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Research article

Qualitative analysis and traveling wave solutions of a predator-prey model with time delay and stage structure

Meng Wang and Naiwei Liu*

School of Mathematics and Information Science, Yantai University, Yantai 264000, China

* **Correspondence:** Email: liunaiwei@aliyun.com.

Abstract: In this paper, we considered a delayed predator-prey model with stage structure and Beddington-DeAngelis type functional response. First, we analyzed the stability of the constant equilibrium points of the model by the linear stability method. Furthermore, we considered the existence of traveling wave solutions connecting the zero equilibrium point and the unique positive equilibrium point. Second, we transformed the existence of traveling wave solutions into the existence of fixed points of an operator by constructing suitable upper and lower solutions, and combined with the Schauder fixed point theorem, we gave the existence of fixed points and obtained the existence of traveling wave solutions of the model.

Keywords: predator-prey model; stage structure; time delay; upper-lower solutions; traveling wave solution

1. Introduction

The relationship among different biological populations is complex and very important and is an essential part of the research on the development of ecology. Due to the prevalence and importance of predation in nature, studying the dynamic relationship between predator and prey has always been one of the dominant topics.

In the 1920s and 1930s, as the pioneers of mathematical ecology, Lotka [1] and Volterra [2] proposed the famous Lotka-Volterra model independently, which is used to discribe the interaction between two groups composed of predators and preys:

$$\begin{cases} \dfrac{du(t)}{dt} = a_1 u(t) - b_1 u(t) v(t), \\ \dfrac{dv(t)}{dt} = -a_2 v(t) + b_2 u(t) v(t). \end{cases}$$

Here, we assume an ecosystem that includes two groups of predators and preys. The predator survives on prey, and the system has no population exchange relationship with the outside world. In order to establish a mathematical model describing the system, the prey and predator population are regarded as the basic variables, which are represented by $u(t)$ and $v(t)$, respectively. The natural increment of the prey population is proportional to the number of itself, and if the proportional constant is $a_1 > 0$, and if the mortality rate of predator population is proportional to its own number, the proportional constant is $a_2 > 0$, and $b_1 > 0$ and $b_2 > 0$ are positive constant. Lotka-Volterra model is a basic model to describe the predator-prey relationship between predator and prey.

In 1975, Landahl and Hanson [3] and Tognetti [4] proposed a stage structure model and used different equations to describe individual behavior at different stages. In the last two decades, Zhang et al. [5] proposed and discussed a delayed predator-prey model with stage structure and nonlocal diffusion, and they studied the existence and exact asymptotic behavior of traveling wave solutions. Zhang and Xu [6, 7] considered the predator-prey model with nonlocal delay and stage structure, and further studied the global stability. One can also see [8–14].

Recently, Hong and Weng [15] studied the delayed predator-prey model with local diffusion and nonlocal spatial effects, and they investigated the stability of the equilibria and the existence of traveling wave solutions connecting the zero equilibrium point and the unique positive equilibrium point.

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = & D_1\dfrac{\partial^2 u_1}{\partial x^2} + a_1u_2(x,t) - d_1u_1(x,t) - a_{11}u_1^2(x,t) - \hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty} G(x,y,\tau)\,u_2(y,t-\tau)\,dy, \\ \dfrac{\partial u_2}{\partial t} = & D_2\dfrac{\partial^2 u_2}{\partial x^2} + \hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty} G(x,y,\tau)\,u_2(y,t-\tau)\,dy - d_2u_2(x,t) \\ & -q_2e_2u_2(x,t) - a_{22}u_2^2(x,t) - \dfrac{a_{23}u_2(x,t)\,v(x,t)}{1+mu_2(x,t)}, \\ \dfrac{\partial v}{\partial t} = & D_3\dfrac{\partial^2 v}{\partial x^2} + [a_2 - b_2v(x,t)]\,v(x,t) - q_3e_3v(x,t) + \dfrac{a_{32}u_2(x,t)\,v(x,t)}{1+mu_2(x,t)}, \end{cases}$$

where $G(x,y,\tau) = \frac{1}{\sqrt{4D_1\pi\tau}}e^{-\frac{(x-y)^2}{4D_1\tau}}$. The model considered the Holling II functional response function. Although Holling type functional response functions are widely used, they do not consider the effect of predator density on predation rate. For this reason, some scholars have proposed a ratio dependent functional response function, and the results are also supported by many experimental facts. For results about stage structure, we refer to [16, 17].

In 2001, Skalaski and Gilliam [18] compared the statistical data in some predator-prey systems, and found that the predator-dependent functional response function model has a high degree of fit with the data. The Beddington-DeAngelis functional response function is more practical in reality. This function maintains all the characteristics of the proportional dependent functional response function and avoids the singular behavior caused by the low density state, so it can better reflect the predator-prey effect (we refer to [19–21] for details).

In 2017, Khajanchi and Banerjee [22] introduced a persistent prey refuge in a stage structured predator-prey model with a ratio dependent functional response and obtained sufficient conditions for permanence and global asymptotic stability by constructing a suitable Lyapunov function.

$$\begin{cases} \dfrac{dx_i(t)}{dt} = \alpha x_m(t) - \beta x_i(t) - \delta_1 x_i(t), \\ \dfrac{dx_m(t)}{dt} = \beta x_i(t) - \delta_2 x_m(t) - \gamma x_m^2(t) - \dfrac{\eta(1-\theta)x_m(t)y(t)}{g(1-\theta)x_m(t) + hy(t)}, \\ \dfrac{dy(t)}{dt} = \dfrac{\mu(1-\theta)x_m(t)y(t)}{g(1-\theta)x_m(t) + hy(t)} - \delta_3 y(t), \end{cases}$$

where α represents the growth rate of juvenile prey. The conversion coefficient from juvenile prey to adult prey is proportional to the existing juvenile prey, and the proportional constant is β. γ represents the intraspecific competition rate of adult prey. δ_1 , δ_2, and δ_3 represent the natural mortality of juvenile prey, adult prey, and predator, respectively. We introduced an adult prey shelter θx_m, $\theta \in (0, 1)$, which measures the strength of the prey shelter. For related work, Cheng and Yuan [23] considered the existence and stability of traveling wave solutions of Holling-Tanner predator-prey model with nonlocal diffusion and Holling type I functional response.

The local Laplacian operator to represent the spatial diffusion phenomenon cannot accurately describe the spatial and temporal behavior of species. In fact, spatial nonlocal effects are ubiquitous in nature. As for a biological population, it will move in a large spatial range than be limited to a small range, which leads to the occurrence of spatial nonlocal effects. Accordingly, many researchers have introduced convolution operators into the research models to describe the movement of individuals in the whole space and used convolution operators to describe the spatial diffusion process (see [24–26]).

In this paper, motivated by the results in [15], we consider the influence of Beddington-DeAngelis functional response function on the existence of traveling wave solutions of the model and consider the stage structure of the prey population and divide the prey population into two categories: Juvenile and adult. For many mammals, the juvenile prey is hidden in the cave and fed by their parents, so they do not have to go out to find food; thus, we have reason to think that the juvenile prey is not at risk of being attacked by predators. Our model is as follows:

$$\begin{cases} \dfrac{\partial u_1}{\partial t} = D_1 \dfrac{\partial^2 u_1}{\partial x^2} + a_1 u_2 - d_1 u_1 - a_{11} u_1^2 - \hat{a} e^{-d_1 \tau} \int_{-\infty}^{+\infty} G(x, y, \tau) u_2(y, t-\tau) dy, \\ \dfrac{\partial u_2}{\partial t} = D_2 \dfrac{\partial^2 u_2}{\partial x^2} + \hat{a} e^{-d_1 \tau} \int_{-\infty}^{+\infty} G(x, y, \tau) u_2(y, t-\tau) dy - d_2 u_2 - q_2 e_2 u_2 \\ \qquad -a_{12} u_2^2 - \dfrac{\beta u_2 v}{1 + m u_2 + w v}, \\ \dfrac{\partial v}{\partial t} = D_5 \dfrac{\partial^2 v}{\partial x^2} + a_2 v - b_2 v - a_{55} v^2 - q_5 e_5 v + \dfrac{\beta_1 u_2 v}{1 + m u_2 + w v}, \end{cases} \tag{1.1}$$

where $G(x, y, \tau) = \frac{1}{\sqrt{4\pi D_1 \tau}} e^{-\frac{(x-y)^2}{4D_1 \tau}}$, $\frac{\beta_1}{\beta}$ is the rate at which nutrients are converted to predators for reproduction. $u_1(x, t)$, $u_2(x, t)$ and $v(x, t)$ are the population density of juvenile prey population, adult prey population and predator population at position x and moment t, respectively. $\hat{a} e^{-d_1 \tau} \int_{-\infty}^{+\infty} G(x, y, \tau) u_2(y, t-\tau) dy$ represents the number of prey species converted from juvenile to adult at position x and moment t. Here, the application of nonlocal Fourier transform and convolution shows that the function value at position x is not only related to this point, but also affected by the surrounding area. $\tau > 0$ is a time delay, indicating that the change rate of the unit population at moment t depends on the number of populations at moment $t - \tau$. $D_1 > 0$, $D_2 > 0$ and $D_5 > 0$ are the diffusion coefficients. $a_1 > 0$ and $a_2 > 0$ are the birth rates of juvenile prey and predator populations respectively. $d_1 > 0$, $d_2 > 0$ and $b_2 > 0$ are the mortality of juvenile prey population, adult prey population and predator population, respectively. $a_{12} > 0$ and $a_{55} > 0$ are the overcrowding rates of adult prey population and predator population respectively. $q_2e_2u_2(x,t) > 0$ and $q_5e_5v(x,t) > 0$ represent capture items of adult prey population and predator population, respectively, and m and w are positive constant.

We take the intial condition

$$u_1(x,0) = \delta_1(x) > 0,\ u_2(x,t) = \delta_2(x,t) \geq 0,\ \delta_2(x,0) > 0,\ v(x,0) = \delta_3(x) > 0,\ x \in \mathbb{R},\ -\tau \leq t \leq 0.$$

Based on the above discussion, we first study the stability of equilibrium points of the delayed predator-prey model with stage structure and Beddington-DeAngelis functional response function using the linear stability method. Then, we establish the existence of traveling wave solutions of (1.1) by constructing a new pair of upper and lower solutions, combined with the Schauder fixed point theorem.

2. Stability of equilibrium points

Note that the second and third equations of system (1.1) are independent of $u_1(x,t)$, and only related to themselves and each other. Thus, it is sufficient to consider the last two equations on their own. For simplicity of notation, we denote $u_2(x,t)$ by $u(x,t)$. Then, we consider the following system:

$$\begin{cases} \dfrac{\partial u}{\partial t} = D_2\dfrac{\partial^2 u}{\partial x^2} + \hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty} G(x,y,\tau)\,u(y,t-\tau)\,dy - d_2u - q_2e_2u - a_{12}u^2 - \dfrac{\beta uv}{1+mu+wv}, \\ \dfrac{\partial v}{\partial t} = D_5\dfrac{\partial^2 v}{\partial x^2} + a_2v - b_2v - a_{55}v^2 - q_5e_5v + \dfrac{\beta_1 uv}{1+mu+wv}. \end{cases} \tag{2.1}$$

In order to facilitate the discussion of subsequent issues, we write here

$$\vartheta_1 := \hat{a}e^{-d_1} - d_2 - q_2e_2, \tag{2.2}$$

$$\vartheta_2 := a_2 - b_2 - q_5e_5. \tag{2.3}$$

Obviously, the system (2.1) has three equilibrium points, which are expressed as

$$E_0(0,0),\quad E_1\left(\frac{\vartheta_1}{a_{12}},0\right),\quad E_2\left(0,\frac{\vartheta_2}{a_{55}}\right).$$

For any constant equilibrium point (u^*,v^*), we linearize the system (2.1) in (u^*,v^*), and obtain

$$\begin{cases} \dfrac{\partial u}{\partial t} = & D_2\dfrac{\partial^2 u}{\partial x^2} + \hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty} G(x,y,\tau)\,u(y,t-\tau)\,dy - d_2u - q_2e_2u - 2a_{12}u^*u \\ & -\dfrac{\beta v^* + w\beta v^{*2}}{(1+mu^*+wv^*)^2}u - \dfrac{\beta u^* + m\beta u^{*2}}{(1+mu^*+wv^*)^2}v, \\ \dfrac{\partial v}{\partial t} = & D_5\dfrac{\partial^2 v}{\partial x^2} + a_2v - b_2v - 2a_{55}v^*v - q_5e_5v + \dfrac{\beta_1 v^* + w\beta_1 v^{*2}}{(1+mu^*+wv^*)^2}u + \dfrac{\beta u^* + m\beta u^{*2}}{(1+mu^*+wv^*)^2}v. \end{cases} \tag{2.4}$$

The system (2.4) has a non-trivial solution in the form of $(c_1,c_2)^T e^{\lambda t + i\sigma x}$ (see [27]) if and only if the corresponding determinant of the system (2.4) coefficient matrix is 0, where λ is a complex number and σ is a real number.

$$\begin{vmatrix} \chi_1(\lambda,\sigma,u^*,v^*) + \dfrac{\beta v^* + w\beta v^{*2}}{(1+mu^*+wv^*)^2} & \dfrac{\beta u^* + m\beta u^{*2}}{(1+mu^*+wv^*)^2} \\ -\dfrac{\beta_1 v^* + w\beta_1 v^{*2}}{(1+mu^*+wv^*)^2} & \chi_2(\lambda,\sigma,u^*,v^*) - \dfrac{\beta u^* + m\beta u^{*2}}{(1+mu^*+wv^*)^2} \end{vmatrix} = 0,$$

equal to

$$\begin{aligned} &\left[\chi_1(\lambda,\sigma,u^*,v^*) + \frac{\beta v^*(1+wv^*)}{(1+mu^*+wv^*)^2}\right]\left[\chi_2(\lambda,\sigma,u^*,v^*) - \frac{\beta u^*(1+mu^*)}{(1+mu^*+wv^*)^2}\right] \\ &\qquad + \frac{\beta\beta_1 u^* v^*(1+mu^*)(1+wv^*)}{(1+mu^*+wv^*)^4} = 0, \end{aligned} \tag{2.5}$$

where

$$\chi_1(\lambda,\sigma,u^*,v^*) := \lambda + D_2\sigma^2 - \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} + d_2 + q_2e_2 + 2a_{12}u^*, \tag{2.6}$$

$$\chi_2(\lambda,\sigma,u^*,v^*) := \lambda + D_5\sigma^2 - a_2 + b_2 + q_5e_5 + 2a_{55}v^*. \tag{2.7}$$

2.1. The zero equilibrium point

Theorem 2.1. *Suppose that $\vartheta_1 \le 0$ and $\vartheta_2 \le 0$, then the zero equilibrium point $E_0(0,0)$ is stable; conversely, assume that either $\vartheta_1 > 0$ or $\vartheta_2 > 0$, then the zero equilibrium point $E_0(0,0)$ is unstable.*

Proof. Substituting $E_0(0,0)$ into (2.5), where $u^* = v^* = 0$, we get

$$\chi_1(\lambda,\sigma,0,0)\chi_2(\lambda,\sigma,0,0) = 0, \tag{2.8}$$

equal to

$$\left[\lambda + D_2\sigma^2 - \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} + d_2 + q_2e_2\right]\left[\lambda + D_5\sigma^2 - a_2 + b_2 + q_5e_5\right] = 0, \tag{2.9}$$

then

$$\lambda + D_2\sigma^2 - \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} + d_2 + q_2e_2 = 0 \quad \text{or} \quad \lambda + D_5\sigma^2 - a_2 + b_2 + q_5e_5 = 0.$$

If $\lambda + D_2\sigma^2 - \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} + d_2 + q_2e_2 = 0$, according to the Eq (2.2), we can get

$$\begin{aligned} \lambda_1 &= -D_2\sigma^2 + \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} - d_2 - q_2e_2 \\ &= -D_2\sigma^2 + \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} + \vartheta_1 - \hat{a}e^{-d_1\tau} \\ &= -D_2\sigma^2 - \hat{a}e^{-d_1\tau}\left(1 - e^{-\lambda\tau}e^{-D_2\sigma^2\tau}\right) + \vartheta_1, \end{aligned}$$

then when $\vartheta_1 \le 0$, $\lambda_1 < 0$.

If $\lambda + D_5\sigma^2 - a_2 + b_2 + q_5e_5 = 0$, according to the Eq (2.3), we can get

$$\lambda_2 = -D_5\sigma^2 + a_2 - b_2 - q_5e_5 = -D_5\sigma^2 + \vartheta_2,$$

then when $\vartheta_2 \le 0$, $\lambda_2 < 0$.

Accordingly, if $\vartheta_1 \le 0$ and $\vartheta_2 \le 0$, the zero equilibrium point $E_0(0,0)$ is stable; if either $\vartheta_1 > 0$ or $\vartheta_2 > 0$, we see that there exists at least a (λ_0, σ_0) satisfying (2.9) such that $\lambda_0 > 0$. Therefore, the zero equilibrium point $E_0(0,0)$ are unstable. □

2.2. The boundary equilibrium point

Theorem 2.2. *Suppose that* $\vartheta_1 \geq 0$ *and* $\vartheta_2 + \frac{\beta_1 \frac{\vartheta_1}{a_{12}}}{1+m\frac{\vartheta_1}{a_{12}}} \leq 0$*, then the boundary equilibrium point* $E_1\left(\frac{\vartheta_1}{a_{12}}, 0\right)$ *is stable; conversely, assume that either* $\vartheta_1 < 0$ *or* $\vartheta_2 + \frac{\beta_1 \frac{\vartheta_1}{a_{12}}}{1+m\frac{\vartheta_1}{a_{12}}} > 0$*, then the boundary equilibrium point* $E_1\left(\frac{\vartheta_1}{a_{12}}, 0\right)$ *is unstable.*

Proof. Substituting $E_1\left(\frac{\vartheta_1}{a_{12}}, 0\right)$ into (2.5), where $u^* = \frac{\vartheta_1}{a_{12}} = \overline{u}$, $v^* = 0$, we get

$$\chi_1(\lambda, \sigma, \overline{u}, 0)\left[\chi_2(\lambda, \sigma, \overline{u}, 0) - \frac{\beta_1 \overline{u}}{1 + m\overline{u}}\right] = 0,$$

equal to

$$\left[\lambda + D_2\sigma^2 - \hat{a}e^{-d_1\tau - \lambda\tau}e^{-D_2\sigma^2\tau} + d_2 + q_2e_2 + 2a_{12}\overline{u}\right]\left[\lambda + D_5\sigma^2 - a_2 + b_2 + q_5e_5 - \frac{\beta_1\overline{u}}{1 + m\overline{u}}\right] = 0, \quad (2.10)$$

then

$$\lambda + D_2\sigma^2 - \hat{a}e^{-d_1\tau - \lambda\tau}e^{-D_2\sigma^2\tau} + d_2 + q_2e_2 + 2a_{12}\overline{u} = 0 \quad \text{or} \quad \lambda + D_5\sigma^2 - a_2 + b_2 + q_5e_5 - \frac{\beta_1\overline{u}}{1 + m\overline{u}} = 0.$$

If $\lambda + D_2\sigma^2 - \hat{a}e^{-d_1\tau - \lambda\tau}e^{-D_2\sigma^2\tau} + d_2 + q_2e_2 + 2a_{12}\overline{u} = 0$, according to the Eq (2.2), we can get

$$\begin{aligned}
\lambda_1 &= -D_2\sigma^2 + \hat{a}e^{-d_1\tau - \lambda\tau}e^{-D_2\sigma^2\tau} - d_2 - q_2e_2 - 2a_{12}\overline{u} \\
&= -D_2\sigma^2 + \hat{a}e^{-d_1\tau - \lambda\tau}e^{-D_2\sigma^2\tau} - d_2 - q_2e_2 - 2a_{12}\frac{\vartheta_1}{a_{12}} \\
&= -D_2\sigma^2 + \hat{a}e^{-d_1\tau - \lambda\tau}e^{-D_2\sigma^2\tau} + \vartheta_1 - \hat{a}e^{-d_1\tau} - 2\vartheta_1 \\
&= -D_2\sigma^2 - \hat{a}e^{-d_1\tau}\left(1 - e^{-\lambda\tau}e^{-D_2\sigma^2\tau}\right) - \vartheta_1,
\end{aligned}$$

then when $\vartheta_1 \geq 0$, $\lambda_1 < 0$.

