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http://www.aimspress.com/journal/era

ERA, 32(3): 2180–2202.
DOI: 10.3934/era.2024099
Received: 31 December 2023
Revised: 26 February 2024
Accepted: 02 March 2024
Published: 15 March 2024

Research article

Global dynamics to a quasilinear chemotaxis system under some critical parameter conditions

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Abstract: In this manuscript, the following chemotaxis system has been considered:

$$\begin{cases} v_t = \nabla\cdot(\phi(v)\nabla v - \varphi(v)\nabla w_1 + \psi(v)\nabla w_2) + av - bv^{\kappa}, & x\in\Omega,\ t>0,\\ 0 = \Delta w_1 + \alpha v^{\gamma_1} - \beta w_1, & x\in\Omega,\ t>0,\\ 0 = \Delta w_2 + \gamma v^{\gamma_2} - \delta w_2, & x\in\Omega,\ t>0, \end{cases}$$

where Ω is a bounded smooth domain of $\mathbb{R}^n(n\ge 1)$, the parameters $a,b,\alpha,\beta,\gamma,\delta,\gamma_1,\gamma_2>0,\kappa>1$, and nonnegative functions $\phi(\varrho)=(\varrho+1)^m$, $\varphi(\varrho)=\chi\varrho(\varrho+1)^{\theta-1}$ and $\psi(\varrho)=\xi\varrho(\varrho+1)^{l-1}$ for $\varrho\ge 0$ with $m,\theta,l\in\mathbb{R}$ and $\chi,\xi>0$. In the present work, we improve the boundedness criteria established in previous work and further show that under the corresponding critical cases, namely, assume that $\theta+\gamma_1=\max\{l+\gamma_2,\kappa\}\ge m+\frac{2}{n}+1$ with $m>-\frac{2}{n}, n\ge 3$, if one of the following conditions holds:

(a) when $\theta+\gamma_1=l+\gamma_2=\kappa$, if $\theta\ge l\ge 1$ and $\frac{[(\kappa-1-m)n-2](2\alpha\chi-\gamma\xi)}{2(l-1)+(\kappa-1-m)n}=b$, or $l\ge\theta\ge 1$ and $\frac{2\alpha\chi[(\kappa-1-m)n-2]}{2(\theta-1)+(\kappa-1-m)n}=b$;
(b) when $\theta+\gamma_1=\kappa>l+\gamma_2$, if $\theta\ge 1$ and $\frac{2\alpha\chi[(\kappa-1-m)n-2]}{2(\theta-1)+(\kappa-1-m)n}=b$,

then the system still possesses at least a global classical solution, which is bounded in $\Omega\times(0,\infty)$. Additionally, we have also explored the long time behavior of the classical solution mentioned above.

Keywords: chemotaxis system; critical parameter conditions; boundedness; long time behavior

1. Introduction

Recently, the following partial differential chemotaxis system has been considered in [1]:

$$\begin{cases} v_t = \nabla\cdot(\phi(v)\nabla v - \varphi(v)\nabla w_1 + \psi(v)\nabla w_2) + av - bv^{\kappa}, & x\in\Omega,\ t>0,\\ 0 = \Delta w_1 + \alpha v^{\gamma_1} - \beta w_1, 0 = \Delta w_2 + \gamma v^{\gamma_2} - \delta w_2, & x\in\Omega,\ t>0, \end{cases} \tag{1.1}$$

under the boundary conditions of $\frac{\partial v}{\partial \nu} = \frac{\partial w_1}{\partial \nu} = \frac{\partial w_2}{\partial \nu}$ on $\partial\Omega$, where Ω is a bounded smooth domain of $\mathbb{R}^n (n \geq 1)$, and ν is a normal vector of $\partial\Omega$. Here, v stands for the density of cell population, w_1 and w_2 represent the concentration of two different chemical signals secreted by cell population, and parameters $a, b, \alpha, \beta, \gamma, \delta, \gamma_1, \gamma_2 > 0, \kappa > 1$. In the system (1.1), the diffusion functions are assumed to satisfy

$$\phi(\varrho) = (\varrho + 1)^m, \ \varphi(\varrho) = \chi\varrho(\varrho + 1)^{\theta-1} \text{ and } \psi(\varrho) = \xi\varrho(\varrho + 1)^{l-1}, \tag{1.2}$$

for all $\varrho \geq 0$ with $m, \theta, l \in \mathbb{R}$ and $\chi, \xi > 0$. Suppose that $\theta + \gamma_1 = \max\{l + \gamma_2, \kappa\} \geq m + \frac{2}{n} + 1$. It has been proven in [1] that if one of the following conditions holds, then the system (1.1) is globally classically solvable

(a) when $\theta + \gamma_1 = l + \gamma_2 = \kappa$, if $\theta \geq l \geq 1$ and $\frac{[(\kappa-1-m)n-2](2\alpha\chi-\gamma\xi)}{2(l-1)+(\kappa-1-m)n} < b$, or $l \geq \theta \geq 1$ and $\frac{2\alpha\chi[(\kappa-1-m)n-2]}{2(\theta-1)+(\kappa-1-m)n} < b$;
(b) when $\theta + \gamma_1 = l + \gamma_2 > \kappa$, if $\theta \geq l \geq 1$ and $2\alpha\chi \leq \gamma\xi$;
(c) when $\theta + \gamma_1 = \kappa > l + \gamma_2$, if $\theta \geq 1$ and $\frac{2\alpha\chi[(\kappa-1-m)n-2]}{2(\theta-1)+(\kappa-1-m)n} < b$.

In the present work, we shall further prove that such conclusions still hold under corresponding critical parameter conditions. Meanwhile, we will analyze the long time behavior of such solutions. Before stating our main conclusions, we shall review some known results regarding this aspect.

Chemotaxis is a universal phenomenon in the real environment, which refers to a reaction of seeking benefits and avoiding harm under the stimulation of chemical substances. The first mathematical model to describe such phenomenon was given by Keller and Segel [2] with the following form:

$$\begin{cases} v_t = \Delta v - \chi\nabla \cdot (v\nabla w), & x \in \Omega, \ t > 0, \\ \tau w_t = \Delta w - w + v, & x \in \Omega, \ t > 0, \\ v(x, 0) = v_0(x), \tau w(x, 0) = \tau w_0(x), & x \in \Omega, \end{cases} \tag{1.3}$$

where the function $v(x, t)$ stands for the cell density, and the function $w(x, t)$ denotes the concentration of signal substance produced by cell population. The constants $\tau \in \{0, 1\}$ and $\chi > 0$. Afterwards, many meaningful results have been studied for system (1.3), such as the global classical solvability of system and the blow-up analysis of classical solutions. When considering that the system (1.3) is a fully parabolic partial system, the conclusions in [3] showed that the classical solutions is globally bounded in one dimensional space. For $n = 2$, the results in [4] imply that if there exists suitable v_0 satisfying $\int_\Omega v_0 dx < \frac{4\pi}{\chi}$, then classical solutions of the system would be globally bounded; otherwise, if $\int_\Omega v_0 dx > \frac{4\pi}{\chi}$, the classical solutions of system system (1.3) would be unbounded in finite time [5]. In the case of $n \geq 3$, Winkler [6] proved that the blow-up solution will occur in finite or infinite time for some suitable initial data v_0 with $\int_\Omega v_0 > 0$. If the second equation was taken with the form of $w_t = \Delta w - w + g(v)$, where $0 \leq g(v) \leq Kv^\alpha$ with $K, \alpha > 0$, Liu and Tao [7] concluded the global boundedness of the classical solutions provided that $0 < \alpha < \frac{2}{n}$. Moreover, if the second equation was taken with the form of $0 = \Delta w - \frac{1}{|\Omega|}\int_\Omega v^\kappa + v^\kappa$ with $\kappa > 0$, Winkler [8] proved that if the number $\kappa > \frac{2}{n}$, then the classical solutions would be unbounded in finite time in radial setting; otherwise, if $\kappa < \frac{2}{n}$ the solutions remain bounded in $\Omega \times (0, \infty)$.

Afterwards, a more general chemotaxis model was considered with the form

$$\begin{cases} v_t = \nabla \cdot (D(v)\nabla v) - \nabla \cdot (S(v)\nabla w) + f(v), & x \in \Omega, \ t > 0, \\ \tau w_t = \Delta w - w + v, & x \in \Omega, \ t > 0, \end{cases} \tag{1.4}$$

where $D(v)$ and $S(v)$ are positive functions, which represent the diffusion intensity and chemoattractant intensity, respectively. Here, $f(v)$ is the logistic term to characterize the proliferation and death of cells. With regard to system (1.4), the existing results imply that there would be colorful dynamic behaviors by taking different forms of $D(v), S(v)$, and $f(v)$. For $\tau = 1$, let $D(v) = 1$ and $S(v) = v$, and if $f(v) \leq a - bv^2$ with $a, b > 0$, Winkler [9] obtained the global existence and boundedness of the solutions in a convex domain. Later on, Cao [10] concluded a similar property when removing the convexity of the domain. Moreover, the convergence of the solutions was also developed therein. For $\tau = 0$, let $D(v) = 1, S(v) = \chi v$, and $f(v) \leq v(a - bv)$ with $a, b, \chi > 0$, Tello and Winkler [11] established global classical solvability of the system provided that the parameters satisfy $\frac{n-2}{n}\chi < b$. For $\tau = 1$, assume that $D(v)$ and $S(v)$ are some nonlinear functions of v. Previous results indicate that global boundedness or blow-up can be determined by the value of $\frac{S(v)}{D(v)}$. For instance, Winkler [12] showed that if the ratio $\frac{S(v)}{D(v)}$ grows faster than $v^{\frac{n}{2}}$ as $v \to \infty$, there will be finite-time or infinite-time blow-up solutions to the system. Tao and Winkler [13] further revealed that such condition is optimal, which means that if $\frac{S(v)}{D(v)}$ grows slower than $v^{\frac{n}{2}}$, the solution would be globally bounded in a classical sense. In addition, some other interesting models related to (1.4), such as chemotaxis-Stokes (see [14]), chemotaxis models with density-suppressed motility (see [15]), and reaction-diffusion equation with a forcing term (see [16]), have been explored and many colorful dynamical behaviors can be found therein.

