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Research article

Least squares solutions of matrix equation $AXB = C$ under semi-tensor product

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Abstract: This paper mainly studies the least-squares solutions of matrix equation $AXB = C$ under a semi-tensor product. According to the definition of the semi-tensor product, the equation is transformed into an ordinary matrix equation. Then, the least-squares solutions of matrix-vector and matrix equations respectively investigated by applying the derivation of matrix operations. Finally, the specific form of the least-squares solutions is given.

Keywords: matrix equations; semi-tensor product; least squares solution

1. Introduction

Matrix equations of the form $AXB = C$ are important research topics in linear algebra. They are widely used in engineering and theoretical studies, such as in image and signal processing, photogrammetry and surface fitting in computer-aided geometric design [1, 2]. In addition, the equation-solving problems are also applied to the numerical solutions of differential equations, signal processing, cybernetics, optimization models, solid mechanics, structural dynamics, and so on [3–7]. So far, there is an abundance of research results on the solutions of matrix equation $AXB = C$, including the existence [8], uniqueness [9], numerical solution [10], and the structure of solutions [11–16]. Moreover, [17] discusses the Hermitian and skew-Hermitian splitting iterative method for solving the equation. The authors of [18] provided the Jacobi and Gauss-Seidel type iterative method to solve the equation.

However, in practical applications, ordinary matrix multiplication can no longer meet the needs. In 2001, Cheng and Zhao constructed a semi-tensor product method, which makes the multiplication of two matrices no longer limited by dimension [19, 20]. After that, the semi-tensor product began to be widely studied and discussed. It is not only applied to problems such as the permutation of high-dimensional data and algebraization of non linear robust stability control of power systems [22], it also provides a new research tool for the study of problems in Boolean networks [23], game theory [24], graph coloring [25], fuzzy control [26] and other fields [27]. However, some of these problems can be reduced to solving linear or matrix equations under the semi-tensor product. Yao et al. studied the solution of the equation under a semi-tensor product (STP equation), i.e., $AX = B$, in [28]. After that, the authors of [29–31] studied the solvability of STP equations $AX^2 = B$, $A \circ_l X = B$ and $AX - XB = C$, respectively.

To date, the application of the STP equation $AXB = C$ has also been reflected in many studies using the matrix semi-tensor product method. For example, in the study of multi-agent distributed cooperative control over finite fields, the authors of [32] transformed nonlinear dynamic equations over finite fields into the form of STP equation $Z(t) = \tilde{L}Z(t+1)$, where $\tilde{L} = \hat{L}QM$ and Q is the control matrix. Thus, if we want to get the right control matrix to realize consensus regarding L, we need to solve the STP equation $AXB = C$. Recently, Ji et al. studied the solutions of STP equation $AXB = C$ and gave the necessary and sufficient conditions for the equation to have a solution; they also formulated the specific solution steps in [33]. Nevertheless, the condition that STP equation $AXB = C$ has a solution is very harsh. On the one hand, parameter matrix C needs to have a specific form; particularly, it should be a block Toeplitz matrix and, even if the matrix C meets certain conditions, the equation may not have a solution. This brings difficulties in practical applications. On the other hand, there is usually a certain error in the data that we measure, which will cause the parameter matrix C of the equation $AXB = C$ to not achieve the required specific form; furthermore, the equation at this time will have no exact solutions.

Therefore, this paper describes a study of the approximate solutions of STP equation $AXB = C$. The main contributions of this paper are as follows: (1) The least-squares (LS) solution of STP equation $AXB = C$ is discussed for the first time. Compared with the existing results indicating that the equation has a solution, it is more general and greatly reduces the requirement of matrix form. (2) On the basis of Moore-Penrose generalized inverse operation and matrix differentiation, the specific forms of the LS solutions under the conditions of the matrix-vector equation and matrix equation are derived.

The paper is organized as follows. First, we study the LS solution problem of the matrix-vector STP equation $AXB = C$, together with a specific form of the LS solutions, where X is an unknown vector. Then, we study the LS solution problem when X is an unknown matrix and give the concrete form of the LS solutions. In addition, several simple numerical examples are given for each case to verify the feasibility of the theoretical results.

2. Preliminaries

This study applies the following notations.

- $\mathbb{R}$: the real number field;
- $\mathbb{R}^n$: the set of n-dimensional vectors over $\mathbb{R}$;
- $\mathbb{R}^{m\times n}$: the set of $m \times n$ matrices over $\mathbb{R}$;
- A^T: the transpose of matrix A;
- $\| A \|$: the Frobenius norm of matrix A;
- $\mathrm{tr}(A)$: the trace of matrix A;
- A^+: the Moore-Penrose generalized inverse of matrix A;
- $lcm\{m, n\}$: the least common multiple of positive integers m and n;
- $\gcd\{m, n\}$: the greatest common divisor of positive integers m and n;

- $\frac{a}{b}$: the formula b divides a;
- $a \mid b$: b is divisible by a;
- $\frac{\partial f(x)}{\partial x}$: differentiation of $f(x)$ with respect to x.

Let $A = [a_{ij}] \in \mathbb{R}^{m\times n}$ and $B = [b_{ij}] \in \mathbb{R}^{p\times q}$. We give the following definitions:

Definition 2.1. *[34] The Kronecker product $A \otimes B$ is defined as follows:*

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{21}B & \dots & a_{1n}B \\ a_{21}B & a_{22}B & \dots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \dots & a_{mn}B \end{bmatrix} \in \mathbb{R}^{mp\times nq}. \tag{2.1}$$

Definition 2.2. *[20] The left semi-tensor product $A \ltimes B$ is defined as follows:*

$$A \ltimes B = (A \otimes I_{t/n})(B \otimes I_{t/p}) \in \mathbb{R}^{(mt/n)\times(qt/p)}, \tag{2.2}$$

where $t = lcm\{n, p\}$.

