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Research article

Exact and least-squares solutions of a generalized Sylvester-transpose matrix equation over generalized quaternions

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Abstract: We have considered a generalized Sylvester-transpose matrix equation $AXB + CX^T D = E$, where A, B, C, D, and E are given rectangular matrices over a generalized quaternion skew-field, and X is an unknown matrix. We have applied certain vectorizations and real representations to transform the matrix equation into a matrix equation over the real numbers. Thus, we have investigated a solvability condition, general exact/least-squares solutions, minimal-norm solutions, and the exact/least-squares solution closest to a given matrix. The main equation included the equation $AXB = E$ and the Sylvester-transpose equation. Our results also covered such matrix equations over the quaternions, and quaternionic linear systems.

Keywords: generalized Sylvester-transpose matrix equation; generalized quaternion matrix; minimal norm solution, least-squares solution; vector operator; Kronecker product

1. Introduction

Linear matrix equations over the field $\mathbb{R}$ of real numbers have a strong connection to certain problems in differential equations, and control and system theory [1–3]. Indeed, the Sylvester-transpose matrix equation

$$AX + X^T D = E, \tag{1.1}$$

is closely related to eigenstructure assignment [4], pole assignment [3], and fault detection in dynamical systems [5]. More generally, many authors investigated a generalized Sylvester-transpose equation:

$$AXB + CX^T D = E, \tag{1.2}$$

and a generalized Sylvester one

$$AXB + CXD = E. \tag{1.3}$$

In the last decade, theory and computational aspects for such equations were investigated for Eq (1.1) [6] and Eq (1.2) [7–13].

Instead of the real number field, we can develop a theory for matrix equations over suitable algebraic structures, e.g., the quaternion skew-field or other skew-fields. Recall that the (Hamilton) quaternions

$$\mathbb{Q} = \left\{ q_1 + q_2 i + q_3 j + q_4 k \;\middle|\; q_1, q_2, q_3, q_4 \in \mathbb{R} \right\},$$

is a non-comutative division ring with respect to the coordinatewise addition and the Hamilton multiplication defined by

$$\begin{aligned} i^2 = j^2 = k^2 = -1,\ ij = -ji = k, \\ jk = -kj = i,\ ki = -ik = j. \end{aligned} \tag{1.4}$$

The quaternions are widely used in quantum physics [14, 15], computer graphics [16], robot trajectory planning [17], and modeling [18], etc., [19–21]. The reader can find more information about quaternions in the survey paper [22]. Moreover, if we generalize the rule (1.4), then we get a generalized quaternion [23]. Let $u, v \in \mathbb{R} - \{0\}$. Let $\mathbb{Q}_{u,v}$ be a four-dimensionl vector space over $\mathbb{R}$ with an ordered basis $\{1, i, j, k\}$, i.e.,

$$\mathbb{Q}_{u,v} = \left\{ x_1 + x_2 i + x_3 j + x_4 k \;\middle|\; x_1, x_2, x_3, x_4 \in \mathbb{R} \right\}.$$

The additon and the scalar multiplication on $\mathbb{Q}_{u,v}$ are defined in usual ways. The multiplication of any two of $1, i, j$, or k is defined so that 1 acts as an identity, and the following rules apply:

$$\begin{aligned} i^2 = u,\ j^2 = v,\ k^2 = ijk = -uv, \\ ij = -ji = k,\ jk = -kj = -vi,\ ik = -ki = uj. \end{aligned}$$

It turns out that $\mathbb{Q}_{u,v}$ becomes a non-commutative division ring. A famous special case $(u, v) = (-1, -1)$ of $\mathbb{Q}_{u,v}$ is known as the Hamilton quaternions. The case $(u, v) = (-1, 1)$, the case $(u, v) = (1, -1)$, and the case $(u, v) = (1, 1)$ are called the split quaternion ring, the nectarine quaternion ring, and the conectarine quaternion ring, respectively.

Matrices over quaternions are one of the main interest topics in linear algebra [22]. Matrix equations over $\mathbb{Q}$ or $\mathbb{Q}_{u,v}$ turn out to be important in various fields, e.g., computer platforms [24], image processing [25, 26], color image restoration [27], image and video inpainting [28, 29], signal processing [30] and quantum mechanics [31]. In the last decade, various authors investigated such matrix equations from theoretical points of view. The work in [32] introduced fast and robust algorithms for the eigenproblem and the QR factorization of matrices over $\mathbb{Q}$. Yuan et al. [33] proposed an explicit expression of the least-squares (LS) solution, the LS pure-imaginary solution, and the real solution of Eq (1.3) with the least norm. Zhang et al. [34] studied special LS solutions of Eq (1.3), and obtained the expressions of the minimal-norm LS solution, the pure-imaginary LS solution, and the real LS solution. Recently, Tian et al. [35] considered Hermitian solutions of Eq (1.3). Indeed, they proposed necessary and sufficient conditions for the existence of a Hermitian solution and provided the explicit general expression of the solution when it was solvable.

In this paper, we investigated the Sylvester-transpose matix Eq (1.2) where A, B, C, D, and E are given generalized quaternion matices with compatible size and X is an unknown. We have measured the associated error of a matrix by the Frobenius norm $\|\cdot\|$. Indeed, we have discussed the following problems.