The Keller-Segel system can be viewed as an attraction-only or repulsion-only chemotaxis system with one kind of signal substance produced by cell. In the real environment, the cell population may simultaneously secrete multiple chemical signals, including attractants and repellents, which will affect the directional movement of cell population. Thus, the more complex chemotaxis (also called attraction-repulsion system [17]) system in the following is considered:

$$\begin{cases} v_t = \Delta v - \chi \nabla \cdot (v \nabla w_1) + \xi \nabla \cdot (v \nabla w_2) + f(v), & x \in \Omega,\ t > 0, \\ 0 = \Delta w_1 - \zeta w_1 + \eta v, & x \in \Omega,\ t > 0, \\ 0 = \Delta w_2 - \theta w_2 + \sigma v, & x \in \Omega,\ t > 0, \end{cases} \tag{1.5}$$

where $\chi, \xi, \eta, \zeta, \sigma, \theta > 0$. Similar to Keller-Segel model, there are also colourful dynamic behaviors in system (1.5) and its variants. For instance, when $f(v) \leq v(\varrho - \iota v)$ with $\varrho, \iota > 0$, for any nonnegative $v_0(x) \in C^0(\overline{\Omega})$, Zhang and Li [18] proved the global classical solvability if the parameters satisfy one of the three conditions: (a) $\eta\chi - \sigma\xi \leq \iota$; or (b) $n \leq 2$; or (c)$\frac{n-2}{n}(\eta\chi - \sigma\xi) < \iota$ with $n \geq 3$. For general logistic term $f(v) \leq v(\varrho - \iota v^s)$, if the second and the third equations were taken with the forms of $0 = \Delta w_1 - \eta w_1 + \zeta v^k$ and $0 = \Delta w_2 - \sigma w_2 + \theta v^l$, respectively, with $\varrho, \iota, k, l, s > 0$, Hong et al. [19] proved the global solvability of the system (1.5) under the condition that $k < \max\{l, s, \frac{2}{n}\}$ in the classical sense. Moreover, when $k = \max\{l, s\} \geq \frac{2}{n}$, the same properties can be also obtained if the parameters satisfy one of the three conditions (a) $k = l = s, \frac{kn-2}{kn}(\eta\chi - \sigma\xi) < \iota$; or (b) $k = l > s, \eta\chi - \sigma\xi < 0$; or (c) $k = s > l, \frac{kn-2}{kn}\eta\chi < \iota$. Based on [19] , Zhou et al. [20] further showed that the boundedness results still hold under the corresponding critical cases (a) $k = l = s, \frac{kn-2}{kn}(\eta\chi - \sigma\xi) = \iota$; or (b) $k = l > s, \eta\chi - \sigma\xi = 0, nk(nk - 2) < 4, 0 < k = l \leq 1$ with $n \geq 2$; or (c) $k = s > l, \frac{kn-2}{kn}\eta\chi = \iota$. The long time behavior of solutions was also studied therein. In addition, some interesting variants of system (1.5) involving nonlinear indirect mechanism of signals can be found in [21, 22].

Inspired by the contributions mentioned above, the present paper aims to further explore the global classical solvability and the long time behavior of the system (1.1) under the corresponding critical cases in [1]. More precisely, we state our conclusions as follows.

Theorem 1.1. *Let $v_0 \in C^0(\overline{\Omega})$ be nonnegative. Suppose that Ω is a bounded smooth domain of $\mathbb{R}^n (n \geq 1)$, and parameters $m, l, \theta \in \mathbb{R}$, $a, \chi, \xi, \alpha, \beta, \gamma, \delta, \gamma_1, \gamma_2 > 0, \kappa > 1$. Assume that $\theta + \gamma_1 = \max\{l + \gamma_2, \kappa\} \geq m + \frac{2}{n} + 1$ with $m > -\frac{2}{n}, n \geq 3$. If one of the following conditions holds, then the system (1.1) has a global and bounded classical solution*

(a) when $\theta+\gamma_1 = l+\gamma_2 = \kappa$, if $\theta \geq l \geq 1$ and $\frac{[(\kappa-1-m)n-2](2\alpha\chi-\gamma\xi)}{2(l-1)+(\kappa-1-m)n} = b$, or $l \geq \theta \geq 1$ and $\frac{2\alpha\chi[(\kappa-1-m)n-2]}{2(\theta-1)+(\kappa-1-m)n} = b$;
(b) when $\theta + \gamma_1 = \kappa > l + \gamma_2$, if $\theta \geq 1$ and $\frac{2\alpha\chi[(\kappa-1-m)n-2]}{2(\theta-1)+(\kappa-1-m)n} = b$.

The main idea to prove Theorem 1.1 comes from [23]. Such idea enables us to deal with a generalized attraction-repulsion system under some critical parameter cases, which is different from the method developed in [19] to handle the sub-critical parameter cases (see Lemma 3.3 in [1]). In a sense, the boundedness criteria in the present work can also be regarded as an extension of [20]. Due to considering the influence of diffusion functions ϕ, φ, and ψ, the techniques used in this paper are more generalized than that in [20, 23] (for instance, please see the definition of $h(p)$ in (3.3) and Lemmas 3.4–3.6), which are more complicated involving a large amount of calculations.

Remark 1.2. *Here, it should be pointed out that the critical parameter conditions in Theorem 1.1 only correspond to the cases where the equality signs hold in the boundedness conditions in [1], which may be not the borderline cases distinguishing the boundedness and blow-up of solutions. However, it seems that we may use the same methods as in this paper to explore the borderline cases for boundedness if we could get them.*

Furthermore, a conclusion on the long time behavior of the classical solutions to the system (1.1) has been developed.

Theorem 1.3. *Assume that the conditions in Theorem 1.1 hold. If the parameter $b > 0$ is sufficiently large, then there exists $C > 0$ such that*

$$\|v - c\|_{L^\infty(\Omega)} + \|w_1 - \frac{\alpha}{\beta}c^{\gamma_1}\|_{L^\infty(\Omega)} + \|w_2 - \frac{\gamma}{\delta}c^{\gamma_2}\|_{L^\infty(\Omega)} \leq Ce^{-\lambda t},$$

for all $t \geq 0$, where $c = (\frac{a}{b})^{\frac{1}{\kappa-1}}$ and $\lambda = \min\{c\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\} > 0$ with

$$\varepsilon_1 = \tfrac{a}{c} - \left[4^{-(\gamma_1+1)}\tfrac{2\lambda_1\chi^2\alpha^2}{\beta}c^{2\gamma_1-1} + 4^{-(\gamma_2+1)}\tfrac{2\lambda_2\xi^2\gamma^2}{\delta}c^{2\gamma_2-1}\right],$$

and

$$\varepsilon_2 = \tfrac{a}{c} - \tfrac{c}{8}\left[\tfrac{\lambda_1\chi^2\alpha^2\gamma_1^2}{\beta}(R+c)^{2\gamma_1-2} + \tfrac{\lambda_2\xi^2\gamma^2\gamma_2^2}{\delta}(R+c)^{2\gamma_2-2}\right],$$

and

$$\varepsilon_3 = \tfrac{a}{c} - \tfrac{c}{8}\left[\tfrac{\lambda_1\chi^2\alpha^2}{\beta}4^{1-\gamma_1}c^{2\gamma_1-2} + \tfrac{\lambda_2\xi^2\gamma^2\gamma_2^2}{\delta}(R+c)^{2\gamma_2-2}\right],$$

as well as

$$\varepsilon_4 = \tfrac{a}{c} - \tfrac{c}{8}\left[\tfrac{\lambda_1\chi^2\alpha^2\gamma_1^2}{\beta}(R+c)^{2\gamma_1-2} + \tfrac{\lambda_2\xi^2\gamma^2}{\delta}4^{1-\gamma_2}c^{2\gamma_2-2}\right],$$

for $R > 0$ and $\lambda_1, \lambda_2 > 0$ as given in (4.1) and (4.2), respectively.

We shall utilize the method developed in [24–26] to prove Theorem 1.3. Compared with [26], our system is more generalized, involving nonlinear diffusion functions and nonlinear signal production mechanisms with general exponents $\gamma_1, \gamma_2 > 0$, so we have to modify the corresponding method [26, Theorem 3.3] to overcome the difficulties arising from these items (please see Lemma 4.2). Moreover, in Theorem 1.3 we also extend the asymptotic behavior result established in [20, Theorem 1.2].

The remaining parts of this paper are carried out as follows. In Section 2, we first show a conclusion involving the local existence of classical solutions and then give a priori estimates of the solutions. In Section 3, we obtain L^p−boundedness for v and prove Theorem 1.1 by using Moser iteration. Finally, we give the stability analysis of solutions to system (1.1).

2. Preliminaries

To begin with, we give a lemma involving local solvability of the system. The proof is quite standard, and it can be derived from [27].

Lemma 2.1. *Let Ω be a bounded domain of $\mathbb{R}^n (n \geq 1)$ with smooth boundary and nonnegative initial data $v_0 \in C^0(\overline{\Omega})$. Then, there exists $T_{\max} \in (0, \infty]$ such that the system* (1.1) *admits a nonnegative classical solution $(v, w_1, w_2) \in C^0(\overline{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\overline{\Omega} \times (0, T_{\max}))$ in $\Omega \times (0, T_{\max})$ with*

$$v, w_1, w_2 \geq 0 \ \text{ in } \ \overline{\Omega} \times (0, T_{\max}). \tag{2.1}$$

Additionally,

$$\text{if } \ T_{\max} < \infty, \ \text{ then } \ \limsup_{t \nearrow T_{\max}} \|v(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \tag{2.2}$$

In order to obtain the proof of the boundedness of $\int_\Omega (v+1)^p$, the following conclusion is useful. The proof is similar to [1, Lemma 2.3].

Lemma 2.2. *(cf. [1, Lemma 2.3]) Assume that (v, w_1, w_2) is a solution of system* (1.1)*. For arbitrary $\tau > 1$ and $\eta > 0$, we have*

$$\int_\Omega w_2^\tau \leq \eta \int_\Omega v^{\gamma_2 \tau} + c_0, \tag{2.3}$$

where $c_0 > 0$ depends only on τ, η, and γ_2, and γ_2 is as in system (1.1)*. Moreover, we have the estimate*

$$\int_\Omega v \leq \max\left\{ \int_\Omega v_0, \left(\frac{a}{b}\right)^{\frac{1}{\kappa-1}} |\Omega| \right\} \ \text{for all } t \in (0, T_{\max}). \tag{2.4}$$

3. Global existence and boundedness

In this section, we shall first study the L^p−boundedness of v under conditions (a) and (b) in Theorem 1.1.

Lemma 3.1. *For any $p > 1$, if the conditions in Theorem 1.1 hold, then we can find $C > 0$ such that the following inequality holds:*

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p + \int_\Omega (v+1)^p + \frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla (v+1)^{\frac{p+m}{2}}|^2 \\
\le& \frac{2\alpha\chi(p-1)}{p+\theta-1}\int_\Omega v^{p+\theta+\gamma_1-1} + \frac{\xi\delta(p-1)}{p+l-1}\int_\Omega (v+1)^{p+l-1}w_2 - \frac{\gamma\xi(p-1)}{p+l-1}\int_\Omega v^{p+l+\gamma_2-1} \\
&+ (a+1)\int_\Omega (v+1)^p - b\int_\Omega v^{p+\kappa-1} + C, \quad t\in(0,T_{\max}).
\end{aligned} \tag{3.1}
$$

Proof. The proof process is similar to [1, Lemma 3.1], and here we omit it.

At the beginning, we study the first case of condition (a) in Theorem1.1: Namely, the parameters satisfy $\theta+\gamma_1 = l+\gamma_2 = \kappa > m+\frac{2}{n}+1$, and $\frac{[(\kappa-1-m)n-2](2\alpha\chi-\gamma\xi)}{2(l-1)+(\kappa-1-m)n} = b$ with $\theta\ge l\ge 1$, $m>-\frac{2}{n}$ and $n\ge 3$. Then, one can get from (3.1) that

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p + \int_\Omega (v+1)^p + \frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla (v+1)^{\frac{p+m}{2}}|^2 \\
\le& \left[\frac{(2\alpha\chi-\gamma\xi)(p-1)}{p+ł-1} - b\right]\int_\Omega v^{p+\theta+\gamma_1-1} + \frac{\delta\xi(p-1)}{p+l-1}\int_\Omega (v+1)^{p+l-1}w_2 \\
&+ (a+1)\int_\Omega (v+1)^p + C, \quad t\in(0,T_{\max}).
\end{aligned} \tag{3.2}
$$

Note that $b>0$, and thus the the equation $\frac{[(\kappa-1-m)n-2](2\alpha\chi-\gamma\xi)}{2(l-1)+(\kappa-1-m)n} = b$ means $2\alpha\chi-\gamma\xi>0$. The following lemma is helpful to prove the L^p-boundedness of v for any $p>1$.

