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Research article

Nontrivial solutions for a Hadamard fractional integral boundary value problem

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Abstract: In this paper, we studied a Hadamard-type fractional Riemann-Stieltjes integral boundary value problem. The existence of nontrivial solutions was obtained by using the fixed-point method when the nonlinearities can be superlinear, suberlinear, and have asymptotic linear growth. Our results improved and generalized some results of the existing literature.

Keywords: Hadamard-type fractional-order differential equations; integral boundary value problems; nontrivial solutions; topological degree

1. Introduction

In this work, we are devoted to studying the existence of nontrivial solutions for the Hadamard-type fractional Riemann-Stieltjes integral boundary value problem

$$\begin{cases} -D^{\beta}z(x) = f(x, z(x)), x \in (1, e), \\ z(1) = \delta z(1) = 0, \delta z(e) = \displaystyle\int_1^e g(x, z(x))\frac{d\eta(x)}{x}, \end{cases} \tag{1.1}$$

where D^{β} is the Hadamard-type fractional derivative with $\beta \in (2, 3]$, $\delta z(x) = x\frac{dz}{dx}$, and the functions f, g, η satisfy the following conditions:

(H1) $f, g \in C([1, e] \times \mathbb{R}, \mathbb{R})$.

(H2) There exist $\gamma_i, \sigma_i \in C([1, e], \mathbb{R}^+)$ and $M_i \in C(\mathbb{R}, \mathbb{R}^+)(i = 1, 2)$ such that

$$f(x, z) \geq -\gamma_1(x) - \gamma_2(x)M_1(z), \quad g(x, z) \geq -\sigma_1(x) - \sigma_2(x)M_2(z), \; x \in [1, e], z \in \mathbb{R}.$$

(H3) $\lim_{|z|\to+\infty} \frac{M_i(z)}{|z|} = 0$, $i = 1, 2$.

(H4) η is a nondecreasing function in $[1, e]$ with $\eta(1) = 0$.

As an important branch of mathematical analysis, fractional calculus can more accurately describe some dynamic processes with memory and heredity characteristics. In view of outstanding advantages of fractional calculus, it has attracted the great attention of many researchers and developed rapidly; we refer the reader to [1–10]. Moreover, we note that originating from the work of Hadamard in 1892, Hadamard fractional calculus is now successfully applied to describe ultra-slow phenomena in the objective world, such as fracture of materials, creep of rocks, etc. Therefore, it is of great significance to study Hadamard-type fractional problems, see [11–19] and the references therein. For example, in [11], the authors used the Guo-Krasnosel'skii fixed point theorem to study the existence and nonexistence of positive solutions for the system of Hadamard fractional differential equations

$$\begin{cases} D^{\alpha_1}\varphi_{p_1}\left(D^{\beta_1}x(s)\right) = \mu f_1(s, x(s), y(s)), \, s \in (1, e), \\ D^{\alpha_2}\varphi_{p_2}\left(D^{\beta_2}y(s)\right) = \nu f_2(s, x(s), y(s)), \, s \in (1, e), \\ \delta x(1) = \delta^2 x(1) = \cdots = \delta^{n_1-2}x(1) = 0, \; D^{\gamma_0}x(e) = \displaystyle\sum_{i=1}^{q_2}\int_1^{\theta_i} h_i(s)D^{\eta_i}y(s)\frac{dH_i(s)}{s}, \\ \delta y(1) = \delta^2 y(1) = \cdots = \delta^{n_2-2}y(1) = 0, \; D^{\eta_0}y(e) = \displaystyle\sum_{j=1}^{q_1}\int_1^{\vartheta_j} k_j(s)D^{\gamma_j}x(s)\frac{dK_j(s)}{s}, \\ D^{\beta_1}x(1) = D^{\beta_1}x(e) = \delta\left(\varphi_{p_1}\left(D^{\beta_1}x(1)\right)\right) = 0, \; D^{\beta_2}y(1) = D^{\beta_2}y(e) = \delta\left(\varphi_{p_2}\left(D^{\beta_2}y(1)\right)\right) = 0, \end{cases}$$

where $f_i (i = 1, 2)$ are nonnegative continuous functions on $[1, e] \times \mathbb{R}^+ \times \mathbb{R}^+$ and satisfy some $(p_i - 1)-$superlinear and $(p_i - 1)-$suberlinear growth conditions.