Problem 1.1. Find the solution set $\mathcal{S}$ of exact solutions to Eq (1.2). In addition, find the minimal-norm element of $\mathcal{S}$, i.e., find a matrix X^* such that

$$\|X^*\| = \min_{X \in \mathcal{S}} \|X\|.$$

Problem 1.2. Find a solution $\bar{X} \in \mathcal{S}$ closest to a given matrix $Y \in \mathbb{Q}_{u,v}^{n\times p}$, i.e., find $\bar{X}$ such that

$$\left\|\bar{X} - Y\right\| = \min_{X \in \mathcal{S}} \|X - Y\|.$$

Problem 1.3. Find the set $\mathcal{L}$ of LS solutions to Eq (1.2). In addition, find $\tilde{X}$ such that

$$\left\|\tilde{X}\right\| = \min_{X \in \mathcal{L}} \|X\|.$$

Problem 1.4. Find an LS solution of Eq (1.2) closest to a given matrix $Y \in \mathbb{Q}_{u,v}^{n\times p}$. That is, find the matrix $\acute{X}$ such that

$$\left\|\acute{X} - Y\right\| = \min_{X \in \mathcal{L}} \|X - Y\|.$$

Moreover, we have discussed certain special cases of Eq (1.2), namely Eq (1.1), the equation $AXB = E$, and the case when $u = v = -1$.

The rest of this paper is structured as follows. In Section 2, we set up basic notations and provide auxiliary tools from matrix theory in order to study matrix equations. In Section 3, we investigate Problems 1.1 and 1.2. In Section 4, we investigate Problems 1.3 and 1.4. In Section 5, we take a look at certain special cases of the main Eq (1.2). In Section 6, we provide numerical examples to illustrate our theory. Finally, we summarize the whole work in the last section.

2. Preliminaries

Let us denote by $\mathbb{R}^{m\times n}$ the set of all $m \times n$ real matrices. The set of n-dimensional real vectors is written by $\mathbb{R}^n := \mathbb{R}^{n\times 1}$. The transpose, the conjugate, the Moor-Penrose inverse, and the Frobenius norm of a matrix A are written by $A^T, \bar{A}, A^\dagger$ and $\|A\|$, respectively. The identity matrix of order n is denoted by I_n. The ith column of a matrix A is denoted by $\mathrm{col}_i(A)$.

2.1. Vectorization and Kronecker products

With each matrix $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{s\times t}$, the (column) vector $\mathrm{V_c}(A)$ is defined as

$$\mathrm{V_c}(A) = \begin{pmatrix} a_{11} & \dots & a_{m1} & a_{12} & \dots & a_{m2} & \dots & a_{1n} & \dots & a_{mn} \end{pmatrix}^T \in \mathbb{R}^{mn},$$

and the Kronecker product of A and B is defined as

$$A \otimes B = (a_{ij}B) = \begin{pmatrix} a_{11}B & a_{12}B & \dots & a_{1j}B \\ a_{21}B & a_{22}B & \dots & a_{2j}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{i1}B & a_{i2}B & \dots & a_{ij}B \end{pmatrix} \in \mathbb{R}^{ms\times nt}.$$

Lemma 2.1. [36] *For any $A \in \mathbb{R}^{m\times n}$, $X \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{p\times q}$, we have*

$$\mathrm{V_c}(AXC) \;=\; (C^T \otimes A)\,\mathrm{V_c}(X).$$

Lemma 2.2. [36] *For any $X \in \mathbb{R}^{n\times p}$, we have*

$$\mathrm{V_c}(X^T) \;=\; \mathrm{P}(n,p)\,\mathrm{V_c}(X).$$

Here, $\mathrm{P}(n,p)$ *is a permutation matrix defined by*

$$\mathrm{P}(n,p) \;=\; \sum_{i=1}^{n}\sum_{j=1}^{p} E_{ij}\otimes E_{ij}^T,$$

where each $E_{ij} \in \mathbb{R}^{n\times p}$ has entry 1 in position (i, j) and all other entries are zero.

2.2. Real representations of generalized quaternion matrices

For any positive integers m and n, we denote the set of all $m \times n$ generalized quaternion matrices by $\mathbb{Q}_{u,v}^{m\times n}$. For each $A \in \mathbb{Q}_{u,v}^{m\times n}$, we can write

$$A \;=\; A_1 + A_2 i + A_3 j + A_4 k,$$

where $A_1, A_2, A_3, A_4 \in \mathbb{R}^{m\times n}$. We define

$$\Gamma(A) \;=\; \begin{pmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{pmatrix} \in \mathbb{R}^{4m\times n}.$$

Now, consider $X = X_1 + X_2 i + X_3 j + X_4 k \in \mathbb{Q}_{u,v}^{n\times p}$, where $X_1, X_2, X_3, X_4 \in \mathbb{R}^{n\times p}$. We have

$$\begin{aligned} AX &= (A_1 + A_2 i + A_3 j + A_4 k)(X_1 + X_2 i + X_3 j + X_4 k) \\ &= A_1(X_1 + X_2 i + X_3 j + X_4 k) + A_2 i(X_1 + X_2 i + X_3 j + X_4 k) \\ &\quad + A_3 j(X_1 + X_2 i + X_3 j + X_4 k) + A_4 k(X_1 + X_2 i + X_3 j + X_4 k) \\ &= (A_1X_1 + uA_2X_2 + vA_3X_3 - uvA_4X_4) + (A_1X_2 + A_2X_1 - vA_3X_4 + vA_4X_3)i \\ &\quad + (A_1X_3 + uA_2X_4 + A_3X_1 - uA_4X_2)j + (A_1X_4 + A_2X_3 - A_3X_2 + A_4X_1)k. \end{aligned}$$