Lemma 3.2. *Assume that the parameters in system* (1.1) *satisfy* $a,b,\alpha,\beta,\chi,\xi,\gamma,\delta,\gamma_1,\gamma_2>0,\kappa>1$, *and* $m,\theta,l\in\mathbb{R}$. *Let* $n\ge 3$ *and* $p>1$. *Under the first case of condition (a) in Theorem 1.1, we define*

$$
p_1 := \frac{(\kappa-1-m)n}{2} \quad \text{and} \quad h(p) := \frac{(2\alpha\chi-\gamma\xi)(p-1)}{p+l-1} - b. \tag{3.3}
$$

Then, one may obtain

$$
h(p)<0 \text{ if } 1<p<p_1, \quad h(p)>0 \text{ if } p>p_1, \text{ and } \lim_{p\to p_1} h(p) = 0. \tag{3.4}
$$

Proof. Since $b = \frac{[(\kappa-1-m)n-2](2\alpha\chi-\gamma\xi)}{2(l-1)+(\kappa-1-m)n}$, we deduce

$$
\begin{aligned}
h(p) &= \frac{(2\alpha\chi-\gamma\xi)(p-1)}{p+l-1} - \frac{[(\kappa-1-m)n-2](2\alpha\chi-\gamma\xi)}{2(l-1)+(\kappa-1-m)n} \\
&= \left[\frac{p-1}{p+l-1} - \frac{(\kappa-1-m)n-2}{2(l-1)+(\kappa-1-m)n}\right](2\alpha\chi-\gamma\xi).
\end{aligned} \tag{3.5}
$$

Thus, the result (3.4) can be directly concluded from (3.5).

Lemma 3.3. *Let* $n\ge 3$ *and* $1<p<p_1$ *with* p_1 *defined in* (3.3). *Under the first case of condition (a) in Theorem 1.1, there exists* $C(p)>0$ *such that*

$$
\int_\Omega (v+1)^p \le C(p), \quad t\in(0,T_{\max}). \tag{3.6}
$$

Proof. From Lemma 3.2, it is easy to see that $h(p) < 0$ for any $1 < p < p_1$. Thus, we can obtain from (3.2) that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p + \int_\Omega (v+1)^p + \frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla (v+1)^{\frac{p+m}{2}}|^2\\ &\le h(p)\int_\Omega v^{p+\theta+\gamma_1-1} + \frac{\delta\xi(p-1)}{p+l-1}\int_\Omega (v+1)^{p+l-1}w_2\\ &\quad + (a+1)\int_\Omega (v+1)^p + C,\quad t\in(0,T_{\max}).\end{aligned}\tag{3.7}$$

Since $\theta+\gamma_1 = l+\gamma_2$, we conclude from Young's inequality and Lemma 2.2 that

$$\begin{aligned}\frac{\delta\xi(p-1)}{p+l-1}\int_\Omega (v+1)^{p+l-1}w_2 &\le \frac{\vartheta}{2}\int_\Omega (v+1)^{p+\theta+\gamma_1-1} + C_\vartheta\int_\Omega w_2^{\frac{p+\theta+\gamma_1-1}{\theta+\gamma_1-l}}\\ &= \frac{\vartheta}{2}\int_\Omega (v+1)^{p+\theta+\gamma_1-1} + C_\vartheta\int_\Omega w_2^{\frac{p+\theta+\gamma_1-1}{\gamma_2}}\\ &\le \frac{\vartheta}{2}\int_\Omega (v+1)^{p+\theta+\gamma_1-1} + \frac{\vartheta}{2}\int_\Omega v^{p+\theta+\gamma_1-1} + \tilde{C}\\ &\le \vartheta\int_\Omega (v+1)^{p+\theta+\gamma_1-1} + \tilde{C},\quad t\in(0,T_{\max}),\end{aligned}\tag{3.8}$$

with any $\vartheta > 0$ and some $\tilde{C} > 0$. Choosing $\vartheta = -\frac{h(p)}{2}$ in (3.8), it is easy to get from (3.7) that

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p + \int_\Omega (v+1)^p + \frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla (v+1)^{\frac{p+m}{2}}|^2\\ &\le \frac{h(p)}{2}\int_\Omega (v+1)^{p+\theta+\gamma_1-1} + (a+1)\int_\Omega (v+1)^p + c_1\\ &\le (a+1)\int_\Omega (v+1)^p + c_1,\quad t\in(0,T_{\max}),\end{aligned}\tag{3.9}$$

with $c_1 = C+\tilde{C} > 0$. Invoking the Gagliardo-Nirenberg inequality and (2.4), one may choose $c_2, c_3 > 0$ such that

$$\begin{aligned}(a+1)\int_\Omega (v+1)^p =&(a+1)\|(v+1)^{\frac{p+m}{2}}\|_{L^{\frac{2p}{p+m}}(\Omega)}^{\frac{2p}{p+m}}\\ \le& c_2\|\nabla(v+1)^{\frac{p+m}{2}}\|_{L^2(\Omega)}^{\frac{2p}{p+m}\cdot b_1}\cdot\|(v+1)^{\frac{p+m}{2}}\|_{L^{\frac{2}{p+m}}(\Omega)}^{\frac{2p}{p+m}\cdot(1-b_1)}\\ &+ c_2\|(v+1)^{\frac{p+m}{2}}\|_{L^{\frac{2}{p+m}}(\Omega)}^{\frac{2p}{p+m}}\\ \le& c_3\|\nabla(v+1)^{\frac{p+m}{2}}\|_{L^2(\Omega)}^{\frac{2p}{p+m}\cdot b_1} + c_3,\quad t\in(0,T_{\max}),\end{aligned}\tag{3.10}$$

where $b_1 = \frac{\frac{m+p}{2}-\frac{p+m}{2p}}{\frac{m+p}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1)$ due to $m > -\frac{2}{n}$. Moreover, the inequality $\frac{2p}{p+m}\cdot b_1 < 2$ can be also ensured by $m+\frac{2}{n} > 0$. From Young's inequality, we obtain that

$$c_3\|\nabla(v+1)^{\frac{p+m}{2}}\|_{L^2(\Omega)}^{\frac{2p}{p+m}\cdot b_1} \le \frac{2(p-1)}{(p+m)^2}\int_\Omega |\nabla(v+1)^{\frac{p+m}{2}}|^2 + c_4,\tag{3.11}$$

with some $c_4 > 0$. We substitute (3.11) into (3.9) to get

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p + \int_\Omega (v+1)^p \le c_1 + c_3 + c_4, \quad t \in (0, T_{\max}). \tag{3.12}$$

With an application of the ODE comparison, we can deduce the desired results of Lemma 3.3, where $C(p) = \max\{c_1 + c_3 + c_4, \int_\Omega (v_0+1)^p\}$.

Lemma 3.4. *For $n \ge 3$, under the first case of condition (a) in Theorem 1.1, there exists $C(p) > 0$ such that*

$$\int_\Omega (v+1)^p \le C(p) \text{ for all } t \in (0, T_{\max}) \text{ with } p = p_1,\ p_1 \text{ defined in (3.3)}. \tag{3.13}$$

Proof. For $p = p_1 = \frac{(\kappa-1-m)n}{2}$ and $\theta + \gamma_1 = l + \gamma_2$, we set $\varepsilon > 0$ sufficiently small to satisfy

$$n\varepsilon(l+\gamma_2-1)\,(p+\theta+\gamma_1-1-\varepsilon) < 2(p-\varepsilon)(\theta+\gamma_1-1-\varepsilon)(p_1 - \frac{n\varepsilon}{2}). \tag{3.14}$$

Adding $np(l+\gamma_2-1)(p+\theta+\gamma_1-1-\varepsilon)$ to both sides of (3.14), we see that

$$\begin{aligned} &np(l+\gamma_2-1)(p+\theta+\gamma_1-1-\varepsilon) + n\varepsilon(l+\gamma_2-1)\,(p+\theta+\gamma_1-1-\varepsilon) \\ &< np(l+\gamma_2-1)(p+\theta+\gamma_1-1-\varepsilon) + 2(p-\varepsilon)(\theta+\gamma_1-1-\varepsilon)(p_1 - \frac{n\varepsilon}{2}), \end{aligned} \tag{3.15}$$

which implies

$$\begin{aligned} &np(l+\gamma_2-1)(p+\theta+\gamma_1-1-\varepsilon) \\ &< n(p-\varepsilon)(l+\gamma_2-1)(p+\theta+\gamma_1-1-\varepsilon) + 2(p-\varepsilon)(p_1 - \frac{n\varepsilon}{2})(\theta+\gamma_1-1-\varepsilon). \end{aligned} \tag{3.16}$$

It is sufficient to obtain $h(p) = 0$ for $p = p_1$. By recalling (3.7), we can obtain

$$\begin{aligned} &\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p + \int_\Omega (v+1)^p + \frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla (v+1)^{\frac{p+m}{2}}|^2 \\ &\le \frac{\delta\xi(p-1)}{p+l-1}\int_\Omega (v+1)^{p+l-1} w_2 + (a+1)\int_\Omega (v+1)^p + C, \quad t \in (0, T_{\max}). \end{aligned} \tag{3.17}$$

The Gagliardo-Nirenberg inequality enables us to find $c_5 > 0$ such that

$$\begin{aligned} \int_\Omega (v+1)^{p+\theta+\gamma_1-1-\varepsilon} &= \|(v+1)^{\frac{p+m}{2}}\|_{L^{\frac{2(p+\theta+\gamma_1-1-\varepsilon)}{p+m}}(\Omega)}^{\frac{2(p+\theta+\gamma_1-1-\varepsilon)}{p+m}} \\ &\le c_5 \|\nabla (v+1)^{\frac{p+m}{2}}\|_{L^2(\Omega)}^{\frac{2(p+\theta+\gamma_1-1-\varepsilon)}{p+m}\cdot b_2} \cdot \|(v+1)^{\frac{p+m}{2}}\|_{L^{\frac{2(p_1-\frac{n\varepsilon}{2})}{p+m}}(\Omega)}^{\frac{2(p+\theta+\gamma_1-1-\varepsilon)}{p+m}\cdot(1-b_2)} \\ &\quad + c_5 \|(v+1)^{\frac{p+m}{2}}\|_{L^{\frac{2(p_1-\frac{n\varepsilon}{2})}{p+m}}(\Omega)}^{\frac{2(p+\theta+\gamma_1-1-\varepsilon)}{p+m}}, \quad t \in (0, T_{\max}), \end{aligned} \tag{3.18}$$

where $b_2 = \frac{\frac{p+m}{2(p_1-\frac{n\varepsilon}{2})} - \frac{p+m}{2(p+\theta+\gamma_1-1-\varepsilon)}}{\frac{p+m}{2(p_1-\frac{n\varepsilon}{2})} + \frac{1}{n} - \frac{1}{2}} = \frac{n(p+m)[(p+\theta+\gamma_1-1-\varepsilon)-(p_1-\frac{n\varepsilon}{2})]}{(p+\theta+\gamma_1-1-\varepsilon)[n(p+m)+2(p_1-\frac{n\varepsilon}{2})-n(p_1-\frac{n\varepsilon}{2})]} \in (0,1)$. By a simple computation, we get

$$\frac{2(p+\theta+\gamma_1-1-\varepsilon)}{p+m}\cdot b_2 = \frac{2[n(p+\theta+\gamma_1-1-\varepsilon) - n(p_1 - \frac{n\varepsilon}{2})]}{n(p+m) + 2(p_1 - \frac{n\varepsilon}{2}) - n(p_1 - \frac{n\varepsilon}{2})} = 2,$$

due to $p_1 = \frac{(\kappa-1-m)n}{2}$ defined in (3.3). The Lemma 3.3 implies that the term $\|(v+1)^{\frac{p+m}{2}}\|_{L^{\frac{2(p_1-\frac{n\varepsilon}{2})}{p+m}}(\Omega)}$ is bounded for $p = p_1 - \frac{n\varepsilon}{2} < p_1$. Thus, there exists $c_6 > 0$ such that

$$\int_\Omega (v+1)^{p+\theta+\gamma_1-1-\varepsilon} \le c_6\|\nabla(v+1)^{\frac{p+m}{2}}\|^2_{L^2(\Omega)} + c_6, \quad t\in(0,T_{\max}). \tag{3.19}$$

