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Research article

Optimal pair of fixed points for a new class of noncyclic mappings under a $(\varphi, \mathcal{R}^t)$-enriched contraction condition

A. Safari-Hafshejani[1], M. Gabeleh[2,*] and M. De la Sen[3]

[1] Department of Pure Mathematics, Payame Noor University (PNU), Tehran, Iran
[2] Department of Mathematics, Ayatollah Boroujerdi University, Boroujerd, Iran
[3] Institute of Research and Development of Processes, Department of Electricity and Electronics, Faculty of Science and Technology, University of the Basque Country, Leioa 48940, Spain

* **Correspondence:** Email: Gabeleh@abru.ac.ir.

Abstract: In the present study, we commenced by presenting a new class of maps, termed noncyclic $(\varphi, \mathcal{R}^t)$-enriched quasi-contractions within metric spaces equipped with a transitive relation $\mathcal{R}^t$. Subsequently, we identified the conditions for the existence of an optimal pair of fixed points pertaining to these mappings, thereby extending and refining a selection of contemporary findings documented in some articles. Specifically, our analysis will encompass the outcomes pertinent to reflexive and strictly convex Banach spaces.

Keywords: optimal pair of fixed points; noncyclic $(\varphi, \mathcal{R}^t)$-enriched quasi-contraction; metric space; transitive relation; reflexive Banach space

1. Introduction

Let $F \neq \emptyset$ and $G \neq \emptyset$ be subsets of a metric space (X, d). A self-mapping Γ on $F \cup G$ is said to be *noncyclic* whenever $\Gamma(F) \subseteq F$ and $\Gamma(G) \subseteq G$. In this situation, a point $(p^*, q^*) \in F \times G$ is called an *optimal pair of fixed points* of Γ provided that

$$(\Gamma p^*, \Gamma q^*) = (p^*, q^*) \quad \text{and} \quad d(p^*, q^*) = \text{Dist}(F, G),$$

where $\text{Dist}(F, G) = \inf\{d(p, q) : \ p \in F, \ q \in G\}$. We denote the set of all optimal pairs of fixed points of Γ in $F \times G$ by $Fix(\Gamma\,|_{F \times G})$.

$\Gamma : F \cup G \to F \cup G$ is called a *noncyclic contraction* if there exists $\lambda \in [0, 1)$ such that

$$d(\Gamma p, \Gamma q) \leq \lambda d(p, q) + (1 - \lambda)\text{Dist}(F, G), \tag{1.1}$$

for all $(p, q) \in F \times G$.

In 2013, Espínola and Gabeleh proved that if F and G are nonempty, weakly compact, and convex subsets of a strictly convex Banach space X, then $Fix(\Gamma\ |_{F\times G}) \neq \emptyset$ for every noncyclic contraction Γ defined on $F \cup G$ (see Theorem 3.10 of [1]).

After that, Gabeleh used the projection operators and proved both existence and convergence of an optimal pair of fixed points for noncyclic contractions in the setting of uniformly convex Banach spaces (see Theorem 3.2 of [2]).

We refer to [3–10] to study the problem of the existence of an optimal pair of fixed points for various classes of noncyclic mappings.

Recently, the authors of [10] introduced a new class of noncyclic mappings called *noncyclic Fisher quasi-contractions*, which contains the class of noncyclic contractions as a subclass, and they surveyed the existence and convergence of an optimal pair of fixed points in metric spaces by using a geometric notion of property WUC (Definition 2.2) on a nonempty pair of subsets of a metric space.

In this article, we extend the main conclusion of the paper [10] by considering an appropriate control function and equipping the metric space (X, d) with a transitive relation $\mathcal{R}^t$. Indeed, we introduce a new class of noncyclic mappings called *noncyclic* $(\varphi, \mathcal{R}^t)$*-enriched quasi-contractions*, which is a kind of contraction at a point defined first in [11] and generalized later on in [12, 13]. We then study the existence, uniqueness, and convergence of an optimal pair of fixed points for such mappings in metric spaces equipped with a transitive relation $\mathcal{R}^t$. This idea to consider a contractive condition only for points in some transitive relations was first introduced in [14] in order to generalize the ideas of coupled fixed points in partially ordered spaces, and further developed in a sequence of articles [15–17]. We will also examine some other existence conclusions of an optimal pair of fixed points in the framework of reflexive and strictly convex Banach spaces.

2. Preliminaries

In this section, we point out some definitions and notations, which will be used in our coming arguments.

In what follows, $\mathcal{B}_X$ and $\mathcal{S}_X$ denote the unit closed ball and the unit sphere in a Banach space X.

Definition 2.1. ([18]) A Banach space X is said to be

(i) uniformly convex provided that for every $\varepsilon \in (0, 2]$, one can find a corresponding $\delta = \delta(\varepsilon)$ with the property that, whenever $p, q \in \mathcal{B}_X$ with $\|p - q\| \geq \varepsilon$, it follows that

$$\|\frac{p+q}{2}\| < 1 - \delta;$$

(ii) strictly convex if for any two distinct elements $p, q \in \mathcal{S}_X$, we have

$$\|\frac{p+q}{2}\| < 1.$$

It is evident that every uniformly convex Banach space X is strictly convex. However, the reverse does not universally hold. For instance, the Banach space ℓ_1, which is equipped with its standard norm

$$\|u\| = \sqrt{\|u\|_1^2 + \|u\|_2^2},\ \forall u \in l^1,$$

where $\|.\|_1$ and $\|.\|_2$ are the norms on l^1 and l^2, respectively, is strictly convex, which is not uniformly convex (see [19] for more details). Also, Hilbert spaces and l^p spaces ($1 < p < \infty$) are well-known examples of uniformly convex Banach spaces. It is worth noticing that by the Milman-Pettis theorem, every uniformly convex Banach space is reflexive, too.

Definition 2.2. ([20,21]) Let $F \neq \emptyset$ and $G \neq \emptyset$ be subsets of a metric space (X, d), then (F, G) is said to satisfy

(i) property UC, if for all sequences $\{p_n\}, \{p'_n\} \subseteq F$ in F and $\{q_n\} \subseteq G$, we have

$$\left.\begin{array}{l}\lim_{n\to\infty} d(p_n, q_n) = \mathrm{Dist}(F, G),\\ \lim_{n\to\infty} d(p'_n, q_n) = \mathrm{Dist}(F, G),\end{array}\right\} \quad \Rightarrow \quad \lim_{n\to\infty} d(p_n, p'_n) = 0;$$

(ii) property WUC, if for any sequence $\{p_n\} \subseteq F$ such that

$$\forall \epsilon > 0,\ \exists q \in G\ ;\ d(p_n, q) \leq \mathrm{Dist}(F, G) + \epsilon,\ \text{ for } n \geq n_0,$$

$\{p_n\}$ is Cauchy.