Thus,

$$\Gamma(AX) \;=\; \begin{pmatrix} A_1X_1 + uA_2X_2 + vA_3X_3 - uvA_4X_4 \\ A_1X_2 + A_2X_1 - vA_3X_4 + vA_4X_3 \\ A_1X_3 + uA_2X_4 + A_3X_1 - uA_4X_2 \\ A_1X_4 + A_2X_3 - A_3X_2 + A_4X_1 \end{pmatrix} \;=\; \mathcal{R}(A)\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix}, \tag{2.1}$$

where

$$\mathcal{R}(A) \;=\; \begin{pmatrix} A_1 & uA_2 & vA_3 & -uvA_4 \\ A_2 & A_1 & vA_4 & -vA_3 \\ A_3 & -uA_4 & A_1 & uA_2 \\ A_4 & -A_3 & A_2 & A_1 \end{pmatrix},$$

is called a real matrix representation of A. From the block columns of A, it is useful to define the following:

$$\Theta(A) = \begin{pmatrix} uA_2 \\ A_1 \\ -uA_4 \\ -A_3 \end{pmatrix}, \quad \Delta(A) = \begin{pmatrix} vA_3 \\ vA_4 \\ A_1 \\ A_2 \end{pmatrix}, \quad \Phi(A) = \begin{pmatrix} -uvA_4 \\ -vA_3 \\ uA_2 \\ A_1 \end{pmatrix} \in \mathbb{R}^{4m\times n}.$$

Clearly, the transformations $\mathrm{V_c}, \Gamma, \Theta, \Delta$, and Φ are injective. It is easy to see that

$$\|A\| = \sqrt{\|A_1\|^2 + \|A_2\|^2 + \|A_3\|^2 + \|A_4\|^2} = \|\Gamma(A)\|. \tag{2.2}$$

Proposition 2.3. [35] *Let $A, B \in \mathbb{Q}_{u,v}^{m\times n}$ and $k \in \mathbb{R}$. Then the following properties hold.*

(i) $\Gamma(A + B) = \Gamma(A) + \Gamma(B)$, $\Gamma(kA) = k(A)$.
(ii) $\mathcal{R}(AB) = \mathcal{R}(A)\mathcal{R}(B)$.
(iii) $\mathcal{R}(I_m) = I_{4m}$.

3. Consistent generalized Sylvester-transpose matrix equation

In this section, we discuss how to solve the Sylvester-transpose matrix equation

$$AXB + CX^T D = E, \tag{3.1}$$

where $A \in \mathbb{Q}_{u,v}^{m\times n}, B \in \mathbb{Q}_{u,v}^{p\times q}, C \in \mathbb{Q}_{u,v}^{m\times p}, D \in \mathbb{Q}_{u,v}^{n\times q}$, and $E \in \mathbb{Q}_{u,v}^{m\times q}$ are given matrices and $X \in \mathbb{Q}_{u,v}^{n\times p}$ is an unknown. Our idea is to transform Eq (3.1) into a real linear system. So, let us recall the following result.

Lemma 3.1. [37] *Given $K \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$, we consider the linear system*

$$Kx = b. \tag{3.2}$$

Then the system (3.2) *has a solution $x \in \mathbb{R}^n$ if and only if $KK^\dagger b = b$, where $K^\dagger$ is the Moore-Penrose inverse of K. For the consistent case, we have the following:*

(i) The general solution of Eq (3.2) *is given by*

$$x = K^\dagger b + (I_n - K^\dagger K)y, \tag{3.3}$$

where $y \in \mathbb{R}^n$ is an arbitrary vector.

(ii) Among the general solution (3.3)*, the minimal-norm solution is given by*

$$x = K^\dagger b. \tag{3.4}$$

(iii) If $\operatorname{rank}(K) = n$*, then the system* (3.2) *has an unique solution given by* (3.4).

The next lemmas are utilized to transform Eq (3.1) into a linear system.

Lemma 3.2. *Let A, B, C, and $D \in \mathbb{R}^{m\times n}$. Then*

$$\mathrm{V_c}\begin{pmatrix} A^T \\ B^T \\ C^T \\ D^T \end{pmatrix} = \mathrm{P}(m, 4n)(\mathrm{P}(4, n) \otimes I_m)\,\mathrm{V_c}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}.$$

Proof. Using Lemma 2.2, we obtain

$$\mathrm{V_c}\begin{pmatrix} A^T \\ B^T \\ C^T \\ D^T \end{pmatrix} = \mathrm{V_c}\begin{pmatrix} A & B & C & D \end{pmatrix}^T = \mathrm{P}(m,4n)\,\mathrm{V_c}\begin{pmatrix} A & B & C & D \end{pmatrix}$$
$$= \mathrm{P}(m,4n)\begin{pmatrix} \mathrm{V_c}(A) \\ \mathrm{V_c}(B) \\ \mathrm{V_c}(C) \\ \mathrm{V_c}(D) \end{pmatrix} = \mathrm{P}(m,4n)(\mathrm{P}(4,n)\otimes I_m)\,\mathrm{V_c}\begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix}.$$