Definition 2.3. *[21] For a matrix $A \in \mathbb{R}^{m\times n}$, the mn column vector $V_c(A)$ is defined as follows:*

$$V_c(A) = [a_{11} \ \cdots \ a_{m1} \ a_{12} \ \cdots \ a_{m2} \ \cdots \ a_{1n} \ \cdots \ a_{mn}]^T. \tag{2.3}$$

Proposition 2.1. *[33, 34] When A, B are two real-valued matrices and X is an unknown variable matrix, we have the following results about matrix differentiation:*

$$\frac{\partial\, \mathrm{tr}(AX)}{\partial X} = A^T, \quad \frac{\partial\, \mathrm{tr}(X^T A)}{\partial X} = A, \frac{\partial\, \mathrm{tr}(X^T AX)}{\partial X} = (A + A^T)X.$$

2.1. Matrix-vector equation

In this subsection, we will consider the following matrix-vector STP equation:

$$AXB = C, \tag{2.4}$$

where $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{r\times l}$, $C \in \mathbb{R}^{h\times k}$ are given matrices, and $X \in \mathbb{R}^p$ is the vector that needs to be solved.

With regard to the requirements of the dimensionality of the matrices in the STP equation (2.4), we have the following properties:

Proposition 2.2. *[33] For matrix-vector STP equation* (2.4)*,*

1) *when $m = h$, the necessary conditions for* (2.4) *with vector solutions of size p are that $\frac{k}{l}$, $\frac{n}{r}$ should be positive integers and $\frac{k}{l} \mid \frac{n}{r}$, $p = \frac{ln}{rk}$;*

2) *when $m \neq h$, the necessary conditions for* (2.4) *with vector solutions of size p are that $\frac{h}{m}$, $\frac{k}{l}$ should be positive integers and $\beta = gcd\{\frac{h}{m}, r\}$, $gcd\{\frac{k}{l},\beta\} = 1$, $gcd\{\frac{h}{m}, \frac{k}{l}\} = 1$ and $p = \frac{nhl}{mrk}$ hold.*

Remark: When Proposition 2.2 is satisfied, matrices A, B, and C are said to be compatible, and the sizes of X are called permissible sizes.

Example 2.1 Consider matrix-vector STP equation $AXB = C$ with the following coefficients:

$$A=\begin{bmatrix}1 & 2 & 0 & 1\end{bmatrix},\quad B=\begin{bmatrix}0\\1\end{bmatrix},\quad C=\begin{bmatrix}1 & 1 & 1\\0 & 2 & 0\end{bmatrix}.$$

It is easy to see that $m = 1$, $n = 4$, $r = 2$, $l = 1$, $h = 2$, and $k = 3$. Although $m \mid h$, $l \mid k, \beta = gcd\{\frac{h}{m}, r\}$, $gcd\{\frac{k}{l}, \beta\} = 1$, and $gcd\{\frac{h}{m}, \frac{k}{l}\} = 1$, $\frac{nb}{ak}$ is not a positive integer. So, A, B, and C are not compatible. At this time, matrix-vector STP equation (2.4) has no solution.

For the case that $m = h$, let $X = [x_1\ x_2\ \cdots\ x_p]^T \in \mathbb{R}^p$, A_s be the s-th column of A, and $\check{A}_1, \check{A}_2, \cdots, \check{A}_p \in \mathbb{R}^{m\times\frac{n}{p}} = \mathbb{R}^{m\times\frac{rk}{l}}$ be the p equal block of the matrix A, i.e., $A = [\check{A}_1\ \check{A}_2\ \cdots\ \check{A}_p]$, and

$$\check{A}_i = \begin{bmatrix}A_{\frac{(i-1)rk}{l}+1} & A_{\frac{(i-1)rk}{l}+2} & \cdots & A_{\frac{irk}{l}}\end{bmatrix}, i = 1, \cdots, p.$$

Let $t_1 = lcm\{n, p\}$, $t_2 = lcm\{\frac{t_1}{p}, r\}$; comparing the relationship of dimensions, we can get that $t_1 = n$ and $t_2 = \frac{rk}{l}$. Then

$$\begin{aligned}
A \ltimes X \ltimes B &= (A \otimes I_{\frac{t_1}{n}})(X \otimes I_{\frac{t_1}{p}}) \ltimes B\\
&= \begin{bmatrix}\check{A}_1 & \check{A}_2 & \cdots & \check{A}_p\end{bmatrix} \ltimes \begin{bmatrix}x_1\\x_2\\\vdots\\x_p\end{bmatrix} \ltimes B\\
&= (x_1\check{A}_1 + x_2\check{A}_2 + \cdots + x_p\check{A}_p) \ltimes B\\
&= x_1\check{A}_1 \ltimes B + x_2\check{A}_2 \ltimes B + \cdots + x_p\check{A}_p \ltimes B\\
&= x_1(\check{A}_1 \otimes I_{\frac{t_2 l}{rk}})(B \otimes I_{\frac{t_2}{r}}) + x_2(\check{A}_2 \otimes I_{\frac{t_2 l}{rk}})(B \otimes I_{\frac{t_2}{r}}) + \cdots + x_p(\check{A}_p \otimes I_{\frac{t_2 l}{rk}})(B \otimes I_{\frac{t_2}{r}})\\
&= x_1\check{A}_1(B \otimes I_{\frac{k}{l}}) + x_2\check{A}_2(B \otimes I_{\frac{k}{l}}) + \cdots + x_p\check{A}_p(B \otimes I_{\frac{k}{l}})\\
&= C \in \mathbb{R}^{m\times k}.
\end{aligned}$$

Denote

$$\begin{aligned}
\check{B}_i &= \check{A}_i(B \otimes I_{\frac{k}{l}})\\
&= \begin{bmatrix}A_{\frac{(i-1)rk}{l}+1} & A_{\frac{(i-1)rk}{l}+2} & \cdots & A_{\frac{irk}{l}}\end{bmatrix}(B \otimes I_{\frac{k}{l}})\\
&= [A_{\frac{(i-1)rk}{l}+1}\ \cdots\ A_{\frac{((ji-1)r+1)k}{l}}](B_1 \otimes I_{\frac{k}{l}}) + \cdots + [A_{\frac{(ir-1)k}{l}+1}\ \cdots\ A_{\frac{irk}{l}}](B_h \otimes I_{\frac{k}{l}}) \in \mathbb{R}^{m\times k}, i = 1, \cdots, p.
\end{aligned}$$

It is easy to see that when the matrices A and C have the same row dimension, the STP equation (2.4) has a better representation.

Proposition 2.3. *Matrix-vector STP equation* (2.4)*, given $m = h$, can be rewritten as follows:*

$$x_1\check{B}_1 + x_2\check{B}_2 + \cdots + x_p\check{B}_p = C. \tag{2.5}$$

Obviously, it can also have the following form:

$$[\check{B}_{1,j}\ \ \check{B}_{2,j}\ \ \cdots\ \ \check{B}_{p,j}]X = C_j,\ i = 1, \cdots, p,\ \ j = 1, \cdots, k,$$

and $\check{B}_{i,j}$ is the j-th column of $\check{B}_i$.