In [22], it was disclosed that each nonempty, closed, and convex pair in a uniformly convex Banach space X possesses the property UC. Additionally, if $F \neq \emptyset$ and $G \neq \emptyset$ are subsets in a metric space (X, d), with F being complete and the pair (F, G) exhibiting the property UC, then the pair (F, G) is also endowed with the property WUC (see [20]). For more information and properties of the geometric notions of UC, we refer to [23] and the most recent results in [24], where the authors have found a connection between the properties UC and uniform convexity and have introduced some generalizations of these properties.

Here, we sate the main result of [10].

Theorem 2.3. *([10]) Given nonempty and complete subsets F and G of a metric space (X, d), suppose that the pairs (F, G) and (G, F) have the property* WUC. *Let noncyclic continuous self-mapping Γ on $F \cup G$, be a noncyclic Fisher quasi-contraction, that is, for some $\alpha, \beta \in \mathbb{N}$, there exists $\lambda \in [0, 1)$ such that*

$$d(\Gamma^\alpha x, \Gamma^\beta q) \leq \lambda \Delta[C^p_\alpha, C^q_\beta] + (1-\lambda)\mathrm{Dist}(F, G) \quad \forall p \in F,\ q \in G, \tag{2.1}$$

where $C^u_n := \{u, \Gamma u, \Gamma^2 u, \cdots, \Gamma^n u\}$ for $u \in X$, $n \in \mathbb{N}$, and

$$\Delta[C^p_\alpha, C^q_\beta] := \sup\{d(p', q') :\ (p', q') \in C^p_\alpha \times C^q_\beta\}.$$

There exists $(p^*, q^*) \in F \times G$ such that $Fix(\Gamma|_{F\times G}) = \{(p^*, q^*)\}$, $(\Gamma^n p_0, \Gamma^n q_0) \to (p^*, q^*)$ as $n \to \infty$ for every $(p_0, q_0) \in F \times G$.

3. Noncyclic $(\varphi, \mathcal{R}^t)$-enriched quasi-contractions

Throughout this section, we assume that I is an identity function defined on $[0, +\infty)$ and $\varphi \in [\phi]$, such that

$$[\phi] := \Big\{\varphi : [0, +\infty) \to [0, +\infty) : \varphi \text{ is a strictly increasing function and } I - \varphi \text{ is increasing}\Big\}.$$

For instance, if we define $\varphi_1(t) = \lambda t$ for some $\lambda \in [0,1)$ and $\varphi_2(t) = (t+2) - \ln(t+2)$ and $\varphi_3(t) = t - \sqrt{t+1} + 3$, then $\varphi_j \in [\phi]$ for $j = 1,2,3$.

It is worth noticing that if $\varphi \in [\phi]$, then for all $t > 0$, we have

$$\varphi(t) > \varphi(\frac{t}{2}) \geq 0. \tag{3.1}$$

So, $(I-\varphi)(t) < t$ for all $t > 0$. Since $I - \varphi$ is increasing, it can be easily proven that φ is continuous.

Also, for given nonempty subsets F and G of a metric space (X,d), we set

$$\begin{aligned} d^*(p,q) &:= d(p,q) - \mathrm{Dist}(F,G), \quad \forall (p,q) \in F \times G, \\ \Delta^*[F,G] &:= \sup\{d^*(p,q) : (p,q) \in F \times G\}. \end{aligned}$$

Definition 3.1. Let $F \neq \emptyset$ and $G \neq \emptyset$ be subsets of a metric space (X,d) and "$\mathcal{R}^t$" be a transitive relation on F. Let Γ be a noncyclic mapping on $F \cup G$, then

(i) we say that Γ is $\mathcal{R}^t$-continuous at $p \in F$ if for every sequence $\{p_n\}$ in F with $p_n \to p$ and $p_n \mathcal{R}^t p_{n+1}$, for all $n \in \mathbb{N}$, we have $\Gamma p_n \to \Gamma p$;

(ii) we say that Γ preserves "$\mathcal{R}^t$" on F whenever $Tu \mathcal{R}^t Tv$ for every $u, v \in F$ with $u \mathcal{R}^t v$;

(iii) we say that "$\mathcal{R}^t$" has a property $(*)$ on F, if for any sequence $\{p_n\}$ in F with $p_n \to p \in F$ and $p_n \mathcal{R}^t p_{n+1}$ for all $n \in \mathbb{N}$, we have $p_n \mathcal{R}^t p$ for all $n \in \mathbb{N}$.

Now, with these prerequisites and inspired by the main existence results of [10], we introduce the following new family of noncyclic mappings. Henceforth, we denote a metric space (X,d) equipped with a transitive relation "$\mathcal{R}^t$" by $X^{d,t}$.

Definition 3.2. Let $\emptyset \neq F, G \subseteq X^{d,t}$. A mapping $\Gamma : F \cup G \to F \cup G$ is said to be a noncyclic $(\varphi, \mathcal{R}^t)$-enriched quasi-contraction if Γ is noncyclic. For some $\alpha, \beta \in \mathbb{N}$,

$$d^*(\Gamma^\alpha p, \Gamma^\beta q) \leq (I-\varphi)\left(\Delta^*[C^p_\alpha, C^q_\beta]\right), \tag{3.2}$$

for all $(p,q) \in F \times G$ that are comparable with respect to "$\mathcal{R}^t$".

Example 3.3. Let $F \neq \emptyset$ and $G \neq \emptyset$ be subsets of a metric space (X,d) and let $\Gamma : F \cup G \to F \cup G$ be a noncyclic Fisher quasi-contraction in the sense of Theorem 2.3, then Γ is a noncyclic $(\varphi, \mathcal{R}^t)$-enriched quasi-contraction with $\mathcal{R}^t := X \times X$ and $\varphi(t) := (1-\lambda)t$ for $t \geq 0$ and $\lambda \in [0,1)$.