Lemma 3.3. *Let $X \in \mathbb{Q}_{u,v}^{n\times p}$. Then*

$$\begin{pmatrix} \mathrm{V_c}(\Gamma(X)) \\ \mathrm{V_c}(\Theta(X)) \\ \mathrm{V_c}(\Delta(X)) \\ \mathrm{V_c}(\Phi(X)) \end{pmatrix} = \mathcal{M}\,\mathrm{V_c}(\Gamma(X)), \quad where \quad \mathcal{M} = \begin{pmatrix} I_{4np} \\ I_p\otimes R_p\otimes I_n \\ I_p\otimes S_p\otimes I_n \\ I_p\otimes T_p\otimes I_n \end{pmatrix} \in \mathbb{R}^{16np\times 4np}, \tag{3.5}$$

and

$$\begin{aligned} R_p &= \begin{pmatrix} e_2^4 & ue_1^4 & -e_4^4 & -ue_3^4 \end{pmatrix} \in \mathbb{R}^{4\times 4}, \\ S_p &= \begin{pmatrix} e_3^4 & e_4^4 & ve_1^4 & ve_2^4 \end{pmatrix} \in \mathbb{R}^{4\times 4}, \\ T_p &= \begin{pmatrix} e_4^4 & ue_3^4 & -ve_2^4 & -uve_1^4 \end{pmatrix} \in \mathbb{R}^{4\times 4}, \end{aligned}$$

where $e_i^4 = \mathrm{col}_i(I_4)$.

Proof. We compute

$$\begin{aligned} \mathrm{V_c}(\Theta(X)) &= \begin{pmatrix} u\,\mathrm{col}_1(X_2) \\ \mathrm{col}_1(X_1) \\ -u\,\mathrm{col}_1(X_4) \\ -\mathrm{col}_1(X_3) \\ \vdots \\ u\,\mathrm{col}_p(X_2) \\ \mathrm{col}_p(X_1) \\ -u\,\mathrm{col}_p(X_4) \\ -\mathrm{col}_p(X_3) \end{pmatrix} = \begin{pmatrix} 0 & uI_n & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ I_n & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -uI_n & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_n & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & uI_n & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & -uI_n \\ 0 & 0 & 0 & 0 & \dots & 0 & 0 & -I_n & 0 \end{pmatrix} \begin{pmatrix} \mathrm{col}_1(X_1) \\ \mathrm{col}_1(X_2) \\ \mathrm{col}_1(X_3) \\ \mathrm{col}_1(X_4) \\ \vdots \\ \mathrm{col}_p(X_1) \\ \mathrm{col}_p(X_2) \\ \mathrm{col}_p(X_3) \\ \mathrm{col}_p(X_4) \end{pmatrix} \\ &= I_p \otimes \left[\begin{pmatrix} e_2^4 & ue_1^4 & -e_4^4 & -ue_3^4 \end{pmatrix} \otimes I_n\right] \mathrm{V_c}(\Gamma(X)) \\ &= (I_p\otimes R_p\otimes I_n)\,\mathrm{V_c}(\Gamma(X)). \end{aligned}$$

With a similar process, we obtain

$$\begin{aligned}\mathrm{V_c}(\Delta(X)) &= I_p \otimes \left[\begin{pmatrix} e_3^4 & e_4^4 & ve_1^4 & ve_2^4 \end{pmatrix} \otimes I_n\right] \mathrm{V_c}(\Gamma(X)) \\ &= (I_p \otimes S_p \otimes I_n)\, \mathrm{V_c}(\Gamma(X)),\end{aligned}$$

and

$$\begin{aligned}\mathrm{V_c}(\Phi(X)) &= I_p \otimes \left[\begin{pmatrix} e_4^4 & ue_3^4 & -ve_2^4 & -uve_1^4 \end{pmatrix} \otimes I_n\right] \mathrm{V_c}(\Gamma(X)) \\ &= (I_p \otimes T_p \otimes I_n)\, \mathrm{V_c}(\Gamma(X)).\end{aligned}$$

Thus, we obtain Eq (3.5).

Theorem 3.4. *Consider Eq* (3.1)*. Let us denote*

$$\mathcal{W} = \left(\Gamma(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma(D)^T \otimes \mathcal{R}(C)\right)(I_4 \otimes \mathrm{P}(n, 4p)(\mathrm{P}(4, p) \otimes I_n)). \tag{3.6}$$

(i) The matrix Eq (3.1) *has a solution if and only if*

$$(\mathcal{W}\mathcal{M})(\mathcal{W}\mathcal{M})^{\dagger} V_c(\Gamma(E)) = V_c(\Gamma(E)).$$

(ii) Then the solution set $\mathcal{S}$ of Problem 1.1 can be expressed as

$$\mathcal{S} = \left\{X \,\middle|\, \mathrm{V_c}(\Gamma(X)) = (\mathcal{W}\mathcal{M})^{\dagger}\, \mathrm{V_c}(\Gamma(E)) + \left[I_{4np} - (\mathcal{W}\mathcal{M})^{\dagger}(\mathcal{W}\mathcal{M})\right] y\right\}, \tag{3.7}$$

where $y \in \mathbb{R}^{4np}$ is an arbitrary vector.