At the same time, applying the V_c operator to both sides of (2.5) yields

$$x_l V_c(\check{B}_1) + x_2 V_c(\check{B}_2) + \cdots + x_p V_c(\check{B}_p) = [V_c(\check{B}_1) \;\; V_c(\check{B}_2) \;\; \cdots \;\; V_c(\check{B}_p)]X = V_c(C).$$

We get the following proposition.

Proposition 2.4. *When $m = h$, matrix-vector STP equation* (2.4) *is equivalent to the linear form equation under the traditional matrix product:*

$$\overline{B}X = V_c(C),$$

where

$$\overline{B} = [V_c(\check{B}_1) \;\; V_c(\check{B}_2) \;\; \cdots \;\; V_c(\check{B}_p)] = \begin{bmatrix} \check{B}_{1,1} & \check{B}_{2,1} & \cdots & \check{B}_{p,1} \\ \check{B}_{1,2} & \check{B}_{2,2} & \cdots & \check{B}_{p,2} \\ \vdots & \vdots & \ddots & \vdots \\ \check{B}_{1,k} & \check{B}_{2,k} & \cdots & \check{B}_{p,k} \end{bmatrix}. \tag{2.6}$$

2.2. Matrix equation

In this subsection, we will consider the following matrix STP equation:

$$AXB = C, \tag{2.7}$$

where $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{r\times l}, C \in \mathbb{R}^{h\times k}$ are given matrices, and $X \in \mathbb{R}^{p\times q}$ is the matrix that needs to be solved.

For matrix STP equation (2.7), the dimensionality of its matrices has the following requirements:

Proposition 2.5. *[33] For matrix STP equation* (2.7)*,*

1) *when $m = h$, the necessary conditions for* (2.7) *with a matrix solution with size $p \times q$ are that $\frac{k}{l}, \frac{n}{r}$ should be positive integers and $p = \frac{n}{\alpha}$, $q = \frac{rk}{l\alpha}$, where α is a common factor of n and $\frac{rk}{l}$;*

2) *when $m \neq h$, the necessary conditions for* (2.7) *with a matrix solution of size $p \times q$ are that $\frac{h}{m}$, $\frac{k}{l}$ should be positive integers, $gcd\{\frac{h}{m\beta}, \frac{\alpha}{\beta}\} = 1$, $gcd\{\frac{h}{m}, \frac{k}{l}\} = 1$, $\beta \mid r$, $p = \frac{nh}{m\alpha}$, $q = \frac{rk}{l\alpha}$, α is the common factor of $\frac{nh}{m}$ and $\frac{rk}{l}$, and $\beta = gcd\{\frac{h}{m}, \alpha\}$.*

Remark: When Proposition 2.5 is satisfied, matrices A, B, and C are said to be compatible, and the sizes of X are called permissible sizes.

Example 2.2 Consider matrix STP equation $AXB = C$ with the following coefficients:

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 & 5 \\ 0 & 2 & 0 \end{bmatrix}.$$

We see that $m = 2$, $n = 4$, $r = 2$, $l = 1$, $h = 2$, and $k = 3$, so A, B and C are compatible. At this time, matrix STP equation (2.7) may have a solution $X \in \mathbb{R}^{2\times 3}$ or $\mathbb{R}^{4\times 6}$. (In fact, by Corollary 4.1 of [33], this equation has no solution.)

When $m = h$, let A_s be the s-th column of A and denote $\check{A}_1, \check{A}_2, \cdots, \check{A}_p \in \mathbb{R}^{m\times\alpha}$ as p blocks of A with the same size, i.e., $A = [\tilde{A}_1\ \check{A}_2\ \cdots\ \check{A}_p]$, where

$$\check{A}_i = \begin{bmatrix} A_{(i-1)\alpha+1} & A_{(i-1)\alpha+2} & \cdots & A_{i\alpha} \end{bmatrix},\ i = 1, \cdots, p.$$

Denote

$$\bar{A} = [V_c(\check{A}_1), V_c(\check{A}_2), \cdots, V_c(\check{A}_2)] = \begin{bmatrix} A_1 & A_{\alpha+1} & \cdots & A_{(p-1)\alpha+1} \\ A_2 & A_{\alpha+2} & \cdots & A_{(p-1)\alpha+2} \\ \vdots & \vdots & \ddots & \vdots \\ A_\alpha & A_{2\alpha} & \cdots & A_{p\alpha} \end{bmatrix},$$

we will have the following proposition.

Proposition 2.6. *[33] When $m = h$, STP equation (2.7) can be rewritten as follows:*

$$(B^T \otimes I_{\frac{km}{l}})(I_q \otimes \bar{A})V_c(X) = V_c(C). \tag{2.8}$$

3. The LS solutions of matrix-vector STP equation $AXB = C$

3.1. The simple case of $m = h$

In this subsection we will consider the LS solutions of the following matrix-vector STP equation:

$$AXB = C, \tag{3.1}$$

where $A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^{r\times l}, C \in \mathbb{R}^{m\times k}$ are given matrices, and $X \in \mathbb{R}^p$ is the vector that needs to be solved. By Proposition 2.2, we know that when $k \mid l,\ n \mid r$, and $\frac{k}{l} \mid \frac{n}{r}$, all matrices are compatible. At this time, the matrix-vector STP equation (3.1) may have solutions in $\mathbb{R}^{\frac{ln}{rk}}$.