In [12], the authors studied the following system of Hadamard fractional differential equations with multipoint Hadamard fractional derivative boundary conditions

$$\begin{cases} D^p u(t) + w_1(t) f\left(t, v(t), D^{q-1}v(t)\right) = 0, n - 1 < p \le n, t \in (1, +\infty), \\ D^q v(t) + w_2(t) g\left(t, u(t), D^{p-1}u(t)\right) = 0, m - 1 < q \le m, t \in (1, +\infty), \\ u(1) = u'(1) = \cdots = u^{(n-2)}(1) = 0, \quad D^{p-1}u(\infty) = \displaystyle\sum_{i=1}^{k_1} a_i D^{r_1} u\left(n_i\right), \\ v(1) = v'(1) = \cdots = v^{(m-2)}(1) = 0, \quad D^{q-1}v(\infty) = \displaystyle\sum_{j=1}^{k_2} b_j D^{r_2} v\left(m_j\right), \end{cases}$$

where D^ϑ are Hadamard-type fractional derivatives of order $\vartheta \in \{p, q, r_1, r_2\}, r_1 \in [0, p - 1], r_2 \in [0, q - 1]$. Using the monotone iterative method, they obtained the existence of monotone positive solutions for their considered problems. In [13], the authors used the fixed-point techniques to study the existence and uniqueness results for the following Riemann-Stieltjes integral boundary value problem involving a Hadamard-type fractional differential equation

$$\begin{cases} D^\nu y(x) = f\left(x, y(x), D^\nu y(x)\right), t \in [1, T], \\ y(1) = 0, \; \displaystyle\int_1^T y(x)dZ(x) = \frac{\mu}{\Gamma(\delta)}\int_1^\eta \left(\ln\frac{\eta}{x}\right)^{\delta-1} y(x)\frac{dx}{x}, \eta \in (1, T), \end{cases}$$

where $f \in ([1,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R})$ satisfies the Lipschitz condition.

Inspired by the aforesaid works, we use the fixed-point methods to study the nontrivial solutions for the Hadamard-type fractional Riemann-Stieltjes integral boundary value problem (1.1). We consider the two-folds: When the nonlinearities f, g are superlinear and suberlinear, we use some conditions concerning the spectral radius of a new linear operator to obtain our existence theorems. When the nonlinearities f, g are asymptotic linear, we use a fixed-point theorem to obtain a nontrivial solution.

2. Preliminaries

We first briefly provide the definition of the Hadamard-type fractional derivative, which can be founded in [11, 14, 15].

Definition 2.1. Let $g : [1, \infty) \to \mathbb{R}$, then the Hadamard-type fractional q-order derivative is defined as

$$D^q g(x) = \frac{1}{\Gamma(n-q)} \left(x \frac{d}{dx} \right)^n \int_1^x (\ln x - \ln y)^{n-q-1} g(y) \frac{dy}{y}, n-1 < q < n,$$

where $n = [q]+1, [q]$ means the integer part of q, and $\ln(\cdot) = \log_e(\cdot)$.

Now, we calculate the Green's function for (1.1).

Lemma 2.2. Let h, V be functions on $[1, e]$, then

$$\begin{cases} -D^\beta z(x) = h(x), x \in (1, e), \\ z(1) = \delta z(1) = 0, \delta z(e) = \int_1^e V(x) \frac{d\eta(x)}{x} \end{cases}$$

has a solution

$$z(x) = \int_1^e G_\beta(x,y) h(y) \frac{dy}{y} + \frac{(\ln x)^{\beta-1}}{\beta-1} \int_1^e V(x) \frac{d\eta(x)}{x},$$

where

$$G_\beta(x,y) = \frac{1}{\Gamma(\beta)} \begin{cases} (\ln x)^{\beta-1}(1-\ln y)^{\beta-2} - (\ln x - \ln y)^{\beta-1}, & 1 \le y \le x \le e, \\ (\ln x)^{\beta-1}(1-\ln y)^{\beta-2}, & 1 \le x \le y \le e. \end{cases}$$