(iii) Among all solutions (3.7)*, the minimal-norm solution is given by*

$$\mathrm{V_c}(\Gamma(X)) = (\mathcal{W}\mathcal{M})^{\dagger}\, \mathrm{V_c}(\Gamma(E)). \tag{3.8}$$

(iv) When $\mathcal{W}\mathcal{M}$ is of full-column rank, Eq (3.1) *has a unique solution given by* (3.8)*.*

Proof. From Eq (3.1), we consider the associated norm-error $\|AXB + CX^T D - E\|$. Using Eq (2.2), Proposition 2.3 and Lemma 2.1, we obtain

$$\begin{aligned}\left\|AXB + CX^T D - E\right\| &= \left\|\Gamma(AXB + CX^T D - E)\right\| \\ &= \left\|\Gamma(AXB) + \Gamma(CX^T D) - \Gamma(E)\right\| \\ &= \left\|\mathcal{R}(A)\mathcal{R}(X)\Gamma(B) + \mathcal{R}(C)\mathcal{R}(X^T)\Gamma(D) - \Gamma(E)\right\| \\ &= \left\|\mathrm{V_c}\left[\mathcal{R}(A)\mathcal{R}(X)\Gamma(B) + \mathcal{R}(C)\mathcal{R}(X^T)\Gamma(D) - \Gamma(E)\right]\right\| \\ &= \left\|\left(\Gamma(B)^T \otimes \mathcal{R}(A)\right)\ \mathrm{V_c}(\mathcal{R}(X)) + \left(\Gamma(D)^T \otimes \mathcal{R}(C)\right)\ \mathrm{V_c}\left(\mathcal{R}(X^T)\right) - \mathrm{V_c}(\Gamma(E))\right\|.\end{aligned}$$

By Lemma 3.2, we have

$$\begin{aligned}
\mathrm{V_c}(\Gamma(X^T)) &= \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Gamma(X)),\\
\mathrm{V_c}(\Theta(X^T)) &= \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Theta(X)),\\
\mathrm{V_c}(\Delta(X^T)) &= \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Delta(X)),\\
\mathrm{V_c}(\Phi(X^T)) &= \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Phi(X)).
\end{aligned}$$

Using Lemma 3.3, we compute

$$\begin{aligned}
&\left(\Gamma(B)^T\otimes\mathcal{R}(A)\right)\,\mathrm{V_c}(\mathcal{R}(X))+\left(\Gamma(D)^T\otimes\mathcal{R}(C)\right)\,\mathrm{V_c}\left(\mathcal{R}(X^T)\right)-\mathrm{V_c}(\Gamma(E))\\
&= \left(\Gamma(B)^T\otimes\mathcal{R}(A)\right)\,\mathrm{V_c}(\mathcal{R}(X))+\left(\Gamma(D)^T\otimes\mathcal{R}(C)\right)\begin{pmatrix}\mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Gamma(X))\\ \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Theta(X))\\ \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Delta(X))\\ \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n)\,\mathrm{V_c}(\Phi(X))\end{pmatrix}\\
&\quad -\mathrm{V_c}(\Gamma(E))\\
&= (\Gamma(B)^T\otimes\mathcal{R}(A))\begin{pmatrix}\mathrm{V_c}(\Gamma(X))\\ \mathrm{V_c}(\Theta(X))\\ \mathrm{V_c}(\Delta(X))\\ \mathrm{V_c}(\Phi(X))\end{pmatrix}\\
&\quad +(\Gamma(D)^T\otimes\mathcal{R}(C))(I_4\otimes \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n))\begin{pmatrix}\mathrm{V_c}(\Gamma(X))\\ \mathrm{V_c}(\Theta(X))\\ \mathrm{V_c}(\Delta(X))\\ \mathrm{V_c}(\Phi(X))\end{pmatrix}-\mathrm{V_c}(\Gamma(E))\\
&= \left[\left(\Gamma(B)^T\otimes\mathcal{R}(A)\right)+\left(\Gamma(D)^T\otimes R(C)\right)(I_4\otimes P(n,4p)(P(4,p)\otimes I_n))\right]\begin{pmatrix}V_c(\Gamma(X))\\ V_c(\Theta(X))\\ V_c(\Delta(X))\\ V_c(\Phi(X))\end{pmatrix}-V_c(\Gamma(E))\\
&= \left[\left(\Gamma(B)^T\otimes\mathcal{R}(A)\right)+\left(\Gamma(D)^T\otimes\mathcal{R}(C)\right)(I_4\otimes \mathrm{P}(n,4p)(\mathrm{P}(4,p)\otimes I_n))\right]\mathcal{M}\,\mathrm{V_c}(\Gamma(X))-\mathrm{V_c}(\Gamma(E))\\
&= \mathcal{W}\mathcal{M}\,\mathrm{V_c}(\Gamma(X))-\mathrm{V_c}(\Gamma(E)).
\end{aligned}$$

So, the generalized quaternion matrix Eq (3.1) is equivalent to a real linear system

$$\mathcal{W}\mathcal{M}\,\mathrm{V_c}(\Gamma(X)) = \mathrm{V_c}(\Gamma(E)). \tag{3.9}$$

By Lemma 3.1, the system (3.9) has the general solution

$$\mathrm{V_c}(\Gamma(X)) = (\mathcal{W}\mathcal{M})^{\dagger}\,\mathrm{V_c}(\Gamma(E))+[I_{4np}-(\mathcal{W}\mathcal{M})^{\dagger}(\mathcal{W}\mathcal{M})]y,$$

where $y\in\mathbb{R}^{4np}$ is an arbitrary vector. The assertions (*iii*) and (*iv*) now follow from Lemma 3.1.