Now, assuming that $k \mid l,\ n \mid r$, and $\frac{k}{l} \mid \frac{n}{r}$ hold and we want to find the LS solutions of matrix-vector STP equation (3.1) on $\mathbb{R}^{\frac{ln}{rk}}$, that is, given $A \in \mathbb{R}^{m\times n},\ B \in \mathbb{R}^{r\times l}$, and $C \in \mathbb{R}^{m\times k}$, we want to find $X^* \in \mathbb{R}^{\frac{ln}{rk}}$ such that

$$\| A \ltimes X^* \ltimes B - C \|^2 = \min_{X\in\mathbb{R}^{\frac{ln}{rk}}} \| A \ltimes X \ltimes B - C \|^2 . \tag{3.2}$$

According to Proposition 2.3, matrix-vector equation (2.4) under the condition that $m = h$ can be rewritten in the column form as follows:

$$[\check{B}_{1,j}\ \ \check{B}_{2,k+j}\ \ \cdots\ \ \check{B}_{p,(p-1)k+j}]X = C_j,\ \ j = 1, \cdots, k.$$

So, we have

$$\begin{aligned} & \| A \ltimes X \ltimes B - C \|^2 \\ = & \sum_{j=1}^{k} \| [\check{B}_{1,j}\ \ \check{B}_{2,k+j}\ \ \cdots\ \ \check{B}_{p,(p-1)k+j}]X - C_j \|^2 \\ = & \sum_{j=1}^{k} \mathrm{tr}[([\check{B}_{1,j}\ \ \check{B}_{2,k+j} \cdots\ \ \check{B}_{p,(p-1)k+j}]X - C_j)^T([\check{B}_{1,j}\ \ \check{B}_{2,k+j} \cdots\ \ \check{B}_{p,(p-1)k+j}]X - C_j)] \end{aligned}$$

$$= \sum_{j=1}^{k} \mathrm{tr}(X^T[\check{B}_{1,j} \ \ \check{B}_{2,k+j} \ \ \cdots \ \ \check{B}_{p,(p-1)k+j}]^T[\check{B}_{1,j} \ \ \check{B}_{2,k+j} \cdots \check{B}_{p,(p-1)k+j}]X - X^T[\check{B}_{1,j} \ \ \check{B}_{2,k+j} \cdots \ \check{B}_{p,(p-1)k+j}]^T$$
$$C_j - C_j^T[\check{B}_{1,j} \ \ \check{B}_{2,k+j} \ \ \cdots \ \ \check{B}_{p,(p-1)k+j}]X + C_j^T C_j).$$

Since $\| A \ltimes X \ltimes B - C \|^2$ is a smooth function for the variables of X, X^* is the minimum point if and only if X^* satisfies the following equation:

$$\frac{d}{dX} \| A \ltimes X \ltimes B - C \|^2 = 0.$$

Then, we derive the following:

$$\frac{d}{dX} \| A \ltimes X \ltimes B - C \|^2 = \sum_{j=1}^{k} (2[\check{B}_{1,j} \cdots \check{B}_{p,(p-1)k+j}][\check{B}_{1,j} \cdots \check{B}_{p,(p-1)k+j}]^T X - 2[\check{B}_{1,j} \ \cdots \ \check{B}_{p,(p-1)k+j}]^T C_j).$$

Taking

$$\frac{d}{dX} \| A \ltimes X \ltimes B - C \|^2 = 0,$$

we have

$$\sum_{j=1}^{k} [\check{B}_{1,j} \ \cdots \ \check{B}_{p,(p-1)k+j}]^T[\check{B}_{1,j} \ \cdots \ \check{B}_{p,(p-1)k+j}]X = \sum_{j=1}^{k} [\check{B}_{1,j} \ \ \cdots \ \ \check{B}_{p,(p-1)k+j}]^T C_j. \tag{3.3}$$

Hence, the minimum point of linear equation (3.2) is given by

$$X^* = (\sum_{j=1}^{k} [\check{B}_{1,j} \cdots \check{B}_{p,(p-1)k+j}]^T[\check{B}_{1,j} \cdots \check{B}_{p,(p-1)k+j}])^+ \cdot (\sum_{j=1}^{k} [\check{B}_{1,j} \ \ \cdots \ \ \check{B}_{p,(p-1)k+j}]^T C_j).$$

And, it is also the LS solution of (3.1).

Meanwhile, we can draw the following result:

Theorem 3.1. *If $\check{B}_1, \check{B}_2, \cdots, \check{B}_p$ are linearly independent and $\overline{B}$ of (2.6) is full rank, then the LS solution of matrix-vector STP equation* (3.1) *is given by*

$$X^* = (\overline{B}^T \overline{B})^{-1} \overline{B}^T V_c(C);$$

If $\check{B}_1, \check{B}_2, \cdots, \check{B}_p$ are linearly related and $\overline{B}$ is not full rank, then the LS solution of matrix-vector STP equation (3.1) *is given by*

$$X^* = (\overline{B}^T \overline{B})^+ \overline{B}^T V_c(C).$$

Proof. According to Proposition 2.4, (3.1) is equals to the following system of linear equations with a traditional matrix product

$$\overline{B} X = V_c(C). \tag{3.4}$$

Therefore, we only need to study the LS solutions of matrix-vector STP equation (3.4). From the conclusion in linear algebra, the LS solutions of matrix-vector STP equation (3.4) must satisfy the following equation:

$$\overline{B}^T\overline{B}X = \overline{B}^T V_c(C). \tag{3.5}$$

When $\overline{B}$ is full rank, $\overline{B}^T\overline{B}$ is invertible and the LS solution of matrix-vector STP equation (3.4) is given by

$$X^* = (\overline{B}^T\overline{B})^{-1}\overline{B}^T V_c(C);$$

When $\overline{B}$ is not full rank, $\overline{B}^T\overline{B}$ is nonsingular and the LS solution of matrix-vector STP equation (3.4) is given by

$$X^* = (\overline{B}^T\overline{B})^{+}\overline{B}^T V_c(C).$$

□

Comparing (3.3) and (3.5), we can see that

$$\overline{B}^T\overline{B} = \sum_{j=1}^{k}[\check{B}_{1,j}\ \ \check{B}_{2,k+j}\cdots\check{B}_{p,(p-1)k+j}]^T[\check{B}_{1,j}\ \ \check{B}_{2,k+j}\cdots\check{B}_{p,(p-1)k+j}],$$
$$\overline{B}^T V_c(C) = \sum_{j=1}^{k}([\check{B}_{1,j}\ \ \check{B}_{2,k+j}\ \ \cdots\ \ \check{B}_{p,(p-1)k+j}]^T C_j,$$

and

$$\| \overline{B}X - V_c(C) \|^2 = \sum_{j=1}^{k} \| [\check{B}_{1,j}\ \ \check{B}_{2,k+j}\cdots\check{B}_{p,(p-1)k+j}]X - C_j \|^2 .$$

Therefore, the two equations are the same, and the LS solution obtained via the two methods is consistent. Obviously, the second method is easier to employ. Below, we only use the second method to find the LS solutions.