Remark 3.4. Let $\emptyset \neq F, G \subseteq X^{d,t}$ and $\Gamma : F \cup G \to F \cup G$ be a noncyclic mapping. Set $\mathfrak{D} := \mathrm{Dist}(F,G)$. If for any $(p,q) \in F \times G$, we have

$$d(\Gamma p, \Gamma q) \leq (I-\varphi)(\max\{d(p,q), d(p,\Gamma q), d(q,\Gamma p)\}) + \varphi(\mathfrak{D}),$$

then

$$\begin{aligned} d(\Gamma p, \Gamma q) \leq & \max\{(I-\varphi)(d(p,q)), (I-\varphi)(d(p,\Gamma q)), (I-\varphi)(d(q,\Gamma p))\} \\ & -(I-\varphi)(\mathfrak{D}) + \mathfrak{D} \\ = & \max\Big\{(I-\varphi)(d^*(p,q)+\mathfrak{D}) - (I-\varphi)(\mathfrak{D}), (I-\varphi)(d^*(p,\Gamma q)+\mathfrak{D}) - (I-\varphi)(\mathfrak{D}), \\ & (I-\varphi)(d^*(q,\Gamma p)+\mathfrak{D}) - (I-\varphi)(\mathfrak{D})\Big\} + \mathfrak{D}. \end{aligned} \tag{3.3}$$

Now, define $\varphi^* : [0,+\infty) \to [0,+\infty)$ with $\varphi^*(t) := \varphi(t+\mathfrak{D}) - \varphi(\mathfrak{D})$ for all $t \geq 0$. In view of the fact that $(I-\varphi^*)(t) = (I-\varphi)(t+\mathfrak{D}) - (I-\varphi)(\mathfrak{D})$, we can see that φ^* is strictly increasing and $I-\varphi^*$ is increasing. So from (3.3), we get

$$\begin{aligned} d(\Gamma p, \Gamma q) \leq & \max\Big\{(I-\varphi^*)(d^*(p,q)), (I-\varphi^*)(d^*(p,\Gamma q)), (I-\varphi^*)(d^*(q,\Gamma p))\Big\} + \mathfrak{D} \\ \leq & (I-\varphi^*)\left(\max\{d^*(p,q), d^*(p,\Gamma q), d^*(q,\Gamma p)\}\right). \end{aligned}$$

Example 3.5. Given complete subsets $F \neq \emptyset$ and $G \neq \emptyset$ of a metric space (X,d), let $\Gamma : F\cup G \to F\cup G$ be a noncyclic φ-contraction ([8]), that is, Γ is noncyclic on $F \cup G$ and

$$\exists\, \varphi \in [\phi];\ d(\Gamma x, \Gamma y) \leq d(p,q) - \varphi(d(p,q)) + \varphi(\mathrm{Dist}(F,G)), \quad \forall (p,q) \in F \times G.$$

From Remark 3.4, Γ is a noncyclic $(\varphi^*, \mathcal{R}^t)$-enriched quasi-contraction with $\mathcal{R}^t := X \times X$.

The following lemmas play essential roles in proving our main result in this section.

Lemma 3.6. *Let $\emptyset \neq F, G \subseteq X^{d,t}$ be complete. Let Γ be a noncyclic $(\varphi, \mathcal{R}^t)$-enriched quasi-contraction mapping on $F \cup G$, and Γ preserves "$\mathcal{R}^t$". Let $p_0 \in F$ and $q_0 \in G$ be such that $p_0\, \mathcal{R}^t\, q_0\, \mathcal{R}^t\, \Gamma p_0$. Define $p_{n+1} := \Gamma p_n$ and $q_{n+1} := \Gamma q_n$ for each $n \geq 0$, then for any $m, n \in \mathbb{N}$, we have*

$$\Delta^*[C_n^{p_0}, C_m^{q_0}] = d^*(\Gamma^k p_0, \Gamma^l q_0), \text{ where } k < \alpha \text{ or } l < \beta. \tag{3.4}$$

Proof. Since Γ preserves "$\mathcal{R}^t$" on $F \cup G$ and $p_0\, \mathcal{R}^t\, q_0\, \mathcal{R}^t\, p_1$, we get

$$p_0\, \mathcal{R}^t\, q_0\, \mathcal{R}^t\, p_1\, \mathcal{R}^t\, q_1\, \mathcal{R}^t\, p_2\, \mathcal{R}^t\, q_2\, \mathcal{R}^t\, p_3\, \mathcal{R}^t\, \cdots. \tag{3.5}$$

So, from transitivity of $\mathcal{R}^t$, for all $i, j \in \mathbb{N}$, we have

$$p_i \text{ and } q_j \text{ are comparable w.r.t. "}\mathcal{R}^t\text{".} \tag{3.6}$$

Suppose that $\Delta^*[C_n^{p_0}, C_m^{q_0}] = d^*(\Gamma^i p_0, \Gamma^j q_0)$, where $\alpha \leq i \leq n$ and $\beta \leq j \leq m$. From (3.2) and (3.6), we have

$$\begin{aligned} d^*(\Gamma^i p_0, \Gamma^j q_0) &= d^*(\Gamma^\alpha p_{i-\alpha}, \Gamma^\beta q_{j-\beta}) \\ &\leq (I-\varphi)\left(\Delta^*[C_\alpha^{p_{i-\alpha}}, C_\beta^{q_{j-\beta}}]\right) \\ &\leq (I-\varphi)\left(\Delta^*[C_n^{p_0}, C_m^{q_0}]\right). \end{aligned} \tag{3.7}$$

Thus, we must have $\varphi\left(\Delta^*[C_n^{p_0}, C_m^{q_0}]\right) \leq 0$. Strictly increasing of the function φ causes $\Delta^*[C_n^{p_0}, C_m^{q_0}] = 0$ and $\Delta^*[C_n^{p_0}, C_m^{q_0}] = d^*(p_0, q_0)$, which ensures that (3.4) holds.