Theorem 3.5. *Consider Eq* (3.1). *Let* $Y\in\mathbb{Q}_{u,v}^{n\times p}$ *be given. Then Problem 1.2 is equivalent to finding the minimal-norm solution* $Z\in\mathbb{Q}_{u,v}^{n\times p}$ *of a matrix equation*

$$AZB+CZ^TD = \hat{E},$$

where $\hat{E}=E-(AYB+CY^TD)$.

Proof. Letting $Z = X - Y$, we consider the following error

$$\begin{aligned} AXB + CX^T D - E &= AXB + CX^T D - E - AYB - CY^T D + AYB + C\mathcal{Y}^T D \\ &= A(X - Y)B + C(X^T - Y^T)D - E + AYB + CY^T D \\ &= AZB + CZ^T D - \hat{E}. \end{aligned}$$

Thus, Problem 1.2 is equivalent to the following minimization:

$$\begin{aligned} \min_{AXB+CX^T D = E} \|X - Y\| &= \min_{AXB+CX^T D = E} \|Z\| \\ &= \min_{AZB+CZ^T D = \hat{E}} \|Z\|, \end{aligned}$$

as desired.

4. Inconsistent generalized Sylvester-transpose matrix equation

In this section, we investigate Eq (3.2) when it is inconsistant. We seek for least-squares (LS) solutions with minimal-norm or the closest solution to a given matrix. Recall the following result:

Lemma 4.1. [37] *Consider the linear system* (3.2) *in the inconsistent case. We have the following:*

(i) The general LS solutions of Eq (3.2) *are given by* (3.3)*, where* $y \in \mathbb{R}^{4np}$ *is an arbitrary vector.*
(ii) Among such LS solutions, the minimal-norm solution is given by (3.4).
(iii) If $\operatorname{rank}(K) = 4np$, *then the system* (3.2) *has a unique LS solution given by* (3.4).

Theorem 4.2. *Suppose that Eq* (3.2) *is inconsistent. Denote* $\mathcal{W}$ *as in* (3.6).

(i) Then the solution set $\mathcal{L}$ *of Problem 1.3 can be expressed as*

$$\mathcal{L} = \left\{X \;\middle|\; \mathrm{V_c}(\Gamma(X)) = (\mathcal{W}\mathcal{M})^{\dagger}\, \mathrm{V_c}(\Gamma(E)) + \left[I_{4np} - (\mathcal{W}\mathcal{M})^{\dagger}(\mathcal{W}\mathcal{M})\right] y\right\}, \tag{4.1}$$

where $y \in \mathbb{R}^{4np}$ *is an arbitrary vector.*
(ii) Among such solutions (4.1)*, the minimal-norm solution is given by* (3.8).
(iii) Moreover, if $\operatorname{rank}(\mathcal{W}\mathcal{S}) = 4np$*, Eq* (3.1) *has a unique LS solution given by* (3.8).

Proof. From the proof of Theorem 3.4, we see that Eq (3.1) is equivalent to the real linear system (3.9). Lemma 4.1 now implies that the LS solutions of Eq (3.1) are given by

$$\mathrm{V_c}(\Gamma(X)) = (\mathcal{W}\mathcal{M})^{\dagger}\, \mathrm{V_c}(\Gamma(E)) + [I_{4np} - (\mathcal{W}\mathcal{M})^{\dagger}(\mathcal{W}\mathcal{M})]y,$$

where $y \in \mathbb{R}^{4np}$ is an arbitrary vector. The assertions (*ii*) and (*iii*) also follow from Lemma 4.1.

Theorem 4.3. *Consider Eq* (3.1). *Let* $Y \in \mathbb{Q}_{u,v}^{n\times p}$ *be given. Then Problem 1.4 is equivalent to finding the minimal-norm least-squares solution* $Z \in \mathbb{Q}_{u,v}^{n\times p} G$ *of a matrix equation*

$$AZB + CZ^T D = \hat{E},$$

where $\hat{E} = E - (AYB + CY^T D)$.

Proof. From the proof of Theorem 3.5, we have

$$\left\|AXB + CX^T D - E\right\| \;=\; \left\|AZB + CZ^T D - \hat{E}\right\|,$$

where $Z = X - Y$. Thus, Problem 1.4 is equivalent to the following:

$$\begin{aligned}\min_{X\in\mathcal{L}} \|X - Y\| &= \min_{\|AXB+CX^T D-E\| \,=\, \min} \|X - Y\| \\ &= \min_{\|AXB+CX^T D-E\| \,=\, \min} \|Z\| \\ &= \min_{\|AZB+CZ^T D-\hat{E}\| \,=\, \min} \|Z\|,\end{aligned}$$

as desired.

5. Special cases

From the Sylvester-transpose Eq (1.2), we can investigate its certain special cases.