Example 3.1 Now, we shall explore the LS solution of the matrix-vector STP equation $AXB = C$ with the following coefficients:

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 0 \end{bmatrix}.$$

By Example 2.1(1), it follows that the matrix-vector STP equation $AXB = C$ has no exact solution. Then, we can investigate the LS solutions of this equation.
First, because A, B, and C are compatible, the matrix-vector equation may have LS solutions on $\mathbb{R}^2$.
Second, divide A into 2 blocks; we have

$$\check{A}_1 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \check{A}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \check{B}_1 = \check{A}_1(B \otimes I_1) = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 0 & 1 \\ 2 & 1 & 2 \end{bmatrix}, \quad \check{B}_2 = \check{A}_2(B \otimes I_1) = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Then, we can get

$$\overline{B} = \begin{bmatrix} 2 & 2 \\ 0 & 0 \\ 2 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 0 \\ 2 & 1 \\ 1 & 0 \\ 2 & -1 \end{bmatrix}, \quad V_c(C) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 2 \\ -1 \\ 1 \\ 0 \end{bmatrix}.$$

Because $\overline{B}$ is full rank, the LS solution of this matrix-vector STP equation is given by

$$X^* = (\overline{B}^T\overline{B})^{-1}\overline{B}^T V_c(C) = \begin{bmatrix} 0.2963 \\ 0.0741 \end{bmatrix}.$$

3.2. The general case

In this subsection we will explore the LS solutions of the following matrix-vector STP equation:

$$AXB = C, \tag{3.6}$$

where $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{r\times l}$ and $C \in \mathbb{R}^{h\times k}$ are given matrices and $X \in \mathbb{R}^p$ is the vector that needs to be solved. By Proposition 2.2, we know that, when $m|h$, $k|l$, $nl \mid rk$, $\beta = gcd\{\frac{h}{m}, r\}$, $gcd\{\frac{k}{l},\beta\} = 1$, and $gcd\{\frac{h}{m}, \frac{k}{l}\} = 1$, A, B, and C are compatible. At this time, STP equation (3.6) may have a solution belonging to $\mathbb{R}^{\frac{nhl}{mrk}}$.

In what follows, we assume that matrix-vector STP equation (3.6) always satisfies the compatibility conditions, and we will find the LS solutions of matrix-vector STP equation (3.6) on $\mathbb{R}^{\frac{nhl}{mrk}}$. Since $\frac{h}{m}$ is a factor of the dimension $\frac{nhl}{mrk}$ of X, it is easy to obtain the matrix-vector STP equation given by

$$A \ltimes X \ltimes B = (A \otimes I_{\frac{h}{m}}) \ltimes X \ltimes B,$$

according to the multiplication rules of semi-tensor products. Let $A' = A \otimes I_{\frac{h}{m}}$; then matrix-vector STP equation (3.6) is transformed into the case of $m = h$, and, from the conclusion of the previous section, one can easily obtain the LS solution of matrix-vector STP equation (3.6).

Below, we give an algorithm for finding the LS solutions of matrix-vector STP equation (3.6):

- **Step one:** Check whether A, B, and C are compatible, that is, whether $m|h$ and $k|l$ hold, and whether $gcd\{\frac{h}{m}, \frac{k}{l}\} = 1$. If not, we can get that the equation has no solution.

- **Step two:** Let $X \in \mathbb{R}^p$, $p = \frac{nhl}{mrk}$, and $A' = A \otimes I_{\frac{h}{m}} \in \mathbb{R}^{h\times\frac{nh}{m}}$. Take $\check{A}'_1, \check{A}'_2, \cdots, \check{A}'_p \in \mathbb{R}^{m\times\frac{nh}{mp}} = \mathbb{R}^{m\times\frac{rk}{l}}$ to have p equal blocks of the matrix A':

$$\check{A}'_i = \begin{bmatrix} A'_{\frac{(i-1)rk}{l}+1} & A'_{\frac{(i-1)rk}{l}+2} & \cdots & A'_{\frac{irk}{l}} \end{bmatrix}, \ i = 1, \cdots, p,$$

  A'_s is the s-th column of A'. Let

$$\check{B}'_1, \check{B}'_2, \cdots, \check{B}'_p \in \mathbb{R}^{m\times k},$$

  where

$$\check{B}'_i = \check{A}'_i(B \otimes I_{\frac{k}{l}}) = \begin{bmatrix} A'_{\frac{(i-1)rk}{l}+1} & A'_{\frac{(i-1)rk}{l}+2} & \cdots & A'_{\frac{irk}{l}} \end{bmatrix}(B \otimes I_{\frac{k}{l}}), \ i = 1, \cdots, p.$$

- **Step three:** Let

$$\overline{B'} = \begin{bmatrix} \check{B}_{1,1} & \check{B}_{2,1} & \cdots & \check{B}_{p,1} \\ \check{B}_{1,2} & \check{B}_{2,2} & \cdots & \check{B}_{p,2} \\ \vdots & \vdots & \ddots & \vdots \\ \check{B}_{1,k} & \check{B}_{2,k} & \cdots & \check{B}_{p,k} \end{bmatrix},$$

and calculate $V_c(C)$.

- **Step four:** Solve the equation $\overline{B'}^T \overline{B'} X = \overline{B'}^T V_c(C)$; if $\overline{B'}$ is full rank and $\overline{B'}^T \overline{B'}$ is reversible, at this time, the LS solution of matrix-vector STP equation (3.6) is given by

$$X^* = (\overline{B'}^T \overline{B'})^{-1} \overline{B'}^T V_c(C);$$

If $\overline{B'}$ is not full rank, the LS solution of matrix-vector STP equation (3.6) is given by

$$X^* = (\overline{B'}^T \overline{B'})^{+} \overline{B'}^T V_c(C).$$

Example 3.2 Now, we shall explore the LS solutions of the matrix-vector STP equation $AXB = C$ with the following coefficients:

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

According to Example 2.1(2), we know that this matrix-vector STP equation has no exact solution. Then, we can investigate the LS solutions of this STP equation.
Step one: $m|h$, $k|l$, $gcd\{\frac{h}{m}$, and $\frac{k}{l}\} = 1$, so A, B, and C are compatible; we proceed to the second step.
Step two: The matrix-vector STP equation may have an LS solution $X \in \mathbb{R}^3$, and

$$A' = A \otimes I_3 = \begin{bmatrix} 1&0&0&0&0&0&1&0&0&1&0&0 \\ 0&1&0&0&0&0&0&1&0&0&1&0 \\ 0&0&1&0&0&0&0&0&1&0&0&1 \end{bmatrix}.$$