If $\lambda + D_5\sigma^2 - a_2 + b_2 + q_5e_5 - \frac{\beta_1\overline{u}}{1+m\overline{u}} = 0$, according to the Eq (2.3), we can get

$$\lambda_2 = -D_5\sigma^2 + a_2 - b_2 - q_5e_5 + \frac{\beta_1\overline{u}}{1 + m\overline{u}} = -D_5\sigma^2 + \vartheta_2 + \frac{\beta_1 \frac{\vartheta_1}{a_{12}}}{1 + m\frac{\vartheta_1}{a_{12}}},$$

then when $\vartheta_2 + \frac{\beta_1 \frac{\vartheta_1}{a_{12}}}{1+m\frac{\vartheta_1}{a_{12}}} \leq 0$, $\lambda_2 < 0$.

Accordingly, if $\vartheta_1 \geq 0$ and $\vartheta_2 + \frac{\beta_1 \frac{\vartheta_1}{a_{12}}}{1+m\frac{\vartheta_1}{a_{12}}} \leq 0$, the boundary equilibrium $E_1\left(\frac{\vartheta_1}{a_{12}}, 0\right)$ is stable ; if either $\vartheta_1 < 0$ or $\vartheta_2 + \frac{\beta_1 \frac{\vartheta_1}{a_{12}}}{1+m\frac{\vartheta_1}{a_{12}}} > 0$, we see that there exists at least a (λ_0, σ_0) satisfying (2.10) such that $\lambda_0 > 0$. Therefore, the boundary equilibrium $E_1\left(\frac{\vartheta_1}{a_{12}}, 0\right)$ is unstable. □

Theorem 2.3. *Suppose that* $\vartheta_1 - \frac{\beta \frac{\vartheta_2}{a_{55}}}{1+w\frac{\vartheta_2}{a_{55}}} \leq 0$ *and* $\vartheta_2 \geq 0$*, then the boundary equilibrium point* $E_2\left(0, \frac{\vartheta_2}{a_{55}}\right)$ *is stable; conversely, assume that either* $\vartheta_1 - \frac{\beta \frac{\vartheta_2}{a_{55}}}{1+w\frac{\vartheta_2}{a_{55}}} > 0$ *or* $\vartheta_2 < 0$*, then the boundary equilibrium point* $E_2\left(0, \frac{\vartheta_2}{a_{55}}\right)$ *is unstable.*

Proof. Substituting $E_2\left(0, \frac{\vartheta_2}{a_{55}}\right)$ into (2.5), where $u^* = 0$, $v^* = \frac{\vartheta_2}{a_{55}} = \overline{v}$, we get

$$\left[\chi_1(\lambda, \sigma, 0, \overline{v}) + \frac{\beta\overline{v}}{1 + w\overline{v}}\right]\chi_2(\lambda, \sigma, 0, \overline{v}) = 0,$$

equal to

$$\left[\lambda + D_2\sigma^2 - \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} + d_2 + q_2e_2 + \frac{\beta\overline{v}}{1 + w\overline{v}}\right]\left[\lambda + D_5\sigma^2 - a_2 + b_2 + q_5e_5 + 2a_{55}\overline{v}\right] = 0, \quad (2.11)$$

then

$$\lambda + D_2\sigma^2 - \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} + d_2 + q_2e_2 + \frac{\beta\overline{v}}{1 + w\overline{v}} = 0 \quad \text{or} \quad \lambda + D_5\sigma^2 - a_2 + b_2 + q_5e_5 + 2a_{55}\overline{v} = 0.$$

If $\lambda + D_2\sigma^2 - \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} + d_2 + q_2e_2 + \frac{\beta\overline{v}}{1+w\overline{v}} = 0$, according to the Eq (2.2), we can get

$$\begin{aligned}
\lambda_1 &= -D_2\sigma^2 + \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} - d_2 - q_2e_2 - \frac{\beta\overline{v}}{1 + w\overline{v}} \\
&= -D_2\sigma^2 + \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} - d_2 - q_2e_2 - \frac{\beta\frac{\vartheta_2}{a_{55}}}{1 + w\frac{\vartheta_2}{a_{55}}} \\
&= -D_2\sigma^2 + \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} + \vartheta_1 - \hat{a}e^{-d_1\tau} - \frac{\beta\frac{\vartheta_2}{a_{55}}}{1 + w\frac{\vartheta_2}{a_{55}}} \\
&= -D_2\sigma^2 - \hat{a}e^{-d_1\tau}\left(1 - e^{-\lambda\tau}e^{-D_2\sigma^2\tau}\right) + \vartheta_1 - \frac{\beta\frac{\vartheta_2}{a_{55}}}{1 + w\frac{\vartheta_2}{a_{55}}},
\end{aligned}$$

then when $\vartheta_1 - \frac{\beta\frac{\vartheta_2}{a_{55}}}{1+w\frac{\vartheta_2}{a_{55}}} \leq 0$, $\lambda_1 < 0$.

If $\lambda + D_5\sigma^2 - a_2 + b_2 + q_5e_5 + 2a_{55}\overline{v} = 0$, according to the Eq (2.3), we can get

$$\lambda_2 = -D_5\sigma^2 + a_2 - b_2 - q_5e_5 - 2a_{55}\overline{v} = -D_5\sigma^2 + \vartheta_2 - 2a_{55}\frac{\vartheta_2}{a_{55}} = -D_5\sigma^2 - \vartheta_2,$$

then when $\vartheta_2 \geq 0$, $\lambda_2 < 0$.

Accordingly, if $\vartheta_1 - \frac{\beta\frac{\vartheta_2}{a_{55}}}{1+w\frac{\vartheta_2}{a_{55}}} \leq 0$ and $\vartheta_2 \geq 0$, the boundary equilibrium $E_2\left(0, \frac{\vartheta_2}{a_{55}}\right)$ is stable; if either $\vartheta_1 - \frac{\beta\frac{\vartheta_2}{a_{55}}}{1+w\frac{\vartheta_2}{a_{55}}} > 0$ or $\vartheta_2 < 0$, we see that there exists at least a (λ_0, σ_0) satisfying (2.11) such that $\lambda_0 > 0$. Therefore, the boundary equilibrium $E_2\left(0, \frac{\vartheta_2}{a_{55}}\right)$ is unstable. □

2.3. *The unique positive equilibrium point*

Considering the actual background of our model, we will assume that $\vartheta_1 > 0$ and $\vartheta_2 > 0$ in the following discussion, so the above three equilibrium points $E_0(0, 0)$, $E_1\left(\frac{\vartheta_1}{a_{12}}, 0\right)$, $E_2\left(0, \frac{\vartheta_2}{a_{55}}\right)$ are non-negative equilibrium points. Now we shall discuss the possibility of the positive equilibrium point.

The positive equilibrium point $E_3(u^+, v^+)$ of system (2.1) satisfies the system

$$\vartheta_1 - a_{12}u - \frac{\beta v}{1 + mu + wv} = 0, \tag{2.12}$$

$$\vartheta_2 - a_{55}v + \frac{\beta_1 u}{1 + mu + wv} = 0. \tag{2.13}$$

From the Eq (2.12), we have form

$$v = \frac{\vartheta_1 + m\vartheta_1 u - a_{12}u - ma_{12}u^2}{wa_{12}u - w\vartheta_1 + \beta},$$

Substituting the above equation into the Eq (2.13), we can get

$$\vartheta_2 - a_{55}\frac{\vartheta_1 + m\vartheta_1 u - a_{12}u - ma_{12}u^2}{wa_{12}u - w\vartheta_1 + \beta} + \frac{\beta_1 u}{1 + mu + wv} = 0,$$

expand and simplify to get the function

$$f(u) = A_0u^3 + A_1u^2 + A_2u + A_3,$$

where

$$\begin{aligned}
A_0 &= m^2a_{55}a_{12}\beta + w^2a_{12}^2\beta_1,\\
A_1 &= 2ma_{12}a_{55}\beta + 2wa_{12}\beta\beta_1 + mwa_{12}\beta\vartheta_2 - 2w^2a_{12}\beta_1\vartheta_1 - m^2a_{55}\beta\vartheta_1,\\
A_2 &= wa_{12}\beta\vartheta_2 + a_{12}a_{55}\beta + m\beta^2\vartheta_2^2 + \beta^2\beta_1 + w^2\beta_1\vartheta_1^2 - 2ma_{55}\beta\vartheta_1 - mw\beta\vartheta_1\vartheta_2 - 2w\beta\beta_1\vartheta_1,\\
A_3 &= \beta^2\vartheta_2 - a_{55}\beta\vartheta_1 - w\beta\vartheta_1\vartheta_2.
\end{aligned}$$

Next we shall analyze the existence of positive roots of the function $f(u)$, and assume that $f(u) = A_0u^3 + A_1u^2 + A_2u + A_3$ has a unique positive root u^+. Obviously, the main part of the function $f(u)$ is $A_0 = m^2a_{55}a_{12}\beta + w^2a_{12}^2\beta_1 > 0$, so we assume $A_3 = \beta^2\vartheta_2 - a_{55}\beta\vartheta_1 - w\beta\vartheta_1\vartheta_2 \le 0$. Therefore,

$$f(+\infty) = +\infty, \quad f(-\infty) = -\infty, \quad f(0) = A_3 \le 0, \quad f'(u) = 3A_0u^2 + 2A_1u + A_2.$$

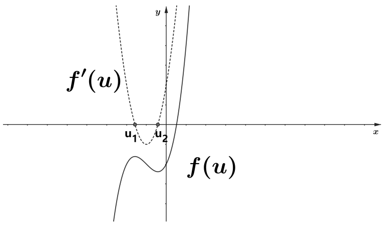
The discriminant of the derivative $f'(u)$ is $\Delta = (2A_1)^2 - 4\times 3A_0A_2 = 4A_1^2 - 12A_0A_2$, let $\Delta_0 = A_1^2 - 3A_0A_2$, thus $\Delta = 4\Delta_0$. The system (2.1) has the unique positive equilibrium $E_3(u^+, v^+)$ if and only if the function $f(u)$ has a unique positive root u^+.

1) If $\Delta_0 > 0$, then the function $f(u)$ has two zero roots u_1 and u_2, which are equivalent to

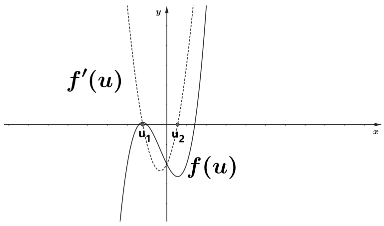
$$u_1 = \frac{-A_1 - \sqrt{\Delta_0}}{3A_0}, \quad u_2 = \frac{-A_1 + \sqrt{\Delta_0}}{3A_0}.$$

(a). If $A_1 > 0$ and $A_2 \ge 0$, then $u_1 < u_2 \le 0$, $f(u)$ is increasing in $[0, +\infty)$. If $f(0) = A_3 < 0$, then $f(u) = 0$ has a unique positive root; if $f(0) = A_3 = 0$, then $f(u) = 0$ has no positive root.

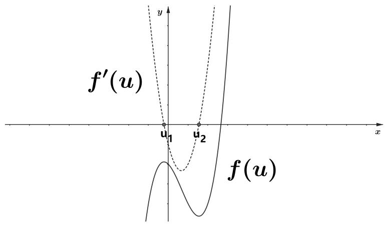
(b). If $A_1 > 0$ and $A_2 < 0$, then $u_1 < 0$, $u_2 > 0$, $f(u)$ is decreasing in $[0, u_2)$, and is increasing in $[u_2, +\infty)$. Since $f(0) = A_3 \le 0$, $f(u) = 0$ has a unique positive root.



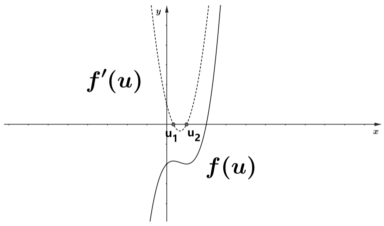
(a) $A_1 > 0$ and $A_2 \geq 0$



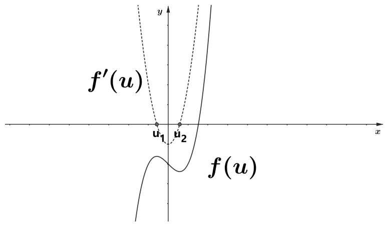
(b) $A_1 > 0$ and $A_2 < 0$



(c) $A_1 < 0$ and $A_2 \leq 0$



(d) $A_1 < 0$ and $A_2 > 0$



(e) $A_1 = 0$ and $A_2 < 0$

Figure 1. Images of $f(u)$ and $f'(u)$ when $\Delta_0 > 0$.

(c). If $A_1 < 0$ and $A_2 \leq 0$, then $u_1 \leq 0$, $u_2 > 0$, $f(u)$ is decreasing in $[0, u_2)$, and is increasing in $[u_2, +\infty)$. Since $f(0) = A_3 \leq 0$, $f(u) = 0$ has a unique positive root.

(d). If $A_1 < 0$ and $A_2 > 0$, then $u_1 > u_2 > 0$, $f(u)$ is increasing in $[0, u_1)$ and $[u_2, +\infty)$, and is decreasing in $[u_1, u_2)$. If $f(0) = A_3 < 0$, $f(u_1) < 0$, $f(u_2) < 0$, then $f(u) = 0$ has a unique positive root; if $f(0) = A_3 = 0$, $f(u_1) > 0$, $f(u_2) = 0$, then $f(u) = 0$ has a unique positive root ; otherwise, $f(u) = 0$ has two positive roots or no positive roots.

(e). If $A_1 = 0$ and $A_2 < 0$, then $u_1 < 0$, $u_2 > 0$, $f(u)$ is decreasing in $[0, u_2)$, and is increasing in $[u_2, +\infty)$. Since $f(0) = A_3 \leq 0$, $f(u) = 0$ has a unique positive root.

2) If $\Delta_0 < 0$, $f(0) = A_3 < 0$, then the function $f(u)$ is monotonically increasing in $[0, +\infty)$, thus $f(u) = 0$ has a unique positive root.
3) If $\Delta_0 = 0$, $f(0) = A_3 < 0$, then the function $f(u)$ is monotonically increasing in $[0, +\infty)$, thus $f(u) = 0$ has a unique positive root.

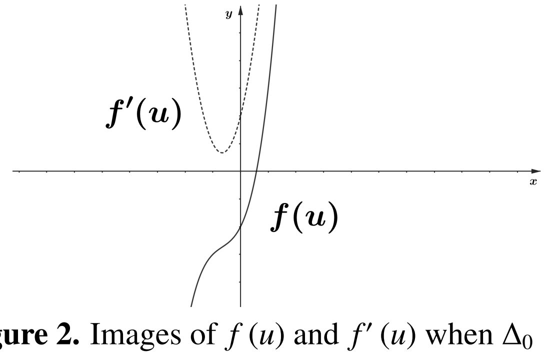


Figure 2. Images of $f(u)$ and $f'(u)$ when $\Delta_0 < 0$.

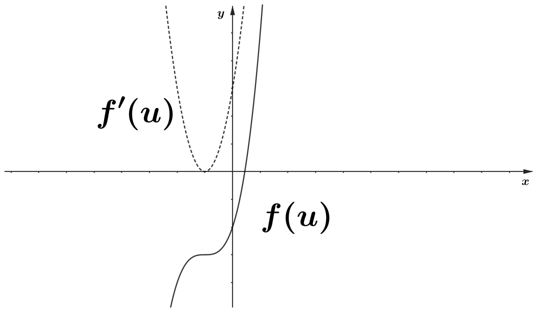


Figure 3. Images of $f(u)$ and $f'(u)$ when $\Delta_0 = 0$.

Summarizing the above discussion, we get the following conclusions.

Lemma 2.1. *Suppose that $A_0 > 0$, $A_3 \leq 0$, equation $f(u) = 0$ has a unique positive root u^+ if and only if one of the following six conditions holds:*

B1). $\Delta_0 > 0$, $A_1 > 0$, $A_2 \geq 0$, $A_3 < 0$;
B2). $\Delta_0 > 0$, $A_1 > 0$, $A_2 < 0$, $A_3 \leq 0$;
B3). $\Delta_0 > 0$, $A_1 < 0$, $A_2 \leq 0$, $A_3 \leq 0$;
B4). $\Delta_0 > 0$, $A_1 < 0$, $A_2 > 0$, $A_3 \leq 0$;
B5). $\Delta_0 > 0$, $A_1 = 0$, $A_2 < 0$, $A_3 \leq 0$;
B6). $\Delta_0 \leq 0$, $A_3 \leq 0$.

$f(u)$ has a unique positive root u^+, through the Eq (2.13) we can get

$$v = \frac{\vartheta_1 + m\vartheta_1 u - a_{12}u - ma_{12}u^2}{wa_{12}u - w\vartheta_1 + \beta} = \frac{(\vartheta_1 - a_{12}u)(1 + mu)}{w(a_{12}u - \vartheta_1) + \beta} = \frac{1 + mu}{-w + \frac{\beta}{\vartheta_1 - a_{12}u}}.$$

As $\frac{\beta}{\vartheta_1 - a_{12}u^+} > w$, that is, $u^+ > \frac{\vartheta_1 - w\beta}{a_{12}}$, there exists a unique corresponding v^+. Thus, the system (2.1) has a unique positive equilibrium point $E_3(u^+, v^+)$.

Theorem 2.4 (The existence condition of E_3)**.** *Suppose $\vartheta_1 > 0$, $\vartheta_2 > 0$ and $a_{12}u^+ < \vartheta_1 < \frac{\beta}{w} + a_{12}u^+$, then the system* (2.1) *has a unique positive equilibrium point $E_3(u^+, v^+)$, where $u^+ > 0$ is the only positive root of $f(u) = 0$.*

Theorem 2.5 (The stability of E_3)**.** *Assume that the unique positive equilibrium point $E_3(u^+, v^+)$ exists, if $a_{12} \geq \frac{m\beta v^+}{(1+mu^+ + wv^+)^2}$, then $E_3(u^+, v^+)$ is stable.*

Proof. Substituting $E_3(u^+, v^+)$ into (2.5), where$u^* = u^+$, $v^* = v^+$, we get

$$\left[\chi_1(\lambda, \sigma, u^+, v^+) + \frac{\beta v^+(1 + wv^+)}{(1 + mu^+ + wv^+)^2}\right]\left[\chi_2(\lambda, \sigma, u^+, v^+) - \frac{\beta_1 u^+(1 + mu^+)}{(1 + mu^+ + wv^+)^2}\right] + \frac{\beta\beta_1 u^+ v^+(1 + mu^+)(1 + wv^+)}{(1 + mu^+ + wv^+)^4} = 0,$$

Here we introduce some representations

$$\gamma_1 = \frac{\beta_1(1 + mu^+)}{(1 + mu^+ + wv^+)^2}, \quad \gamma_2 = \frac{\beta(1 + wv^+)}{(1 + mu^+ + wv^+)^2},$$

such that

$$\left[\lambda + D_2\sigma^2 - \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} + d_2 + q_2e_2 + 2a_{12}u^+ + \gamma_2v^+\right]\left[\lambda + D_5\sigma^2 - a_2 + b_2 + q_5e_5 + 2a_{55}v^+ - \gamma_1u^+\right] + \gamma_1\gamma_2u^+v^+ = 0. \tag{2.14}$$

where $\gamma_1,\ \gamma_2,\ u^+,\ v^+ > 0$.