5.1. Matrix equations

Corollary 5.1. *Let $A \in \mathbb{Q}_{u,v}^{m\times n}$, $B \in \mathbb{Q}_{u,v}^{p\times q}$, and $E \in \mathbb{Q}_{u,v}^{m\times q}$. Consider the matrix equation*

$$AXB \;=\; E$$

in an unknown $X \in \mathbb{Q}_{u,v}^{n\times p}$. Then the conclusions of Theorems 3.4, 3.5, 4.2 and 4.3 hold, where the matrix $\mathcal{W}$ is given by

$$\mathcal{W} \;=\; \Gamma(B)^T \otimes \mathcal{R}(A).$$

Proof. We set $C = 0$ and $D = 0$ in those theorems.

The next special case is the Sylvester-transpose matrix equation

$$AX + X^T D \;=\; E. \tag{5.1}$$

Corollary 5.2. *Let $A \in \mathbb{Q}_{u,v}^{p\times n}$, $D \in \mathbb{Q}_{u,v}^{n\times p}$, and $E \in \mathbb{Q}_{u,v}^{p\times p}$. Consider Eq* (5.1) *in an unknown $X \in \mathbb{Q}_{u,v}^{n\times p}$. Then the conclusions of Theorems 3.4, 3.5, 4.2 and 4.3 hold, where*

$$\mathcal{W} \;=\; \left(\Gamma(I_p) \otimes \mathcal{R}(A)\right) + \left(\Gamma(D)^T \otimes I_{4p}\right)(I_4 \otimes \mathrm{P}(n, 4p)(\mathrm{P}(4, p) \otimes I_n)).$$

Proof. We set $B = C = I_p$ in those theorems.

In the next result, we consider Eq (1.2) over the quaternions.

Corollary 5.3. *Let $A \in \mathbb{Q}^{m\times n}$, $B \in \mathbb{Q}^{p\times q}$, $C \in \mathbb{Q}^{m\times p}$, $D \in \mathbb{Q}^{n\times q}$, and $E \in \mathbb{Q}^{m\times q}$. Consider the matrix equation*

$$AXB + CX^T D \;=\; E.$$

Then the conclusions of Theorems 3.4, 3.5, 4.2 and 4.3 hold, where the matrix $\mathcal{M}$ is given explicitly by

$$\mathcal{M} = \begin{pmatrix} I_{4np} \\ I_p \otimes \acute{R}_P \\ I_p \otimes \acute{S}_P \\ I_p \otimes \acute{T}_P \end{pmatrix} \in \mathbb{R}^{16np \times 4np}, \tag{5.2}$$

and

$$\acute{R}_p = \begin{pmatrix} 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \\ 0 & 0 & 0 & I_n \\ 0 & 0 & -I_n & 0 \end{pmatrix}, \quad \acute{S}_p = \begin{pmatrix} 0 & 0 & -I_n & 0 \\ 0 & 0 & 0 & -I_n \\ I_n & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 \end{pmatrix}, \quad \acute{T}_p = \begin{pmatrix} 0 & 0 & 0 & -I_n \\ 0 & 0 & I_n & 0 \\ 0 & -I_n & 0 & 0 \\ I_n & 0 & 0 & 0 \end{pmatrix}.$$

Proof. Set $u = v = -1$ in those theorems.

5.2. Linear systems over the quaternions

In this subsection, we consider a quaternion linear system

$$Ax = b, \tag{5.3}$$

where $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$ are given, and $x \in \mathbb{Q}^n$ is an unknown.

Corollary 5.4. *Consider the linear system* (5.3)*. Denote $\mathcal{M}$ as in* (5.2) *where $p = 1$.*

(i) Then the system (5.3) *has a solution if and only if*

$$\left(\mathcal{R}(A)\mathcal{M}\right)\left(\mathcal{R}(A)\mathcal{M}\right)^{\dagger}\Gamma(b) = \Gamma(b).$$

(ii) The general exact/LS solution of Eq (5.3) *can be expressed as*

$$\Gamma(x) = \left(\mathcal{R}(A)\mathcal{M}\right)^{\dagger}\Gamma(b) + \left[I_{4n} - \left(\mathcal{R}(A)\mathcal{M}\right)^{\dagger}\left(\mathcal{R}(A)\mathcal{M}\right)\right]y, \tag{5.4}$$

where $y \in \mathbb{R}^{4n}$ is an arbitrary vector.

(iii) Among all solutions (5.4)*, the minimal-norm solution is given by*

$$\Gamma(x) = \left(\mathcal{R}(A)\mathcal{M}\right)^{\dagger}\Gamma(b). \tag{5.5}$$

(iv) When $\mathcal{R}(A)\mathcal{M}$ is of full-column rank, Eq (5.3) *has a unique exact/LS solution given by* (5.5).

Proof. From Theorems 3.4 and 4.2, set $p = 1, B = I_1$, and $C = 0$.

A conjugate gradient type to solve the quaternion linear system (5.3) is the quaternion generalized minimal residual method (QGMRES) [38]. Now, we discuss the following problem.