Proof. We first consider the problem

$$\begin{cases} -D^\beta z(x) = h(x), x \in (1, e), \\ z(1) = \delta z(1) = \delta z(e) = 0. \end{cases} \tag{2.1}$$

By the methods of [15], we can obtain

$$z(x) = c_1(\ln x)^{\beta-1} + c_2(\ln x)^{\beta-2} + c_3(\ln x)^{\beta-3} - \frac{1}{\Gamma(\beta)} \int_1^x (\ln x - \ln y)^{\beta-1} h(y) \frac{dy}{y},$$

where $c_i \in \mathbb{R}, i = 1, 2, 3$. The condition $z(1) = \delta z(1) = 0$ implies that $c_2 = c_3 = 0$. Furthermore, from $\delta z(e) = 0$, we find

$$z(e) = c_1 - \frac{1}{\Gamma(\beta)} \int_1^e (1 - \ln s)^{\beta-2} h(s) \frac{ds}{s} = 0,$$

then

$$z(x) = \frac{1}{\Gamma(\beta)}\int_1^e (\ln x)^{\beta-1}(1-\ln y)^{\beta-2}h(y)\frac{dy}{y} - \frac{1}{\Gamma(\beta)}\int_1^x (\ln x - \ln y)^{\beta-1}h(y)\frac{dy}{y} = \int_1^e G_\beta(x,y)h(y)\frac{dy}{y}. \tag{2.2}$$

Next, we consider the problem

$$\begin{cases} -D^\beta z(x) = 0, x \in (1,e), \\ z(1) = \delta z(1) = 0, \delta z(e) = \int_1^e V(x)\frac{d\eta(x)}{x}, \end{cases} \tag{2.3}$$

which yields that

$$z(x) = \widetilde{c}_1(\ln x)^{\beta-1} + \widetilde{c}_2(\ln x)^{\beta-2} + \widetilde{c}_3(\ln x)^{\beta-3},$$

where $\widetilde{c}_i \in \mathbb{R}$, $i = 1,2,3$. Similarly, $\widetilde{c}_2 = \widetilde{c}_3 = 0$. Consequently, we get

$$\delta z(e) = (\beta-1)\widetilde{c}_1 = \int_1^e V(x)\frac{d\eta(x)}{x},$$

and

$$z(x) = \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e V(x)\frac{d\eta(x)}{x}. \tag{2.4}$$

Combining (2.1)–(2.4), we can obtain the conclusion of this lemma. This completes the proof.

Lemma 2.3 (see [1]). The function G_β satisfies the following properties:

(I1) $G_\beta(x,y) \geq 0$ for $x,y \in [1,e]$;

(I2) $(\ln x)^{\beta-1}G_\beta(e,y) \leq G_\beta(x,y) \leq G_\beta(e,y)$ for $x,y \in [1,e]$.

Let $E := C[1,e], \|z\| := \max_{x\in[1,e]}|z(x)|, P := \{z \in E : z(x) \geq 0, \forall x \in [1,e]\}$, then $(E, \|\cdot\|)$ is a real Banach space and P a cone on E. By Lemma 2.2, we can define an operator as follows:

$$(Tz)(x) = \int_1^e G_\beta(x,y)f(y,z(y))\frac{dy}{y} + \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e g(x,z(x))\frac{d\eta(x)}{x},\ x \in [1,e],\ z \in E.$$

Moreover, it is easy to find that if there is a $z^* \in E\backslash\{0\}$ such that $Tz^* = z^*$, then this z^* is a nontrivial solution for (1.1). Hence, we only need to study the existence of nontrivial fixed points of T. For $\xi_i > 0$ $(i = 1,2)$, let $L_{\xi_1,\xi_2} : E \to E$ be defined as

$$(L_{\xi_1,\xi_2}z)(x) = \xi_1\int_1^e G_\beta(x,y)z(y)\frac{dy}{y} + \xi_2\frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e z(x)\frac{d\eta(x)}{x}, x \in [1,e], z \in E. \tag{2.5}$$

We then see that L_{ξ_1,ξ_2} is a linear operator, and we also obtain the following lemma.