Due to $\lambda + D_2\sigma^2 - \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} + d_2 + q_2e_2 + 2a_{12}u^+ + \gamma_2v^+ \neq 0$, with the help of Eqs (2.3) and (2.13), the Eq (2.14) can be transformed into

$$\begin{aligned}
\lambda &= -\frac{\gamma_1\gamma_2u^+v^+}{\lambda + D_2\sigma^2 - \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} + d_2 + q_2e_2 + 2a_{12}u^+ + \gamma_2v^+} \\
&\quad - \left(D_5\sigma^2 - a_2 + b_2 + q_5e_5 + 2a_{55}v^+ - \gamma_1u^+\right) \\
&= -\frac{\gamma_1\gamma_2u^+v^+}{\lambda + D_2\sigma^2 - \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} + d_2 + q_2e_2 + 2a_{12}u^+ + \gamma_2v^+} \\
&\quad - \left(D_5\sigma^2 + \vartheta_2 + \frac{2\beta_1u^+}{1 + mu^+ + wv^+} - \frac{\beta_1\left(1 + mu^+\right)u^+}{\left(1 + mu^+ + wv^+\right)^2}\right) \\
&= -\frac{\gamma_1\gamma_2u^+v^+}{\lambda + D_2\sigma^2 - \hat{a}e^{-d_1\tau-\lambda\tau}e^{-D_2\sigma^2\tau} + d_2 + q_2e_2 + 2a_{12}u^+ + \gamma_2v^+} \\
&\quad - \left(D_5\sigma^2 + \vartheta_2 + \frac{\beta_1u^+}{1 + mu^+ + wv^+} + \frac{w\beta_1u^+v^+}{\left(1 + mu^+ + wv^+\right)^2}\right).
\end{aligned}$$

$\lambda = \mu + i\omega$ is a complex number, that is, $Re\lambda = \mu$, $Im\lambda = \omega$, $(\lambda, \sigma) = (\mu + i\omega, \sigma)$. The real part of λ is $Re\lambda = \mu < 0$. Otherwise, $Re\lambda = \mu \geq 0$ is not established, and the counter-proof method proves as follows.

Suppose that there exists $(\lambda_1, \sigma_1) = (\mu_1 + i\omega_1, \sigma_1)$, $\mu_1 \geq 0$. Using Euler formula to split the real and imaginary parts of λ. Let

$$\begin{aligned}
A\left(\mu_1, \omega_1, \sigma_1\right) &= \mu_1 + D_2\sigma_1^2 - \hat{a}e^{-d_1\tau-\mu_1\tau}e^{-D_2\sigma_1^2\tau}\cos\left(\omega_1\tau\right) + d_2 + q_2e_2 + 2a_{12}u^+ + \gamma_2v^+, \\
B\left(\mu_1, \omega_1, \sigma_1\right) &= \omega_1 + \hat{a}e^{-d_1\tau-\mu_1\tau}e^{-D_2\sigma_1^2\tau}\sin\left(\omega_1\tau\right),
\end{aligned}$$

then

$$
\begin{aligned}
0 \le \mu_1 =& -\frac{A\gamma_1\gamma_2 u^+ v^+}{A^2+B^2} - \left(D_5\sigma^2 + \vartheta_2 + \frac{\beta_1 u^+}{1+mu^+ + wv^+} + \frac{w\beta_1 u^+ v^+}{(1+mu^+ + wv^+)^2}\right) \\
\le& -\frac{\left[-\hat{a}e^{-d_1\tau-\mu_1\tau}e^{-D_2\sigma_1^2\tau}\cos(\omega_1\tau) + d_2 + q_2e_2 + 2a_{12}u^+ + \gamma_2 v^+\right]\gamma_1\gamma_2 u^+ v^+}{A^2+B^2} \\
& - \left(D_5\sigma^2 + \vartheta_2 + \frac{\beta_1 u^+}{1+mu^+ + wv^+} + \frac{w\beta_1 u^+ v^+}{(1+mu^+ + wv^+)^2}\right) \\
=& (-\gamma_1\gamma_2 u^+ v^+)\frac{\hat{a}e^{-d_1\tau} - \hat{a}e^{-d_1\tau-\mu_1\tau}e^{-D_2\sigma_1^2\tau}\cos(\omega_1\tau) + a_{12}u^+ + \gamma_2 v^+ - \frac{\beta v^+}{1+mu^+ + wv^+}}{A^2+B^2} \\
& - \left(D_5\sigma^2 + \vartheta_2 + \frac{\beta_1 u^+}{1+mu^+ + wv^+} + \frac{w\beta_1 u^+ v^+}{(1+mu^+ + wv^+)^2}\right) \\
\le& (-\gamma_1\gamma_2 u^+ v^+)\frac{a_{12}u^+ + \gamma_2 v^+ - \frac{\beta v^+}{1+mu^+ + wv^+}}{A^2+B^2} - \left(D_5\sigma^2 + \vartheta_2 + \frac{\beta_1 u^+}{1+mu^+ + wv^+} + \frac{w\beta_1 u^+ v^+}{(1+mu^+ + wv^+)^2}\right) \\
=& (-\gamma_1\gamma_2 u^+ v^+)\frac{a_{12}u^+ - \frac{m\beta u^+ v^+}{(1+mu^+ + wv^+)^2}}{A^2+B^2} - \left(D_5\sigma^2 + \vartheta_2 + \frac{\beta_1 u^+}{1+mu^+ + wv^+} + \frac{w\beta_1 u^+ v^+}{(1+mu^+ + wv^+)^2}\right) \\
<& 0,
\end{aligned}
$$

as $a_{12} \ge \frac{m\beta v^+}{(1+mu^+ + wv^+)^2}$. This is a contradiction.

Consequently, assume that the unique positive equilibrium point $E_3(u^+, v^+)$ exists, if $a_{12} \ge \frac{m\beta v^+}{(1+mu^+ + wv^+)^2}$, then $E_3(u^+, v^+)$ is stable. □

3. Existence of traveling wave solutions

In this section, by using the Schauder fixed point theorem and the method of constructing upper and lower solutions by cross iteration, the existence of traveling wave solutions of the connecting equilibrium points E_0 and E_3 of the system (2.1) is obtained. The traveling wave solution of the system (2.1) is a special translation invariant solution in the form of $(u(x,t), v(x,t)) = (\phi(x+ct), \psi(x+ct))$, where the wave velocity $c > 0$, ϕ and ψ are wave profile functions, and the wave profile propagates in one-dimensional space domain at a constant speed $c > 0$. Substituting $(u(x,t), v(x,t)) = (\phi(x+ct), \psi(x+ct))$ into system (2.1) and replacing $x + ct$ with t, we get

$$
\begin{cases}
D_2\phi''(t) - c\phi'(t) + f_2(\phi,\psi)(t) = 0, \\
D_5\psi''(t) - c\psi'(t) + f_5(\phi,\psi)(t) = 0,
\end{cases}
\tag{3.1}
$$

satisfy the following asymptotic boundary value conditions

$$
\lim_{t\to-\infty}(\phi(t), \psi(t)) = (0,0), \quad \lim_{t\to+\infty}(\phi(t), \psi(t)) = (u^+, v^+),
$$

where

$$
f_2(\phi,\psi)(t) = \hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\phi(t-y-c\tau)\,dy - (d_2+q_2e_2)\phi(t)
$$

$$
-a_{12}\phi^2(t) - \frac{\beta\phi(t)\psi(t)}{1+m\phi(t)+w\psi(t)},
$$

$$
f_5(\phi,\psi)(t) = (a_2 - b_2 - q_5e_5)\psi(t) - a_{55}\psi^2(t) + \frac{\beta_1\phi(t)\psi(t)}{1+m\phi(t)+w\psi(t)}.
$$

3.1. The construction of upper and lower solutions of the system

In this section, we discuss the existence of upper and lower solutions. Firstly, we give the definition of the upper and lower solutions of the system (3.1).

Definition 3.1. *Let $\overline{\rho}(t) = \left(\overline{\phi}(t), \overline{\psi}(t)\right)$, $\underline{\rho}(t) = \left(\underline{\phi}(t), \underline{\psi}(t)\right)$, $t \in \mathbb{R}$ be two continuous functions, then $\overline{\rho}(t) = \left(\overline{\phi}(t), \overline{\psi}(t)\right)$ and $\underline{\rho}(t) = \left(\underline{\phi}(t), \underline{\psi}(t)\right)$, $t \in \mathbb{R}$ are the upper and lower solutions of the system* (3.1)*, respectively. If there exists a finite set of points $S = \{s_i \in \mathbb{R}, i = 1, 2, \cdots, n\}$, where $s_1 < s_2 < \cdots < s_n$, such that $\overline{\rho}(t)$ and $\underline{\rho}(t)$ are twice continuously differentiable on $\mathbb{R}\backslash S$, and for any $t \in \mathbb{R}\backslash S$, satisfy*

$$
\begin{aligned}
&D_2\overline{\phi}''(t) - c\overline{\phi}'(t) + f_2\left(\overline{\phi}, \underline{\psi}\right)(t) \le 0,\\
&D_5\overline{\psi}''(t) - c\overline{\psi}'(t) + f_5\left(\overline{\phi}, \overline{\psi}\right)(t) \le 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&D_2\underline{\phi}''(t) - c\underline{\phi}'(t) + f_2\left(\underline{\phi}, \overline{\psi}\right)(t) \ge 0,\\
&D_5\underline{\psi}''(t) - c\underline{\psi}'(t) + f_5\left(\underline{\phi}, \underline{\psi}\right)(t) \ge 0.
\end{aligned}
$$

Now linearizing the system (3.1) at $(0,0)$, we obtain

$$
\begin{cases}
D_2\phi''(t) - c\phi'(t) + \hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\phi(t-y-c\tau)\,dy - (d_2+q_2e_2)\phi(t) = 0,\\
D_5\psi''(t) - c\psi'(t) + (a_2 - b_2 - q_5e_5)\psi(t) = 0.
\end{cases}
\tag{3.2}
$$

Substituting $\phi(t) = e^{\lambda t}$ and $\psi(t) = e^{\lambda t}$ into the system (3.2), due to $\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}e^{-\lambda(y+c\tau)}dy = e^{\left(D_1\lambda^2 - c\lambda\right)\tau}$, we get

$$
\Delta_1(\lambda, c) = 0, \quad \Delta_2(\lambda, c) = 0,
$$

where

$$
\begin{cases}
\Delta_1(\lambda, c) = D_2\lambda^2 - c\lambda + \hat{a}e^{-d_1\tau}e^{\left(D_1\lambda^2 - c\lambda\right)\tau} - d_2 - q_2e_2,\\
\Delta_2(\lambda, c) = D_5\lambda^2 - c\lambda + a_2 - b_2 - q_5e_5.
\end{cases}
$$

Lemma 3.1. *If $\hat{a}e^{-d_1\tau}e^{\left(D_1\lambda^2 - c\lambda\right)\tau} - d_2 - q_2e_2 > 0$, let $c_1^* = \sqrt{4D_2\left(\hat{a}e^{-d_1\tau}e^{\left(D_1\lambda^2 - c\lambda\right)\tau} - d_2 - q_2e_2\right)}$, then the following conclusions hold.*

1). If $c > c_1^*$, then $\Delta_1(\lambda, c) = 0$ has two different positive roots $\lambda_1(c)$ and $\lambda_2(c)$, we may set $0 < \lambda_1(c) < \lambda_2(c)$;

2). If $0 < c < c_1^*$*, then* $\Delta_1(\lambda, c) = 0$ *has no real root.*

Lemma 3.2. *If* $a_2 - b_2 - q_5 e_5 > 0$*, write* $c_2^* = \sqrt{4D_5(a_2 - b_2 - q_5 e_5)}$*, then the following conclusions hold.*

1). If $c > c_2^*$*, then* $\Delta_2(\lambda, c) = 0$ *has two different positive roots* $\lambda_3(c)$ *and* $\lambda_4(c)$*, we may set* $0 < \lambda_3(c) < \lambda_4(c)$*;*

2). If $0 < c < c_2^*$*, then* $\Delta_2(\lambda, c) = 0$ *has no real root.*

Proof. We regard $\Delta_1(\lambda, c) = 0$ and $\Delta_2(\lambda, c) = 0$ as a quadratic equation with one variable λ, and consider the existence of the solution of the equation according to the size of the respective discriminant Δ and 0. □

Lemma 3.3. *Assume that* $a_{12}u^+ \geq \frac{(3+2\sqrt{2})\beta v^+}{1+mu^++wv^+}$ *and* $a_{55}v^+ \geq \frac{2\sqrt{2}\beta_1 u^+}{1+mu^++wv^+}$ *hold, there exist* $\varepsilon_1 \in \left(0, \left(\sqrt{2}-1\right)u^+\right)$ *and* $\varepsilon_2 \in \left(0, \frac{v^+}{2}\right)$ *such that*

$$\begin{cases} -a_{12}\varepsilon_1^2 + \left(2\sqrt{2}-2\right)a_{12}u^+\varepsilon_1 + \dfrac{\beta u^+ v^+}{1+mu^++wv^+} - \dfrac{2\beta(u^+-\varepsilon_1)v^+}{1+m(u^+-\varepsilon_1)+2wv^+} > \varepsilon_0, \\ -a_{55}\varepsilon_2^2 + a_{55}v^+\varepsilon_2 - \dfrac{\beta_1 u^+ v^+}{1+mu^++wv^+} + \dfrac{\beta_1(u^+-\varepsilon_1)(v^+-\varepsilon_2)}{1+m(u^+-\varepsilon_1)+w(v^+-\varepsilon_2)} > \varepsilon_0, \end{cases} \tag{3.3}$$

where $\varepsilon_0 > 0$ *is a constant.*

Proof. Let

$$\begin{aligned} g_1(\varepsilon_1) &= -a_{12}\varepsilon_1^2 + \left(2\sqrt{2}-2\right)a_{12}u^+\varepsilon_1, \\ g_2(\varepsilon_1) &= -\frac{\beta u^+ v^+}{1+mu^++wv^+} + \frac{2\beta(u^+-\varepsilon_1)v^+}{1+m(u^+-\varepsilon_1)+2wv^+}, \\ g_3(\varepsilon_2) &= -a_{55}\varepsilon_2^2 + a_{55}v^+\varepsilon_2, \\ g_4(\varepsilon_2) &= \frac{\beta_1 u^+ v^+}{1+mu^++wv^+} - \frac{\beta_1(u^+-\varepsilon_1)(v^+-\varepsilon_2)}{1+m(u^+-\varepsilon_1)+w(v^+-\varepsilon_2)}. \end{aligned}$$

$g_1(\varepsilon_1)$ is a quadratic function with respect to ε_1. The image opens down through the origin, and the symmetry axis is $x = \left(\sqrt{2}-1\right)u^+ > 0$, so that $g_1(\varepsilon_1)$ is increasing in $\left(0, \left(\sqrt{2}-1\right)u^+\right)$. Thus, $g_1(0) = 0$ and the maximum value is $max\{g_1(\varepsilon_1)\} = g_1\left(\left(\sqrt{2}-1\right)u^+\right) = \left(3-2\sqrt{2}\right)a_{12}(u^+)^2$; $g_2(\varepsilon_1)$ is decreasing with respect to ε_1 and the maximum value is $max\{g_2(\varepsilon_1)\} < g_2(0) = \frac{\beta u^+ v^+}{1+mu^++2wv^+}$. According to the assumption of $a_{12}u^+ \geq \frac{(3+2\sqrt{2})\beta v^+}{1+mu^++wv^+}$, then $\left(3-2\sqrt{2}\right)a_{12}(u^+)^2 \geq \frac{\beta u^+ v^+}{1+mu^++2wv^+}$, there exists $\varepsilon_1 \in \left(0, \left(\sqrt{2}-1\right)u^+\right)$, so that $g_1(\varepsilon_1) > g_2(\varepsilon_1)$. The first inequality is proved.

$g_3(\varepsilon_2)$ is a quadratic function with respect to ε_2. The image opening goes down through the origin, and the symmetry axis is $x = \frac{v^+}{2} > 0$, so that $g_3(\varepsilon_2)$ is increasing in $\left(0, \frac{v^+}{2}\right)$. Thus, $g_3(0) = 0$ and the maximum value is $max\{g_3(\varepsilon_2)\} = g_3\left(\frac{v^+}{2}\right) = \frac{1}{4}a_{55}(v^+)^2$; $g_4(\varepsilon_2)$ is increasing in $\left(0, \frac{v^+}{2}\right)$ with respect to ε_2, then the maximum value is $max\{g_4(\varepsilon_2)\} < g_4\left(\frac{v^+}{2}\right) = \frac{\beta_1 u^+ v^+}{1+mu^++wv^+} - \frac{\beta_1(u^+-\varepsilon_1)\frac{v^+}{2}}{1+m(u^+-\varepsilon_1)+w\frac{v^+}{2}}$, here $-\frac{\beta_1(u^+-\varepsilon_1)\frac{v^+}{2}}{1+m(u^+-\varepsilon_1)+w\frac{v^+}{2}}$ is increasing for $\varepsilon_1 \in \left(0, \left(\sqrt{2}-1\right)u^+\right)$ with respect to ε_1, such that

$$max\{g_4(\varepsilon_2)\} < g_4\left(\frac{v^+}{2}\right) = \frac{\beta_1 u^+ v^+}{1+mu^++wv^+} - \frac{\beta_1(u^+-\varepsilon_1)\frac{v^+}{2}}{1+m(u^+-\varepsilon_1)+w\frac{v^+}{2}}$$

$$
\begin{aligned}
&< \frac{\beta_1 u^+ v^+}{1 + mu^+ + wv^+} - \frac{\beta_1 \left(u^+ - \left(\sqrt{2} - 1\right) u^+\right) \frac{v^+}{2}}{1 + m\left(u^+ - \left(\sqrt{2} - 1\right) u^+\right) + w\frac{v^+}{2}} \\
&< \frac{\beta_1 u^+ v^+}{1 + mu^+ + wv^+} - \frac{\beta_1 \left(\frac{2-\sqrt{2}}{2}\right) u^+ v^+}{1 + mu^+ + wv^+} \\
&= \frac{\frac{\sqrt{2}}{2}\beta_1 u^+ v^+}{1 + mu^+ + wv^+}.
\end{aligned}
$$

According to the assumption of $a_{55}v^+ \geq \frac{2\sqrt{2}\beta_1 u^+}{1+mu^++wv^+}$, then $\frac{1}{4}a_{55}\left(v^+\right)^2 \geq \frac{\frac{\sqrt{2}}{2}\beta_1 u^+ v^+}{1+mu^++wv^+}$, there exists $\varepsilon_2 \in \left(0, \frac{v^+}{2}\right)$, so that $g_3(\varepsilon_2) > g_4(\varepsilon_2)$. The second inequality is proved. □

For the unique positive equilibrium (u^+, v^+), we know that $\vartheta_1 - a_{12}u^+ - \frac{\beta v^+}{1+mu^++wv^+} = 0$ and $\vartheta_2 - a_{55}v^+ + \frac{\beta_1 u^+}{1+mu^++wv^+} = 0$, thereby

1). If $\vartheta_1 > \frac{(4+2\sqrt{2})\beta v^+}{1+mu^++wv^+}$ holds, then $a_{12}u^+ = \vartheta_1 - \frac{\beta v^+}{1+mu^++wv^+} > \frac{(3+2\sqrt{2})\beta v^+}{1+mu^++wv^+}$;
2). If $\vartheta_2 > \frac{(2\sqrt{2}-1)\beta_1 u^+}{1+mu^++wv^+}$ holds, then $a_{55}v^+ = \vartheta_2 + \frac{\beta_1 u^+}{1+mu^++wv^+} > \frac{2\sqrt{2}\beta_1 u^+}{1+mu^++wv^+}$.

Remark 3.1. *Suppose* $\vartheta_1 > \frac{(4+2\sqrt{2})\beta v^+}{1+mu^++wv^+}$ *and* $\vartheta_2 > \frac{(2\sqrt{2}-1)\beta_1 u^+}{1+mu^++wv^+}$ *hold, Lemma* 3.3 *holds.*

In addition, from $a_{12}u^+ \geq \frac{(3+2\sqrt{2})\beta v^+}{1+mu^++wv^+}$ we can deduce

$$
a_{12} \geq \frac{\beta v^+}{(1 + mu^+ + wv^+) u^+} = \frac{m\beta v^+}{(1 + mu^+ + wv^+) mu^+} \geq \frac{m\beta v^+}{(1 + mu^+ + wv^+)^2}.
$$

As $a_{12} \geq \frac{m\beta v^+}{(1+mu^++wv^+)^2}$, the unique positive equilibrium (u^+, v^+) is stable.

Let $c^* = max\left\{c_1^*, c_2^*\right\}$. For fixed $c > c^*$, take constant $\eta \in \left(1, min\left\{2, \frac{\lambda_2}{\lambda_1}, \frac{\lambda_4}{\lambda_3}, \frac{\lambda_1+\lambda_3}{\lambda_1}, \frac{\lambda_1+\lambda_3}{\lambda_3}\right\}\right)$, then there are $\Delta_1(\eta\lambda_1, c) < 0$ and $\Delta_2(\eta\lambda_3, c) < 0$.