Let

$$\check{A'}_1 = \begin{bmatrix} 1&0&0&0 \\ 0&1&0&0 \\ 0&0&1&0 \end{bmatrix}, \quad \check{A'}_2 = \begin{bmatrix} 0&0&1&0 \\ 0&0&0&1 \\ 0&0&0&0 \end{bmatrix}, \quad \check{A'}_3 = \begin{bmatrix} 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&1 \end{bmatrix},$$

be three equal blocks of the matrix A'. We have

$$\check{B'}_1 = \check{A'}_1(B \otimes I_2) = \begin{bmatrix} 2&0 \\ 0&2 \\ 0&0 \end{bmatrix}, \quad \check{B'}_2 = \check{A'}_2(B \otimes I_2) = \begin{bmatrix} 0&0 \\ 0&0 \\ 0&0 \end{bmatrix}, \quad \check{B'}_3 = \check{A'}_3(B \otimes I_2) = \begin{bmatrix} 0&2 \\ 0&0 \\ 2&0 \end{bmatrix}.$$

Step three: Let

$$\overline{B'} = \begin{bmatrix} 2&0&0 \\ 0&0&0 \\ 0&0&2 \\ 0&0&2 \\ 2&0&0 \\ 0&0&0 \end{bmatrix}, \quad V_c(C) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}.$$

Step four: Because $\overline{B'}$ is not full rank, the LS solution of this matrix-vector STP equation is given by

$$X^* = (\overline{B'}^T \overline{B'})^{+} \overline{B'}^T V_c(C) = \begin{bmatrix} 0.7500 \\ 0 \\ 0.5000 \end{bmatrix}.$$

4. The LS solutions of matrix STP equation $AXB = C$

4.1. The simple case of $m = h$

In this subsection we will explore the LS solutions of the following matrix STP equation

$$AXB = C, \tag{4.1}$$

where $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{r\times l}$, and $C \in \mathbb{R}^{m\times k}$ are given matrices and $X \in \mathbb{R}^{p\times q}$ is the matrix that needs to be solved. By Proposition 2.5, we have that, when $l \mid k$, all matrices are compatible. At this time, matrix STP equation (4.1) may have solutions in $\mathbb{R}^{\frac{n}{\alpha}\times\frac{rk}{l\alpha}}$, and α is a common factor of n and $\frac{rk}{l}$.

Now, we assume that $l \mid k$, and we want to find the LS solutions of matrix STP equation (4.1) on $\mathbb{R}^{\frac{n}{\alpha}\times\frac{rk}{l\alpha}}$; the problem is as follows: Given $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{r\times l}$, and $C \in \mathbb{R}^{m\times k}$, we want to find $X^* \in \mathbb{R}^{p\times q}$ such that

$$\| A \ltimes X^* \ltimes B - C \|^2 = \min_{X\in\mathbb{R}^{p\times q}} \| A \ltimes X \ltimes B - C \|^2 . \tag{4.2}$$

By Proposition 2.6, matrix STP equation (4.1) can be rewritten as

$$(B^T \otimes I_{\frac{km}{l}})(I_q \otimes \bar{A})V_c(X) = V_c(C). \tag{4.3}$$

So, finding the LS solution of (4.1) is equivalent to finding $X^* \in \mathbb{R}^{p\times q}$ such that

$$\| (B^T \otimes I_{\frac{km}{l}})(I_q \otimes \bar{A})V_c(X) - V_c(C) \|^2 = \min_{X\in\mathbb{R}^{p\times q}} \| (B^T \otimes I_{\frac{km}{l}})(I_q \otimes \bar{A})V_c(X^*) - V_c(C) \|^2 . \tag{4.4}$$

Then, we have the following theorem.

Theorem 4.1. *When $B''A'' = (B^T \otimes I_{\frac{km}{l}})(I_q \otimes \bar{A})$ is full rank and $B''A''$ is invertible, the LS solution of matrix STP equation* (4.1) *is given by*

$$V_c(X^*) = (B''A'')^+C'';$$

When $B''A''$ is not full rank and $B''A''$ is nonsingular, the LS solution of matrix STP equation (4.1) *is given by*

$$V_c(X^*) = (B''A'')^{-1}C''.$$

Proof. Let $B'' = B^T \otimes I_{\frac{km}{l}}$, $A'' = I_q \otimes \bar{A}$, $X'' = V_c(X)$, and $C'' = V_c(C)$; then (4.4) can be rewritten as

$$\begin{aligned}
&\| (B^T \otimes I_{\frac{km}{l}})(I_q \otimes \bar{A})V_c(X) - V_c(C) \|^2 \\
=&\| B''A''X'' - C'' \|^2 \\
=& \operatorname{tr}[(B''A''X'' - C'')^T(B''A''X'' - C'')] \\
=& \operatorname{tr}[(X''^T A''^T B''^T - C''^T)(B''A''X'' - C'')] \\
=& \operatorname{tr}[(X''^T A''^T B''^T B''A''X'') - (X''^T A''^T B''^T C'') - (C''^T B''A''X'') + (C''^T C'')].
\end{aligned}$$

Because $\| A \ltimes X \ltimes B - C \|^2$ is a smooth function for the variables of X, it follows that X^* is the minimum point if and only if X^* satisfies

$$\frac{d}{dX} \| (B^T \otimes I_{\frac{km}{l}})(I_q \otimes \bar{A})V_c(X) - V_c(C) \|^2 = 0.$$

Given that

$$\frac{d}{dX} \parallel (B^T \otimes I_{\frac{km}{l}})(I_q \otimes \bar{A})V_c(X) - V_c(C) \parallel^2 = 2A''^T B''^T B'' A'' X'' - 2A''^T B''^T C'',$$

let

$$\frac{d}{dX} \parallel (B^T \otimes I_{\frac{km}{l}})(I_q \otimes \bar{A})V_c(X) - V_c(C) \parallel^2 = 0.$$