Based on (3.17) and Young's inequality, it is easy to see

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p + \int_\Omega (v+1)^p + \frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla(v+1)^{\frac{p+m}{2}}|^2\\ &\le \frac{p-1}{(m+p)^2c_6}\int_\Omega (v+1)^{p+\theta+\gamma_1-1-\varepsilon} + c_7\int_\Omega w_2^{\frac{p+\theta+\gamma_1-1-\varepsilon}{\theta+\gamma_1-l-\varepsilon}} + (a+1)\int_\Omega (v+1)^p + C,\end{aligned} \tag{3.20}$$

where

$$c_7 = \frac{\delta\xi(p-1)}{p+l-1}\left(\frac{p+\theta+\gamma_1-1-\varepsilon}{\delta\xi(p+m)^2c_6}\right)^{-\frac{p+l-1}{\theta+\gamma_1-l-\varepsilon}}\cdot\frac{\theta+\gamma_1-l-\varepsilon}{p+\theta+\gamma_1-1-\varepsilon} > 0.$$

Next we deal with the term $c_7\int_\Omega w_2^{\frac{p+\theta+\gamma_1-1-\varepsilon}{\theta+\gamma_1-l-\varepsilon}}$. Multiplying the third equation of system (1.1) with $w_2^{\frac{p}{\theta+\gamma_1-1-\varepsilon}}$, it is not difficult to get from Young's inequality that

$$\begin{aligned}&\frac{4p(\theta+\gamma_1-1-\varepsilon)}{(p+\theta+\gamma_1-1-\varepsilon)^2}\int_\Omega |\nabla w_2^{\frac{p+\theta+\gamma_1-1-\varepsilon}{2(\theta+\gamma_1-1-\varepsilon)}}|^2 + \delta\int_\Omega w_2^{\frac{p+\theta+\gamma_1-1-\varepsilon}{\theta+\gamma_1-1-\varepsilon}}\\ &= \gamma\int_\Omega v^{\gamma_2}w_2^{\frac{p}{\theta+\gamma_1-1-\varepsilon}}\\ &\le \frac{\delta(p-1)}{(m+p)^2c_6c_7}\int_\Omega (v+1)^{p+\theta+\gamma_1-1-\varepsilon} + c_8\int_\Omega w_2^{\frac{p}{\theta+\gamma_1-1-\varepsilon}\cdot\frac{p+\theta+\gamma_1-1-\varepsilon}{p+\theta+\gamma_1-1-\varepsilon-\gamma_2}}\\ &\le \frac{\delta(p-1)}{(m+p)^2c_6c_7}\int_\Omega (v+1)^{p+\theta+\gamma_1-1-\varepsilon} + c_9\int_\Omega w_2^{\frac{p(p+\theta+\gamma_1-1-\varepsilon)}{(\theta+\gamma_1-1-\varepsilon)(p-\varepsilon)}} + c_{10},\end{aligned} \tag{3.21}$$

with $c_8, c_9, c_{10} > 0$. An application of the Gagliardo-Nirenberg inequality implies that there exists $c_{11} > 0$ such that

$$\begin{aligned}\int_\Omega w_2^{\frac{p(p+\theta+\gamma_1-1-\varepsilon)}{(\theta+\gamma_1-1-\varepsilon)(p-\varepsilon)}} &= \|w_2^{\frac{p+\theta+\gamma_1-1-\varepsilon}{2(\theta+\gamma_1-1-\varepsilon)}}\|^{\frac{2p}{p-\varepsilon}}_{L^{\frac{2p}{p-\varepsilon}}(\Omega)}\\ &\le c_{11}\|\nabla w_2^{\frac{p+\theta+\gamma_1-1-\varepsilon}{2(\theta+\gamma_1-1-\varepsilon)}}\|^{\frac{2p}{p-\varepsilon}\cdot b_3}_{L^2(\Omega)}\cdot\|w_2^{\frac{p+\theta+\gamma_1-1-\varepsilon}{2(\theta+\gamma_1-1-\varepsilon)}}\|^{\frac{2p}{p-\varepsilon}\cdot(1-b_3)}_{L^{\frac{2(\theta+\gamma_1-1-\varepsilon)(p_1-\frac{n\varepsilon}{2})}{(p+\theta+\gamma_1-1-\varepsilon)(l+\gamma_2-1)}}(\Omega)}\\ &\quad + c_{11}\|w_2^{\frac{p+\theta+\gamma_1-1-\varepsilon}{2(\theta+\gamma_1-1-\varepsilon)}}\|^{\frac{2p}{p-\varepsilon}}_{L^{\frac{2(\theta+\gamma_1-1-\varepsilon)(p_1-\frac{n\varepsilon}{2})}{(p+\theta+\gamma_1-1-\varepsilon)(l+\gamma_2-1)}}(\Omega)}, \quad t\in(0,T_{\max}),\end{aligned} \tag{3.22}$$

where $b_3 = \frac{\frac{(p+\theta+\gamma_1-1-\varepsilon)(l+\gamma_2-1)}{2(\theta+\gamma_1-1-\varepsilon)(p_1-\frac{n\varepsilon}{2})} - \frac{p-\varepsilon}{2p}}{\frac{(p+\theta+\gamma_1-1-\varepsilon)(l+\gamma_2-1)}{2(\theta+\gamma_1-1-\varepsilon)(p_1-\frac{n\varepsilon}{2})} + \frac{1}{n} - \frac{1}{2}} \in (0,1)$ for $\varepsilon > 0$ small enough. Let $\zeta = \theta+\gamma_1-1$. Since $p = p_1 = \frac{(\kappa-1-m)n}{2}$ and (3.16), one may obtain

$$\begin{aligned}&\frac{2p}{p-\varepsilon}\cdot b_3\\&=\frac{2[np(l+\gamma_2-1)(p+\zeta-\varepsilon)-n(\zeta-\varepsilon)(p_1-\frac{n\varepsilon}{2})(p-\varepsilon)]}{n(p-\varepsilon)(l+\gamma_2-1)(p+\zeta-\varepsilon)+2(\zeta-\varepsilon)(p-\varepsilon)(p_1-\frac{n\varepsilon}{2})-n(\zeta-\varepsilon)(p-\varepsilon)(p_1-\frac{n\varepsilon}{2})}\\&<2.\end{aligned}\tag{3.23}$$

By applying a classical L^p-estimate for the second derivatives of the elliptic equation (see [28, Theorems 9 and 11]) and Lemma 3.3, we can find $c_{12}, c_{13} > 0$ large enough such that

$$\|w_2^{\frac{p+\theta+\gamma_1-1-\varepsilon}{2(\theta+\gamma_1-1-\varepsilon)}}\|_{L^{\frac{2(\theta+\gamma_1-1-\varepsilon)(p_1-\frac{n\varepsilon}{2})}{(p+\theta+\gamma_1-1-\varepsilon)(l+\gamma_2-1)}}(\Omega)} \le c_{12}\|v^{\frac{\gamma_2(p+\theta+\gamma_1-1-\varepsilon)}{2(\theta+\gamma_1-1-\varepsilon)}}\|_{L^{\frac{2(\theta+\gamma_1-1-\varepsilon)(p_1-\frac{n\varepsilon}{2})}{(p+\theta+\gamma_1-1-\varepsilon)(l+\gamma_2-1)}}(\Omega)} \le c_{13}.\tag{3.24}$$

Combining (3.22)–(3.24), for any $\epsilon_1 > 0$ one may choose $c_{14} = c_{14}(\epsilon_1) > 0$ such that

$$\begin{aligned}\int_\Omega w_2^{\frac{p(p+\theta+\gamma_1-1-\varepsilon)}{(\theta+\gamma_1-1-\varepsilon)(p-\varepsilon)}} &\le c_{13}\left(\int_\Omega |\nabla w_2^{\frac{p+\theta+\gamma_1-1-\varepsilon}{2(\theta+\gamma_1-1-\varepsilon)}}|^2\right)^{\frac{p}{p-\varepsilon}\cdot b_2}+c_{13}\\&\le \epsilon_1\int_\Omega |\nabla w_2^{\frac{p+\theta+\gamma_1-1-\varepsilon}{2(\theta+\gamma_1-1-\varepsilon)}}|^2+c_{14}.\end{aligned}\tag{3.25}$$

Let $\epsilon_1 = \frac{2p(\theta+\gamma_1-1-\varepsilon)}{(p+\theta+\gamma_1-1-\varepsilon)^2 c_9}$. Then, a combination of (3.21) and (3.25) leads to

$$\int_\Omega w_2^{\frac{p+\theta+\gamma_1-1-\varepsilon}{\theta+\gamma_1-1-\varepsilon}} \le \frac{p-1}{(p+m)^2c_6c_7}\int_\Omega (v+1)^{p+\theta+\gamma_1-1-\varepsilon}+c_{15},\quad t\in(0,T_{\max}),\tag{3.26}$$

with some $c_{15} > 0$. From (3.19) and (3.26), we write the inequality (3.20) as follows:

$$\begin{aligned}&\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p+\int_\Omega (v+1)^p+\frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla (v+1)^{\frac{p+m}{2}}|^2\\&\le \frac{2(p-1)}{(p+m)^2c_6}\int_\Omega (v+1)^{p+\theta+\gamma_1-1-\varepsilon}+(a+1)\int_\Omega (v+1)^p+c_{16}\\&\le \frac{2(p-1)}{(p+m)^2c_6}(c_6\int_\Omega |\nabla (v+1)^{\frac{p+m}{2}}|^2+c_6)+(a+1)\int_\Omega (v+1)^p+c_{16},\end{aligned}\tag{3.27}$$

with $c_{16} > 0$. Therefore, we can find $c_{17} > 0$ such that

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p+\int_\Omega (v+1)^p \le -\frac{2(p-1)}{(p+m)^2}\int_\Omega |\nabla (v+1)^{\frac{p+m}{2}}|^2+(a+1)\int_\Omega (v+1)^p+c_{17}.\tag{3.28}$$