Let $\eta > 1$, $q > 1$ be large enough and $\lambda > 0$ be small enough. Here $\varepsilon_1 \in \left(0, \left(\sqrt{2} - 1\right) u^+\right)$ and $\varepsilon_2 \in \left(0, \frac{v^+}{2}\right)$. We write $\lambda_i = \lambda_i(c) > 0\,(i = 1, 2, 3, 4)$. The continuous functions $\left(\overline{\phi}(t), \overline{\psi}(t)\right)$ and $\left(\underline{\phi}(t), \underline{\psi}(t)\right)$ are defined as follows:

$$
\overline{\phi}(t) = \begin{cases} e^{\lambda_1 t}, & t \leq t_1, \\ u^+ + u^+ e^{-\lambda t}, & t \geq t_1, \end{cases} \quad \overline{\psi}(t) = \begin{cases} e^{\lambda_3 t} + qe^{\eta\lambda_3 t}, & t \leq t_2, \\ v^+ + v^+ e^{-\lambda t}, & t \geq t_2, \end{cases}
$$

$$
\underline{\phi}(t) = \begin{cases} e^{\lambda_1 t} - qe^{\eta\lambda_1 t}, & t \leq t_3, \\ u^+ - \varepsilon_1 e^{-\lambda t}, & t \geq t_3, \end{cases} \quad \underline{\psi}(t) = \begin{cases} e^{\lambda_3 t} - qe^{\eta\lambda_3 t}, & t \leq t_4, \\ v^+ - \varepsilon_2 e^{-\lambda t}, & t \geq t_4. \end{cases}
$$

It is easy to know that $\left(\overline{\phi}(t), \overline{\psi}(t)\right)$ and $\left(\underline{\phi}(t), \underline{\psi}(t)\right)$ have the following properties:

1) There are two constants $N_1 > 0$ and $N_2 > 0$ such that $(0, 0) \leq \left(\underline{\phi}(t), \underline{\psi}(t)\right) \leq \left(\overline{\phi}(t), \overline{\psi}(t)\right) \leq (N_1, N_2)$;
2) $\lim\limits_{t\to-\infty}\left(\overline{\phi}(t), \overline{\psi}(t)\right) = (0, 0)$, $\lim\limits_{t\to+\infty}\left(\underline{\phi}(t), \underline{\psi}(t)\right) = \lim\limits_{t\to+\infty}\left(\overline{\phi}(t), \overline{\psi}(t)\right) = (u^+, v^+)$;

3) For all $t \in \mathbb{R}$, $\overline{\phi}'(t+) \le \overline{\phi}'(t-)$, $\underline{\phi}'(t+) \ge \underline{\phi}'(t-)$.

Remark 3.2. *According to the definition of* $\left(\overline{\phi}(t), \overline{\psi}(t)\right)$ *and* $\left(\underline{\phi}(t), \underline{\psi}(t)\right)$, *we know*

$$\overline{\phi}(t) \le min\left\{e^{\lambda_1 t}, u^+ + u^+ e^{-\lambda t}\right\}, \quad \overline{\psi}(t) \le min\left\{e^{\lambda_3 t} + qe^{\eta\lambda_3 t}, v^+ + v^+ e^{-\lambda t}\right\},$$
$$\underline{\phi}(t) \ge max\left\{e^{\lambda_1 t} - qe^{\eta\lambda_1 t}, u^+ - \varepsilon_1 e^{-\lambda t}\right\}, \quad \underline{\psi}(t) \ge max\left\{e^{\lambda_3 t} - qe^{\eta\lambda_3 t}, v^+ - \varepsilon_2 e^{-\lambda t}\right\}.$$

Remark 3.3. *If* $q > 1$ *is large enough, then it is clear that* $t_1 \ge max\{t_2, t_3, t_4\}$.

Lemma 3.4. *Assume that* $\vartheta_1 > \frac{(4+2\sqrt{2})\beta v^+}{1+mu^+ + wv^+}$ *and* $\vartheta_2 > \frac{(2\sqrt{2}-1)\beta_1 u^+}{1+mu^+ + wv^+}$ *hold, and* $q > 1$ *is large enough, then* $\left(\overline{\phi}(t), \overline{\psi}(t)\right)$ *and* $\left(\underline{\phi}(t), \underline{\psi}(t)\right)$ *are a pair of upper and lower solutions of system* (3.1).

Proof. We first consider $\overline{\phi}(t)$. According to the function definition, we have

$$\overline{\phi}(t) = \begin{cases} e^{\lambda_1 t}, & t \le t_1 \\ u^+ + u^+ e^{-\lambda t}, & t \ge t_1 \end{cases}, \quad \overline{\phi}(t) \le min\left\{e^{\lambda_1 t}, u^+ + u^+ e^{-\lambda t}\right\}.$$

According to the definition of upper and lower solutions Definition 3.1, we want to prove that $D_2\overline{\phi}''(t) - c\overline{\phi}'(t) + f_2\left(\overline{\phi}, \underline{\psi}\right)(t) \le 0$, where $\underline{\psi}(t) \ge max\left\{e^{\lambda_3 t} - qe^{\eta\lambda_3 t}, v^+ - \varepsilon_2 e^{-\lambda t}\right\}$.

If $t \le t_1$, then $\overline{\phi}(t) = e^{\lambda_1 t}$, $\overline{\phi}'(t) = \lambda_1 e^{\lambda_1 t}$ and $\overline{\phi}''(t) = \lambda_1^2 e^{\lambda_1 t}$. In addition, we note that if $t-y-c\tau \le t_1$, then $\overline{\phi}(t-y-c\tau) = e^{\lambda_1(t-y-c\tau)}$; if $t-y-c\tau \ge t_1$, then $\overline{\phi}(t-y-c\tau) \le e^{\lambda_1(t-y-c\tau)}$. Thus,

$$\begin{aligned}
&D_2\overline{\phi}''(t) - c\overline{\phi}'(t) + f_2\left(\overline{\phi}, \overline{\psi}\right)(t) \\
=&D_2\overline{\phi}''(t) - c\overline{\phi}'(t) + \hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\overline{\phi}(t-y-c\tau)\,dy - (d_2 + q_2 e_2)\overline{\phi}(t) \\
&- a_{12}\overline{\phi}^2(t) - \frac{\beta\overline{\phi}(t)\underline{\psi}(t)}{1 + m\overline{\phi}(t) + w\underline{\psi}(t)} \\
\le&D_2\overline{\phi}''(t) - c\overline{\phi}'(t) + \hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\overline{\phi}(t-y-c\tau)\,dy - (d_2 + q_2 e_2)\overline{\phi}(t) \\
\le&D_2\left(\lambda_1^2 e^{\lambda_1 t}\right) - c\left(\lambda_1 e^{\lambda_1 t}\right) + \hat{a}e^{-d_1\tau}e^{\lambda_1 t}e^{\left(D_1\lambda^2 - c\lambda\right)\tau} - (d_2 + q_2 e_2)e^{\lambda_1 t} \\
=&e^{\lambda_1 t}\left(D_2\lambda_1^2 - c\lambda_1 + \hat{a}e^{-d_1\tau}e^{\left(D_1\lambda^2 - c\lambda\right)\tau} - d_2 - q_2 e_2\right) \\
=&e^{\lambda_1 t}\Delta_1(\lambda_1, c) \\
=&0.
\end{aligned}$$

If $t \ge t_1$, then $\overline{\phi}(t) = u^+ + u^+ e^{-\lambda t}$, here $\underline{\psi}(t) \ge v^+ - \varepsilon_2 e^{-\lambda t}$, $\overline{\phi}'(t) = -\lambda u^+ e^{-\lambda t}$ and $\overline{\phi}''(t) = \lambda^2 u^+ e^{-\lambda t}$. In addition, we note that if $t-y-c\tau \ge t_1$, then $\overline{\phi}(t-y-c\tau) = u^+ + u^+ e^{-\lambda(t-y-c\tau)}$; if $t-y-c\tau \le t_1$, then

$\overline{\phi}(t-y-c\tau) \le u^+ + u^+e^{-\lambda(t-y-c\tau)}$. Thus,

$$\begin{aligned}
&D_2\overline{\phi}''(t) - c\overline{\phi}'(t) + f_2\left(\overline{\phi}, \underline{\psi}\right)(t)\\
=&D_2\overline{\phi}''(t) - c\overline{\phi}'(t) + \hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\overline{\phi}(t-y-c\tau)\,dy - (d_2+q_2e_2)\overline{\phi}(t)\\
&-a_{12}\overline{\phi}^2(t) - \frac{\beta\overline{\phi}(t)\underline{\psi}(t)}{1+m\overline{\phi}(t)+w\underline{\psi}(t)}\\
\le&D_2\left(\lambda^2u^+e^{-\lambda t}\right) - c\left(-\lambda u^+e^{-\lambda t}\right) + \hat{a}e^{-d_1\tau}u^+ + \hat{a}e^{-d_1\tau}u^+e^{-\lambda t}e^{\left(D_1\lambda^2+c\lambda\right)\tau} - (d_2+q_2e_2)\left(u^+ + u^+e^{-\lambda t}\right)\\
&-a_{12}\left(u^+ + u^+e^{-\lambda t}\right)^2 - \frac{\beta\left(u^+ + u^+e^{-\lambda t}\right)\underline{\psi}(t)}{1+m\left(u^+ + u^+e^{-\lambda t}\right)+w\underline{\psi}(t)}\\
=&u^+e^{-\lambda t}\left(D_2\lambda^2 + c\lambda + \hat{a}e^{-d_1\tau}e^{\left(D_1\lambda^2+c\lambda\right)\tau} - d_2 - q_2e_2\right) + u^+\left(\hat{a}e^{-d_1\tau} - d_2 - q_2e_2 - a_{12}u^+\right)\\
&-2a_{12}\left(u^+\right)^2e^{-\lambda t} - a_{12}\left(u^+\right)^2e^{-2\lambda t} - \frac{\beta\left(u^+ + u^+e^{-\lambda t}\right)\underline{\psi}(t)}{1+m\left(u^+ + u^+e^{-\lambda t}\right)+w\underline{\psi}(t)}\\
=&u^+e^{-\lambda t}\Delta_1(-\lambda,c) - 2a_{12}\left(u^+\right)^2e^{-\lambda t} - a_{12}\left(u^+\right)^2e^{-2\lambda t} - \frac{\beta\left(u^+ + u^+e^{-\lambda t}\right)\underline{\psi}(t)}{1+m\left(u^+ + u^+e^{-\lambda t}\right)+w\underline{\psi}(t)} + \frac{\beta u^+v^+}{1+mu^++wv^+}\\
=&u^+e^{-\lambda t}\left[\Delta_1(-\lambda,c) - 2a_{12}u^+\right] - u^+\left[a_{12}u^+e^{-2\lambda t} + \frac{\beta\left(1+e^{-\lambda t}\right)\underline{\psi}(t)}{1+m\left(u^+ + u^+e^{-\lambda t}\right)+w\underline{\psi}(t)} - \frac{\beta v^+}{1+mu^++wv^+}\right].
\end{aligned}$$

According to the premise assumption, $\Delta_1(0,c) - 2a_{12}u^+ = \vartheta_1 - 2a_{12}u^+ = \frac{\beta u^+v^+}{1+mu^++wv^+} - a_{12}u^+ < 0$ can be obtained, so there is a constant $\widetilde{\lambda}_1 > 0$, which makes $\Delta_1(-\lambda,c) - 2a_{12}u^+ < 0$ for $\forall\lambda \in \left(0, \widetilde{\lambda}_1\right)$.

Let $I_1(\lambda,t) := a_{12}u^+e^{-2\lambda t} + \frac{\beta\left(1+e^{-\lambda t}\right)\underline{\psi}(t)}{1+m\left(u^++u^+e^{-\lambda t}\right)+w\underline{\psi}(t)} - \frac{\beta v^+}{1+mu^++wv^+}$, where $t \ge t_1$ and $\underline{\psi}(t) \ge v^+ - \varepsilon_2e^{-\lambda t} \ge 0$. $\frac{\beta\left(1+e^{-\lambda t}\right)\underline{\psi}(t)}{1+m\left(u^++u^+e^{-\lambda t}\right)+w\underline{\psi}(t)}$ is increasing with respect to $\underline{\psi}(t)$, thus

$$I_1(\lambda,t) \ge a_{12}u^+e^{-2\lambda t} + \frac{\beta\left(1+e^{-\lambda t}\right)\left(v^+ - \varepsilon_2e^{-\lambda t}\right)}{1+m\left(u^+ + u^+e^{-\lambda t}\right)+w\left(v^+ - \varepsilon_2e^{-\lambda t}\right)} - \frac{\beta v^+}{1+mu^++wv^+}.$$

Here $t \ge t_1$, $(t_1 < 0)$. Therefore, t is divided into $t \in [t_1, 0]$ and $t > 0$ for discussion.

If $t \in [t_1, 0]$, from the hypothesis we know that

$$\begin{aligned}
I_1(\lambda,t) \ge& a_{12}u^+e^{-2\lambda t} + \frac{\beta\left(1+e^{-\lambda t}\right)\left(v^+ - \varepsilon_2e^{-\lambda t}\right)}{1+m\left(u^+ + u^+e^{-\lambda t}\right)+w\left(v^+ - \varepsilon_2e^{-\lambda t}\right)} - \frac{\beta v^+}{1+mu^++wv^+}\\
\ge& a_{12}u^+ + \frac{\beta\left(1+e^{-\lambda t}\right)\left(v^+ - \varepsilon_2e^{-\lambda t}\right)}{1+m\left(u^+ + u^+e^{-\lambda t}\right)+w\left(v^+ - \varepsilon_2e^{-\lambda t}\right)} - \frac{\beta v^+}{1+mu^++wv^+}\\
>& \frac{\left(2+2\sqrt{2}\right)\beta v^+}{1+mu^++wv^+} + \frac{\beta\left(1+e^{-\lambda t}\right)\left(v^+ - \varepsilon_2e^{-\lambda t}\right)}{1+m\left(u^+ + u^+e^{-\lambda t}\right)+w\left(v^+ - \varepsilon_2e^{-\lambda t}\right)}\\
>&0.
\end{aligned}$$

If $t > 0$, here $\varepsilon_2 \in \left(0, \frac{v^+}{2}\right)$, we have that

$$\begin{aligned}\frac{\beta\left(1+e^{-\lambda t}\right)\left(v^+-\varepsilon_2 e^{-\lambda t}\right)}{1+m\left(u^++u^+e^{-\lambda t}\right)+w\left(v^+-\varepsilon_2 e^{-\lambda t}\right)} &\geq \frac{\beta\left(1+e^{-\lambda t}\right)\left(v^+-\varepsilon_2 e^{-\lambda t}\right)_{min}}{1+m\left(u^++u^+e^{-\lambda t}\right)+w\left(v^+-\varepsilon_2 e^{-\lambda t}\right)_{min}}\\ &\geq \frac{\beta\left(1+e^{-\lambda t}\right)\frac{v^+}{2}}{1+mu^+\left(1+e^{-\lambda t}\right)+w\frac{v^+}{2}}\\ &> 0.\end{aligned}$$

Since $a_{12}u^+e^{-2\lambda t}$ and $\frac{\beta\left(1+e^{-\lambda t}\right)\frac{v^+}{2}}{1+mu^+\left(1+e^{-\lambda t}\right)+w\frac{v^+}{2}}$ are decreasing about the variable t on $t > 0$, furthermore, $I_1(\lambda, 0) = a_{12}u^+ + \frac{2\beta\left(v^+-\varepsilon_2\right)}{1+2mu^++w\left(v^+-\varepsilon_2\right)} - \frac{\beta v^+}{1+mu^++wv^+} > \frac{\left(2+2\sqrt{2}\right)\beta v^+}{1+mu^++wv^+} + \frac{2\beta\left(v^+-\varepsilon_2\right)}{1+2mu^++w\left(v^+-\varepsilon_2\right)} > 0$ and $I_1(\lambda, +\infty) = 0$, then $I_1(\lambda, t) > 0$ for $\forall t \geq t_1$.

In consequence, $\overline{\phi}$ satisfies the upper solution definition, that is, $D_2\overline{\phi}''(t) - c\overline{\phi}'(t) + f_2\left(\overline{\phi}, \underline{\psi}\right)(t) \leq 0$.

Next we consider $\overline{\psi}(t)$. According to the function definition, we have

$$\overline{\psi}(t) = \begin{cases} e^{\lambda_3 t} + qe^{\eta\lambda_3 t}, & t \leq t_2 \\ v^+ + v^+e^{-\lambda t}, & t \geq t_2 \end{cases}, \quad \overline{\psi}(t) \leq min\left\{e^{\lambda_3 t} + qe^{\eta\lambda_3 t}, v^+ + v^+e^{-\lambda t}\right\}.$$

According to the definition of upper and lower solutions Definition 3.1, we want to prove that $D_5\overline{\psi}''(t) - c\overline{\psi}'(t) + f_5\left(\overline{\phi}, \overline{\psi}\right)(t) \leq 0$, where $\overline{\phi}(t) \leq min\left\{e^{\lambda_1 t}, u^+ + u^+e^{-\lambda t}\right\}$.

If $t \leq t_2$, then $\overline{\psi}(t) = e^{\lambda_3 t} + qe^{\eta\lambda_3 t}$, here $\overline{\phi}(t) \leq e^{\lambda_1 t}$, $\overline{\psi}'(t) = \lambda_3 e^{\lambda_3 t} + q\eta\lambda_3 e^{\eta\lambda_3 t}$ and $\overline{\psi}''(t) = \lambda_3^2 e^{\lambda_3 t} + q\eta^2\lambda_3^2 e^{\eta\lambda_3 t}$. Thus,

$$\begin{aligned}&D_5\overline{\psi}''(t) - c\overline{\psi}'(t) + f_5\left(\overline{\phi}, \overline{\psi}\right)(t)\\ =&D_5\overline{\psi}''(t) - c\overline{\psi}'(t) + \left(a_2 - b_2 - q_5e_5\right)\overline{\psi}(t) - a_{55}\overline{\psi}^2(t) + \frac{\beta_1\overline{\phi}(t)\overline{\psi}(t)}{1+m\overline{\phi}(t)+w\overline{\psi}(t)}\\ \leq&D_5\left(\lambda_3^2e^{\lambda_3 t} + q\eta^2\lambda_3^2e^{\eta\lambda_3 t}\right) - c\left(\lambda_3e^{\lambda_3 t} + q\eta\lambda_3e^{\eta\lambda_3 t}\right) + \left(a_2 - b_2 - q_5e_5\right)\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right) - a_{55}\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right)^2\\ &+ \frac{\beta_1e^{\lambda_1 t}\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right)}{1+me^{\lambda_1 t}+w\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right)}\\ =&e^{\lambda_3 t}\left(D_5\lambda_3^2 - c\lambda_3 + a_2 - b_2 - q_5e_5\right) + qe^{\eta\lambda_3 t}\left(D_5\eta^2\lambda_3^2 - c\eta\lambda_3 + a_2 - b_2 - q_5e_5\right) - a_{55}\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right)^2\\ &+ \frac{\beta_1e^{\lambda_1 t\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right)}}{1+me^{\lambda_1 t}+w\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right)}\\ =&e^{\lambda_3 t}\Delta_2\left(\lambda_3, c\right) + qe^{\eta\lambda_3 t}\Delta_2\left(\eta\lambda_3, c\right) - a_{55}\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right)^2 + \frac{\beta_1e^{\lambda_1 t\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right)}}{1+me^{\lambda_1 t}+w\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right)}\\ \leq&qe^{\eta\lambda_3 t}\Delta_2\left(\eta\lambda_3, c\right) + \beta_1e^{\lambda_1 t}\left(e^{\lambda_3 t} + qe^{\eta\lambda_3 t}\right)\\ =&e^{\eta\lambda_3 t}\left[q\Delta_2\left(\eta\lambda_3, c\right) + \beta_1e^{\left(\lambda_1+\lambda_3-\eta\lambda_3\right)t} + q\beta_1e^{\lambda_1 t}\right]\\ \leq&e^{\eta\lambda_3 t}\left\{q\left[\Delta_2\left(\eta\lambda_3, c\right) + \beta_1e^{\lambda_1 t}\right] + \beta_1\right\}.\end{aligned}$$

Here $q > 1$ is large enough, then $-t_2 > 0$ is also large enough. $q\left[\Delta_2(\eta\lambda_3, c) + \beta_1 e^{\lambda_1 t}\right] + \beta_1$ is increasing about the variable t on $t \le t_2$ for $\forall t \le t_2$, so there exists $q\left[\Delta_2(\eta\lambda_3, c) + \beta_1 e^{\lambda_1 t}\right] + \beta_1 < 0$, thus $e^{\eta\lambda_3 t}\left\{q\left[\Delta_2(\eta\lambda_3, c) + \beta_1 e^{\lambda_1 t}\right] + \beta_1\right\} < 0$ for $\forall t \le t_2$.