Lemma 3.7. *Under the assumptions and notations of Lemma 3.6, for every $m, n \in \mathbb{N}$, we have*

$$\Delta^*[C_n^{p_0}, C_m^{q_0}] \leq M_{p_0,q_0}, \tag{3.8}$$

where

$$M_{p_0,q_0} = \max_{0 \leq i,j \leq \max\{\alpha,\beta\}} \left\{ d^*(\Gamma^i p_0, \Gamma^j q_0), \varphi^{-1}\left(d(\Gamma^i p_0, \Gamma^\alpha p_0)\right) \varphi^{-1}\left(d(\Gamma^i q_0, \Gamma^\beta q_0)\right) \right\}.$$

Proof. From Lemma 3.6, we have $\Delta^*[C_n^{p_0}, C_m^{q_0}] = d^*(\Gamma^i p_0, \Gamma^j q_0)$, for some $i, j \geq 0$ where $i < \alpha$ or $j < \beta$. In the case that $i < \alpha$ and $j < \beta$, (3.8) clearly holds. Therefore, without loss of generality, it can be assumed that $0 \leq i < \alpha$ and $\beta \leq j \leq m$. Using (3.7), we obtain

$$\begin{aligned} \Delta^*[C_n^{p_0}, C_m^{q_0}] &= d^*(\Gamma^i p_0, \Gamma^j q_0) \\ &\leq d(\Gamma^i p_0, \Gamma^\alpha p_0) + d^*(\Gamma^\alpha p_0, \Gamma^j q_0) \\ &\leq d(\Gamma^i p_0, \Gamma^\alpha p_0) + (I - \varphi)\left(\Delta^*[C_n^{p_0}, C_m^{q_0}]\right), \end{aligned}$$

which deduces that

$$\varphi\left(\Delta^*[C_n^{p_0}, C_m^{q_0}]\right) \leq d(\Gamma^i p_0, \Gamma^\alpha p_0).$$

Since $\varphi \in [\phi]$, φ^{-1} exists. Therefore,

$$\Delta^*[C_n^{p_0}, C_m^{q_0}] \leq \varphi^{-1}\left(d(\Gamma^i p_0, \Gamma^\alpha p_0)\right),$$

and so (3.8) holds.

Lemma 3.8. *Under the assumptions and notations of Lemma 3.6, for each $m, n, r, s \geq 0$ with $m, n \geq \max\{\alpha, \beta\}$, we have*

$$\Delta^*[C_r^{p_n}, C_s^{q_m}] \leq (I - \varphi)\left(\Delta^*[C_{r+\alpha}^{p_{n-\alpha}}, C_{s+\beta}^{q_{m-\beta}}]\right). \tag{3.9}$$

Proof. It follows from the relation (3.7) that for some $0 \leq r' \leq r, 0 \leq s' \leq s$,

$$\begin{aligned} \Delta^*[C_r^{p_n}, C_s^{q_m}] &= d^*(\Gamma^{r'} p_n, \Gamma^{s'} q_m) \\ &= d(\Gamma^{p+r'} p_{n-\alpha}, \Gamma^{q+s'} q_{m-\beta}) \\ &\leq (I - \varphi)\left(\Delta^*[C_{r+\alpha}^{p_{n-\alpha}}, C_{s+\beta}^{q_{m-\beta}}]\right). \end{aligned}$$

Hence, (3.9) holds.

Lemma 3.9. *Under the assumptions and notations of Lemma 3.6,*

$$\forall \epsilon > 0, \quad \exists m \in \mathbb{N}; \quad d(p_n, q_m) \leq \text{Dist}(F, G) + \epsilon, \quad \textit{for} \quad n \geq m.$$

Proof. From Lemma 3.8, for $n, m \geq \max\{2\alpha, 2\beta\}$, we have

$$\begin{aligned} d^*(p_n, q_m) &= \Delta^*[C_0^{p_n}, C_0^{q_m}] \\ &\leq (I-\varphi)\left(\Delta^*[C_\alpha^{p_{n-\alpha}}, C_\beta^{q_{m-\beta}}]\right) \\ &\leq (I-\varphi)\left((I-\varphi)\left(\Delta^*[C_{2\alpha}^{p_{n-2\alpha}}, C_{2\beta}^{q_{m-2\beta}}]\right)\right) \\ &= (I-\varphi)^2\left(\Delta^*[C_{2\alpha}^{p_{n-2\alpha}}, C_{2q}^{\beta_{m-2\beta}}]\right). \end{aligned}$$

Continuing this process and using Lemma 3.7, we get

$$\begin{aligned} 0 &\leq d^*(p_n, q_m) \\ &\leq (I-\varphi)^{k_{n,m}}\left(\Delta^*[C_{k_{n,m}\alpha}^{p_{n-k_{n,m}\alpha}}, C_{k_{n,m}\beta}^{q_{m-k_{n,m}\beta}}]\right) \\ &\leq (I-\varphi)^{k_{n,m}}\left(\Delta^*[C_n^{p_0}, C_m^{q_0}]\right) \\ &\leq (I-\varphi)^{k_{n,m}}\left(M_{p_0,q_0}\right), \end{aligned} \tag{3.10}$$

where $k_{n,m} = \min\{\lfloor \frac{n}{\alpha} \rfloor, \lfloor \frac{m}{\beta} \rfloor\}$. On the other hand, for the purposes of this discussion, it is permissible to presume that $M_{p_0,q_0} > 0$. Since $I - \varphi$ is increasing and $(I - \varphi)(t) < t$ for all $t > 0$, we obtain

$$M_{p_0,q_0} \geq (I-\varphi)\left(M_{p_0,q_0}\right) \geq (I-\varphi)^2\left(M_{p_0,q_0}\right) \geq \cdots. \tag{3.11}$$

Additionally, from (3.10), for every $i \in \mathbb{N}$ there exist $n_i, m_i \in \mathbb{N}$ such that $k_{n_i,m_i} \geq i$, and so (3.11) implies that

$$(I-\varphi)^i\left(M_{p_0,q_0}\right) \geq (I-\varphi)^{k_{n_i,m_i}}\left(M_{p_0,q_0}\right) \geq 0.$$

Thus,

$$M_{p_0,q_0} \geq (I-\varphi)\left(M_{p_0,q_0}\right) \geq (I-\varphi)^2\left(M_{p_0,q_0}\right) \geq \cdots \geq 0,$$

which deduces that the sequence $\{(I-\varphi)^k\left(M_{p_0,q_0}\right)\}$ is decreasing. Since $\{(I-\varphi)^k\left(M_{p_0,q_0}\right)\}$ is bounded below, we assume that

$$\lim_{k\to\infty}(I-\varphi)^k\left(M_{p_0,q_0}\right) = s,$$

for some $s \geq 0$. If $(I-\varphi)^{k_0}\left(M_{p_0,q_0}\right) = 0$ for some $k_0 \geq 1$, then $s = 0$. Otherwise, if $(I-\varphi)^k\left(M_{p_0,q_0}\right) > 0$ for each $k \in \mathbb{N}$, from continuity of $I - \varphi$, we get

$$(I-\varphi)(s) = s,$$

hence, $\varphi(s) = 0$, and from (3.1), we get $s = 0$. Therefore, from (3.10), we conclude that

$$\forall\, \epsilon > 0, \quad \exists\, m \in \mathbb{N}: \quad d^*(p_n, q_m) \leq \epsilon, \quad \text{for} \quad n \geq m,$$

and, in addition, the lemma.