In view of the Gagliardo-Nirenberg inequality and (2.4), we get

$$
\begin{aligned}
(a+1)\int_\Omega (v+1)^p &= (a+1)\|(v+1)^{\frac{p+m}{2}}\|_{L^{\frac{2}{p+m}}(\Omega)}^{\frac{2p}{p+m}} \\
&\le c_{18}\|\nabla(v+1)^{\frac{p+m}{2}}\|_{L^2(\Omega)}^{\frac{2p}{p+m}\cdot b_4}\cdot\|(v+1)^{\frac{p+m}{2}}\|_{L^{\frac{2}{p+m}}(\Omega)}^{\frac{2p}{p+m}\cdot(1-b_4)} \\
&\quad + c_{18}\|(v+1)^{\frac{p+m}{2}}\|_{L^{\frac{2}{p+m}}(\Omega)}^{\frac{2p}{p+m}} \\
&\le c_{19}\|\nabla(v+1)^{\frac{p+m}{2}}\|_{L^2(\Omega)}^{\frac{2p}{p+m}\cdot b_4} + c_{19}, \quad t\in(0,T_{\max}),
\end{aligned}
\tag{3.29}
$$

with some $c_{18}, c_{19} > 0$, where $b_4 = \frac{\frac{m+p}{2}-\frac{p+m}{2p}}{\frac{m+p}{2}+\frac{1}{n}-\frac{1}{2}} \in (0,1)$ due to $p = p_1 = \frac{(\kappa-m-1)n}{2}$. Since $m > -\frac{2}{n}$, we know that $\frac{2p}{p+m}\cdot b_4 < 2$. Thus, by Young's inequality, there exists $c_{20} > 0$ such that

$$
c_{19}\|\nabla(v+1)^{\frac{p+m}{2}}\|_{L^2(\Omega)}^{\frac{2p}{p+m}\cdot b_4} \le \frac{2(p-1)}{(p+m)^2}\int_\Omega |\nabla(v+1)^{\frac{p+m}{2}}|^2 + c_{20}. \tag{3.30}
$$

Collecting (3.30) and (3.28), we gain that

$$
\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p + \int_\Omega (v+1)^p \le c_{21}, \quad t\in(0,T_{\max}),
$$

with some $c_{21} > 0$. Hence, the proof of Lemma 3.4 is complete.

Lemma 3.5. *Let $n \ge 3$ and $p_1 < p \le p_1 + \sigma$ with p_1 defined in* (3.3) *and $\sigma > 0$ small enough. Under the first case of condition (a) in Theorem 1.1, one may find $C = C(p) > 0$ satisfying*

$$
\int_\Omega (v+1)^p \le C, \quad t\in(0,T_{\max}). \tag{3.31}
$$

Proof. Recalling Lemma 3.2, it is clear to get that $h(p) > 0$ if $p > p_1$. Moreover, Lemma 3.4 implies that $\|(v+1)\|_{L^{p_1}(\Omega)} \le c_{22}$ with some $c_{22} > 0$. Taking $\vartheta = \frac{h(p)}{2}$ in (3.8) and substituting this into (3.7), we deduce

$$
\begin{aligned}
&\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p + \int_\Omega (v+1)^p + \frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla(v+1)^{\frac{p+m}{2}}|^2 \\
&\le \frac{3}{2}h(p)\int_\Omega (v+1)^{p+\theta+\gamma_1-1} + (a+1)\int_\Omega (v+1)^p + c_{23}, \quad t\in(0,T_{\max}),
\end{aligned}
\tag{3.32}
$$

with some $c_{23} > 0$. By Young's inequality, it is not difficult to check that

$$
(a+1)\int_\Omega (v+1)^p \le \frac{h(p)}{2}\int_\Omega (v+1)^{p+\theta+\gamma_1-1} + c_{24}, \quad t\in(0,T_{\max}), \tag{3.33}
$$

with some $c_{24} > 0$. Thus, we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p + \int_\Omega (v+1)^p + \frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla (v+1)^{\frac{p+m}{2}}|^2$$
$$\leq 2h(p)\int_\Omega (v+1)^{p+\theta+\gamma_1-1} + c_{25}, \quad t\in(0,T_{\max}), \tag{3.34}$$

where $c_{25} = c_{23} + c_{24} > 0$. By the Gagliardo-Nirenberg inequality, there exists $c_{26} > 0$ such that

$$2h(p)\int_\Omega (v+1)^{p+\theta+\gamma_1-1} = 2h(p)\|(v+1)^{\frac{p+m}{2}}\|^{\frac{2(p+\theta+\gamma_1-1)}{p+m}}_{L^{\frac{2(p+\theta+\gamma_1-1)}{p+m}}(\Omega)}$$
$$\leq h(p)\cdot c_{26}\|\nabla(v+1)^{\frac{p+m}{2}}\|^{\frac{2(p+\theta+\gamma_1-1)}{p+m}\cdot b_5}_{L^2(\Omega)}\cdot \|(v+1)^{\frac{p+m}{2}}\|^{\frac{2(p+\theta+\gamma_1-1)}{p+m}\cdot(1-b_5)}_{L^{\frac{2p_1}{p+m}}(\Omega)}$$
$$+ h(p)\cdot c_{26}\|(v+1)^{\frac{p+m}{2}}\|^{\frac{2(p+\theta+\gamma_1-1)}{p+m}}_{L^{\frac{2p_1}{p+m}}(\Omega)}$$
$$\leq h(p)\cdot c_{26}\|\nabla(v+1)^{\frac{p+m}{2}}\|^{\frac{2(p+\theta+\gamma_1-1)}{p+m}\cdot b_5}_{L^2(\Omega)}\cdot \|(v+1)\|^{(p+\theta+\gamma_1-1)(1-b_5)}_{L^{p_1}(\Omega)}$$
$$+ h(p)\cdot c_{26}\|(v+1)\|^{p+\theta+\gamma_1-1}_{L^{p_1}(\Omega)}$$
$$\leq h(p)\cdot c_{26}\cdot c_{22}^{(p+\theta+\gamma_1-1)\cdot(1-b_5)}\|\nabla(v+1)^{\frac{p+m}{2}}\|^{\frac{2(p+\theta+\gamma_1-1)}{p+m}\cdot b_5}_{L^2(\Omega)} + h(p)\cdot c_{26}\cdot c_{22}^{p+\theta+\gamma_1-1}, \tag{3.35}$$

where $b_5 := \frac{\frac{p+m}{2p_1}-\frac{p+m}{2(p+\theta+\gamma_1-1)}}{\frac{p+m}{2p_1}+\frac{1}{n}-\frac{1}{2}} = \frac{n(p+m)(p+\theta+\gamma_1-1-p_1)}{(p+\theta+\gamma_1-1)[(p+m)n+2p_1-np_1]} \in (0,1)$. By the definition of $p_1 = \frac{(\kappa-1-m)n}{2}$ in (3.3), we directly compute that

$$\frac{2(p+\theta+\gamma_1-1)}{p+m}\cdot b_5 = \frac{2(p+\theta+\gamma_1-1)}{p+m}\cdot\frac{n(p+m)(p+\theta+\gamma_1-1-p_1)}{(p+\theta+\gamma_1-1)[(p+m)n+2p_1-np_1]}$$
$$= \frac{2[np+n(\theta+\gamma_1-1)-np_1]}{np+nm+2p_1-np_1}$$
$$= \frac{2[np+n(\theta+\gamma_1-1)-\frac{(\kappa-1-m)n}{2}\cdot n]}{np+nm+(\kappa-1-m)n-\frac{(\kappa-1-m)n}{2}\cdot n} = 2. \tag{3.36}$$

Combining (3.35) with (3.36), we get

$$2h(p)\int_\Omega (v+1)^{p+\theta+\gamma_1-1} \leq h(p)\cdot c_{26}\cdot c_{22}^{(p+\theta+\gamma_1-1)(1-b_5)}\int_\Omega |\nabla(v+1)^{\frac{p+m}{2}}|^2 + c_{27}, \tag{3.37}$$

where $c_{27} = h(p)\cdot c_{26}\cdot c_{22}^{p+\theta+\gamma_1-1}$. Due to $\lim_{p\to p_1} h(p) = 0$ for any $\sigma > 0$ sufficiently small satisfying $p_1 < p \leq p_1+\sigma$, we get

$$h(p)\cdot c_{26}\cdot c_{22}^{(p+\theta+\gamma_1-1)\cdot(1-b_5)} \leq \frac{4(p-1)}{(p+m)^2}. \tag{3.38}$$

Furthermore, collecting (3.37), (3.38), and (3.34), we deduce

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p + \int_\Omega (v+1)^p \leq c_{28} \text{ for all } p\in(p_1,p_1+\sigma], \quad t\in(0,T_{\max}),$$

where $c_{28} = c_{25} + c_{27} > 0$. Hence, we finish the proof of Lemma 3.5.

Lemma 3.6. *Let $n \geq 3$ and $p_1 + \sigma < p < +\infty$ with p_1 defined in* (3.3)*. Under the first case of condition (a) in Theorem 1.1, there exists $C = C(p) > 0$ such that*

$$\int_\Omega (v+1)^p \leq C, \quad t \in (0, T_{\max}), \tag{3.39}$$

where $\sigma > 0$ is given in Lemma 3.5.

Proof. Thanks to Lemma 3.5, there exists $c_{29} > 0$ such that

$$\|(v+1)\|_{L^{\bar{p}}(\Omega)} \leq c_{29} \text{ for all } t \in (0, T_{\max}),$$

with $\bar{p} = p_1 + \sigma$ and $\sigma > 0$ small enough. In view of (3.34), we have

$$\frac{1}{p}\frac{d}{dt}\int_\Omega (v+1)^p + \int_\Omega (v+1)^p + \frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla(v+1)^{\frac{p+m}{2}}|^2 \leq 2h(p)\int_\Omega (v+1)^{p+\theta+\gamma_1-1} + c_{30}, \tag{3.40}$$

for all $p > \bar{p}$ with some $c_{30} > 0$. Applying the boundedness of $\|v(\cdot,t)\|_{L^{\bar{p}}(\Omega)}$, it can be seen from the Gagliardo-Nirenberg inequality that

$$\begin{aligned}
2h(p)\int_\Omega (v+1)^{p+\theta+\gamma_1-1} &= 2h(p)\|(v+1)^{\frac{p+m}{2}}\|_{L^{\frac{2(p+\theta+\gamma_1-1)}{p+m}}(\Omega)}^{\frac{2(p+\theta+\gamma_1-1)}{p+m}} \\
&\leq h(p)\cdot c_{31}\|\nabla(v+1)^{\frac{p+m}{2}}\|_{L^2(\Omega)}^{\frac{2(p+\theta+\gamma_1-1)}{p+m}\cdot b_6} \cdot \|(v+1)\|_{L^{\bar{p}}(\Omega)}^{(p+\theta+\gamma_1-1)\cdot(1-b_6)} \\
&\quad + h(p)\cdot c_{31}\|(v+1)\|_{L^{\bar{p}}(\Omega)}^{p+\theta+\gamma_1-1} \\
&\leq h(p)\cdot c_{31}\cdot c_{29}^{(p+\theta+\gamma_1-1)\cdot(1-b_6)}\|\nabla(v+1)^{\frac{p+m}{2}}\|_{L^2(\Omega)}^{\frac{2(p+\theta+\gamma_1-1)}{p+m}\cdot b_6} \\
&\quad + h(p)\cdot c_{31}\cdot c_{29}^{p+\theta+\gamma_1-1} \\
&\leq c_{32}\|\nabla(v+1)^{\frac{p+m}{2}}\|_{L^2(\Omega)}^{\frac{2(p+\theta+\gamma_1-1)}{p+m}\cdot b_6} + c_{32},
\end{aligned} \tag{3.41}$$

with some $c_{31} = c_{31}(p) > 0$, where

$$c_{32} = \max\left\{h(p)\cdot c_{31}\cdot c_{29}^{(p+\theta+\gamma_1-1)(1-b_6)}, h(p)\cdot c_{31}\cdot c_{29}^{p+\theta+\gamma_1-1}\right\},$$

and

$$b_6 = \frac{\frac{p+m}{2\bar{p}} - \frac{p+m}{2(p+\theta+\gamma_1-1)}}{\frac{p+m}{2\bar{p}} + \frac{1}{n} - \frac{1}{2}} = \frac{n(p+m)(p+\theta+\gamma_1-1-\bar{p})}{(p+\theta+\gamma_1-1)[n(p+m)+2\bar{p}-n\bar{p}]} \in (0,1).$$

