)^2}ab\sqrt{\beta}-3a\beta c+\beta c}{(a-1)a} & \frac{(1-2a)\sqrt{1-a}\sqrt{\beta}}{\sqrt{a-1}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{(1-2a)\sqrt{1-a}}{\sqrt{a-1}\sqrt{\beta}} & \frac{\sqrt{a-1}a^{3/2}b+(a-1)ab+2\sqrt{-(a-1)^2}a\sqrt{\beta}c-\sqrt{-((a-1)a)}\sqrt{\beta}c}{\sqrt{a-1}a^{3/2}} & 2\sqrt{a-1}\sqrt{a} \end{pmatrix},$$

$$(x_1, x_2, x_3) = \begin{pmatrix} 0 & \frac{\sqrt{a-1}\sqrt{\beta}}{\sqrt{1-a}} & 0 \end{pmatrix}.$$

As shown in Theorems 4.2 and 4.3, if $\alpha \neq 0, \beta = 0$ or $\alpha = 0, \beta \neq 0$, then $\mathbb{H}_{\alpha\beta}$ has the different Hopf algebra structures. Next, we shall show that they are all isomorphic to Sweedler Hopf algebra. Thus, they are mutually isomorphic to each other.

Theorem 4.4. *All Hopf generalized quaternion algebras are isomorphic to the Sweedler Hopf algebra* $\mathbb{H}_4$.

Proof. Firstly, we consider the case (I'). The desired map Φ is given by

$$\Phi(1, i, j, k) = (1, g, v, gv) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{\beta} & 0 \\ 0 & 1 & -ia\sqrt{\beta} & 0 \\ 0 & 0 & -b\beta & -i\sqrt{\beta} \end{pmatrix}.$$

Since the determinant of $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{\beta} & 0 \\ 0 & 1 & -ia\sqrt{\beta} & 0 \\ 0 & 0 & -b\beta & -i\sqrt{\beta} \end{pmatrix}$ is equal to $-\beta$, it follows that Φ is bijective. It can be proved directly that Φ is a Hopf algebra homomorphism.

For the other types, let the desired map $\Phi : \mathbb{H}_{\alpha\beta} \to \mathbb{H}_4$ as follows:

$$\Phi(1, i, j, k) = (1, g, v, gv)\mathbf{U}.$$

Notice that Φ is uniquely determined by the matrix $\mathbf{U}$. See the following tables for the desired $\mathbf{U}$.

Types	Range of a, b	$\mathbf{U}$	Determinant of $\mathbf{U}$
I	$a=1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{i}{\sqrt{\alpha}} & 0 \end{pmatrix}$	-1
	$a=0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i\sqrt{\alpha} \end{pmatrix}$	$-\alpha$
	$a\neq 0,1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 1 & \frac{ia\sqrt{\alpha}}{\sqrt{(a-1)a}} \\ 0 & 0 & \frac{a}{\sqrt{(a-1)a}} & i\sqrt{\alpha} \end{pmatrix}$	$\frac{\alpha}{a-1}$
II	$a=1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{i}{\sqrt{\alpha}} & 0 \end{pmatrix}$	-1
	$a=0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i\sqrt{\alpha} \end{pmatrix}$	$-\alpha$
	$a\neq 0,1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 1 & -\frac{ia\sqrt{\alpha}}{\sqrt{(a-1)a}} \\ 0 & 0 & \frac{a}{\sqrt{(a-1)a}} & -i\sqrt{\alpha} \end{pmatrix}$	$\frac{\alpha}{a-1}$
III	$a=1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{i}{\sqrt{\alpha}} & 0 \end{pmatrix}$	-1
	$a=0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i\sqrt{\alpha} \end{pmatrix}$	$-\alpha$
	$a\neq 0,1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 1 & -\frac{ia\sqrt{\alpha}}{\sqrt{(a-1)a}} \\ 0 & 0 & -\frac{a}{\sqrt{(a-1)a}} & i\sqrt{\alpha} \end{pmatrix}$	$\frac{\alpha}{a-1}$