Lemma 2.4. Let $P_0 = \{z \in E : z(x) \geq (\ln x)^{\beta-1}\|z\|, x \in [1,e]\}$, then $L_{\xi_1,\xi_2}(P) \subset P_0$.

This can be easily obtained from Lemma 2.3 (I2), so we don't need to offer its proof.

Lemma 2.5. $r(L_{\xi_1,\xi_2}) > 0$, where $r(L_{\xi_1,\xi_2})$ stands for the spectral radius of L_{ξ_1,ξ_2}.

Proof. Let $(L_{\xi_1}z)(x) = \xi_1\int_1^e G_\beta(x,y)z(y)\frac{dy}{y}, x \in [1,e], z \in E$, then for all $n \in \mathbb{N}_+$, we have

$$(L^n_{\xi_1}z)(x) = \xi_1^n \underbrace{\int_1^e \cdots \int_1^e}_{n} G_\beta(x,y_1)G_\beta(y_1,y_2)\cdots G_\beta(y_{n-1},y_n)z(y_n)\frac{dy_1}{y_1}\frac{dy_2}{y_2}\cdots\frac{dy_n}{y_n}.$$

By Lemma 2.3 (I2), we have

$$\|L_{\xi_1}^n\| \ge \max_{x\in[1,e]}(L_{\xi_1}^n \mathbf{1})(x) \ge \xi_1^n \max_{x\in[1,e]}(\ln x)^{\beta-1}\left[\int_1^e (\ln y)^{\beta-1}G_\beta(e,y)\frac{dy}{y}\right]^{n-1}\int_1^e G_\beta(e,y)\frac{dy}{y},$$

where $\mathbf{1}(x) \equiv 1, x \in [1,e]$. Consequently, the Gelfand's theorem implies that

$$r(L_{\xi_1}) = \liminf_{n\to\infty} \sqrt[n]{\|L_{\xi_1}^n\|} \ge \xi_1 \int_1^e (\log y)^{\beta-1}G_\beta(e,y)\frac{dy}{y} = \frac{\xi_1\Gamma(\beta-1)}{\Gamma(2\beta-1)} > 0 \text{ for } \xi_1 > 0.$$

Note that $r(L_{\xi_1,\xi_2}) \ge r(L_{\xi_1}) > 0$. This completes the proof.

Lemma 2.5 and the Krein-Rutman theorem [20] imply that there exists $\zeta_{\xi_1,\xi_2} \in P\backslash\{0\}$ such that

$$L_{\xi_1,\xi_2}\zeta_{\xi_1,\xi_2} = r(L_{\xi_1,\xi_2})\zeta_{\xi_1,\xi_2}. \tag{2.6}$$

Moreover, by Lemma 2.4, we have

$$\zeta_{\xi_1,\xi_2} \in P_0. \tag{2.7}$$

Lemma 2.6 (see [21]). Let E be a Banach space, $W \subset E$ a bounded open set, and $T : W \to E$ a continuous compact operator. If there exists $z_0 \in E\backslash\{0\}$ such that

$$z - Tz \ne \mu z_0, \forall z \in \partial W, \mu \ge 0,$$

then the topological degree $\deg(I - T, W, 0) = 0$.

Lemma 2.7 (see [21]). Let E be a Banach space, $W \subset E$ a bounded open set with $0 \in W$, and $T : W \to E$ a continuous compact operator. If

$$Tz \ne \mu z, \forall z \in \partial W, \mu \ge 1,$$

then the topological degree $\deg(I - T, W, 0) = 1$.

Lemma 2.8 (see [22]). Let $T : E \to E$ be a completely continuous operator, and $L : E \to E$ a bounded linear operator. Suppose that 1 isn't an eigenvalue of L and

$$\lim_{\|z\|\to\infty} \frac{\|Tz - Lz\|}{\|z\|} = 0,$$

then there exists $z^{**} \in E$ such that $Tz^{**} = z^{**}$.

3. Main results

Now, we list some assumptions on f and g, which we need in this section.