If $t \ge t_2$, then $\overline{\psi}(t) = v^+ + v^+ e^{-\lambda t}$, here $\overline{\phi}(t) \le u^+ + u^+ e^{-\lambda t}$, $\overline{\psi}'(t) = -\lambda v^+ e^{-\lambda t}$ and $\overline{\psi}''(t) = \lambda^2 v^+ e^{-\lambda t}$. Thus,

$$\begin{aligned}
& D_5\overline{\psi}''(t) - c\overline{\psi}'(t) + f_5\left(\overline{\phi}, \overline{\psi}\right)(t) \\
=& D_5\overline{\psi}''(t) - c\overline{\psi}'(t) + (a_2 - b_2 - q_5 e_5)\overline{\psi}(t) - a_{55}\overline{\psi}^2(t) + \frac{\beta_1\overline{\phi}(t)\overline{\psi}(t)}{1 + m\overline{\phi}(t) + w\overline{\psi}(t)} \\
\le& D_5\left(\lambda^2 v^+ e^{-\lambda t}\right) - c\left(-\lambda v^+ e^{-\lambda t}\right) + (a_2 - b_2 - q_5 e_5)\left(v^+ + v^+ e^{-\lambda t}\right) - a_{55}\left(v^+ + v^+ e^{-\lambda t}\right)^2 \\
& + \frac{\beta_1\left(u^+ + u^+ e^{-\lambda t}\right)\left(v^+ + v^+ e^{-\lambda t}\right)}{1 + m(u^+ + u^+ e^{-\lambda t}) + w(v^+ + v^+ e^{-\lambda t})} \\
=& v^+ e^{-\lambda t}\left(D_5\lambda^2 + c\lambda + a_2 - b_2 - q_5 e_5\right) + v^+(a_2 - b_2 - q_5 e_5 - a_{55}v^+) - 2a_{55}(v^+)^2 e^{-\lambda t} - a_{55}(v^+)^2 e^{-2\lambda t} \\
& + \frac{\beta_1\left(u^+ + u^+ e^{-\lambda t}\right)\left(v^+ + v^+ e^{-\lambda t}\right)}{1 + m(u^+ + u^+ e^{-\lambda t}) + w(v^+ + v^+ e^{-\lambda t})} \\
=& v^+ e^{-\lambda t}\Delta_2(-\lambda, c) - 2a_{55}(v^+)^2 e^{-\lambda t} - a_{55}(v^+)^2 e^{-2\lambda t} + \frac{\beta_1 u^+ v^+\left(1 + e^{-\lambda t}\right)^2}{1 + mu^+(1 + e^{-\lambda t}) + wv^+(1 + e^{-\lambda t})} - \frac{\beta_1 u^+ v^+}{1 + mu^+ + wv^+} \\
=& v^+ e^{-\lambda t}\left[\Delta_2(-\lambda, c) - a_{55}v^+\right] - v^+\left[a_{55}v^+ e^{-\lambda t} + a_{55}v^+ e^{-2\lambda t} - \frac{\beta_1 u^+\left(1 + e^{-\lambda t}\right)^2}{1 + mu^+(1 + e^{-\lambda t}) + wv^+(1 + e^{-\lambda t})}\right. \\
& \left. + \frac{\beta_1 u^+}{1 + mu^+ + wv^+}\right].
\end{aligned}$$

Because of $\Delta_2(0, c) - a_{55}v^+ = \vartheta_2 - a_{55}v^+ = -\frac{\beta_1 u^+}{1+mu^++wv^+} < 0$, there is a constant $\widetilde{\lambda}_2 > 0$, which makes $\Delta_2(-\lambda, c) - a_{55}v^+ < 0$ for $\forall\lambda \in \left(0, \widetilde{\lambda}_2\right)$.

Let $I_2(\lambda, t) := a_{55}v^+ e^{-\lambda t} + a_{55}v^+ e^{-2\lambda t} - \frac{\beta_1 u^+\left(1+e^{-\lambda t}\right)^2}{1+mu^+\left(1+e^{-\lambda t}\right)+wv^+\left(1+e^{-\lambda t}\right)} + \frac{\beta_1 u^+}{1+mu^++wv^+}$, we have that

$$\begin{aligned}
I_2(\lambda, t) =& a_{55}v^+ e^{-\lambda t} + a_{55}v^+ e^{-2\lambda t} - \frac{\beta_1 u^+\left(1 + e^{-\lambda t}\right)^2}{1 + mu^+(1 + e^{-\lambda t}) + wv^+(1 + e^{-\lambda t})} + \frac{\beta_1 u^+}{1 + mu^+ + wv^+} \\
>& a_{55}v^+ e^{-\lambda t} + a_{55}v^+ e^{-2\lambda t} - \frac{\beta_1 u^+\left(1 + e^{-\lambda t}\right)^2}{1 + mu^+ + wv^+} + \frac{\beta_1 u^+}{1 + mu^+ + wv^+} \\
=& a_{55}v^+ e^{-\lambda t} + a_{55}v^+ e^{-2\lambda t} - \frac{\beta_1 u^+}{1 + mu^+ + wv^+}\left(2e^{-\lambda t} + e^{-2\lambda t}\right) \\
=& e^{-\lambda t}\left\{\left(a_{55}v^+ - \frac{2\beta_1 u^+}{1 + mu^+ + wv^+}\right) + \left(a_{55}v^+ - \frac{\beta_1 u^+}{1 + mu^+ + wv^+}\right)e^{-\lambda t}\right\}.
\end{aligned}$$

According to the premise hypothesis, $a_{55}v^+ > \frac{2\sqrt{2}\beta_1 u^+}{1+mu^++wv^+}$ can be obtained, thus $I_2(\lambda, t) > 0$ for $\forall t \ge t_2$.

In consequence, $\overline{\psi}$ satisfies the upper solution definition, that is, $D_5\overline{\psi}''(t) - c\overline{\psi}'(t) + f_5\left(\overline{\phi}, \overline{\psi}\right)(t) \le 0$.

We next consider $\underline{\phi}(t)$. According to the function definition, we have

$$\underline{\phi}(t)=\begin{cases}e^{\lambda_1 t}-qe^{\eta\lambda_1 t}, & t\le t_3\\ u^+-\varepsilon_1 e^{-\lambda t}, & t\ge t_3\end{cases},\quad \underline{\phi}(t)\ge max\left\{e^{\lambda_1 t}-qe^{\eta\lambda_1 t},u^+-\varepsilon_1 e^{-\lambda t}\right\}.$$

According to the definition of upper and lower solutions Definition 3.1, we want to prove that $D_2\underline{\phi}''(t)-c\underline{\phi}'(t)+f_2\left(\underline{\phi},\overline{\psi}\right)(t)\ge 0$, where $\overline{\psi}(t)\le min\left\{e^{\lambda_3 t}+qe^{\eta\lambda_3 t},v^+ + v^+e^{-\lambda t}\right\}$.

If $t\le t_3$, then $\underline{\phi}(t)=e^{\lambda_1 t}-qe^{\eta\lambda_1 t}$, here $\overline{\psi}(t)\le e^{\lambda_3 t}+qe^{\eta\lambda_3 t}$, $\underline{\phi}'(t)=\lambda_1 e^{\lambda_1 t}-q\eta\lambda_1 e^{\eta\lambda_1 t}$ and $\underline{\phi}''(t)=\lambda_1^2 e^{\lambda_1 t}-q\eta^2\lambda_1^2 e^{\eta\lambda_1 t}$. In addition, we note that if $t-y-c\tau\ge t_3$, then $\overline{\underline{\phi}}(t-y-c\tau)=e^{\lambda_1(t-y-c\tau)}-qe^{\eta\lambda_1(t-y-c\tau)}$; if $t-y-c\tau\le t_3$, then $\overline{\phi}(t-y-c\tau)\ge e^{\lambda_1(t-y-c\tau)}-qe^{\eta\lambda_1(t-y-c\tau)}$. Thus,

$$\begin{aligned}
&D_2\underline{\phi}''(t)-c\underline{\phi}'(t)+f_2\left(\underline{\phi},\overline{\psi}\right)(t)\\
=&D_2\underline{\phi}''(t)-c\underline{\phi}'(t)+\hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\underline{\phi}(t-y-c\tau)\,dy-(d_2+q_2e_2)\underline{\phi}(t)\\
&-a_{12}\underline{\phi}^2(t)-\frac{\beta\underline{\phi}(t)\overline{\psi}(t)}{1+m\underline{\phi}(t)+w\overline{\psi}(t)}\\
\ge&D_2\left(\lambda_1^2e^{\lambda_1 t}-q\eta^2\lambda_1^2e^{\eta\lambda_1 t}\right)-c\left(\lambda_1 e^{\lambda_1 t}-q\eta\lambda_1 e^{\eta\lambda_1 t}\right)+\hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\left(e^{\lambda_1(t-y-c\tau)}-qe^{\eta\lambda_1(t-y-c\tau)}\right)dy\\
&-(d_2+q_2e_2)\left(e^{\lambda_1 t}-qe^{\eta\lambda_1 t}\right)-a_{12}\left(e^{\lambda_1 t}-qe^{\eta\lambda_1 t}\right)^2-\frac{\beta\left(e^{\lambda_1 t}-qe^{\eta\lambda_1 t}\right)\overline{\psi}(t)}{1+m(e^{\lambda_1 t}-qe^{\eta\lambda_1 t})+w\overline{\psi}(t)}\\
=&e^{\lambda_1 t}\left(D_2\lambda_1^2-c\lambda_1+\hat{a}e^{-d_1\tau}e^{\left(D_1\lambda_1^2-c\lambda_1\right)\tau}-d_2-q_2e_2\right)-qe^{\eta\lambda_1 t}\left(D_2\eta^2\lambda_1^2-c\eta\lambda_1+\hat{a}e^{-d_1\tau}e^{\left(D_1\eta^2\lambda_1^2-c\eta\lambda_1\right)\tau}\right.\\
&\left.-d_2-q_2e_2\right)-a_{12}\left(e^{\lambda_1 t}-qe^{\eta\lambda_1 t}\right)^2-\frac{\beta\left(e^{\lambda_1 t}-qe^{\eta\lambda_1 t}\right)\overline{\psi}(t)}{1+m(e^{\lambda_1 t}-qe^{\eta\lambda_1 t})+w\overline{\psi}(t)}\\
=&e^{\lambda_1 t}\Delta_1(\lambda_1,c)-qe^{\eta\lambda_1 t}\Delta_1(\eta\lambda_1,c)-a_{12}\left(e^{\lambda_1 t}-qe^{\eta\lambda_1 t}\right)^2-\frac{\beta\left(e^{\lambda_1 t}-qe^{\eta\lambda_1 t}\right)\overline{\psi}(t)}{1+m(e^{\lambda_1 t}-qe^{\eta\lambda_1 t})+w\overline{\psi}(t)}\\
\ge&e^{\lambda_1 t}\Delta_1(\lambda_1,c)-qe^{\eta\lambda_1 t}\Delta_1(\eta\lambda_1,c)-a_{12}\left(e^{\lambda_1 t}-qe^{\eta\lambda_1 t}\right)^2-\frac{\beta\left(e^{\lambda_1 t}-qe^{\eta\lambda_1 t}\right)\left(e^{\lambda_3 t}+qe^{\eta\lambda_3 t}\right)}{1+m(e^{\lambda_1 t}-qe^{\eta\lambda_1 t})+w(e^{\lambda_3 t}+qe^{\eta\lambda_3 t})}\\
\ge&-qe^{\eta\lambda_1 t}\Delta_1(\eta\lambda_1,c)-a_{12}e^{2\lambda_1 t}-\beta e^{\lambda_1 t}\left(e^{\lambda_3 t}+qe^{\eta\lambda_3 t}\right)\\
=&-e^{\eta\lambda_1 t}\left[q\Delta_1(\eta\lambda_1,c)+a_{12}e^{(2\lambda_1-\eta\lambda_1)t}+\beta e^{(\lambda_1+\lambda_3-\eta\lambda_1)t}+q\beta e^{(\lambda_1+\eta\lambda_3-\eta\lambda_1)t}\right]\\
\ge&-e^{\eta\lambda_1 t}\left[q\Delta_1(\eta\lambda_1,c)+a_{12}+\beta+q\beta e^{(\lambda_1+\eta\lambda_3-\eta\lambda_1)t}\right]\\
=&-e^{\eta\lambda_1 t}\left\{q\left[\Delta_1(\eta\lambda_1,c)+\beta e^{(\lambda_1+\eta\lambda_3-\eta\lambda_1)t}\right]+(a_{12}+\beta)\right\}.
\end{aligned}$$

Here $q>1$ is large enough, then $-t_3>0$ is also large enough. $q\left[\Delta_1(\eta\lambda_1,c)+\beta e^{(\lambda_1+\eta\lambda_3-\eta\lambda_1)t}\right]+(a_{12}+\beta)$ is increasing about the variable t on $t\le t_3$ for $\forall t\le t_3$, so there exists $q\left[\Delta_1(\eta\lambda_1,c)+\beta e^{(\lambda_1+\eta\lambda_3-\eta\lambda_1)t}\right]+(a_{12}+\beta)<0$, thus $-e^{\eta\lambda_1 t}\left\{q\left[\Delta_1(\eta\lambda_1,c)+\beta e^{(\lambda_1+\eta\lambda_3-\eta\lambda_1)t}\right]+(a_{12}+\beta)\right\}>0$ for $\forall t\le t_3$.

If $t>t_3$,then $\underline{\phi}(t)=u^+-\varepsilon_1 e^{-\lambda t}$, here $\overline{\psi}(t)\le v^+ + v^+e^{-\lambda t}$, $\underline{\phi}'(t)=\lambda\varepsilon_1 e^{-\lambda t}$ and $\underline{\phi}''(t)=-\lambda^2\varepsilon_1 e^{-\lambda t}$. In addition, we note that if $t-y-c\tau\ge t_3$, then $\underline{\phi}(t-y-c\tau)=u^+-\varepsilon_1 e^{-\lambda(t-y-c\tau)}$; if $t-y-c\tau\le t_3$, then

$\underline{\phi}(t-y-c\tau) \geq u^+ - \varepsilon_1 e^{-\lambda(t-y-c\tau)}$. Thus,

$$
\begin{aligned}
&D_2\underline{\phi}''(t) - c\underline{\phi}'(t) + f_2\left(\underline{\phi}, \overline{\psi}\right)(t) \\
=&D_2\underline{\phi}''(t) - c\underline{\phi}'(t) + \hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\underline{\phi}(t-y-c\tau)\,dy - (d_2+q_2e_2)\,\underline{\phi}(t) \\
&-a_{12}\underline{\phi}^2(t) - \frac{\beta\underline{\phi}(t)\,\overline{\psi}(t)}{1+m\underline{\phi}(t)+w\overline{\psi}(t)} \\
\geq&D_2\left(-\lambda^2\varepsilon_1e^{-\lambda t}\right) - c\left(\lambda\varepsilon_1e^{-\lambda t}\right) + \hat{a}e^{-d_1\tau}u^+ - \hat{a}e^{-d_1\tau}\varepsilon_1e^{-\lambda t}e^{\left(D_1\lambda^2+c\lambda\right)\tau} - (d_2+q_2e_2)\left(u^+-\varepsilon_1e^{-\lambda t}\right) \\
&-a_{12}\left(u^+-\varepsilon_1e^{-\lambda t}\right)^2 - \frac{\beta\left(u^+-\varepsilon_1e^{-\lambda t}\right)\overline{\psi}(t)}{1+m\left(u^+-\varepsilon_1e^{-\lambda t}\right)+w\overline{\psi}(t)} \\
=&-\varepsilon_1e^{-\lambda t}\left(D_2\lambda^2+c\lambda+\hat{a}e^{-d_1\tau}e^{\left(D_1\lambda^2+c\lambda\right)\tau}-d_2-q_2e_2\right) + u^+\left(\hat{a}e^{-d_1\tau}-d_2-q_2e_2\right) + 2a_{12}u^+\varepsilon_1e^{-\lambda t} \\
&-a_{12}\varepsilon_1^2e^{-2\lambda t} - \frac{\beta\left(u^+-\varepsilon_1e^{-\lambda t}\right)\overline{\psi}(t)}{1+m\left(u^+-\varepsilon_1e^{-\lambda t}\right)+w\overline{\psi}(t)} \\
=&-\varepsilon_1e^{-\lambda t}\Delta_1(-\lambda,c) + 2a_{12}u^+\varepsilon_1e^{-\lambda t} - a_{12}\varepsilon_1^2e^{-2\lambda t} + \frac{\beta u^+v^+}{1+mu^++wv^+} - \frac{\beta\left(u^+-\varepsilon_1e^{-\lambda t}\right)\overline{\psi}(t)}{1+m\left(u^+-\varepsilon_1e^{-\lambda t}\right)+w\overline{\psi}(t)} \\
=&\varepsilon_1e^{-\lambda t}\left[-\Delta_1(-\lambda,c) + \left(4-2\sqrt{2}\right)a_{12}u^+\right] \\
&+\left(2\sqrt{2}-2\right)a_{12}u^+\varepsilon_1e^{-\lambda t} - a_{12}\varepsilon_1^2e^{-2\lambda t} + \frac{\beta u^+v^+}{1+mu^++wv^+} - \frac{\beta\left(u^+-\varepsilon_1e^{-\lambda t}\right)\overline{\psi}(t)}{1+m\left(u^+-\varepsilon_1e^{-\lambda t}\right)+w\overline{\psi}(t)}.
\end{aligned}
$$

According to the premise assumption, we can obtain that $-\Delta_1(-\lambda,c) + \left(4-2\sqrt{2}\right)a_{12}u^+ = -\vartheta_1 + \left(4-2\sqrt{2}\right)a_{12}u^+ = \left(3-2\sqrt{2}\right)a_{12}u^+ - \frac{\beta v^+}{1+mu^++wv^+} > 0$, so there is a constant $\widetilde{\lambda}_3 > 0$, which makes $-\Delta_1(-\lambda,c) + \left(4-2\sqrt{2}\right)a_{12}u^+ > 0$ for $\forall\lambda \in \left(0,\widetilde{\lambda}_3\right)$.

Let $I_3(\lambda,t) = \left(2\sqrt{2}-2\right)a_{12}u^+\varepsilon_1e^{-\lambda t} - a_{12}\varepsilon_1^2e^{-2\lambda t} + \frac{\beta u^+v^+}{1+mu^++wv^+} - \frac{\beta\left(u^+-\varepsilon_1e^{-\lambda t}\right)\overline{\psi}(t)}{1+m\left(u^+-\varepsilon_1e^{-\lambda t}\right)+w\overline{\psi}(t)}$, we have that

$$
\begin{aligned}
I_3(\lambda,t) =& \left(2\sqrt{2}-2\right)a_{12}u^+\varepsilon_1e^{-\lambda t} - a_{12}\varepsilon_1^2e^{-2\lambda t} + \frac{\beta u^+v^+}{1+mu^++wv^+} - \frac{\beta\left(u^+-\varepsilon_1e^{-\lambda t}\right)\overline{\psi}(t)}{1+m\left(u^+-\varepsilon_1e^{-\lambda t}\right)+w\overline{\psi}(t)} \\
\geq& \left(2\sqrt{2}-2\right)a_{12}u^+\varepsilon_1e^{-\lambda t} - a_{12}\varepsilon_1^2e^{-2\lambda t} + \frac{\beta u^+v^+}{1+mu^++wv^+} - \frac{\beta\left(u^+-\varepsilon_1e^{-\lambda t}\right)\left(v^++v^+e^{-\lambda t}\right)}{1+m\left(u^+-\varepsilon_1e^{-\lambda t}\right)+w\left(v^++v^+e^{-\lambda t}\right)},
\end{aligned}
$$

where $t \geq t_3$. Therefore, $I_3(\lambda,0) = -a_{12}\varepsilon_1^2 + \left(2\sqrt{2}-2\right)a_{12}u^+\varepsilon_1 + \frac{\beta u^+v^+}{1+mu^++wv^+} - \frac{2\beta\left(u^+-\varepsilon_1\right)v^+}{1+m(u^+-\varepsilon_1)+2wv^+}$.