Types	Range of a, b	$\mathbf{U}$	Determinant of $\mathbf{U}$
IV	$a = 1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{i}{\sqrt{\alpha}} & 0 \end{pmatrix}$	-1
	$a = 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i\sqrt{\alpha} \end{pmatrix}$	$-\alpha$
	$a \neq 0, 1$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 1 & \frac{ia\sqrt{\alpha}}{\sqrt{(a-1)a}} \\ 0 & 0 & -\frac{a}{\sqrt{(a-1)a}} & -i\sqrt{\alpha} \end{pmatrix}$	$\frac{\alpha}{a-1}$
V	$b = 0, a \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{ia}{\sqrt{\alpha}} & -\frac{i}{\sqrt{\alpha}} & 0 \end{pmatrix}$	-1
	$b \neq 0, a \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\sqrt{\alpha} & 0 & 0 \\ 0 & 1 & 0 & \frac{ia}{\sqrt{\alpha}b} \\ 0 & \frac{a^2}{\alpha b} & \frac{a}{\alpha b} & 0 \end{pmatrix}$	$\frac{a^2}{\alpha b^2}$
VI	$b = 0, a \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{ia}{\sqrt{\alpha}} & \frac{i}{\sqrt{\alpha}} & 0 \end{pmatrix}$	-1
	$b \neq 0, a \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i\sqrt{\alpha} & 0 & 0 \\ 0 & 1 & 0 & -\frac{ia}{\sqrt{\alpha}b} \\ 0 & \frac{a^2}{\alpha b} & \frac{a}{\alpha b} & 0 \end{pmatrix}$	$\frac{a^2}{\alpha b^2}$
VII	$a^2 + \alpha b^2 \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\sqrt{\alpha} & 0 & 0 \\ 0 & 1 & \frac{a}{a^2+\alpha b^2} & \frac{\alpha b}{a^2+\alpha b^2} \\ 0 & 0 & -\frac{i\sqrt{\alpha}b}{a^2+\alpha b^2} & \frac{ia\sqrt{\alpha}}{a^2+\alpha b^2} \end{pmatrix}$	$-\frac{\alpha}{a^2+\alpha b^2}$
VIII	$a^2 + \alpha b^2 \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i\sqrt{\alpha} & 0 & 0 \\ 0 & 1 & \frac{a}{a^2+\alpha b^2} & \frac{\alpha b}{a^2+\alpha b^2} \\ 0 & 0 & \frac{i\sqrt{\alpha}b}{a^2+\alpha b^2} & -\frac{ia\sqrt{\alpha}}{a^2+\alpha b^2} \end{pmatrix}$	$-\frac{\alpha}{a^2+\alpha b^2}$
IX	$c = 0, b \neq 0, a^2 + \alpha b^2 \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 1 & -\frac{a}{b} \\ 0 & \frac{i\left(a^2+\alpha b^2\right)}{\sqrt{\alpha}b} & \frac{ia}{\sqrt{\alpha}b} & i\sqrt{\alpha} \end{pmatrix}$	$-\frac{\left(a^2+\alpha b^2\right)}{b^2}$
	$(a^2 + \alpha b^2)bc \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\sqrt{\alpha} & 0 & 0 \\ 0 & 1 & \frac{\alpha b^2}{c\left(a^2+\alpha b^2\right)} & -\frac{a\alpha b}{c\left(a^2+\alpha b^2\right)} \\ 0 & \frac{i\sqrt{\alpha}b}{c} & \frac{ia\sqrt{\alpha}b}{c\left(a^2+\alpha b^2\right)} & \frac{i\alpha^{3/2}b^2}{c\left(a^2+\alpha b^2\right)} \end{pmatrix}$	$-\frac{\alpha^2 b^2}{c^2\left(a^2+\alpha b^2\right)}$
X	$c = 0, b \neq 0, a^2 + \alpha b^2 \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -i\sqrt{\alpha} & 0 & 0 \\ 0 & 0 & 1 & -\frac{a}{b} \\ 0 & -\frac{i\left(a^2+\alpha b^2\right)}{\sqrt{\alpha}b} & -\frac{ia}{\sqrt{\alpha}b} & -i\sqrt{\alpha} \end{pmatrix}$	$-\frac{\left(a^2+\alpha b^2\right)}{b^2}$
	$(a^2 + \alpha b^2)bc \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i\sqrt{\alpha} & 0 & 0 \\ 0 & 1 & \frac{\alpha b^2}{c\left(a^2+\alpha b^2\right)} & -\frac{a\alpha b}{c\left(a^2+\alpha b^2\right)} \\ 0 & \frac{i\sqrt{\alpha}b}{c} & \frac{ia\sqrt{\alpha}b}{c\left(a^2+\alpha b^2\right)} & \frac{i\alpha^{3/2}b^2}{c\left(a^2+\alpha b^2\right)} \end{pmatrix}$	$-\frac{\alpha^2 b^2}{c^2\left(a^2+\alpha b^2\right)}$