Problem 5.5. Let $A \in \mathbb{Q}^{m \times n}$ and $b \in \mathbb{Q}^m$ be given. Find an LS solution of Eq (5.3) closest to a given vector $h \in \mathbb{Q}^n$. That is, find the vector $\tilde{x}$ such that

$$\|\tilde{x} - h\| = \min_{\|Ax-b\| = \min} \|x - h\|.$$

Corollary 5.6. *Consider Eq* (5.3)*. Let $h \in \mathbb{Q}^n$ be given. Then the solution of Problem 5.5 is given by $x = h + z$ where*

$$\Gamma(z) = \left(\mathcal{R}(A)\mathcal{M}\right)^{\dagger}\left[\Gamma(b) - \mathcal{R}(A)\Gamma(h)\right].$$

Here, the matrix $\mathcal{M}$ is given by (5.2) *where $p = 1$.*

Proof. From the case $B = 1$ and $C = 0$ in Theorem 4.3, we see that Problem 5.5 is equivalent to finding a minimal-norm LS solution z of the linear system

$$Az = b - Ah.$$

Indeed, the desired solution is $x = h + z$. From Corollary 5.4 and Eq (2.1), we obtain

$$\begin{aligned}\Gamma(z) &= \left(\mathcal{R}(A)\mathcal{M}\right)^{\dagger}\Gamma(b - Ah) \\ &= \left(\mathcal{R}(A)\mathcal{M}\right)^{\dagger}\left[\Gamma(b) - \mathcal{R}(A)\Gamma(h)\right].\end{aligned}$$

6. Numerical examples

In this section, we provide numerical examples to illustrate our results.

Example 6.1. Consider the generalized Sylvester-transpose matrix equation $AXB + CX^TD = E$ over the split quaternions (i.e., $(u, v) = (-1, 1)$),

$$A = \begin{pmatrix}1 & i+2j\end{pmatrix}_{1\times 2}, \quad C = \begin{pmatrix}-1 & -i+j+k\end{pmatrix}_{1\times 2},$$
$$B = \begin{pmatrix}i+k \\ 2+3j\end{pmatrix}_{2\times 1}, \quad D = \begin{pmatrix}2i \\ 3-k\end{pmatrix}_{2\times 1}, \quad E = \begin{pmatrix}-1+4i+3j+k\end{pmatrix}_{1\times 1}.$$

Then we have

$$\mathcal{R}(A) = \begin{pmatrix}1 & 0 & 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & -2 \\ 0 & 2 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & -2 & 0 & 1 & 1 & 0\end{pmatrix}, \quad \Gamma(B)^T = \begin{pmatrix}0 & 2 & 1 & 0 & 0 & -3 & 1 & 0\end{pmatrix},$$

$$\mathcal{R}(C) = \begin{pmatrix}-1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & -1 & -1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0\end{pmatrix}, \quad \Gamma(D)^T = \begin{pmatrix}0 & 3 & 2 & 0 & 0 & 0 & 0 & -1\end{pmatrix},$$

and

$$\begin{aligned}\mathcal{W} &= \left(\Gamma(B)^T \otimes \mathcal{R}(A)\right) + \left(\Gamma(D)^T \otimes \mathcal{R}(C)\right), \\ \mathcal{M} &= I_4 \otimes \mathrm{P}(2,8)\left(\mathrm{P}(4,2) \otimes I_2\right), \quad \Gamma(E) = \begin{pmatrix}-1 & 4 & 3 & 1\end{pmatrix}^T.\end{aligned}$$

According to Theorem 3.4, the matrix equation has a unique solution, computed via MATLAB as follows:

$$X = \begin{pmatrix} 0 & 0 \\ 0 & -0.2709 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0.0739 \end{pmatrix} i + \begin{pmatrix} 0 & 0 \\ 0 & -0.8079 \end{pmatrix} j + \begin{pmatrix} 0 & 0 \\ 0 & -0.8079 \end{pmatrix} k.$$

Example 6.2. Consider the matrix equation $AXB + CX^T D = E$ over the split quaternions, i.e., $(u, v) = (-1, 1)$. Here, we are given the matrices A, B, C, D, and E as in Example 6.1, and we will find a solution X closest to a given matrix

$$Y = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}.$$

We obtain

$$\hat{E} = E - (AYB + CY^T D) \text{ and } \Gamma(\hat{E}) = \begin{pmatrix} -2 & 4 & 2 & -8 \end{pmatrix}^T.$$

Using Theorem 3.5 and MATLAB, we obtain:

$$Z = \begin{pmatrix} 0 & 0 \\ 0 & 0.3481 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0.6404 \end{pmatrix} i + \begin{pmatrix} 0 & 0 \\ 0 & 0.3350 \end{pmatrix} j + \begin{pmatrix} 0 & 0 \\ 0 & -0.1938 \end{pmatrix} k.$$

Thus, we get the desired solution:

$$X = Z + Y = \begin{pmatrix} 1 & 0 \\ 0 & 0.3481 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -0.3596 \end{pmatrix} i + \begin{pmatrix} 0 & 0 \\ 0 & 0.3350 \end{pmatrix} j + \begin{pmatrix} 0 & 0 \\ 0 & -0.1938 \end{pmatrix} k.$$

7. Conclusions

We investigated a generalized Sylvester-transpose matrix equation $AXB + CX^T D = E$, where A, B, C, D, E, and X are matrices over a generalized quaternion skew-field. When all matrix dimensions were compatible, we provided a criterion for the equation to have a solution, involving Moore-Penrose inverses of associated matrices. Applying vectorizations and real representations of generalized quaternion matrices, we derived formulas of general exact/least-squares solutions, the minimal-norm solution, and the solution closest to a given matrix. Our results included the equation $AXB = E$ and the Sylvester-transpose equation, quaternionic matrix equations, and quaternionic linear systems.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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