The next result is a direct consequence of Lemma 3.9.

Corollary 3.10. *Under the assumptions and notations of Lemma 3.6, if (F, G) has the property* WUC*, then the sequence $\{p_n\}$ is Cauchy.*

We have now reached a level of preparedness that allows us to demonstrate the main existential finding of this segment, an expanded variant of Theorem 2.3.

Theorem 3.11. *Under the assumptions and notations of Lemma 3.6, the following statements hold:*

(i) *If the pair (F, G) satisfies the property* WUC*, the set F is complete, and $\Gamma\mid_F \colon F \to F$ is $\mathcal{R}^t$-continuous on F, then there exists $p^* \in F$ such that $\Gamma p^* = p^*$;*
(ii) *If the pair (G, F) satisfies the property* WUC*, the set G is complete, and $\Gamma\mid_G \colon G \to G$ is $\mathcal{R}^t$-continuous on G, then there exists $q^* \in G$ such that $\Gamma q^* = q^*$;*
(iii) *If, in addition to* (i) *and* (ii)*, every pair of elements $(p, q) \in F \times G$ are comparable w.r.t. "$\mathcal{R}^t$", then $Fix(\Gamma\mid_{F\times G}) = \{(p^*, q^*)\}$.*

Proof. (i) Let $p_{n+1} := \Gamma p_n$ for each $n \geq 0$. From Corollary 3.10 and completeness of F, the sequence $\{p_n\}$ converges to some $p^* \in F$. Also from (3.5), we have $p_n \mathcal{R}^t p_{n+1}$ for each $n \geq 0$. Since $\Gamma\mid_F$ is $\mathcal{R}^t$-continuous, it follows that $\Gamma p^* = p^*$.

(ii) By using a similar argument (i), the result is obtained.

(iii) If $p^* \in F$ and $q^* \in G$ are the fixed points of T, then from Lemma 3.9 we have

$$d(p^*, q^*) = \lim_{n\to\infty} d(\Gamma^n p_0, \Gamma^n q_0) = \text{Dist}(F, G),$$

that is, $(p^*, q^*) \in Fix(\Gamma\mid_{F\times G})$. Now, assume that each elements $p \in F$ and $q \in G$ are comparable with respect to "$\mathcal{R}^t$". Suppose $\overline{p}$ is another fixed point of Γ in F and let $q_0 \in G$. From Lemma 3.9, we have

$$\begin{aligned}\lim_{n\to\infty} d(p^*, \Gamma^n q_0) &= \lim_{n\to\infty} d(\Gamma^n p^*, \Gamma^n q_0)\\ &= \text{Dist}(F, G)\\ &= \lim_{n\to\infty} d(\Gamma^n \overline{p}, \Gamma^n q_0)\\ &= \lim_{n\to\infty} d(\overline{p}, \Gamma^n q_0).\end{aligned}$$

Since (F, G) satisfies the property WUC, we get $p^* = \overline{p}$. In a similar fashion, it becomes apparent that q^* is a unique fixed point of Γ in G.

Example 3.12. Consider $X := \mathbb{R}$ with the usual metric and let

$$\mathcal{R}^t := \Big\{\big(\pm\frac{1}{n+1}, \pm\frac{1}{m+1}\big) \in X \times X : \quad n, m \in \mathbb{N}\Big\}.$$

For $F = [0, 1]$ and $G = [-1, 0]$, define a noncyclic mapping $\Gamma : F \cup G \to F \cup G$ with

$$\Gamma(p) = \begin{cases} \dfrac{p}{1+2p} & \text{if } p \in \{\frac{1}{n+1} : \quad n \in \mathbb{N}\},\\ 1 & \text{if } p \in F \setminus \{0, \frac{1}{n+1} : \quad n \in \mathbb{N}\},\\ 0 & \text{if } p = 0.\end{cases}$$

$$\Gamma(q) = \begin{cases} \dfrac{q}{1-2q} & \text{if } q \in \{-\frac{1}{m+1} : \quad m \in \mathbb{N}\},\\ -1 & \text{if } p \in G \setminus \{0, -\frac{1}{m+1} : \quad m \in \mathbb{N}\},\\ 0 & \text{if } q = 0.\end{cases}$$

If $\varphi(t)=\frac{t^2}{1+2t}$ for $t\geq 0$, then $(I-\varphi)(t)=\frac{t+t^2}{1+2t}$ and $\varphi\in[\phi]$. Let $(p,q)\in F\times G$ be comparable w.r.t. "$\mathcal{R}^t$", then we must have $(p,q)=(\frac{1}{n+1},-\frac{1}{m+1})$ for some $n,m\in\mathbb{N}$, which implies that

$$\begin{aligned}
d^*(\Gamma p,\Gamma q) &= \left|\frac{\frac{1}{n+1}}{1+\frac{2}{n+1}}+\frac{\frac{1}{m+1}}{1+\frac{2}{m+1}}\right| \\
&= \frac{\frac{1}{n+1}+\frac{1}{m+1}+\frac{4}{(n+1)(m+1)}}{1+2(\frac{1}{n+1}+\frac{1}{m+1})+\frac{4}{(n+1)(m+1)}} \\
&\leq \frac{\frac{1}{n+1}+\frac{1}{m+1}+\frac{4}{(n+1)(m+1)}}{1+2(\frac{1}{n+1}+\frac{1}{m+1})} \\
&\leq \frac{(\frac{1}{n+1}+\frac{1}{m+1})+(\frac{1}{n+1}+\frac{1}{m+1})^2}{1+2(\frac{1}{n+1}+\frac{1}{m+1})} \\
&= (I-\varphi)(\frac{1}{n+1}+\frac{1}{m+1}) \\
&= (I-\varphi)(d^*(p,q)),
\end{aligned}$$

that is, Γ is a noncyclic $(\varphi,\mathcal{R}^t)$-enriched quasi-contraction map, which is not a noncyclic φ-contraction. It is not difficult to see that all conditions of the part (i) of Theorem 3.11 are satisfied, and $p^*=0$ is a fixed point of Γ in F. Note that since every pair of elements $(p,q)\in F\times G$ are not comparable w.r.t. "$\mathcal{R}^t$", the fixed point of Γ in F is not unique.