Moreover, we have

$$\begin{aligned}
\frac{2(p+\theta+\gamma_1-1)}{p+m}\cdot b_6 &= \frac{2(p+\theta+\gamma_1-1)}{p+m}\cdot\frac{n(p+m)(p+\theta+\gamma_1-1-\bar{p})}{(p+\theta+\gamma_1-1)[n(p+m)+2\bar{p}-n\bar{p}]} \\
&= \frac{2[np + n(\theta+\gamma_1-1) - \frac{(\kappa-1-m)n}{2}\cdot n - n\sigma]}{np + n(\kappa-1) - \frac{(\kappa-1-m)n}{2}\cdot n + 2\sigma - n\sigma} < 2,
\end{aligned} \tag{3.42}$$

due to $\bar{p} > p_1 = \frac{(\kappa-1-m)n}{2}$ defined in (3.3) and $\kappa = \theta + \gamma_1$. Thus, in light of Young's inequality, there exists $c_{33} > 0$ such that

$$2h(p)\int_\Omega (v+1)^{p+\theta+\gamma_1-1} \leq \frac{4(p-1)}{(p+m)^2}\int_\Omega |\nabla(v+1)^{\frac{p+m}{2}}|^2 + c_{33}, \quad t \in (0, T_{\max}). \tag{3.43}$$

Collecting (3.43) and (3.40), we arrive at

$$\frac{1}{p}\frac{d}{dt}\int_{\Omega}(v+1)^p+\int_{\Omega}(v+1)^p\leq c_{34}$$

for all $p\in(\bar{p},+\infty]$ and $t\in(0,T_{\max})$, with $c_{34}=c_{30}+c_{33}>0$, which implies (3.39). Thus, we conclude the proof of Lemma 3.6.

In fact, a similar proof process can be applied to the second case of conditions (a) and (b) in Theorem 1.1 to obtain the estimate of $\|v+1\|_{L^p(\Omega)}$ for any $p>1$. We omit them here.

Based on the above preparation work, it is sufficient to prove Theorem 1.1.

The proof of Theorem 1.1. Suppose that the conditions in Theorem 1.1 hold. Let $p>\max\{1,n\gamma_1,n\gamma_2\}$. Invoking the elliptic L^p−estimate, one may get

$$\|w_1(\cdot,t)\|_{W^{2,p/\gamma_1}},\ \|w_2(\cdot,t)\|_{W^{2,p/\gamma_2}}\leq C,\quad t\in(0,T_{\max}).\tag{3.44}$$

Based on the Sobolev embedding theorem, we obtain

$$\|w_1(\cdot,t)\|_{C^1(\overline{\Omega})},\ \|w_2(\cdot,t)\|_{C^1(\overline{\Omega})}\leq C,\quad t\in(0,T_{\max}).\tag{3.45}$$

From Moser iteration in [13], we can infer the boundedness of $\|(v+1)\|_{L^\infty(\Omega)}$ for all $t\in(0,T_{\max})$. Hence, we conclude from Lemma 2.1 that $T_{\max}=\infty$. Obviously, (v,w_1,w_2) solves the system (1.1) in the classical sense in $\Omega\times(0,\infty)$.

4. Large time behavior

In the following, we further explore the long time behavior of the classical solutions obtained in Theorem 1.1. It can be inferred from Theorem 1.1 that there exist constants $R>0$ and $\lambda_1,\lambda_2>0$ such that

$$0<v(x,t)\leq R,\tag{4.1}$$

and

$$(v+1)^{2\theta-m-2}\leq\lambda_1 \text{ and } (v+1)^{2l-m-2}\leq\lambda_2,\tag{4.2}$$

hold on $\overline{\Omega}\times[0,\infty)$, where R,λ_1,λ_2 are independent of the parameters of the system.

Lemma 4.1. *(cf. [24, Lemma 3.1.]) Assume that* $h:(t_0,\infty)\to[0,\infty)$ *is a uniformly continuous function satisfying* $\int_{t_0}^{\infty}h(t)dt<\infty$ *with* $t_0>0$*. Then,*

$$h(t)\to 0,\ as\ t\to\infty.\tag{4.3}$$

To begin with, we construct an energy functional as follows:

$$W(t)=\int_{\Omega}\left(v(\cdot,t)-c-c\ln\frac{v(\cdot,t)}{c}\right),\tag{4.4}$$

with $c=(\frac{a}{b})^{\frac{1}{\kappa-1}}$.

Lemma 4.2. *Suppose that the conditions in Theorem 1.1 are true. Then, the following properties hold:*

$$\frac{d}{dt}W(t) \leq \frac{\lambda_1\chi^2 c}{8}\int_\Omega N_1(v)(v-c)^2 + \frac{\lambda_2\xi^2 c}{8}\int_\Omega N_2(v)(v-c)^2 - \frac{a}{c}\int_\Omega (v-c)^2, \tag{4.5}$$

with λ_1, λ_2 *defined in* (4.2) *for all* $t > 0$, *where*

$$\begin{cases} N_1(v) = \frac{\alpha^2}{\beta}4^{1-\gamma_1}c^{2\gamma_1-2} & if\ \gamma_1 \in (0,1), \\ N_1(v) = \frac{\alpha^2}{\beta}\gamma_1^2(v+c)^{2\gamma_1-2} & if\ \gamma_1 \in [1,\infty), \end{cases} \tag{4.6}$$

and

$$\begin{cases} N_2(v) = \frac{\gamma^2}{\delta}4^{1-\gamma_2}c^{2\gamma_2-2} & if\ \gamma_2 \in (0,1), \\ N_2(v) = \frac{\gamma^2}{\delta}\gamma_2^2(v+c)^{2\gamma_2-2} & if\ \gamma_2 \in [1,\infty). \end{cases} \tag{4.7}$$

Proof. It is not difficult to see that $v = c$ is the minimum point of $W(t)$, which means that $W(t) \geq 0$. By direct computation, we arrive at

$$\begin{aligned} \frac{d}{dt}W(t) =& \frac{d}{dt}\int_\Omega v - c - c\ln(\frac{v}{c}) = \int_\Omega (1-\frac{c}{v})v_t \\ =& -c\int_\Omega (v+1)^m\frac{|\nabla v|^2}{v^2} + c\chi\int_\Omega (v+1)^{\theta-1}\frac{\nabla v\cdot\nabla w_1}{v} - c\xi\int_\Omega (v+1)^{l-1}\frac{\nabla v\cdot\nabla w_2}{v} \\ &+ \int_\Omega (1-\frac{c}{v})(av - bv^\kappa). \end{aligned} \tag{4.8}$$

An application of Young's inequality enables us to get from (4.2) that

$$\begin{aligned} c\chi\int_\Omega (v+1)^{\theta-1}\frac{\nabla v\cdot\nabla w_1}{v} \leq& \frac{c}{2\lambda_1}\int_\Omega (v+1)^{2\theta-2}\frac{|\nabla v|^2}{v^2} + \frac{\lambda_1\chi^2 c}{2}\int_\Omega |\nabla w_1|^2 \\ \leq& \frac{c}{2\lambda_1}\int_\Omega (v+1)^{2\theta-m-2}(v+1)^m\frac{|\nabla v|^2}{v^2} + \frac{\lambda_1\chi^2 c}{2}\int_\Omega |\nabla w_1|^2 \\ \leq& \frac{c}{2}\int_\Omega (v+1)^m\frac{|\nabla v|^2}{v^2} + \frac{\lambda_1\chi^2 c}{2}\int_\Omega |\nabla w_1|^2, \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} -c\xi\int_\Omega (v+1)^{l-1}\frac{\nabla v\cdot\nabla w_2}{v} \leq& \frac{c}{2\lambda_2}\int_\Omega (v+1)^{2l-2}\frac{|\nabla v|^2}{v^2} + \frac{\lambda_2\xi^2 c}{2}\int_\Omega |\nabla w_2|^2 \\ \leq& \frac{c}{2\lambda_2}\int_\Omega (v+1)^{2l-m-2}(v+1)^m\frac{|\nabla v|^2}{v^2} + \frac{\lambda_2\xi^2 c}{2}\int_\Omega |\nabla w_2|^2 \\ \leq& \frac{c}{2}\int_\Omega (v+1)^m\frac{|\nabla v|^2}{v^2} + \frac{\lambda_2\xi^2 c}{2}\int_\Omega |\nabla w_2|^2. \end{aligned} \tag{4.10}$$

In addition, we infer that

$$\int_\Omega (1-\frac{c}{v})(av-bv^\kappa) = -b\int_\Omega (v-c)(v^{\kappa-1}-c^{\kappa-1})$$
$$\leq -bc^{\kappa-2}\int_\Omega (v-c)^2 \leq -\frac{a}{c}\int_\Omega (v-c)^2. \quad (4.11)$$

Therefore, we conclude from (4.8)–(4.11) that

$$\frac{d}{dt}W(t) \leq \frac{\lambda_1\chi^2 c}{2}\int_\Omega |\nabla w_1|^2 + \frac{\lambda_2\xi^2 c}{2}\int_\Omega |\nabla w_2|^2 - \frac{a}{c}\int_\Omega (v-c)^2. \quad (4.12)$$

From the second equation of system (1.1), employing Young's inequality, we deduce

$$\int_\Omega |\nabla w_1|^2 = -\beta\int_\Omega (w_1-\frac{\alpha}{\beta}c^{\gamma_1})^2 + \alpha\int_\Omega (w_1-\frac{\alpha}{\beta}c^{\gamma_1})(v^{\gamma_1}-c^{\gamma_1})$$
$$\leq -\beta\int_\Omega (w_1-\frac{\alpha}{\beta}c^{\gamma_1})^2 + \beta\int_\Omega (w_1-\frac{\alpha}{\beta}c^{\gamma_1})^2 + \frac{\alpha^2}{4\beta}\int_\Omega (v^{\gamma_1}-c^{\gamma_1})^2$$
$$\leq \frac{\alpha^2}{4\beta}\int_\Omega (v^{\gamma_1}-c^{\gamma_1})^2. \quad (4.13)$$

By the same process as in (4.13), we can also obtain

$$\int_\Omega |\nabla w_2|^2 \leq \frac{\gamma^2}{4\delta}\int_\Omega (v^{\gamma_2}-c^{\gamma_2})^2. \quad (4.14)$$

In the following, we shall divide the parameters γ_1 and γ_2 into two different cases to obtain the better estimates of (4.13) and (4.14).