From Remark 3.1, we can see that $\vartheta_1 > \frac{\left(4+2\sqrt{2}\right)\beta v^+}{1+mu^++wv^+}$is established, there is $\varepsilon_1 \in \left(0,\left(\sqrt{2}-1\right)u^+\right)$, making $-a_{12}\varepsilon_1^2 + \left(2\sqrt{2}-2\right)a_{12}u^+\varepsilon_1 + \frac{\beta u^+v^+}{1+mu^++wv^+} - \frac{2\beta\left(u^+-\varepsilon_1\right)v^+}{1+m(u^+-\varepsilon_1)+2wv^+} > \varepsilon_0 > 0$. We have $I_3(\lambda,0) > 0$, here $\varepsilon_1 \in \left(0,\left(\sqrt{2}-1\right)u^+\right)$. We can choose a small enough $\delta_1 > 0$, such that $\delta^* := \varepsilon_1 + \delta_1$ for $\forall\delta \in [\varepsilon_1,\delta^*]$ satisfying

$$
-a_{12}\delta^2 + \left(2\sqrt{2}-2\right)a_{12}u^+\delta + \frac{\beta u^+v^+}{1+mu^++wv^+} - \frac{\beta(u^+-\delta)(2v^++\delta)}{1+m(u^+-\delta)+w(2v^++\delta)} > \frac{\varepsilon_0}{2} > 0.
$$

We want to prove that $I_3(\lambda,t) > 0$ for $\forall t > t_3$. Here $t > t_3\,(t_3 < 0)$, therefore, t is divided into $t \in (t_3, 0]$ and $t > 0$ two parts to discuss.

If $t \in (t_3, 0]$, let $\nu(t) := \varepsilon_1 e^{-\lambda t}$, $\mu(t) := v^+ + v^+ e^{-\lambda t}$. Select $\widetilde{\lambda}_3 > 0$ small enough such that for any given $\lambda \in \left(0, \widetilde{\lambda}_3\right)$, we have

$$\nu(t_3) = \varepsilon_1 e^{-\lambda t_3} = \delta^*, \qquad \mu(t_3) = v^+ + v^+ e^{-\lambda t_3} = \delta^*,$$

which leads to $\varepsilon_1 \le \nu(t) \le \delta^*$ and $\varepsilon_1 \le \mu(t) \le \delta^*$. So we get $I_3(\lambda,t) > 0$ for $\forall t \in (t_3, 0]$.

If $t \ge 0$, here $-\frac{\beta\left(u^+ - \varepsilon_1 e^{-\lambda t}\right)\overline{\psi}(t)}{1+m\left(u^+ - \varepsilon_1 e^{-\lambda t}\right)+w\overline{\psi}(t)} < 0$, based on the assumption that $\lambda > 0$ is small enough,

$$\left[-\frac{\beta\left(u^+ - \varepsilon_1 e^{-\lambda t}\right)\overline{\psi}(t)}{1 + m\left(u^+ - \varepsilon_1 e^{-\lambda t}\right) + w\overline{\psi}(t)}\right]_t' = \frac{-\beta\varepsilon_1\lambda e^{-\lambda t}\overline{\psi}(t)\left[1 + \omega\overline{\psi}(t)\right] - \beta\left(u^+ - \varepsilon_1 e^{-\lambda t}\right)\overline{\psi}'(t)\left[1 + m\left(u^+ - \varepsilon_1 e^{-\lambda t}\right)\right]}{\left[1 + m\left(u^+ - \varepsilon_1 e^{-\lambda t}\right) + w\overline{\psi}(t)\right]^2},$$

where $\overline{\psi}(t) > 0$ and $\overline{\psi}'(t) < 0$. Thus we have the function $-\frac{\beta\left(u^+ - \varepsilon_1 e^{-\lambda t}\right)\overline{\psi}(t)}{1+m\left(u^+ - \varepsilon_1 e^{-\lambda t}\right)+w\overline{\psi}(t)}$ is increasing about the variable t on $t \ge 0$; the function $\left(2\sqrt{2} - 2\right)a_{12}u^+\varepsilon_1 e^{-\lambda t} - a_{12}\varepsilon_1^2 e^{-2\lambda t} > 0$ is decreasing about the variable t on $t \ge 0$. We can get that

$$I_3(\lambda, 0) > 0, \qquad I_3(\lambda, +\infty) = 0.$$

So we get $I_3(\lambda,t) > 0$ for $\forall t \ge 0$. Thus, $I_3(\lambda,t) > 0$ for $\forall t > t_3$. In consequence, $\underline{\phi}$ satisfies lower solution definition, that is, $D_2\underline{\phi}''(t) - c\underline{\phi}'(t) + f_2\left(\underline{\phi}, \overline{\psi}\right)(t) \ge 0$.

We finally consider $\underline{\psi}(t)$. According to the function definition, we have

$$\underline{\psi}(t) = \begin{cases} e^{\lambda_3 t} - q e^{\eta\lambda_3 t}, & t \le t_4 \\ v^+ - \varepsilon_2 e^{-\lambda t}, & t \ge t_4 \end{cases}, \quad \underline{\psi}(t) \ge max\left\{e^{\lambda_3 t} - q e^{\eta\lambda_3 t}, v^+ - \varepsilon_2 e^{-\lambda t}\right\}.$$

According to the definition of upper and lower solutions Definition 3.1, we want to prove that $D_5\underline{\psi}''(t) - c\underline{\psi}'(t) + f_5\left(\underline{\phi}, \underline{\psi}\right)(t) \ge 0$, where $\underline{\phi}(t) \ge max\left\{e^{\lambda_1 t} - q e^{\eta\lambda_1 t}, u^+ - \varepsilon_1 e^{-\lambda t}\right\}$.

If $t \le t_4$, then $\underline{\psi}(t) = e^{\lambda_3 t} - q e^{\eta\lambda_3 t}$, $\underline{\psi}'(t) = \lambda_3 e^{\lambda_3 t} - q\eta\lambda_3 e^{\eta\lambda_3 t}$ and $\underline{\psi}''(t) = \lambda_3^2 e^{\lambda_3 t} - q\eta^2\lambda_3^2 e^{\eta\lambda_3 t}$. Thus,

$$\begin{aligned}
& D_5\underline{\psi}''(t) - c\underline{\psi}'(t) + f_5\left(\underline{\phi}, \underline{\psi}\right)(t) \\
=& D_5\underline{\psi}''(t) - c\underline{\psi}'(t) + (a_2 - b_2 - q_5 e_5)\underline{\psi}(t) - a_{55}\underline{\psi}^2(t) + \frac{\beta_1\underline{\phi}(t)\underline{\psi}(t)}{1 + m\underline{\phi}(t) + w\underline{\psi}(t)} \\
=& D_5\left(\lambda_3^2 e^{\lambda_3 t} - q\eta^2\lambda_3^2 e^{\eta\lambda_3 t}\right) - c\left(\lambda_3 e^{\lambda_3 t} - q\eta\lambda_3 e^{\eta\lambda_3 t}\right) + (a_2 - b_2 - q_5 e_5)\left(e^{\lambda_3 t} - q e^{\eta\lambda_3 t}\right) \\
& - a_{55}\left(e^{\lambda_3 t} - q e^{\eta\lambda_3 t}\right)^2 + \frac{\beta_1\underline{\phi}(t)\left(e^{\lambda_3 t} - q e^{\eta\lambda_3 t}\right)}{1 + m\underline{\phi}(t) + w\left(e^{\lambda_3 t} - q e^{\eta\lambda_3 t}\right)} \\
=& e^{\lambda_3 t}\left(D_5\lambda_3^2 - c\lambda_3 + a_2 - b_2 - q_5 e_5\right) - q e^{\eta\lambda_3 t}\left(D_5\eta^2\lambda_3^2 - c\eta\lambda_3 + a_2 - b_2 - q_5 e_5\right) \\
& - a_{55}\left(e^{\lambda_3 t} - q e^{\eta\lambda_3 t}\right)^2 + \frac{\beta_1\underline{\phi}(t)\left(e^{\lambda_3 t} - q e^{\eta\lambda_3 t}\right)}{1 + m\underline{\phi}(t) + w\left(e^{\lambda_3 t} - q e^{\eta\lambda_3 t}\right)}
\end{aligned}$$

$$
\begin{aligned}
&\geq -qe^{\eta\lambda_3 t}\Delta_2(\eta\lambda_3, c) - a_{55}e^{2\lambda_3 t}\\
&\geq e^{\eta\lambda_3 t}\left[-q\Delta_2(\eta\lambda_3, c) - a_{55}\right]\\
&> 0,
\end{aligned}
$$

here $\eta \in \left(1, min\left\{2, \frac{\lambda_2}{\lambda_1}, \frac{\lambda_4}{\lambda_3}, \frac{\lambda_1+\lambda_3}{\lambda_1}, \frac{\lambda_1+\lambda_3}{\lambda_3}\right\}\right)$ and $q > 1$is large enough .

If $t \geq t_4$, then $\underline{\psi}(t) = v^+ - \varepsilon_2 e^{-\lambda t}$, here $\underline{\phi}(t) \geq u^+ - \varepsilon_1 e^{-\lambda t}$, $\underline{\psi}'(t) = \lambda\varepsilon_2 v^+ e^{-\lambda t}$ and $\underline{\psi}''(t) = -\lambda^2\varepsilon_2 v^+ e^{-\lambda t}$. Thus,

$$
\begin{aligned}
&D_5\underline{\psi}''(t) - c\underline{\psi}'(t) + f_5\left(\underline{\phi}, \underline{\psi}\right)(t)\\
=&D_5\underline{\psi}''(t) - c\underline{\psi}'(t) + (a_2 - b_2 - q_5e_5)\underline{\psi}(t) - a_{55}\underline{\psi}^2(t) + \frac{\beta_1\underline{\phi}(t)\underline{\psi}(t)}{1 + m\underline{\phi}(t) + w\underline{\psi}(t)}\\
\geq&D_5\left(-\lambda^2\varepsilon_2 v^+ e^{-\lambda t}\right) - c\left(\lambda\varepsilon_2 v^+ e^{-\lambda t}\right) + (a_2 - b_2 - q_5e_5)\left(v^+ - \varepsilon_2 e^{-\lambda t}\right) - a_{55}\left(v^+ - \varepsilon_2 e^{-\lambda t}\right)^2\\
&+ \frac{\beta_1\left(u^+ - \varepsilon_1 e^{-\lambda t}\right)\left(v^+ - \varepsilon_2 e^{-\lambda t}\right)}{1 + m(u^+ - \varepsilon_1 e^{-\lambda t}) + w(v^+ - \varepsilon_2 e^{-\lambda t})}\\
=&-\varepsilon_2 e^{-\lambda t}\left(D_5\lambda^2 + c\lambda + a_2 - b_2 - q_5e_5\right) + v^+(a_2 - b_2 - q_5e_5 - a_{55}v^+) + 2a_{55}\varepsilon_2(v^+)^2 e^{-\lambda t} - a_{55}(\varepsilon_2)^2 e^{-2\lambda t}\\
&+ \frac{\beta_1\left(u^+ - \varepsilon_1 e^{-\lambda t}\right)\left(v^+ - \varepsilon_2 e^{-\lambda t}\right)}{1 + m(u^+ - \varepsilon_1 e^{-\lambda t}) + w(v^+ - \varepsilon_2 e^{-\lambda t})}\\
=&-\varepsilon_2 e^{-\lambda t}\Delta_2(-\lambda, c) + 2a_{55}\varepsilon_2(v^+)^2 e^{-\lambda t} - a_{55}(\varepsilon_2)^2 e^{-2\lambda t} - \frac{\beta_1 u^+ v^+}{1 + mu^+ + wv^+}\\
&+ \frac{\beta_1\left(u^+ - \varepsilon_1 e^{-\lambda t}\right)\left(v^+ - \varepsilon_2 e^{-\lambda t}\right)}{1 + m(u^+ - \varepsilon_1 e^{-\lambda t}) + w(v^+ - \varepsilon_2 e^{-\lambda t})}\\
=&\varepsilon_2 e^{-\lambda t}\left[-\Delta_2(-\lambda, c) + a_{55}v^+\right] + a_{55}\varepsilon_2(v^+)^2 e^{-\lambda t} - a_{55}(\varepsilon_2)^2 e^{-2\lambda t} - \frac{\beta_1 u^+ v^+}{1 + mu^+ + wv^+}\\
&+ \frac{\beta_1\left(u^+ - \varepsilon_1 e^{-\lambda t}\right)\left(v^+ - \varepsilon_2 e^{-\lambda t}\right)}{1 + m(u^+ - \varepsilon_1 e^{-\lambda t}) + w(v^+ - \varepsilon_2 e^{-\lambda t})}.
\end{aligned}
$$

Because of $-\Delta_2(0, c) + a_{55}v^+ = -\vartheta_2 + a_{55}v^+ = \frac{\beta_1 u^+}{1+mu^++wv^+} > 0$, there is a constant $\widetilde{\lambda}_4 > 0$, which makes $-\Delta_2(-\lambda, c) + a_{55}v^+ > 0$ for $\forall\lambda \in \left(0, \widetilde{\lambda}_4\right)$.

Let $I_4(\lambda, t) := a_{55}\varepsilon_2(v^+)^2 e^{-\lambda t} - a_{55}(\varepsilon_2)^2 e^{-2\lambda t} - \frac{\beta_1 u^+ v^+}{1+mu^++wv^+} + \frac{\beta_1\left(u^+-\varepsilon_1 e^{-\lambda t}\right)\left(v^+-\varepsilon_2 e^{-\lambda t}\right)}{1+m\left(u^+-\varepsilon_1 e^{-\lambda t}\right)+w\left(v^+-\varepsilon_2 e^{-\lambda t}\right)}$, where $t \geq t_4$. Therefore, $I_4(\lambda, 0) = -a_{55}\varepsilon_2^2 + a_{55}v^+\varepsilon_2 - \frac{\beta_1 u^+ v^+}{1+mu^++wv^+} + \frac{\beta_1\left(u^+-\varepsilon_1\right)\left(v^+-\varepsilon_2\right)}{1+m(u^+-\varepsilon_1)+w(v^+-\varepsilon_2)} > 0$ and $I_4(\lambda, +\infty) = 0$.

From Remark 3.1, we can see that $\vartheta_2 > \frac{(2\sqrt{2}-1)\beta_1 u^+}{1+mu^++wv^+}$, there are $\varepsilon_1 \in \left(0, \left(\sqrt{2} - 1\right)u^+\right)$ and $\varepsilon_2 \in \left(0, \frac{v^+}{2}\right)$, making $-a_{55}\varepsilon_2^2 + a_{55}v^+\varepsilon_2 - \frac{\beta_1 u^+ v^+}{1+mu^++wv^+} + \frac{\beta_1\left(u^+-\varepsilon_1\right)\left(v^+-\varepsilon_2\right)}{1+m(u^+-\varepsilon_1)+w(v^+-\varepsilon_2)} > \varepsilon_0 > 0$. We have that $I_4(\lambda, 0) > 0$, here $\varepsilon_1 \in \left(0, \left(\sqrt{2} - 1\right)u^+\right)$ and $\varepsilon_2 \in \left(0, \frac{v^+}{2}\right)$. We can choose a small enough $\delta_2 > 0$ such that $\delta^{**} := \varepsilon_2 + \delta_2$ for $\forall\delta \in [\varepsilon_2, \delta^{**}]$ satisfying

$$
-a_{55}\delta^2 + a_{55}v^+\delta - \frac{\beta_1 u^+ v^+}{1 + mu^+ + wv^+} + \frac{\beta_1(u^+ - \varepsilon_1)(v^+ - \delta)}{1 + m(u^+ - \varepsilon_1) + w(v^+ - \delta)} > 0.
$$

We want to prove that $I_4(\lambda, t) > 0$ for $\forall t \geq t_4$. Here $t \geq t_4$, $(t_4 < 0)$, therefore, t is divided into $t \in [t_4, 0]$ and $t > 0$ to discuss.

If $t \in (t_4, 0]$, let $\nu(t) := \varepsilon_2 e^{-\lambda t}$. Select $\widetilde{\lambda}_4 > 0$ small enough such that for any given $\lambda \in \left(0, \widetilde{\lambda}_4\right)$, we have $\nu(t_4) = \varepsilon_2 e^{-\lambda t_4} = \delta^{**}$, which leads to $\varepsilon_2 \le \nu(t) \le \delta^{**}$. So we get $I_4(\lambda, t) > 0$ for $\forall t \in (t_4, 0]$.

If $t \ge 0$, here $\frac{\beta_1\left(u^+ - \varepsilon_1 e^{-\lambda t}\right)\left(v^+ - \varepsilon_2 e^{-\lambda t}\right)}{1 + m\left(u^+ - \varepsilon_1 e^{-\lambda t}\right) + w\left(v^+ - \varepsilon_2 e^{-\lambda t}\right)} > 0$, according to the assumption that $\lambda > 0$ is small enough, we have the function $\frac{\beta_1\left(u^+ - \varepsilon_1 e^{-\lambda t}\right)\left(v^+ - \varepsilon_2 e^{-\lambda t}\right)}{1 + m\left(u^+ - \varepsilon_1 e^{-\lambda t}\right) + w\left(v^+ - \varepsilon_2 e^{-\lambda t}\right)}$ is increasing about the variable t on $t \ge 0$; the function $a_{55}\varepsilon_2 \left(v^+\right)^2 e^{-\lambda t} - a_{55} \left(\varepsilon_2\right)^2 e^{-2\lambda t} > 0$ is decreasing about the variable t on $t \ge 0$. We can get

$$I_4(\lambda, 0) > 0, \qquad I_4(\lambda, +\infty) = 0.$$

So we get $I_4(\lambda, t) > 0$ for $\forall t \ge 0$. Thus, $I_4(\lambda, t) > 0$ for $\forall t > t_4$. In consequence, $\underline{\psi}$ satisfies the following solution definition, that is, $D_5 \underline{\psi}''(t) - c\underline{\psi}'(t) + f_5\left(\underline{\phi}, \underline{\psi}\right)(t) \ge 0$.

Let $\widetilde{\lambda} = min\left\{\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{\lambda}_3, \widetilde{\lambda}_4\right\}$, select $\lambda \in \left(0, \widetilde{\lambda}\right)$, so that $\left(\overline{\phi}(t), \overline{\psi}(t)\right)$ and $\left(\underline{\phi}(t), \underline{\psi}(t)\right)$ satisfy the upper and lower solution definition Definition 3.1, which proves the existence of the upper and lower solutions of the system (3.1). □

3.2. Existence of traveling wave solutions

For $\mu > 0$, define

$$B_\mu\left(\mathbb{R}, \mathbb{R}^2\right) = \left\{(\phi, \psi) \in C\left(\mathbb{R}, \mathbb{R}^2\right) : \sup_{t \in \mathbb{R}} |(\phi, \psi)(t)| e^{-\mu|t|} < \infty\right\}$$

and

$$|(\phi, \psi)|_\mu = \sup_{t \in \mathbb{R}} |(\phi, \psi)(t)| e^{-\mu|t|},$$

it is easy to know that $\left(B_\mu\left(\mathbb{R}, \mathbb{R}^2\right), |\cdot|_\mu\right)$ is a Banach space.