Types	Range of a, b	$\mathbf{U}$	Determinant of $\mathbf{U}$
I'	$a \in \mathbb{R}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{\beta} & 0 \\ 0 & 1 & -ia\sqrt{\beta} & 0 \\ 0 & 0 & -b\beta & -i\sqrt{\beta} \end{pmatrix}$	$-\beta$
II'	$a \in \mathbb{R}$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{\beta} & 0 \\ 0 & 1 & ia\sqrt{\beta} & 0 \\ 0 & 0 & -b\beta & i\sqrt{\beta} \end{pmatrix}$	$-\beta$
III'	$b = 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{\beta} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{i}{\sqrt{\beta}} & a & 0 \end{pmatrix}$	-1
	$b \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & i\sqrt{\beta} & 0 \\ 0 & 0 & 1 & \frac{i}{b\sqrt{\beta}} \\ 0 & -\frac{1}{b\beta} & \frac{ia}{b\sqrt{\beta}} & 0 \end{pmatrix}$	$\frac{1}{b^2\beta}$
IV'	$b = 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{\beta} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\frac{i}{\sqrt{\beta}} & a & 0 \end{pmatrix}$	-1
	$b \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -i\sqrt{\beta} & 0 \\ 0 & 0 & 1 & -\frac{i}{b\sqrt{\beta}} \\ 0 & -\frac{1}{b\beta} & -\frac{ia}{b\sqrt{\beta}} & 0 \end{pmatrix}$	$\frac{1}{b^2\beta}$
V'	$a(a-1) \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{a-1}\sqrt{\beta}}{\sqrt{1-a}} & 0 \\ 0 & 1 & \frac{\beta c}{\sqrt{a-1}\sqrt{a}} & \frac{a\sqrt{\beta}}{\sqrt{-((a-1)a)}} \\ 0 & -\frac{1}{\sqrt{\frac{a-1}{a}}} & -\frac{\sqrt{-((a-1)a)}b\sqrt{\beta}+(a-1)\beta c}{(a-1)a} & \frac{\sqrt{1-a}\sqrt{\beta}}{\sqrt{a-1}} \end{pmatrix}$	$\frac{\beta}{a-1}$
VI'	$a(a-1) \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{1-a}\sqrt{\beta}}{\sqrt{a-1}} & 0 \\ 0 & 1 & \frac{\beta c}{\sqrt{a-1}\sqrt{a}} & -\frac{a\sqrt{\beta}}{\sqrt{-((a-1)a)}} \\ 0 & -\frac{1}{\sqrt{\frac{a-1}{a}}} & \frac{\sqrt{-((a-1)a)}b\sqrt{\beta}-(a-1)\beta c}{(a-1)a} & \frac{\sqrt{a-1}\sqrt{\beta}}{\sqrt{1-a}} \end{pmatrix}$	$\frac{\beta}{a-1}$
VII'	$a(a-1) \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{1-a}\sqrt{\beta}}{\sqrt{a-1}} & 0 \\ 0 & 1 & -\frac{\beta c}{\sqrt{a-1}\sqrt{a}} & \frac{a\sqrt{\beta}}{\sqrt{-((a-1)a)}} \\ 0 & \frac{1}{\sqrt{\frac{a-1}{a}}} & -\frac{\sqrt{-((a-1)a)}b\sqrt{\beta}+(a-1)\beta c}{(a-1)a} & \frac{\sqrt{a-1}\sqrt{\beta}}{\sqrt{1-a}} \end{pmatrix}$	$\frac{\beta}{a-1}$
VIII'	$a(a-1) \neq 0$	$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{\sqrt{a-1}\sqrt{\beta}}{\sqrt{1-a}} & 0 \\ 0 & 1 & -\frac{\beta c}{\sqrt{a-1}\sqrt{a}} & -\frac{a\sqrt{\beta}}{\sqrt{-((a-1)a)}} \\ 0 & \frac{1}{\sqrt{\frac{a-1}{a}}} & \frac{\sqrt{-((a-1)a)}b\sqrt{\beta}-(a-1)\beta c}{(a-1)a} & \frac{\sqrt{1-a}\sqrt{\beta}}{\sqrt{a-1}} \end{pmatrix}$	$\frac{\beta}{a-1}$

□

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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