Example 3.13. Again, consider $X:=\mathbb{R}$ with the usual metric and let $\mathcal{R}^t:=X\times X$. For $F=[0,1]$ and $G=[-1,0]$, define a noncyclic mapping $\Gamma: F\cup G\to F\cup G$ by

$$\Gamma(p)=\begin{cases}\dfrac{p}{1+2p} & \text{if } p\in F,\\ \dfrac{q}{1-2q} & \text{if } q\in G.\end{cases}$$

If $\varphi(t)=\frac{t^2}{1+2t}$ for $t\geq 0$, then $\varphi\in[\phi]$. A similar argument of the previous example shows that $d^*(\Gamma p,\Gamma q)\leq (I-\varphi)(d^*(p,q))$ for all $(p,q)\in F\times G$. Hence, Γ is a noncyclic $(\varphi,\mathcal{R}^t)$-enriched quasi-contraction map. It now follows from Theorem 3.11 that $p^*=0$ is a unique fixed point of Γ in F.

The next theorem shows that if $\alpha=1$ (resp., $\beta=1$) in Definition 3.2, then we can drop the continuity of $T|_F$ (resp., $T|_G$) in Theorem 3.11. In this way, we obtain a real generalization of Theorem 3 in [6] as well as Theorem 2.7 in [10].

Theorem 3.14. *Let $\emptyset\neq F,G\subseteq X^{d,t}$ be such that F is complete and (F,G) satisfies the property* WUC. *Let "$\mathcal{R}^t$" be a transitive relation on $F\cup G$ with the property $(*)$ on F, and Γ is a noncyclic $(\varphi,\mathcal{R}^t)$-enriched quasi-contraction mapping on $F\cup G$ with $\alpha=1$, for which Γ preserves "$\mathcal{R}^t$" on $F\cup G$. Let $p_0\in F$ and $q_0\in G$ be such that $p_0\,\mathcal{R}^t\,q_0\,\mathcal{R}^t\,\Gamma p_0$, then there exists $p^*\in F$ such that $\Gamma p^*=p^*$. If every pair of elements $p\in F$ and $q\in G$ are comparable with respect to "$\mathcal{R}^t$", then Γ has a unique fixed point in F.*

Proof. From the proof of Theorem 3.11, the sequence $\{\Gamma^n p_0\}$ is convergent to some $p^*\in F$. By Lemma 3.9, $p_n\,\mathcal{R}^t\,p_{n+1}$ for each $n\geq 0$. By using property $(*)$, we get $p_n\,\mathcal{R}^t\,p^*$ for each $n\geq 0$. Now, from the relation (3.6), we obtain $q_n \mathcal{R}^t p_{n+1} \mathcal{R}^t p^*$ for each $n \geq 0$. Thus, $q_n \mathcal{R}^t p^*$ for each $n \geq 0$, and by the fact that Γ is a noncyclic $(\varphi, \mathcal{R}^t)$-enriched quasi-contraction from (3.2), we have

$$d^*(\Gamma p^*, \Gamma^n q_0) = d^*(\Gamma p^*, \Gamma^\beta q_{n-\beta}) \leq (I - \varphi)\left(\Delta^*[C_1^{p^*}, C_\beta^{q_{n-\beta}}]\right).$$

Therefore,

$$\limsup_{n\to\infty} d^*(\Gamma p^*, \Gamma^n q_0) \leq (I - \varphi)\Big(\max\{\limsup_{m\to\infty} d^*(p^*, \Gamma^m q_0), \limsup_{m\to\infty} d^*(\Gamma p^*, \Gamma^m q_0)\}\Big).$$

By Lemma 3.9, we get

$$\limsup_{n\to\infty} d^*(\Gamma p^*, \Gamma^n q_0) \leq (I - \varphi)\left(\max\{0, \limsup_{n\to\infty} d^*(\Gamma p^*, \Gamma^n q_0)\}\right).$$

Hence,

$$\varphi\left(\limsup_{n\to\infty} d^*(\Gamma p^*, \Gamma^n q_0)\right) = 0.$$

So, from (3.1), we obtain

$$\lim_{n\to\infty} d(\Gamma p^*, \Gamma^n q_0) = \mathrm{Dist}(F, G). \tag{3.12}$$

Since $\lim_{n\to\infty} d(p^*, \Gamma^n q_0) = \mathrm{Dist}(F, G)$, from (3.12) and by taking into account that (F, G) has the property WUC, we conclude that $\Gamma p^* = p^*$. The uniqueness of a fixed point of Γ in F follows from an equivalent discussion of Theorem 3.11.

Corollary 3.15. *Let $F \neq \emptyset$ and $G \neq \emptyset$ be complete subsets of a metric space (X, d) such that (F, G) and (G, F) satisfy the property* WUC. *Let "$\mathcal{R}^t$" be a transitive relation on $F \cup G$ with the property $(*)$ on $F \cup G$. Assume that Γ is a noncyclic mapping on $F \cup G$ satisfying*

$$d^*(\Gamma p, \Gamma q) \leq (I - \varphi)(\max\{d^*(p, q), d^*(p, \Gamma q), d^*(q, \Gamma p)\}),$$

for each $(p, q) \in F \times G$ that are comparable with respect to "$\mathcal{R}^t$". Let $(p_0, q_0) \in F \times G$ be such that $p_0 \mathcal{R}^t q_0 \mathcal{R}^t \Gamma p_0$ and Γ preserves "$\mathcal{R}^t$" on $F \cup G$, then there exists $(p^*, q^*) \in Fix(\Gamma|_{F\times G})$. If every pair of elements $p \in F$ and $q \in G$ are comparable with respect to "$\mathcal{R}^t$", then $Fix(\Gamma|_{F\times G}) = \{(p^*, q^*)\}$.

Building upon the foundations laid by the preceding theorem, we arrive at a subsequent finding that serves as a generalization of Corollary 2.8 of [10].