Case (a) $\gamma_1, \gamma_2 \in (0,1)$. Considering that $(\tilde{x},\tilde{t}) \in \Omega\times(0,\infty)$ fulfills $v(\tilde{x},\tilde{t}) \leq \frac{c}{2}$, thus we can obtain

$$|v^{\gamma_i}-c^{\gamma_i}| \leq |v-c|^{\gamma_i} \leq 2^{1-\gamma_i}c^{\gamma_i-1}|v-c| \;\; i=1,2. \quad (4.15)$$

Furthermore, the mean value theorem enables us to find $\xi_j \in (0,1)$ with $j=1,2$ satisfying

$$|v^{\gamma_i}-c^{\gamma_i}| \leq \gamma_i(v-\xi_j v+\xi_j c)^{\gamma_i-1}|v-c|. \quad (4.16)$$

Clearly, $(v-\xi_j v+\xi_j c)^{\gamma_i-1}$ is monotone decreasing with respect to v on $[\frac{c}{2},\infty)$, and $v-\xi_j v+\xi_j c > \frac{c}{2}$ if $v \geq \frac{c}{2}$. Thus, we deduce from (4.16) that

$$|v^{\gamma_i}-c^{\gamma_i}| \leq \gamma_i 2^{1-\gamma_i}c^{\gamma_i-1}|v-c|. \quad (4.17)$$

Case (b) $\gamma_1, \gamma_2 \in [1,\infty)$. Thanks to $\gamma_i \in [1,\infty)(i=1,2)$ for $\xi_j \in (0,1)$ with $j=3,4$, we deduce that the function $(v-\xi_j v+\xi_j c)^{\gamma_i-1}$ is monotone increasing with respect to v. Employing the mean value theorem again, one may get

$$|v^{\gamma_i}-c^{\gamma_i}| \leq \gamma_i(v-\xi_j v+\xi_j c)^{\gamma_i-1}|v-c| \leq \gamma_i(v+c)^{\gamma_i-1}|v-c|. \quad (4.18)$$

Collecting (4.13)–(4.18), for any $\gamma_1, \gamma_2 > 0$, we can obtain

$$\int_\Omega |\nabla w_1|^2 \leq \frac{\alpha^2}{4\beta}\int_\Omega (v^{\gamma_1}-c^{\gamma_1})^2 = \frac{1}{4}\int_\Omega N_1(v)(v-c)^2, \quad (4.19)$$

and

$$\int_{\Omega}|\nabla w_2|^2 \leq \frac{\gamma^2}{4\delta}\int_{\Omega}(v^{\gamma_2}-c^{\gamma_2})^2 = \frac{1}{4}\int_{\Omega}N_2(v)(v-c)^2, \tag{4.20}$$

with $N_1(v)$ and $N_2(v)$ defined in (4.6) and (4.7), respectively. Substituting (4.19) and (4.20) into (4.12), we can infer that

$$\frac{d}{dt}W(t) \leq \frac{\lambda_1\chi^2 c}{8}\int_{\Omega}N_1(v)(v-c)^2 + \frac{\lambda_2\xi^2 c}{8}\int_{\Omega}N_2(v)(v-c)^2 - \frac{a}{c}\int_{\Omega}(v-c)^2. \tag{4.21}$$

Thus, Lemma 4.2 is a direct result by collecting **Cases (a) and (b)**.

Now, it is sufficient to conclude the proof of Theorem 1.3.

The proof of Theorem 1.3. Based on Theorem 1.1, by applying the parabolic and elliptic regularity (see [28, 29]) and the global boundedness of (v, w_1, w_2), one can find $\sigma_1 \in (0,1)$ and $C > 0$ such that

$$\|v\|_{C^{2+\sigma_1,1+\frac{\sigma_1}{2}}(\overline{\Omega}\times[t,t+1])} + \|w_1\|_{C^{2+\sigma_1,1+\frac{\sigma_1}{2}}(\overline{\Omega}\times[t,t+1])} + \|w_2\|_{C^{2+\sigma_1,1+\frac{\sigma_1}{2}}(\overline{\Omega}\times[t,t+1])} \leq C, \tag{4.22}$$

where $t \geq 1$. In the sequent, we divide the proof into four cases.

Case (i) $\gamma_1, \gamma_2 \in (0,1)$. Combining (4.6) and (4.7), we get

$$\begin{aligned}
&\frac{\lambda_1\chi^2 c}{8}\int_{\Omega}N_1(v)(v-c)^2 + \frac{\lambda_2\xi^2 c}{8}\int_{\Omega}N_2(v)(v-c)^2 \\
&\leq \frac{\lambda_1\chi^2 c}{8}\cdot\frac{\alpha^2}{\beta}4^{1-\gamma_1}c^{2\gamma_1-2}\int_{\Omega}(v-c)^2 + \frac{\lambda_2\xi^2 c}{8}\cdot\frac{\gamma^2}{\delta}4^{1-\gamma_2}c^{2\gamma_2-2}\int_{\Omega}(v-c)^2 \\
&= \left[4^{-(\gamma_1+1)}\frac{2\lambda_1\chi^2\alpha^2}{\beta}c^{2\gamma_1-1} + 4^{-(\gamma_2+1)}\frac{2\lambda_2\xi^2\gamma^2}{\delta}c^{2\gamma_2-1}\right]\int_{\Omega}(v-c)^2.
\end{aligned} \tag{4.23}$$

We substitute (4.23) into (4.5) to have

$$\begin{aligned}
\frac{d}{dt}W(t) &\leq -\left[\frac{a}{c} - (4^{-(\gamma_1+1)}\frac{2\lambda_1\chi^2\alpha^2}{\beta}c^{2\gamma_1-1} + 4^{-(\gamma_2+1)}\frac{2\lambda_2\xi^2\gamma^2}{\delta}c^{2\gamma_2-1})\right]\int_{\Omega}(v-c)^2 \\
&= -\left[b^{\frac{1}{\kappa-1}}\cdot a^{1-\frac{1}{\kappa-1}} - (4^{-(\gamma_1+1)}\frac{2\lambda_1\chi^2\alpha^2}{\beta}c^{2\gamma_1-1} + 4^{-(\gamma_2+1)}\frac{2\lambda_2\xi^2\gamma^2}{\delta}c^{2\gamma_2-1})\right]\int_{\Omega}(v-c)^2.
\end{aligned} \tag{4.24}$$

Recalling $c = (\frac{a}{b})^{\frac{1}{\kappa-1}}$, we can find $b_0 > 0$ large enough such that

$$b^{\frac{1}{\kappa-1}}\cdot a^{1-\frac{1}{\kappa-1}} - (4^{-(\gamma_1+1)}\frac{2\lambda_1\chi^2\alpha^2}{\beta}c^{2\gamma_1-1} + 4^{-(\gamma_2+1)}\frac{2\lambda_2\xi^2\gamma^2}{\delta}c^{2\gamma_2-1}) > 0,$$

whenever $b > b_0$. Thus, by taking

$$\varepsilon_1 = b^{\frac{1}{\kappa-1}}a^{1-\frac{1}{\kappa-1}} - (4^{-(\gamma_1+1)}\frac{2\lambda_1\chi^2\alpha^2}{\beta}c^{2\gamma_1-1} + 4^{-(\gamma_2+1)}\frac{2\lambda_2\xi^2\gamma^2}{\delta}c^{2\gamma_2-1}),$$

we have

$$\frac{d}{dt}W(t) \leq -\varepsilon_1\int_{\Omega}(v-c)^2. \tag{4.25}$$

Integrating (4.25) from t_0 to ∞, due to $W(t) \geq 0$ we get

$$\int_{t_0}^{\infty}\int_{\Omega}(v-c)^2 \leq \frac{W(t_0)}{\varepsilon_1}. \tag{4.26}$$

In view of Lemma 4.1, we conclude from (4.26) that

$$\lim_{t\to\infty}\int_{\Omega}(v-c)^2 = 0. \tag{4.27}$$

In light of the Gagliardo-Nirenberg inequality, we conclude from (4.22) and (4.27) that

$$\|v-c\|_{L^\infty(\Omega)} \leq C_{GN}\|v-c\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}}\|v-c\|_{L^2(\Omega)}^{\frac{n}{n+2}} \leq C\|v-c\|_{L^2(\Omega)}^{\frac{n}{n+2}} \to 0, \quad as\ t\to\infty. \tag{4.28}$$

With an application of L'Hôpital's rule, we conclude

$$\lim_{v\to c}\frac{v-c-c\ln\frac{v}{c}}{(v-c)^2} = \frac{1}{2c}. \tag{4.29}$$

Thus, from (4.29) we have

$$\frac{1}{4c}\int_{\Omega}(v-c)^2 \leq W(t) \leq \frac{1}{c}\int_{\Omega}(v-c)^2, \quad t > T_1, \tag{4.30}$$

with some $T_1 > 0$. According to (4.25) and (4.30), we can infer that

$$\frac{d}{dt}W(t) \leq -c\varepsilon_1 W(t), \quad t > T_1.$$

Using the Gronwall inequality, we derive

$$W(t) \leq W(T_1)e^{-c\varepsilon_1(t-T_1)}, \quad t > T_1.$$

Thus,

$$\frac{1}{4c}\int_{\Omega}(v-c)^2 \leq W(t) \leq W(T_1)e^{-c\varepsilon_1(t-T_1)}, \quad t > T_1. \tag{4.31}$$

In accordance with (4.13), we have

$$\begin{aligned}\int_{\Omega}|\nabla w_1|^2 &= -\beta\int_{\Omega}(w_1-\frac{\alpha}{\beta}c^{\gamma_1})^2 + \alpha\int_{\Omega}(w_1-\frac{\alpha}{\beta}c^{\gamma_1})(v^{\gamma_1}-c^{\gamma_1})\\ &\leq -\frac{\beta}{2}\int_{\Omega}(w_1-\frac{\alpha}{\beta}c^{\gamma_1})^2 + \frac{\alpha^2}{2\beta}\int_{\Omega}(v^{\gamma_1}-c^{\gamma_1})^2,\end{aligned} \tag{4.32}$$

and thus

$$\int_{\Omega}(w_1-\frac{\alpha}{\beta}c^{\gamma_1})^2 \leq \frac{\alpha^2}{\beta^2}\int_{\Omega}(v^{\gamma_1}-c^{\gamma_1})^2. \tag{4.33}$$

According to (4.17), we obtain

$$\int_\Omega (w_1 - \frac{\alpha}{\beta}c^{\gamma_1})^2 \le \frac{\alpha^2}{\beta^2}4^{1-\gamma_1}c^{2\gamma_1-2}\int_\Omega (v-c)^2. \tag{4.34}$$

Analogously, for component w we can obtain

$$\int_\Omega (w_2 - \frac{\gamma}{\delta}c^{\gamma_2})^2 \le \frac{\gamma^2}{\delta^2}4^{1-\gamma_2}c^{2\gamma_2-2}\int_\Omega (v-c)^2. \tag{4.35}$$

Thus, we can get from (4.31)

$$\int_\Omega (w_1 - \frac{\alpha}{\beta}c^{\gamma_1})^2 \le \frac{\alpha^2}{\beta^2}4^{2-\gamma_1}c^{2\gamma_1-1}W(T_1)e^{-c\varepsilon_1(t-T_1)}, \tag{4.36}$$

and

$$\int_\Omega (w_2 - \frac{\gamma}{\delta}c^{\gamma_2})^2 \le \frac{\gamma^2}{\delta^2}4^{2-\gamma_2}c^{2\gamma_2-1}W(T_1)e^{-c\varepsilon_1(t-T_1)}, \quad t > T_1. \tag{4.37}$$

We apply the Gagliardo-Nirenberg inequality to (4.31), (4.36), and (4.37), respectively, and then conclude from (4.22) that

$$\|v-c\|_{L^\infty(\Omega)} + \|w_1 - \frac{\alpha}{\beta}c^{\gamma_1}\|_{L^\infty(\Omega)} + \|w_2 - \frac{\gamma}{\delta}c^{\gamma_2}\|_{L^\infty(\Omega)} \le Ce^{-\frac{c\varepsilon_1}{n+2}(t-T_1)} \tag{4.38}$$

for all $t > T_1$ with some $C > 0$.