Then, we have

$$A''^T B''^T B'' A'' X'' = A''^T B''^T C''.$$

Thus, the minimum point of linear equation (4.2) is also the LS solution of matrix STP equation (4.1). That is to say, $\parallel A \ltimes X \ltimes B - C \parallel^2$ is the smallest if and only if $\parallel (B^T \otimes I_{\frac{km}{l}})(I_q \otimes \bar{A})V_c(X^*) - V_c(C) \parallel^2$ gets the minimum value. And, the statement is naturally proven. □

Now, we shall examine the relationship between the LS solutions of different compatible sizes. Let $p_1 \times q_1, p_2 \times q_2$ be two different compatible sizes and $1 < \frac{p_1}{q_1} = \frac{p_2}{q_2} \in Z$. If $X \in \mathbb{R}^{p_1 \times q_1}$, we should have that $X \otimes I_{\frac{p_2}{p_1}} \in \mathbb{R}^{p_2 \times q_2}$; we can get the following formula:

$$\min_{X \in \mathbb{R}^{p_2 \times q_2}} \parallel A \ltimes X \ltimes B - C \parallel^2 \leq \min_{X \in \mathbb{R}^{p_1 \times q_1}} \parallel A \ltimes X \ltimes B - C \parallel^2 .$$

Therefore, if we consider (4.1) to take the LS solutions among all compatible sizes of matrices, then it should be the LS solutions of the equation on $\mathbb{R}^{n \times k}$.

Example 4.1 Now, we shall explore the LS solutions of matrix STP equation $AXB = C$ with the following coefficients:

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 & 5 \\ 0 & 2 & 0 \end{bmatrix}.$$

Example 2.2(1) shows that the matrix STP equation $AXB = C$ has no exact solution. Now, we can investigate the LS solutions of this equation.

First, given that A, B, and C are compatible, the matrix STP equation may have LS solutions on $\mathbb{R}^{2\times 3}$ or $\mathbb{R}^{4\times 6}$.

(1) The case that $\alpha = 2$, $X \in \mathbb{R}^{2\times 3}$:

Let

$$\check{A}_1 = [\, A_1 \; A_2 \,] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \check{A}_2 = [\, A_3 \; A_4 \,] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then, we have

$$\bar{A} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad A'' = I_3 \otimes \bar{A} = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}.$$

Let

$$B'' = B^T \otimes I_6 = \begin{bmatrix} 2&0&0&0&0&0&1&0&0&0&0&0\\ 0&2&0&0&0&0&0&1&0&0&0&0\\ 0&0&2&0&0&0&0&0&1&0&0&0\\ 0&0&0&2&0&0&0&0&0&1&0&0\\ 0&0&0&0&2&0&0&0&0&0&1&0\\ 0&0&0&0&0&2&0&0&0&0&0&1 \end{bmatrix}, \quad C'' = V_c(C) = \begin{bmatrix} 3\\1\\5\\0\\2\\0 \end{bmatrix}.$$

Because $B''A''$ is full rank, the LS solution of this matrix STP equation satisfies

$$V_c(X^*) = (B''A'')^{-1}C'' = \begin{bmatrix} 0\\1.1667\\-1.0000\\0.6667\\0\\2.6667 \end{bmatrix}, \quad \text{then } X^* = \begin{bmatrix} 0 & 1.1667 & -1.0000\\ 0.6667 & 0 & 2.6667 \end{bmatrix}.$$

(2) The case that $\alpha = 1,\ X \in \mathbb{R}^{4\times 6}$:
Let

$$\bar{A} = \begin{bmatrix} 1&0&1&1\\ 0&-1&0&0 \end{bmatrix},$$

$$A'' = I_6 \otimes \bar{A} = \begin{bmatrix}
1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0&0&0&1\\
0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&-1&0&0&0&0&0&0&0&0&0&0&0&0
\end{bmatrix}.$$

Let

$$B'' = B^T \otimes I_6 = \begin{bmatrix} 2&0&0&0&0&0&1&0&0&0&0&0\\ 0&2&0&0&0&0&0&1&0&0&0&0\\ 0&0&2&0&0&0&0&0&1&0&0&0\\ 0&0&0&2&0&0&0&0&0&1&0&0\\ 0&0&0&0&2&0&0&0&0&0&1&0\\ 0&0&0&0&0&2&0&0&0&0&0&1 \end{bmatrix}, \quad C'' = V_c(C) = \begin{bmatrix} 3\\1\\5\\0\\2\\0 \end{bmatrix}.$$

Because $B''A''$ is full rank, the LS solution of this matrix STP equation satisfies

$$V_c(X^*) = (B''A'')^{+}C'' = \begin{bmatrix} 0.4615\\0.1538\\0.7692\\0\\0.3077\\0\\-0.2308\\-0.0769\\-0.3846\\0\\-0.1538\\0\\0.4615\\0.1538\\0.7692\\0\\0.3077\\0\\0.4615\\0.1538\\0.7692\\0\\0.3077\\0 \end{bmatrix}, \text{ then } X^* = \begin{bmatrix} 0.4615 & 0.1538 & 0.7692 & 0 & 0.3077 & 0\\ -0.2308 & -0.0769 & -0.3846 & 0 & -0.1538 & 0\\ 0.4615 & 0.1538 & 0.7692 & 0 & 0.3077 & 0\\ 0.4615 & 0.1538 & 0.7692 & 0 & 0.3077 & 0 \end{bmatrix}.$$

4.2. The general case

This section focuses on the LS solutions of the following matrix STP equation:

$$AXB = C, \tag{4.5}$$

where $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{r\times l}$ and $C \in \mathbb{R}^{h\times k}$ are given matrices and $X \in \mathbb{R}^{p\times q}$ is the matrix that needs to be solved. By Proposition 2.5, we have that, when $m|h$, $l|k$, $gcd\{\frac{h}{m\beta}, \frac{\alpha}{\beta}\} = 1, gcd\{\beta, \frac{k}{l}\} = 1$, and $\beta|r$, where $\beta = gcd\{\frac{h}{m}, \alpha\}$, all matrices are compatible. At this time, the matrix-vector equation (4.5) may have solutions in $\mathbb{R}^{\frac{nh}{m\alpha}\times\frac{rk}{l\alpha}}$ and α is a common factor of $\frac{nh}{m}$ and $\frac{rk}{l}$.