Corollary 3.16. *Let $F \neq \emptyset$ and $G \neq \emptyset$ be complete subsets of a metric space (X, d) such that (F, G) and (G, F) satisfy the property* WUC. *Assume that Γ is a noncyclic mapping on $F \cup G$ satisfying*

$$d^*(\Gamma p, \Gamma q) \leq (I - \varphi)(\max\{d^*(p, q), d^*(p, \Gamma q), d^*(q, \Gamma p)\}),$$

for each $p \in F$ and $q \in G$. There exists $(p^*, q^*) \in F \times G$ such that $Fix(\Gamma|_{F\times G}) = \{(p^*, q^*)\}$, and for every $p_0 \in F$ and $q_0 \in G$, the sequences $\{\Gamma^n p_0\}$ and $\{\Gamma^n q_0\}$ converge to p^* and q^*, respectively.

The following common fixed point results are obtained from Theorem 3.11 and Corollary 3.15, immediately. These results are extensions of Corollaries 2.10 and 2.11 of [10].

Corollary 3.17. *Let* Γ *and* Λ *be two continuous self-mappings on a complete metric space* (X, d) *such that for some* $\alpha, \beta \in \mathbb{N}$,

$$d(\Gamma^{\alpha} p, \Lambda^{\beta} q) \leq (I - \varphi)\Big(\max\{d(\Gamma^{i} p, \Lambda^{j} q) : \ 0 \leq i \leq \alpha, \ 0 \leq j \leq \beta\}\Big),$$

for all $p, q \in X$*, then* Λ *and* Γ *have a unique common fixed point* $p^* \in X$ *such that* $\lim_{n\to\infty} \Gamma^n p_0 = \lim_{n\to\infty} \Lambda^n p_0 = p^*$ *for every* $p_0 \in X$.

Corollary 3.18. *Let* Γ *and* Λ *be two self-mappings on a complete metric space* (X, d) *satisfying*

$$d(\Gamma p, \Lambda q) \leq (I - \varphi)(\max\{d(p, q), d(p, \Lambda q), d(q, \Gamma p)\}),$$

for all $p, q \in X$*, then* Λ *and* Γ *have a unique common fixed point in* X.

4. More results in reflexive and strictly convex Banach spaces

In the latest section of this article, motivated by the results of [25, 26], we present some other existence, convergence, and uniqueness of an optimal pair of fixed points of noncyclic φ-quasi-contractions in the setting of reflexive and strictly convex Banach spaces. We also refer to [27–29] for different approaches to the same problems for cyclic mappings and some interesting applications in game theory.

Throughout this section, we assume that $\varphi \in [\phi]$. Also, by " $\xrightarrow{w}$ ", we mean the weak convergence in a Banach space X.

Theorem 4.1. *Suppose that* $F \neq \emptyset$ *and* $G \neq \emptyset$ *are weakly closed subsets of a reflexive Banach space* X *and let* $\Gamma : F \cup G \to F \cup G$ *be a noncyclic* φ*-quasi-contraction map, that is,*

$$\|\Gamma p - \Gamma q\| \leq (I - \varphi)(\max\{\|x - y\|, \|x - \Gamma q\|, \|\Gamma p - y\|\}) + \varphi(\mathrm{Dist}(F, G)),$$

for all $(p, q) \in F \times G$*. There exists* $(p^*, q^*) \in F \times G$ *such that* $\|p^* - q^*\| = \mathrm{Dist}(F, G)$.

Proof. In the case that $\mathrm{Dist}(F, G) = 0$, the result follows from Theorem 3.14. Otherwise, if $\mathrm{Dist}(F, G) > 0$, for an arbitrary element $(p_0, q_0) \in F \times G$, define

$$(p_{n+1}, q_{n+1}) := (\Gamma p_n, \Gamma q_n), \quad \forall n \geq 0.$$

From Lemma 3.9, the sequence $\{(p_n, q_n)\}$ is bounded in $F \times G$. Since F is weakly closed in a reflexive Banach space X, there exists a subsequence $\{p_{n_k}\}$ of $\{p_n\}$ with $p_{n_k} \xrightarrow{w} p^* \in F$. As $\{q_{n_k}\}$ is a bounded sequence in a weakly closed set G, without loss of generality, one may assume that $q_{n_k} \xrightarrow{w} q^* \in G$ as $k \to \infty$. Since $p_{n_k} - q_{n_k} \xrightarrow{w} p^* - q^* \neq 0$ as $k \to \infty$, one can find a bounded linear functional $f : X \to [0, +\infty)$ with the property that

$$\|f\| = 1 \quad \text{and} \quad f(p^* - q^*) = \|p^* - q^*\|.$$

It follows from Lemma 3.9 that

$$\begin{aligned}\|p^* - q^*\| &= |f(p^* - q^*)| \\ &= \lim_{k\to\infty} |f(p_{n_k} - q_{n_k})| \\ &\le \lim_{k\to\infty} \|f\|\|p_{n_k} - q_{n_k}\| \\ &= \lim_{k\to\infty} \|p_{n_k} - q_{n_k}\| \\ &= \mathrm{Dist}(F, G).\end{aligned}$$

So, $\|p^* - q^*\| = \mathrm{Dist}(F, G)$.

Definition 4.2. Suppose that F and G are subsets of a normed linear space X and Γ is a noncyclic self-mapping on $F \cup G$. We say that Γ satisfies the D-property on F if $\{p_n\}$ is a sequence in F and $\{q_n\}$ is a sequence in G, such that

$$p_n \xrightarrow{w} p^* \in F, \quad \|p_n - q_n\| \to \mathrm{Dist}(F, G), \text{ and } \|\Gamma p_n - q_n\| \to \mathrm{Dist}(F, G),$$

then $\Gamma p^* = p^*$.

Note that if $\mathrm{Dist}(F, G) = 0$ or (F, G) has the property UC, then the conditions of the above definition require that

$$p_n \xrightarrow{w} p^* \in F, \quad \text{and} \quad \|\Gamma p_n - p_n\| \to 0.$$

Therefore, in these cases, the D-property of Γ on F is equal to demiclosedness property of $I - \Gamma|_F$ at 0.

Theorem 4.3. *Suppose that $F \neq \emptyset$ and $G \neq \emptyset$ are weakly closed subsets of a reflexive and strictly convex Banach space X and let $\Gamma : F \cup G \to F \cup G$ be a noncyclic φ-quasi-contraction map. Assume that one of the following conditions is satisfied:*

(a) F is convex and Γ is weakly continuous on F;
(b) Γ satisfies the D-property on F.

Thus Γ has a fixed point in F.