Case (ii) $\gamma_1, \gamma_2 \in [1, +\infty)$. Combining (4.6) and (4.7), we gain that

$$\begin{aligned}
&\frac{\lambda_1\chi^2 c}{8}\int_\Omega N_1(v)(v-c)^2 + \frac{\lambda_2\xi^2 c}{8}\int_\Omega N_2(v)(v-c)^2 \\
&\le \frac{\lambda_1\chi^2 c}{8}\cdot\frac{\alpha^2}{\beta}\gamma_1^2\int_\Omega (R+c)^{2\gamma_1-2}(v-c)^2 + \frac{\lambda_2\xi^2 c}{8}\cdot\frac{\gamma^2}{\delta}\gamma_2^2\int_\Omega (R+c)^{2\gamma_2-2}(v-c)^2 \\
&\le \left[\frac{\lambda_1\chi^2\alpha^2\gamma_1^2}{8\beta}c(R+c)^{2\gamma_1-2} + \frac{\lambda_2\xi^2\gamma^2\gamma_2^2}{8\delta}c(R+c)^{2\gamma_2-2}\right]\int_\Omega (v-c)^2.
\end{aligned} \tag{4.39}$$

Thus, substituting (4.39) into (4.5), we have

$$\begin{aligned}
\frac{d}{dt}W(t) &\le -\left[\frac{a}{c} - \frac{c}{8}\left(\frac{\lambda_1\chi^2\alpha^2\gamma_1^2}{\beta}(R+c)^{2\gamma_1-2} + \frac{\lambda_2\xi^2\gamma^2\gamma_2^2}{\delta}(R+c)^{2\gamma_2-2}\right)\right]\int_\Omega (v-c)^2 \\
&\le -\left[b^{\frac{1}{\kappa-1}}\cdot a^{1-\frac{1}{\kappa-1}} - \frac{c}{8}\left(\frac{\lambda_1\chi^2\alpha^2\gamma_1^2}{\beta}(R+c)^{2\gamma_1-2} + \frac{\lambda_2\xi^2\gamma^2\gamma_2^2}{\delta}(R+c)^{2\gamma_2-2}\right)\right]\int_\Omega (v-c)^2.
\end{aligned} \tag{4.40}$$

Thanks to $c = (\frac{a}{b})^{\frac{1}{\kappa-1}}$, we can find $b_0 > 0$ large enough such that

$$b^{\frac{1}{\kappa-1}}\cdot a^{1-\frac{1}{\kappa-1}} - \frac{c}{8}\left(\frac{\lambda_1\chi^2\alpha^2\gamma_1^2}{\beta}(R+c)^{2\gamma_1-2} + \frac{\lambda_2\xi^2\gamma^2\gamma_2^2}{\delta}(R+c)^{2\gamma_2-2}\right) > 0,$$

whenever $b > b_0$. Setting

$$\varepsilon_2 = b^{\frac{1}{\kappa-1}} a^{1-\frac{1}{\kappa-1}} - \frac{c}{8}\left(\frac{\lambda_1\chi^2\alpha^2\gamma_1^2}{\beta}(R+c)^{2\gamma_1-2} + \frac{\lambda_2\xi^2\gamma^2\gamma_2^2}{\delta}(R+c)^{2\gamma_2-2}\right),$$

we can get

$$\frac{d}{dt}W(t) \le -\varepsilon_2 \int_\Omega (v-c)^2. \tag{4.41}$$

By a similar discussion as in **Case (i)**, we have

$$\frac{1}{4c}\int_\Omega (v-c)^2 \le W(t) \le W(T_1)e^{-\varepsilon_2(t-T_1)}, \quad t > T_1. \tag{4.42}$$

Repeating the processes in (4.32)–(4.35), one may obtain that

$$\int_\Omega (w_1 - \frac{\alpha}{\beta}c^{\gamma_1})^2 \le \frac{4\alpha^2\gamma_1^2}{\beta^2}c(R+c)^{2\gamma_1-2}W(T_1)e^{-\varepsilon_2(t-T_1)}, \tag{4.43}$$

and

$$\int_\Omega (w_2 - \frac{\gamma}{\delta}c^{\gamma_2})^2 \le \frac{4\gamma^2\gamma_2^2}{\delta^2}c(R+c)^{2\gamma_2-2}W(T_1)e^{-\varepsilon_2(t-T_1)}, \tag{4.44}$$

for $t > T_1$. In view of (4.22) and (4.28), we conclude from (4.42)–(4.44) that

$$\|v-c\|_{L^\infty(\Omega)} + \|w_1 - \frac{\alpha}{\beta}c^{\gamma_1}\|_{L^\infty(\Omega)} + \|w_2 - \frac{\gamma}{\delta}c^{\gamma_2}\|_{L^\infty(\Omega)} \le Ce^{-\frac{\varepsilon_2}{n+2}(t-T_1)}, \tag{4.45}$$

for $t > T_1$, with some $C > 0$.

Case (iii) $\gamma_1 \in (0,1), \gamma_2 \in [1,+\infty)$. Using (4.1), (4.6), and (4.7), we easily have

$$\begin{aligned} &\frac{\lambda_1\chi^2 c}{8}\int_\Omega N_1(v)(v-c)^2 + \frac{\lambda_2\xi^2 c}{8}\int_\Omega N_2(v)(v-c)^2 \\ &\le \left[\frac{c}{8}\left(\frac{\lambda_1\chi^2\alpha^2}{\beta}4^{1-\gamma_1}c^{2\gamma_1-2} + \frac{\lambda_2\xi^2\gamma^2\gamma_2^2}{\delta}(R+c)^{2\gamma_2-2}\right)\right]\int_\Omega (v-c)^2. \end{aligned} \tag{4.46}$$

Thus, substituting (4.46) into (4.5), we have

$$\frac{d}{dt}W(t) \le -\left[b^{\frac{1}{\kappa-1}} \cdot a^{1-\frac{1}{\kappa-1}} - \frac{c}{8}\left(\frac{\lambda_1\chi^2\alpha^2}{\beta}4^{1-\gamma_1}c^{2\gamma_1-2} + \frac{\lambda_2\xi^2\gamma^2\gamma_2^2}{\delta}(R+c)^{2\gamma_2-2}\right)\right]\int_\Omega (v-c)^2. \tag{4.47}$$

By the same discussion as in **Case (i)**, we can find $b_0 > 0$ large enough such that

$$\varepsilon_3 = b^{\frac{1}{\kappa-1}} \cdot a^{1-\frac{1}{\kappa-1}} - \frac{c}{8}\left[\frac{\lambda_1\chi^2\alpha^2}{\beta}4^{1-\gamma_1}c^{2\gamma_1-2} + \frac{\lambda_2\xi^2\gamma^2\gamma_2^2}{\delta}(R+c)^{2\gamma_2-2}\right] > 0, \tag{4.48}$$

and

$$\frac{1}{4c}\int_\Omega (v-c)^2 \le W(t) \le W(T_1)e^{-\varepsilon_3(t-T_1)}, \quad t > T_1, \tag{4.49}$$

with $b > b_0$. Similarly, we can obtain

$$\int_\Omega (w_1 - \frac{\alpha}{\beta} c^{\gamma_1})^2 \leq \frac{\alpha^2}{\beta^2} 4^{2-\gamma_1} c^{2\gamma_1 - 1} W(T_1) e^{-\varepsilon_3 (t - T_1)}, \tag{4.50}$$

and

$$\int_\Omega (w_2 - \frac{\gamma}{\delta} c^{\gamma_2})^2 \leq \frac{4\gamma^2 \gamma_2^2}{\delta^2} c(R + c)^{2\gamma_2 - 2} W(T_1) e^{-\varepsilon_3 (t - T_1)}, \quad t > T_1. \tag{4.51}$$

Hence, from (4.22) and (4.28), we can derive

$$\|v - c\|_{L^\infty(\Omega)} + \|w_1 - \frac{\alpha}{\beta} c^{\gamma_1}\|_{L^\infty(\Omega)} + \|w_2 - \frac{\gamma}{\delta} c^{\gamma_2}\|_{L^\infty(\Omega)} \leq C e^{-\frac{\varepsilon_3}{n+2}(t - T_1)}, \tag{4.52}$$

for $t > T_1$, with $C > 0$.

Case (iv) $\gamma_1 \in [1, \infty), \gamma_2 \in (0, 1)$. We can compute

$$\frac{d}{dt} W(t) \leq -\varepsilon_4 \int_\Omega (v - c)^2, \tag{4.53}$$

where

$$\varepsilon_4 = b^{\frac{1}{\kappa - 1}} \cdot a^{1 - \frac{1}{\kappa - 1}} - \frac{c}{8} \left[\frac{\lambda_1 \chi^2 \alpha^2 \gamma_1^2}{\beta} (R + c)^{2\gamma_1 - 2} + \frac{\lambda_2 \xi^2 \gamma^2}{\delta} 4^{1 - \gamma_2} c^{2\gamma_2 - 2} \right] > 0,$$

due to $b > 0$ large enough. Using similar processes as in **Case (iii)**, we can conclude the proof for this case. Therefore, based on the above analysis, we finish the proof of Theorem 1.3.

5. Conclusions and outlook

In this paper, we continued to study the model established in [1] and further showed that the results on global existence and boundedness of the classical solutions still hold under the corresponding critical cases. Moreover, we have also explored the long time behavior of the classical solution. In fact, it should be pointed out that the critical cases mentioned here are be not the borderline cases distinguishing the boundedness and blow-up of solutions. Naturally, there leaves an interesting problem that how can we get the genuinely critical conditions in the sense of separating ranges of distinct solution behavior. We will consider this problem in future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

We would like to thank the anonymous referees for many useful comments and suggestions that greatly improved the work. We also deeply thank Professor Li-Ming Cai for his support. This work was partially supported by the Scientific and Technological Key Projects of Henan Province No. 232102310227 and Nanhu Scholars Program for Young Scholars of XYNU No. 2020017.

Conflict of interest

The authors declare that there is no conflict of interest.

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