Define $\Omega = \left\{(\phi, \psi) \in C\left(\mathbb{R}, \mathbb{R}^2\right) : 0 \le \phi(t) \le N_1, 0 \le \psi(t) \le N_2, t \in \mathbb{R}\right\}$, let η_1 and η_2 be two constants, satisfying

$$\eta_1 \ge d_2 + q_2 e_2 + 2a_{12} N_1 + \beta N_2 (1 + \omega N_2), \quad \eta_2 \ge 2a_{55} N_2 + b_2 + q_5 e_5 - a_2. \tag{3.4}$$

Define the operator $\boldsymbol{H} = (H_1, H_2) : \Omega \longrightarrow C\left(\mathbb{R}, \mathbb{R}^2\right)$ as

$$H_1(\phi, \psi)(t) = f_2(\phi, \psi)(t) + \eta_1 \phi(t),$$
$$H_2(\phi, \psi)(t) = f_5(\phi, \psi)(t) + \eta_2 \psi(t),$$

the system (3.1) becomes the following form

$$\begin{cases} D_2 \phi''(t) - c\phi'(t) - \eta_1 \phi(t) + H_1(\phi, \psi)(t) = 0, \\ D_5 \psi''(t) - c\psi'(t) - \eta_2 \psi(t) + H_2(\phi, \psi)(t) = 0. \end{cases} \tag{3.5}$$

Let

$$\lambda_{11} = \frac{c - \sqrt{c^2 + 4\eta_1 D_2}}{2D_2} < 0, \qquad \lambda_{12} = \frac{c + \sqrt{c^2 + 4\eta_1 D_2}}{2D_2} > 0,$$

$$\lambda_{21}=\frac{c-\sqrt{c^2+4\eta_2 D_5}}{2D_5}<0,\qquad \lambda_{22}=\frac{c+\sqrt{c^2+4\eta_2 D_5}}{2D_5}>0.$$

In this paper, we take the above μ to satisfy $0<\mu<\min\{-\lambda_{11},\lambda_{12},-\lambda_{21},\lambda_{22}\}$.

Define the operator $\boldsymbol{F}=(F_1,F_2):\Omega\longrightarrow C\left(\mathbb{R},\mathbb{R}^2\right)$ as

$$\begin{aligned}
F_1(\phi,\psi)(t)&=\frac{1}{D_2(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{t}e^{\lambda_{11}(t-s)}H_1(\phi,\psi)(s)\,ds+\int_{t}^{+\infty}e^{\lambda_{12}(t-s)}H_1(\phi,\psi)(s)\,ds\right],\\
F_2(\phi,\psi)(t)&=\frac{1}{D_5(\lambda_{22}-\lambda_{21})}\left[\int_{-\infty}^{t}e^{\lambda_{21}(t-s)}H_2(\phi,\psi)(s)\,ds+\int_{t}^{+\infty}e^{\lambda_{22}(t-s)}H_2(\phi,\psi)(s)\,ds\right].
\end{aligned}$$

Define the set $\Gamma=\left\{(\phi,\psi)\in\Omega:\left(\underline{\phi}(t),\underline{\psi}(t)\right)\le(\phi(t),\psi(t))\le\left(\overline{\phi}(t),\overline{\psi}(t)\right)\right\}$. Obviously, Γ is not empty and is a bounded closed convex set. The operator $\boldsymbol{F}=(F_1,F_2)$ for $\forall(\phi,\psi)\in\Gamma$ satisfying

$$\begin{cases}
D_2F_1(\phi,\psi)''(t)-cF_1(\phi,\psi)'(t)-\eta_1F_1(\phi,\psi)(t)+H_1(\phi,\psi)(t)=0,\\
D_5F_2(\phi,\psi)''(t)-cF_2(\phi,\psi)'(t)-\eta_2F_2(\phi,\psi)(t)+H_2(\phi,\psi)(t)=0.
\end{cases}$$

The fixed point of $\boldsymbol{F}$ is the solution of system (3.5), which is the solution of system (3.1).

Previously, we find a pair of upper and lower solutions $\left(\overline{\phi}(t),\overline{\psi}(t)\right)$ and $\left(\underline{\phi}(t),\underline{\psi}(t)\right)$ of the system (3.1) satisfying properties $P1$), $P2$) and $P3$). We will find the traveling wave solutions of the system (3.1) in the profile set Γ.

Lemma 3.5. *For sufficiently large η_1 and η_2 satisfying* (3.4)*, we have*

$$\begin{aligned}
&H_1(\phi_1,\psi_1)(t)\ge H_1(\phi_2,\psi_1)(t), && H_1(\phi_1,\psi_1)(t)\le H_1(\phi_1,\psi_2)(t),\\
&H_2(\phi_1,\psi_1)(t)\ge H_2(\phi_2,\psi_1)(t), && H_2(\phi_1,\psi_1)(t)\ge H_2(\phi_1,\psi_2)(t),
\end{aligned}$$

for $t\in\mathbb{R}$ with $0\le\phi_2(t)\le\phi_1(t)\le N_1, 0\le\psi_2(t)\le\psi_1(t)\le N_2$.

Proof. According to the definition of operator $\boldsymbol{H}=(H_1,H_2)$, we know that

$$\begin{aligned}
H_1(\phi,\psi)(t)=&f_2(\phi,\psi)(t)+\eta_1\phi(t)\\
=&\hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\phi(t-y-c\tau)\,dy-(d_2+q_2e_2)\phi(t)-a_{12}\phi^2(t)\\
&-\frac{\beta\phi(t)\psi(t)}{1+m\phi(t)+w\psi(t)}+\eta_1\phi(t),\\
H_2(\phi,\psi)(t)=&f_5(\phi,\psi)(t)+\eta_2\psi(t)\\
=&(a_2-b_2-q_5e_5)\psi(t)-a_{55}\psi^2(t)+\frac{\beta_1\phi(t)\psi(t)}{1+m\phi(t)+w\psi(t)}+\eta_2\psi(t).
\end{aligned}$$

The derivative of $H_2(\phi,\psi)(t)$ with respect to variable ϕ is obtained

$$\frac{\partial H_2(\phi,\psi)}{\partial\phi}=\frac{\beta_1\psi(1+\omega\psi)}{[1+m\phi+w\psi]^2}>0,$$

the derivative $\frac{\partial H_2(\phi,\psi)}{\partial\phi}>0$, so that H_2 is increasing with respect to the variable ϕ.

The derivative of $H_2(\phi,\psi)(t)$ with respect to variable ψ is obtained

$$\frac{\partial H_2(\phi,\psi)}{\partial\psi}=a_2-b_2-q_5e_5-2a_{55}\psi(t)+\frac{\beta_1\phi(1+m\phi)}{[1+m\phi+w\psi]^2}+\eta_2,$$

Since (3.4) knows $\eta_2\geq 2a_{55}N_2+b_2+q_5e_5-a_2>2a_{55}\psi(t)+b_2+q_5e_5-a_2-\frac{\beta_1\phi(1+m\phi)}{[1+m\phi+w\psi]^2}$, the derivative $\frac{\partial H_2(\phi,\psi)}{\partial\psi}>0$, thus H_2 is increasing with respect to the variable ψ.

The derivative of $H_1(\phi,\psi)(t)$ with respect to variable ψ is obtained

$$\frac{\partial H_1(\phi,\psi)}{\partial\psi}=\frac{-\beta\phi(1+m\phi)}{[1+m\phi+w\psi]^2}<0,$$

the derivative $\frac{\partial H_1(\phi,\psi)}{\partial\psi}<0$, so that $H_1$is decreasing with respect to the variable ψ.

The derivative of $H_1(\phi,\psi)(t)$ with respect to variable ϕ is obtained

$$\frac{\partial H_1(\phi,\psi)}{\partial\phi}=\left[\hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\phi(t-y-c\tau)\,dy\right]_\phi'-d_2-q_2e_2-2a_{12}\phi(t)-\frac{\beta\psi(1+\omega\psi)}{[1+m\phi+w\psi]^2}+\eta_1,$$

where $\left[\hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\phi(t-y-c\tau)\,dy\right]_\phi'>0$, $0\leq\phi(t)\leq N_1$, $0\leq\psi(t)\leq N_2$ and $\frac{\beta\psi(1+\omega\psi)}{[1+m\phi+w\psi]^2}\leq\beta\psi(1+\omega\psi)\leq\beta N_2(1+\omega N_2)$. Since (3.4) knows $\eta_1\geq d_2+q_2e_2+2a_{12}N_1+\beta N_2(1+\omega N_2)>d_2+q_2e_2+2a_{12}\phi(t)+\frac{\beta\psi(1+\omega\psi)}{[1+m\phi+w\psi]^2}-\left[\hat{a}e^{-d_1\tau}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{4\pi D_1\tau}}e^{-\frac{y^2}{4D_1\tau}}\phi(t-y-c\tau)\,dy\right]_\phi$, the derivative $\frac{\partial H_1(\phi,\psi)}{\partial\phi}>0$, thus H_1 is increasing with respect to the variable ϕ.

In consequence, H_1 is increasing with respect to the variable ϕ and is decreasing with respect to the variable ψ; H_2 is increasing with respect to the variable ϕ and is increasing with respect to the variable ψ. The above lemma holds. □

Lemma 3.6. *For sufficiently large η_1 and η_2 satisfying* (3.4)*, we have*

$$F_1(\phi_1,\psi_1)(t)\geq F_1(\phi_2,\psi_1)(t),\qquad F_1(\phi_1,\psi_1)(t)\leq F_1(\phi_1,\psi_2)(t),$$
$$F_2(\phi_1,\psi_1)(t)\geq F_2(\phi_2,\psi_1)(t),\qquad F_2(\phi_1,\psi_1)(t)\geq F_2(\phi_1,\psi_2)(t),$$

for $t\in\mathbb{R}$ with $0\leq\phi_2(t)\leq\phi_1(t)\leq N_1, 0\leq\psi_2(t)\leq\psi_1(t)\leq N_2$.

Proof. According to the definition of the operator $\boldsymbol{F}=(F_1,F_2)$ and the lemma 3.5, we know that

$$\begin{aligned}&F_1(\phi_1,\psi_1)(t)-F_1(\phi_2,\psi_1)(t)\\=&\frac{1}{D_2(\lambda_{12}-\lambda_{11})}\left\{\int_{-\infty}^{t}e^{\lambda_{11}(t-s)}[H_1(\phi_1,\psi_1)(s)-H_1(\phi_2,\psi_1)(s)]\,ds\right.\\&\left.+\int_t^{+\infty}e^{\lambda_{12}(t-s)}[H_1(\phi_1,\psi_1)(s)-H_1(\phi_2,\psi_1)(s)]\,ds\right\}\\\geq&0,\end{aligned}$$

thus $F_1(\phi_1,\psi_1)(t)\geq F_1(\phi_2,\psi_1)(t)$. Similarly, other conclusions can be proved. The above lemma is established. □

Lemma 3.7. $\boldsymbol{F} : \Gamma \longrightarrow \Gamma$ *is completely continuous.*

Proof. The proof process is divided into three parts.

Step1. $\boldsymbol{F} = (F_1, F_2)$ is continuous with respect to the norm $|\cdot|_\mu$ on $B_\mu\left(\mathbb{R}, \mathbb{R}^2\right)$. We first prove the continuity of $\boldsymbol{H}$.

We can notice that

$$\begin{aligned}\int_{-\infty}^{+\infty} G(\tau, y)\, e^{\mu|y+c\tau|} dy &\leq \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi D_1 \tau}} e^{-\frac{y^2}{4D_1\tau}} e^{\mu(|y|+c\tau)} dy \\ &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi D_1 \tau}} e^{-\frac{(|y|-2D_1\mu\tau)^2}{4D_1\tau}} e^{\left(D_1\mu^2+c\mu\right)\tau} dy \\ &= e^{\left(D_1\mu^2+c\mu\right)\tau}.\end{aligned}$$

If $\Phi = (\phi_1, \psi_1), \Psi = (\phi_2, \psi_2) \in B_\mu\left(\mathbb{R}, \mathbb{R}^2\right)$. We get

$$\begin{aligned}&\left|H_1(\phi_1, \psi_1)(t) - H_1(\phi_2, \psi_2)(t)\right| e^{-\mu|t|} \\ =& \left|f_2(\phi_1, \psi_1)(t) - f_2(\phi_2, \psi_2)(t) + \eta_1\left[\phi_1(t) - \phi_2(t)\right]\right| e^{-\mu|t|} \\ \leq& \left|f_2(\phi_1, \psi_1)(t) - f_2(\phi_2, \psi_2)(t)\right| e^{-\mu|t|} + \eta_1\left|\phi_1(t) - \phi_2(t)\right| e^{-\mu|t|} \\ =& \left|\hat{a} e^{-d_1\tau} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi D_1\tau}} e^{-\frac{y^2}{4D_1\tau}} \left[\phi_1(t-y-c\tau) - \phi_2(t-y-c\tau)\right] dy - (d_2 + q_2 e_2)\left[\phi_1(t) - \phi_2(t)\right]\right. \\ &\left. - a_{12}\left[\phi_1^2(t) - \phi_2^2(t)\right] - \left[\frac{\beta\phi_1(t)\psi_1(t)}{1 + m\phi_1(t) + w\psi_1(t)} - \frac{\beta\phi_2(t)\psi_2(t)}{1 + m\phi_2(t) + w\psi_2(t)}\right]\right| e^{-\mu|t|} + \eta_1\left|\phi_1(t) - \phi_2(t)\right| e^{-\mu|t|} \\ \leq& \left\{\hat{a} e^{-d_1\tau} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi D_1\tau}} e^{-\frac{y^2}{4D_1\tau}} \left|\phi_1(t-y-c\tau) - \phi_2(t-y-c\tau)\right| dy + (d_2 + q_2 e_2)\left|\phi_1(t) - \phi_2(t)\right|\right. \\ &\left. + a_{12}\left|\phi_1(t) + \phi_2(t)\right|\left|\phi_1(t) - \phi_2(t)\right| + \left|\frac{\beta\phi_1(t)\psi_1(t)}{1 + m\phi_1(t) + w\psi_1(t)} - \frac{\beta\phi_2(t)\psi_2(t)}{1 + m\phi_2(t) + w\psi_2(t)}\right|\right\} e^{-\mu|t|} \\ &+ \eta_1\left|\phi_1(t) - \phi_2(t)\right| e^{-\mu|t|} \\ \leq& \hat{a} e^{-d_1\tau} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi D_1\tau}} e^{-\frac{y^2}{4D_1\tau}} e^{\mu|y+c\tau|} dy \left|\phi_1(t) - \phi_2(t)\right|_\mu + (d_2 + q_2 e_2)|\Phi - \Psi|_\mu + 2a_{12}N_1|\Phi - \Psi|_\mu \\ &+ \left(\beta N_1 + m\beta N_1^2 + \beta N_2 + m\beta N_2^2\right)|\Phi - \Psi|_\mu + \eta_1|\Phi - \Psi|_\mu \\ \leq& \kappa_1|\Phi - \Psi|_\mu,\end{aligned}$$

where $\kappa_1 = \hat{a} e^{-d_1\tau} e^{\left(D_1\mu^2+c\mu\right)\tau} + d_2 + q_2 e_2 + 2a_{12}N_1 + \beta N_1 + m\beta N_1^2 + \beta N_2 + m\beta N_2^2 + \eta_1$. Thus, $H_1 : B_\mu\left(\mathbb{R}, \mathbb{R}^2\right) \longrightarrow B_\mu\left(\mathbb{R}, \mathbb{R}^2\right)$ is continuous with respect to the norm $|\cdot|_\mu$.

Similarly, we can prove that $H_2 : B_\mu\left(\mathbb{R}, \mathbb{R}^2\right) \longrightarrow B_\mu\left(\mathbb{R}, \mathbb{R}^2\right)$ is continuous with respect to the norm $|\cdot|_\mu$.

$$\begin{aligned}&\left|H_2(\phi_1, \psi_1)(t) - H_2(\phi_2, \psi_2)(t)\right| e^{-\mu|t|} \\ =& \left|f_5(\phi_1, \psi_1)(t) - f_5(\phi_2, \psi_2)(t) + \eta_2\left[\phi_1(t) - \phi_2(t)\right]\right| e^{-\mu|t|} \\ \leq& \left|f_5(\phi_1, \psi_1)(t) - f_5(\phi_2, \psi_2)(t)\right| e^{-\mu|t|} + \eta_2\left|\phi_1(t) - \phi_2(t)\right| e^{-\mu|t|} \\ =& \left|(a_2 - b_2 - q_5 e_5)\left[\phi_1(t) - \phi_2(t)\right] - a_{55}\left[\psi_1^2(t) - \psi_2^2(t)\right]\right.\end{aligned}$$

$$+\left[\frac{\beta_1\phi_1(t)\psi_1(t)}{1+m\phi_1(t)+w\psi_1(t)}-\frac{\beta_1\phi_2(t)\psi_2(t)}{1+m\phi_2(t)+w\psi_2(t)}\right]\Bigg| e^{-\mu|t|}+\eta_2|\phi_1(t)-\phi_2(t)|e^{-\mu|t|}$$

$$\leq\{(a_2+b_2+q_5e_5)|\phi_1(t)-\phi_2(t)|+a_{55}|\phi_1(t)+\phi_2(t)||\phi_1(t)-\phi_2(t)|$$

$$+\left|\frac{\beta_1\phi_1(t)\psi_1(t)}{1+m\phi_1(t)+w\psi_1(t)}-\frac{\beta_1\phi_2(t)\psi_2(t)}{1+m\phi_2(t)+w\psi_2(t)}\right|\right\} e^{-\mu|t|}+\eta_2|\phi_1(t)-\phi_2(t)|e^{-\mu|t|}$$

$$\leq(a_2+b_2+q_5e_5)|\Phi-\Psi|_\mu+2a_{55}N_2|\Phi-\Psi|_\mu+\left(\beta_1N_1+m\beta_1N_1^2+\beta_1N_2+m\beta_1N_2^2\right)|\Phi-\Psi|_\mu+\eta_2|\Phi-\Psi|_\mu$$

$$\leq\kappa_2|\Phi-\Psi|_\mu,$$

where $\kappa_2=a_2+b_2+q_5e_5+2a_{55}N_2+\beta_1N_1+m\beta_1N_1^2+\beta_1N_2+m\beta_1N_2^2+\eta_2$. Thus $H_2:B_\mu\left(\mathbb{R},\mathbb{R}^2\right)\longrightarrow B_\mu\left(\mathbb{R},\mathbb{R}^2\right)$ is continuous with respect to the norm $|\cdot|_\mu$.

Next we prove that F_1 and F_2 are continuous with respect to the norm $|\cdot|_\mu$ on $B_\mu\left(\mathbb{R},\mathbb{R}^2\right)$. Because $t\in\mathbb{R}$, we divide it into $t>0$ and $t\leq 0$ for discussion.