Proof. In the case that $\mathrm{Dist}(F, G) = 0$, there is nothing to prove by Theorem 3.14, so assume that $\mathrm{Dist}(F, G) > 0$. Let $(p_0, q_0) \in F \times G$ be an arbitrary element and define

$$(p_{n+1}, q_{n+1}) := (\Gamma p_n, \Gamma q_n), \quad \forall n \ge 0.$$

From Theorem 4.1, there exists a point $(p^*, q^*) \in F \times G$ and subsequences $\{p_{n_k}\}$ and $\{q_{n_k}\}$ such that $\|p^* - q^*\| = \mathrm{Dist}(F, G)$, $p_{n_k} \xrightarrow{w} p^* \in F$, and $q_{n_k} \xrightarrow{w} q^* \in G$ as $k \to \infty$.

(a) Since Γ is weakly continuous on F and $\Gamma(F) \subseteq F$, we have $p_{n_k+1} \xrightarrow{w} \Gamma p^* \in F$ as $k \to \infty$. Since $p_{n_k+1} - q_{n_k} \xrightarrow{w} \Gamma p^* - q^* \neq 0$ as $k \to \infty$, one can find a bounded linear functional $f : X \to [0, +\infty)$ with the property that

$$\|f\| = 1, \quad \text{and} \quad f(\Gamma p^* - q^*) = \|\Gamma p^* - q^*\|.$$

It follows from Lemma 3.9 that

$$\begin{aligned}\|\Gamma p^* - q^*\| &= |f(\Gamma p^* - q^*)| \\ &= \lim_{k\to\infty} |f(p_{n_k+1} - q_{n_k})| \\ &\leq \lim_{k\to\infty} \|f\|\|p_{n_k+1} - q_{n_k}\| \\ &= \lim_{k\to\infty} \|p_{n_k+1} - q_{n_k}\| \\ &= \mathrm{Dist}(F, G).\end{aligned}$$

So, $\|\Gamma p^* - q^*\| = \mathrm{Dist}(F, G)$. We assume the contrary, $\Gamma p^* \neq p^*$, and it follows from the strict convexity of X that

$$\|\frac{p^* + \Gamma p^*}{2} - q^*\| < \mathrm{Dist}(F, G). \tag{4.1}$$

Since F is convex, $\frac{p^*+\Gamma p^*}{2} \in F$, so (4.1) is a contradiction.

(b) It follows from Lemma 3.9 that

$$\lim_{k\to\infty} \|p_{n_k} - q_{n_k}\| = \lim_{k\to\infty} \|\Gamma p_{n_k} - q_{n_k}\| = \mathrm{Dist}(F, G),$$

and by the D-property of Γ on F, we get $\Gamma p^* = p^*$.

Theorem 4.4. *Suppose that $F \neq \emptyset$ and $G \neq \emptyset$ are weakly closed and convex subsets of a reflexive and strictly convex Banach space X, and let $\Gamma : F \cup G \to F \cup G$ be a noncyclic φ-quasi-contraction map. Let one of the following conditions be satisfied:*

(a) Γ *is weakly continuous on $F \cup G$;*
(b) Γ *satisfies the D-property on $F \cup G$.*

Thus, $Fix(\Gamma|_{F\times G}) \neq \emptyset$. Also, if $(F - F) \cap (G - G) = \{0\}$, then $Fix(\Gamma|_{F\times G}) = \{(p^*, q^*)\}$ for some $(p^*, q^*) \in F \times G$.

Proof. According to Theorems 4.1 and 4.3, it is enough to prove the uniqueness of an optimal pair of fixed points $(p^*, q^*) \in F \times G$. Suppose that there exists another point $(p', q') \in F \times G$ for which $\|p'-q'\| = \mathrm{Dist}(F, G)$. As $(F-F)\cap(G-G) = \{0\}$, we obtain that $p'-p^* \neq q'-q^*$ (since $(p', q') \neq (p^*, q^*)$, we have $p' \neq p^*$ or $q' \neq q^*$. Hence, $p' - p^* \neq 0$ or $q' - q^* \neq 0$), so $p^* - q^* \neq p' - q'$. From the strict convexity of X, we have

$$\|\frac{p' + p^*}{2} - \frac{q' + q^*}{2}\| < \mathrm{Dist}(F, G). \tag{4.2}$$

which is a contradiction.

The next result guarantees the uniqueness of an optimal pair of fixed points in Theorem 3.5 of [5].

Theorem 4.5. *Suppose that $F \neq \emptyset$ and $G \neq \emptyset$ are closed and convex subsets of a reflexive and strictly convex Banach space X and let $\Gamma : F \cup G \to F \cup G$ be a noncyclic φ-contraction map, that is,*

$$\|\Gamma p - \Gamma q\| \leq \|p - q\| - \varphi(\|p - q\|) + \varphi(\mathrm{Dist}(F, G)), \tag{4.3}$$

for all $(p, q) \in F \times G$. If $(F - F) \cap (G - G) = \{0\}$, then there exists $(p^*, q^*) \in F \times G$ such that $Fix(\Gamma|_{F\times G}) = \{(p^*, q^*)\}$.

Proof. In the case that $\mathrm{Dist}(F,G) = 0$, the result concludes from Theorem 3.14 directly. Otherwise, if $\mathrm{Dist}(F,G) > 0$, since F is closed and convex, it is weakly closed. It follows from Theorem 4.1 that there exists $(p^*, q^*) \in F\times G$ such that $\|p^* - q^*\| = \mathrm{Dist}(F,G)$. The proof of uniqueness of $(p^*, q^*) \in F\times G$ with $\|p^* - q^*\| = \mathrm{Dist}(F,G)$ is concluded from a similar discussion of Theorem 4.4. It follows from (4.3) that

$$\|\Gamma p^* - \Gamma q^*\| = \|p^* - q^*\| = \mathrm{Dist}(F,G),$$

which ensures that $(\Gamma p^*, \Gamma q^*) = (p^*, q^*)$. Thus, $\Gamma p^* = p^*$ and $\Gamma q^* = q^*$, and we are finished.

5. Conclusions

In this paper, we defined a new class of noncyclic mappings and investigated the existence, uniqueness, and convergence of an optimal pair fixed point for such maps in the framework of metric spaces equipped with a transitive relation. We also presented the counterpart results under some other sufficient conditions in strictly convex and reflexive Banach spaces. In this way, we obtained some real extensions of previous results that appeared in [2, 10, 22, 25].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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