Now, we assume that matrix STP equation (4.5) always satisfies the compatibility conditions. Since $\frac{h}{m}$ is a factor of the row dimension $\frac{nh}{m\alpha}$ of X, it is easy to obtain the matrix STP equation

$$A \ltimes X \ltimes B = (A \otimes I_{\frac{h}{m}}) \ltimes X \ltimes B,$$

according to the multiplication rules of STP. Let $A' = A \otimes I_{\frac{h}{m}}$: then (4.5) can be transformed into the case of $m = h$, and we can easily obtain the LS solution of matrix STP equation (4.5).

Below, we give an algorithm for finding the LS solutions of matrix STP equation (4.5):

- **Step one:** Check whether $m|h$ and $l|k$ hold. If not, we can get that the equation has no solution.

- **Step two:** Find all values of α that satisfy that $gcd\{\frac{r}{m}, h\} = 1$, $gcd\{\frac{h}{m\beta}, \frac{\alpha}{\beta}\} = 1$, $gcd\{\beta, \frac{k}{l}\} = 1$, and $\beta|r$, $\beta = gcd\{\frac{h}{m}, \alpha\}$; correspondingly, find all compatible sizes $p \times q$ and perform the following steps for each compatible size.

- **Step three:** Let $A' = A \otimes I_{\frac{h}{m}} \in \mathbb{R}^{h\times\frac{nh}{m}}$. We have

$$\bar{A}' = [V_c(\check{A}'_1), V_c(\check{A}'_2), \cdots, V_c(\check{A}'_2)] = \begin{bmatrix} A'_1 & A'_{\alpha+1} & \cdots & A'_{(p-1)\alpha+1} \\ A'_2 & A'_{\alpha+2} & \cdots & A'_{(p-1)\alpha+2} \\ \vdots & \vdots & \ddots & \vdots \\ A'_\alpha & A'_{2\alpha} & \cdots & A'_{p\alpha} \end{bmatrix},$$

  where $\check{A}'_1, \check{A}'_2, \cdots, \check{A}'_p \in \mathbb{R}^{m\times\alpha}$ are p blocks of A' of the same size, and A'_i is the i-th column of A'. Let $B'' = B^T \otimes I_{\frac{kh}{l}}$, $A'' = I_q \otimes \bar{A}'$, $X'' = V_c(X)$, and $C'' = V_c(C)$.

- **Step four:** Solve the equation $A''^T B''^T B'' A'' X'' = A''^T B''^T C''$; if $B''A''$ is full rank and $B''A''$ is reversible, the LS solution of matrix STP equation (4.5) is given by

$$V_c(X^*) = (B''A'')^{-1}C'';$$

  if $B''A''$ is not full rank, the LS solution of matrix STP equation (4.5) is given by

$$V_c(X^*) = (B''A'')^{+}C''.$$

Example 4.2 Now, we shall explore the LS solutions of matrix STP equation $AXB = C$ with the following coefficients:

$$A = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad C = \begin{bmatrix} 3 & 1 & 5 \\ 0 & 2 & 0 \end{bmatrix}.$$

According to Example 2.2(2), matrix STP equation $AXB = C$ has no exact solution. Now, we can investigate the LS solutions of this equation:
Step one: $m|r$ and $l|k$ hold.
Step two: $gcd\{\frac{r}{m}, h\} = 1,\ gcd\{\frac{h}{m\beta},\ \frac{\alpha}{\beta}\} = 1,\ gcd\{\beta, \frac{k}{l}\} = 1,\ \beta|r$, and $\beta = gcd\{\frac{h}{m}, \alpha\}$ hold. The matrix STP equation $AXB = C$ may have the solution $X \in \mathbb{R}^{2\times 2}$ or $\mathbb{R}^{4\times 4}$.
Step three: (1) The case that $\alpha = 2$:
Let

$$A' = A \otimes I_2 = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right].$$

Then, we have

$$\bar{A}' = [V_c(\check{A}'_1), V_c(\check{A}'_2)] = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Let

$$B'' = B^T \otimes I_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix},\ A'' = I_2 \otimes \bar{A}' = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},\ \ C'' = V_c(C) = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 0 \\ 2 \\ 0 \end{bmatrix}.$$

(2) The case that $\alpha = 1$:
Let

$$A' = A \otimes I_2 = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right].$$

Then, we have

$$\bar{A}' = [V_c(\check{A}'_1), V_c(\check{A}'_2), V_c(\check{A}'_3), V_c(\check{A}'_4)] = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{smallmatrix}\right].$$

Let

$$B'' = B^T \otimes I_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 \end{bmatrix},\ \ A'' = I_4 \otimes \bar{A}' = \begin{bmatrix} 1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&0 \end{bmatrix},$$

$$C'' = V_c(C) = \begin{bmatrix} 3 \\ 1 \\ 5 \\ 0 \\ 2 \\ 0 \end{bmatrix}.$$

Step four: Because $B''A''$ is not full rank, the LS solution of this matrix STP equation is obtained as follows:
(1) The case that $\alpha = 2$:

$$V_c(X^*) = (B''A'')^+C'' = \begin{bmatrix} 1.0000 \\ 0 \\ 1.0000 \\ 0 \end{bmatrix} \Longrightarrow X^* = \left[\begin{smallmatrix} 1.0000 & 0 \\ 1.0000 & 0 \end{smallmatrix}\right].$$

(2) The case that $\alpha = 1$:

$$V_c(X^*) = (B''A'')^+C'' = \begin{bmatrix} 1.5000 \\ 0.5000 \\ -5.0000 \\ 0 \\ 1.5000 \\ 0.5000 \\ 1.0000 \\ 1.0000 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \Longrightarrow \quad X^* = \begin{bmatrix} 1.5000 & 0.5000 & -5.0000 & 0 \\ 1.5000 & 0.5000 & 1.0000 & 1.0000 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

5. Conclusions

In this paper, we applied the semi-tensor product to solve the matrix equation $AXB = C$ and studied the LS solutions of the matrix equation under the semi-tensor product. By appling the definition of semi-tensor products, the equation can be transformed into the matrix equation under the ordinary matrix product and then combined with the Moore-Penrose generalized inverse operation and matrix differentiation. The specific forms of the LS solutions under the conditions of the matrix-vector equation and matrix equation were also respectively derived. Investigating the solution of Sylvester equations under a semi-tensor product, as well as the LS solution problem, will be future research work.

Use of AI tools declaration

No artificial intelligence tools were usded in the creation of this article.

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Conflict of interest

No potential conflict of interest was reported by the author.

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