For $t>0$, due to $|H_1(\phi_1,\psi_1)(t)-H_1(\phi_2,\psi_2)(t)|e^{-\mu|t|}\leq\kappa_1|\Phi-\Psi|_\mu$, thus

$$\begin{aligned}
&|F_1(\phi_1,\psi_1)(t)-F_1(\phi_2,\psi_2)(t)|e^{-\mu|t|}\\
=&\frac{e^{-\mu t}}{D_2(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{t}e^{\lambda_{11}(t-s)}+\int_{t}^{+\infty}e^{\lambda_{12}(t-s)}\right]|H_1(\phi_1,\psi_1)(s)-H_1(\phi_2,\psi_2)(s)|ds\\
\leq&\frac{\kappa_1e^{-\mu t}}{D_2(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{0}e^{\lambda_{11}(t-s)}+\int_{0}^{t}e^{\lambda_{11}(t-s)}+\int_{t}^{+\infty}e^{\lambda_{12}(t-s)}\right]e^{\mu|s|}ds\,|\Phi-\Psi|_\mu\\
=&\frac{\kappa_1}{D_2(\lambda_{12}-\lambda_{11})}\left[e^{(\lambda_{11}-\mu)t}\int_{-\infty}^{0}e^{(-\lambda_{11}-\mu)s}ds+e^{(\lambda_{11}-\mu)t}\int_{0}^{t}e^{(-\lambda_{11}+\mu)s}ds+e^{(\lambda_{12}-\mu)t}\int_{t}^{+\infty}e^{(-\lambda_{12}+\mu)s}ds\right]|\Phi-\Psi|_\mu\\
=&\frac{\kappa_1}{D_2(\lambda_{12}-\lambda_{11})}\left[\frac{2\mu}{\lambda_{11}^2-\mu^2}e^{(\lambda_{11}-\mu)t}+\frac{\lambda_{12}-\lambda_{11}}{(\mu-\lambda_{11})(\lambda_{12}-\mu)}\right]|\Phi-\Psi|_\mu\\
\leq&\frac{\kappa_1}{D_2(\lambda_{12}-\lambda_{11})}\left[\frac{2\mu}{\lambda_{11}^2-\mu^2}+\frac{\lambda_{12}-\lambda_{11}}{(\mu-\lambda_{11})(\lambda_{12}-\mu)}\right]|\Phi-\Psi|_\mu.
\end{aligned}$$

For $t\leq 0$, due to $|H_1(\phi_1,\psi_1)(t)-H_1(\phi_2,\psi_2)(t)|e^{-\mu|t|}\leq\kappa_1|\Phi-\Psi|_\mu$, thus

$$\begin{aligned}
&|F_1(\phi_1,\psi_1)(t)-F_1(\phi_2,\psi_2)(t)|e^{-\mu|t|}\\
=&\frac{e^{-\mu(-t)}}{D_2(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{t}e^{\lambda_{11}(t-s)}+\int_{t}^{+\infty}e^{\lambda_{12}(t-s)}\right]|H_1(\phi_1,\psi_1)(s)-H_1(\phi_2,\psi_2)(s)|ds\\
\leq&\frac{\kappa_1e^{\mu t}}{D_2(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{t}e^{\lambda_{11}(t-s)}+\int_{t}^{0}e^{\lambda_{12}(t-s)}+\int_{0}^{+\infty}e^{\lambda_{12}(t-s)}\right]e^{\mu|s|}ds\,|\Phi-\Psi|_\mu\\
=&\frac{\kappa_1}{D_2(\lambda_{12}-\lambda_{11})}\left[e^{(\lambda_{11}+\mu)t}\int_{-\infty}^{t}e^{(-\lambda_{11}-\mu)s}ds+e^{(\lambda_{12}+\mu)t}\int_{t}^{0}e^{(-\lambda_{12}-\mu)s}ds+e^{(\lambda_{12}+\mu)t}\int_{0}^{+\infty}e^{(-\lambda_{12}+\mu)s}ds\right]|\Phi-\Psi|_\mu\\
=&\frac{\kappa_1}{D_2(\lambda_{12}-\lambda_{11})}\left[\frac{2\mu}{\lambda_{12}^2-\mu^2}e^{(\lambda_{12}+\mu)t}+\frac{\lambda_{11}-\lambda_{12}}{(\lambda_{11}+\mu)(\lambda_{12}+\mu)}\right]|\Phi-\Psi|_\mu\\
\leq&\frac{\kappa_1}{D_2(\lambda_{12}-\lambda_{11})}\left[\frac{2\mu}{\lambda_{12}^2-\mu^2}-\frac{\lambda_{12}-\lambda_{11}}{(\lambda_{11}+\mu)(\lambda_{12}+\mu)}\right]|\Phi-\Psi|_\mu.
\end{aligned}$$

Therefore, $F_1:B_\mu\left(\mathbb{R},\mathbb{R}^2\right)\longrightarrow B_\mu\left(\mathbb{R},\mathbb{R}^2\right)$ is continuous with respect to the norm $|\cdot|_\mu$ on $B_\mu\left(\mathbb{R},\mathbb{R}^2\right)$.

Similarly, $F_2 : B_\mu\left(\mathbb{R},\mathbb{R}^2\right) \longrightarrow B_\mu\left(\mathbb{R},\mathbb{R}^2\right)$ is continuous with respect to the norm $|\cdot|_\mu$ on $B_\mu\left(\mathbb{R},\mathbb{R}^2\right)$.

Step2. $\boldsymbol{F}(\Gamma) \subset \Gamma$, that is, for any $(\phi,\psi) \in \Gamma$, we have $\boldsymbol{F}(\phi,\psi) \in \Gamma$.

Since $\left(\underline{\phi}(t), \underline{\psi}(t)\right) \le (\phi(t), \psi(t)) \le \left(\overline{\phi}(t), \overline{\psi}(t)\right)$, according to the Lemma 3.6 can get

$$
\begin{aligned}
F_1\left(\underline{\phi},\overline{\psi}\right) &\le F_1(\phi,\psi) \le F_1\left(\overline{\phi},\underline{\psi}\right),\\
F_2\left(\underline{\phi},\underline{\psi}\right) &\le F_2(\phi,\psi) \le F_2\left(\overline{\phi},\overline{\psi}\right).
\end{aligned}
$$

Next we prove that $F_1\left(\overline{\phi},\underline{\psi}\right) \le \overline{\phi}$.

Without losing generality, we assume the finite point set $S = \{s_i \in \mathbb{R}, i = 1,2,\ldots,n\}$, where $s_1 < s_2 < \cdots < s_n$, and define $s_0 = 0$, $s_{n+1} = +\infty$.

According to the definition of Definition 3.1, we have $H_1\left(\overline{\phi},\underline{\psi}\right)(t) \le -D_2\overline{\phi}''(t) + c\overline{\phi}'(t) + \eta_1\overline{\phi}(t)$ for $t \in \mathbb{R}\backslash S$.

Due to the properties $P3$) of $\left(\overline{\phi}(t), \overline{\psi}(t)\right)$ and $\left(\underline{\phi}(t), \underline{\psi}(t)\right)$, that is, $\overline{\phi'}(t+) \le \overline{\phi'}(t-)$ and $\underline{\phi'}(t+) \ge \underline{\phi'}(t-)$ for $\forall t \in \mathbb{R}$. Then,

$$
\begin{aligned}
F_1\left(\overline{\phi},\underline{\psi}\right)(t) =& \frac{1}{D_2(\lambda_{12}-\lambda_{11})}\left[\int_{-\infty}^{t} e^{\lambda_{11}(t-s)} + \int_{t}^{+\infty} e^{\lambda_{12}(t-s)}\right] H_1\left(\overline{\phi},\underline{\psi}\right) ds\\
=& \frac{1}{D_2(\lambda_{12}-\lambda_{11})} \times \sum_{j=0}^{n}\int_{s_j}^{s_{j+1}} \min\left\{e^{\lambda_{11}(t-s)}, e^{\lambda_{12}(t-s)}\right\} H_1\left(\overline{\phi},\underline{\psi}\right)(t)\, ds\\
\le& \frac{1}{D_2(\lambda_{12}-\lambda_{11})} \times \sum_{j=0}^{n}\int_{s_j}^{s_{j+1}} \min\left\{e^{\lambda_{11}(t-s)}, e^{\lambda_{12}(t-s)}\right\} \left[-D_2\overline{\phi}''(t) + c\overline{\phi}'(t) + \eta_1\overline{\phi}(t)\right] ds\\
=& \overline{\phi}(t) + \frac{1}{\lambda_{12}-\lambda_{11}} \times \left\{\sum_{j=0}^{n}\int_{s_j}^{s_{j+1}} \min\left\{e^{\lambda_{11}(t-s)}, e^{\lambda_{12}(t-s)}\right\}\left[\overline{\phi}'\left(s_{j+}\right) - \overline{\phi}'\left(s_{j-}\right)\right]\right\}\\
\le& \overline{\phi}(t).
\end{aligned}
$$

In fact, through the continuity of $F_1\left(\overline{\phi},\underline{\psi}\right)(t)$ and $\overline{\phi}(t)$, the above inequality holds for $t \in \mathbb{R}$.

Similarly, we can get $F_1\left(\underline{\phi},\overline{\psi}\right) \ge \underline{\phi}$, $F_2\left(\underline{\phi},\underline{\psi}\right) \ge \underline{\psi}$, $F_2\left(\overline{\phi},\overline{\psi}\right) \le \overline{\psi}$, then $\underline{\phi} \le F_1(\phi,\psi) \le \overline{\phi}$, $\underline{\psi} \le F_2(\phi,\psi) \le \overline{\psi}$. Therefore, $\boldsymbol{F}(\phi,\psi) \in \Gamma$ for $\forall(\phi,\psi) \in \Gamma$.

Step3. $\boldsymbol{F} : \Gamma \longrightarrow \Gamma$ is compact.

For any $(\phi,\psi) \in \Gamma$,

$$
F_2'(\phi,\psi)(t) = \frac{\lambda_{21}e^{\lambda_{21}t}}{D_5(\lambda_{22}-\lambda_{21})}\int_{-\infty}^{t} e^{-\lambda_{21}-s} H_2(\phi,\psi)(s)\, ds + \frac{\lambda_{22}e^{\lambda_{22}t}}{D_5(\lambda_{22}-\lambda_{21})}\int_{t}^{+\infty} e^{-\lambda_{22}s} H_2(\phi,\psi)(s)\, ds.
$$

Therefore, we have

$$
\begin{aligned}
\left|F_2'(\phi,\psi)(t)\right|_\mu =& \sup_{t\in\mathbb{R}}\left|\frac{\lambda_{21}e^{\lambda_{21}t}}{D_5(\lambda_{22}-\lambda_{21})}\int_{-\infty}^{t} e^{-\lambda_{21}s} H_2(\phi,\psi)(s)\, ds\right.\\
&\left.+\frac{\lambda_{22}e^{\lambda_{22}t}}{D_5(\lambda_{22}-\lambda_{21})}\int_{t}^{+\infty} e^{-\lambda_{22}s} H_2(\phi,\psi)(s)\, ds\right| e^{-\mu|t|}\\
\le& \frac{|\lambda_{21}|}{D_5(\lambda_{22}-\lambda_{21})}\sup_{t\in\mathbb{R}}\left\{e^{\lambda_{21}t-\mu|t|}\int_{-\infty}^{t} e^{-\lambda_{21}s} e^{\mu|s|} e^{-\mu|s|} H_2(\phi,\psi)(s)\, ds\right\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\lambda_{22}}{D_5(\lambda_{22}-\lambda_{21})}\sup_{t\in\mathbb{R}}\left\{e^{\lambda_{22}t-\mu|t|}\int_t^{+\infty}e^{-\lambda_{22}s}e^{\mu|s|}e^{-\mu|s|}H_2(\phi,\psi)(s)\,ds\right\}\\
\leq&\frac{|\lambda_{21}|}{D_5(\lambda_{22}-\lambda_{21})}|H_2(\phi,\psi)|_\mu\sup_{t\in\mathbb{R}}\left\{e^{\lambda_{21}t-\mu|t|}\int_{-\infty}^{t}e^{-\lambda_{21}s}e^{\mu|s|}ds\right\}\\
&+\frac{\lambda_{22}}{D_5(\lambda_{22}-\lambda_{21})}|H_2(\phi,\psi)|_\mu\sup_{t\in\mathbb{R}}\left\{e^{\lambda_{22}t-\mu|t|}\int_t^{+\infty}e^{-\lambda_{22}s}e^{\mu|s|}ds\right\}.
\end{aligned}
$$

If $t>0$, then

$$
\begin{aligned}
\left|F_2'(\phi,\psi)(t)\right|_\mu\leq&\frac{|\lambda_{21}|}{D_5(\lambda_{22}-\lambda_{21})}|H_2(\phi,\psi)|_\mu\sup_{t\in\mathbb{R}}\left\{e^{(\lambda_{21}-\mu)t}\int_{-\infty}^{t}e^{-\lambda_{21}s}e^{\mu|s|}ds\right\}\\
&+\frac{\lambda_{22}}{D_5(\lambda_{22}-\lambda_{21})}|H_2(\phi,\psi)|_\mu\sup_{t\in\mathbb{R}}\left\{e^{(\lambda_{22}-\mu)t}\int_t^{+\infty}e^{-\lambda_{22}s}e^{\mu|s|}ds\right\}\\
=&\frac{|\lambda_{21}|}{D_5(\lambda_{22}-\lambda_{21})}|H_2(\phi,\psi)|_\mu\sup_{t\in\mathbb{R}}\left\{e^{(\lambda_{21}-\mu)t}\left[\int_{-\infty}^{0}e^{(-\lambda_{21}-\mu)s}ds+\int_0^{t}e^{(\mu-\lambda_{21})s}ds\right]\right\}\\
&+\frac{\lambda_{22}}{D_5(\lambda_{22}-\lambda_{21})}|H_2(\phi,\psi)|_\mu\sup_{t\in\mathbb{R}}\left\{e^{(\lambda_{22}-\mu)t}\int_t^{+\infty}e^{(\mu-\lambda_{22})s}ds\right\}\\
=&\frac{|\lambda_{21}|}{D_5(\lambda_{22}-\lambda_{21})}|H_2(\phi,\psi)|_\mu\sup_{t\in\mathbb{R}}\left\{\frac{e^{(\lambda_{21}-\mu)t}}{-\lambda_{21}-\mu}+\frac{1-e^{(\lambda_{21}-\mu)t}}{\mu-\lambda_{21}}\right\}\\
&+\frac{\lambda_{22}}{D_5(\lambda_{22}-\lambda_{21})}|H_2(\phi,\psi)|_\mu\sup_{t\in\mathbb{R}}\left\{\frac{1}{\lambda_{22}-\mu}\right\}\\
\leq&\frac{1}{D_5(\lambda_{22}-\lambda_{21})}\left[\frac{\lambda_{21}}{\lambda_{21}+\mu}+\frac{\lambda_{22}}{\lambda_{22}-\mu}\right]|H_2(\phi,\psi)|_\mu.
\end{aligned}
$$

If $t<0$, then

$$
\begin{aligned}
\left|F_2'(\phi,\psi)(t)\right|_\mu\leq&\frac{|\lambda_{21}|}{D_5(\lambda_{22}-\lambda_{21})}|H_2(\phi,\psi)|_\mu\sup_{t\in\mathbb{R}}\left\{e^{(\lambda_{21}+\mu)t}\int_{-\infty}^{t}e^{-\lambda_{21}s}e^{\mu|s|}ds\right\}\\
&+\frac{\lambda_{22}}{D_5(\lambda_{22}-\lambda_{21})}|H_2(\phi,\psi)|_\mu\sup_{t\in\mathbb{R}}\left\{e^{(\lambda_{22}+\mu)t}\int_t^{+\infty}e^{-\lambda_{22}s}e^{\mu|s|}ds\right\}\\
=&\frac{|\lambda_{21}|}{D_5(\lambda_{22}-\lambda_{21})}|H_2(\phi,\psi)|_\mu\sup_{t\in\mathbb{R}}\left\{e^{(\lambda_{21}+\mu)t}\int_{-\infty}^{t}e^{(-\lambda_{21}-\mu)s}ds\right\}\\
&+\frac{\lambda_{22}}{D_5(\lambda_{22}-\lambda_{21})}|H_2(\phi,\psi)|_\mu\sup_{t\in\mathbb{R}}\left\{e^{(\lambda_{22}+\mu)t}\left[\int_t^{0}e^{(-\lambda_{22}-\mu)s}ds+\int_0^{+\infty}e^{(\mu-\lambda_{22})s}ds\right]\right\}\\
=&\frac{|\lambda_{21}|}{D_5(\lambda_{22}-\lambda_{21})}|H_2(\phi,\psi)|_\mu\sup_{t\in\mathbb{R}}\left\{-\frac{1}{\lambda_{21}+\mu}\right\}\\
&+\frac{\lambda_{22}}{D_5(\lambda_{22}-\lambda_{21})}|H_2(\phi,\psi)|_\mu\sup_{t\in\mathbb{R}}\left\{\frac{1}{\lambda_{22}+\mu}+\frac{2\mu}{\lambda_{22}^2-\mu^2}e^{(\lambda_{22}+\mu)t}\right\}\\
\leq&\frac{1}{D_5(\lambda_{22}-\lambda_{21})}\left[\frac{\lambda_{21}}{\lambda_{21}+\mu}+\frac{\lambda_{22}}{\lambda_{22}-\mu}\right]|H_2(\phi,\psi)|_\mu.
\end{aligned}
$$

Therefore, $F_2:B_\mu\left(\mathbb{R},\mathbb{R}^2\right)\longrightarrow B_\mu\left(\mathbb{R},\mathbb{R}^2\right)$ is continuous with respect to the norm $|\cdot|_\mu$, and the set Γ is uniformly bounded.

Thus there exists the constant M_2 such that $\left|F_2'(\phi,\psi)(t)\right|_\mu \le M_2$; Similarly, there exists constant M_1 such that $\left|F_1'(\phi,\psi)(t)\right|_\mu \le M_1$. Therefore, $\boldsymbol{F}$ is equicontinuous on Γ and $\boldsymbol{F}(\Gamma)$ is uniformly bounded.

Next we prove that $\boldsymbol{F}:\Gamma \longrightarrow \Gamma$ is compact. Define $\boldsymbol{F}^n(\phi,\psi)$ as follows

$$\boldsymbol{F}^n(\phi,\psi) = \begin{cases} \boldsymbol{F}(\phi,\psi)(t), & t \in [-n,n], \\ \boldsymbol{F}(\phi,\psi)(n), & t \in (n,+\infty), \\ \boldsymbol{F}(\phi,\psi)(-n), & t \in (-\infty,-n). \end{cases}$$

For any $n \ge 1$, $\boldsymbol{F}^n(\Gamma)$ is uniformly bounded equicontinuous.

Now, in the interval $[-n,n]$, it follows from Ascoli-Arzela Theorem that $\boldsymbol{F}^n$ is compact.

In addition, in $B_\mu\left(\mathbb{R},\mathbb{R}^2\right)$ we have $\boldsymbol{F}^n \longrightarrow \boldsymbol{F}$, as $n \longrightarrow +\infty$. For any $(\phi,\psi) \in \Gamma$,

$$\sup_{t\in\mathbb{R}} \left|\boldsymbol{F}^n(\phi,\psi)(t) - \boldsymbol{F}(\phi,\psi)(t)\right| e^{-\mu|t|} = \sup_{t\in(-\infty,-n)\bigcup(n,+\infty)} \left|\boldsymbol{F}^n(\phi,\psi)(t) - \boldsymbol{F}(\phi,\psi)(t)\right| e^{-\mu|t|} \le 2(N_1+N_2)e^{-\mu n}, \quad n \longrightarrow +\infty.$$

In consequence, $\boldsymbol{F}$ is compact. □

Theorem 3.1. *Assume that the unique positive equilibrium $E_3(u^+,v^+)$ exists and satisfies*

$$\vartheta_1 > \frac{\left(4+2\sqrt{2}\right)\beta v^+}{1+mu^+ + wv^+}, \quad \vartheta_2 > \frac{\left(2\sqrt{2}-1\right)\beta_1 u^+}{1+mu^+ + wv^+},$$

then for every $c > c^*$, There is always a traveling wave solution $(\phi^*(t),\psi^*(t))$ connecting the equilibrium points $(0,0)$ and (u^+,v^+) with the wave velocity c in the system (2.1). Moreover

$$\lim_{t\to-\infty} \phi^*(t)e^{-\lambda_1 t} = \lim_{t\to+\infty} \psi^*(t)e^{-\lambda_3 t} = 1.$$

Proof. According to the Lemma 3.5, Lemma 3.6 and Schauder fixed point theorem, it can be concluded that the operator $\boldsymbol{F}$ has a fixed point $(\phi^*(t),\psi^*(t))$ in Γ, so $(\phi^*(t),\psi^*(t))$ is the solution of the system (3.1).

In order to prove that the solution is a traveling wave solution, only the asymptotic conditions need to be verified. According to the property $P2$) of $\left(\overline{\phi}(t),\overline{\psi}(t)\right)$ and $\left(\underline{\phi}(t),\underline{\psi}(t)\right)$, and $\left(\underline{\phi}(t),\underline{\psi}(t)\right) \le (\phi^*(t),\psi^*(t)) \le \left(\overline{\phi}(t),\overline{\psi}(t)\right)$, we have

$$\lim_{t\to-\infty}(\phi^*(t),\psi^*(t)) = (0,0),$$
$$\lim_{t\to+\infty}(\phi^*(t),\psi^*(t)) = (u^+,v^+).$$

Because of $\underline{\phi} \le \phi^* \le \overline{\phi}$ and $\underline{\psi} \le \psi^* \le \overline{\psi}$, then

$$e^{\lambda_1 t} - qe^{\eta\lambda_1 t} \le \phi^*(t) \le e^{\lambda_1 t}, \quad t < \min\{t_1,t_3\},$$
$$e^{\lambda_3 t} - qe^{\eta\lambda_3 t} \le \psi^*(t) \le e^{\lambda_3 t} + qe^{\eta\lambda_3 t}, \quad t < \min\{t_2,t_4\}.$$

Consequently,

$$1 - qe^{(\eta-1)\lambda_1 t} \le \phi^* e^{-\lambda_1 t}(t) \le 1, \quad t < \min\{t_1,t_3\},$$
$$1 - qe^{(\eta-1)\lambda_3 t} \le \psi^*(t)e^{-\lambda_3 t} \le 1 + qe^{(\eta-1)\lambda_3 t}, \quad t < \min\{t_2,t_4\}.$$

The above conclusion is proved. □

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there is no conflicts of interest.

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