(H5) There exist $\xi_1, \xi_2 > 0$ with $r(L_{\xi_1,\xi_2}) \ge 1$ such that

$$\liminf_{|z|\to+\infty} \frac{f(x,z)}{|z|} > \xi_1, \liminf_{|z|\to+\infty} \frac{g(x,z)}{|z|} > \xi_2, \text{ uniformly for } x \in [1,e].$$

(H6) There exist $\xi_3, \xi_4 > 0$ with $r(L_{\xi_3,\xi_4}) < 1$ such that

$$\limsup_{|z|\to0} \frac{|f(x,z)|}{|z|} < \xi_3, \limsup_{|z|\to0} \frac{|g(x,z)|}{|z|} < \xi_4, \text{ uniformly for } x \in [1,e].$$

(H7) There exist $\xi_5, \xi_6 > 0$ with $r(L_{\xi_5,\xi_6}) > 1$ such that

$$\liminf_{|z|\to 0^+} \frac{f(x,z)}{|z|} > \xi_5, \liminf_{|z|\to 0^+} \frac{g(x,z)}{|z|} > \xi_6, \text{ uniformly for } x \in [1,e].$$

(H8) There exist $\xi_7, \xi_8 > 0$ with $r(L_{\xi_7,\xi_8}) < 1$ such that

$$\limsup_{|z|\to +\infty} \frac{|f(x,z)|}{|z|} < \xi_7, \limsup_{|z|\to +\infty} \frac{|g(x,z)|}{|z|} < \xi_8, \text{ uniformly for } x \in [1,e].$$

(H9) $f, g(x,0) \not\equiv 0, x \in [1,e]$, and there exist ξ_9, ξ_{10} with $\frac{|\xi_9|}{\beta(\beta-1)\Gamma(\beta)} + |\xi_{10}|\frac{\int_1^e \frac{d\eta(t)}{t}}{\beta-1} < 1$ such that

$$\lim_{z\to\infty} \frac{f(x,z)}{z} = \xi_9, \lim_{z\to\infty} \frac{g(x,z)}{z} = \xi_{10}, \text{ uniformly for } \in x \in [1,e].$$

Theorem 3.1. Let (H1)–(H6) hold, then (1.1) has a nontrivial solution.

Proof. From (H5), there exist $\epsilon_i > 0$ $(i = 1, 2)$ and $Z_0 > 0$ such that

$$f(x,z) \geq (\xi_1 + \epsilon_1)|z|, \ g(x,z) \geq (\xi_2 + \epsilon_2)|z|, \text{ for } |z| > Z_0, x \in [1,e].$$

From (H3), for any $\varepsilon_i > 0$ $(i = 1, 2)$, there exists $Z_1 > Z_0$ such that

$$M_1(z) \leq \varepsilon_1|z|, \ M_2(z) \leq \varepsilon_2|z|, \text{ for } |z| > Z_1.$$

Let $M_1^* = \max_{|z|\leq Z_1} M_1(z), M_2^* = \max_{|z|\leq Z_1} M_2(z)$, then we have

$$M_1(z) \leq \varepsilon_1|z| + M_1^*, \ M_2(z) \leq \varepsilon_2|z| + M_2^*, z \in \mathbb{R}. \tag{3.1}$$

Note that by (H2), for $|z| > Z_1, x \in [1,e]$, and we obtain

$$\begin{aligned} f(x,z) &\geq (\xi_1 + \epsilon_1)|z| - \gamma_1(x) - \gamma_2(x)M_1(z) \geq (\xi_1 + \epsilon_1 - \varepsilon_1\|\gamma_2\|)|z| - \gamma_1(x), \\ g(x,z) &\geq (\xi_2 + \epsilon_2)|z| - \sigma_1(x) - \sigma_2(x)M_2(z) \geq (\xi_2 + \epsilon_2 - \varepsilon_2\|\sigma_2\|)|z| - \sigma_1(x). \end{aligned}$$

Note that $f, g(x,z)$ are bounded on $[1,e] \times [-Z_1, Z_1]$, then let $C_f = (\xi_1 + \epsilon_1 - \varepsilon_1\|\gamma_2\|)Z_1 + \max_{x\in[1,e],|z|\leq Z_0} |f(x,z)|$, $C_g = (\xi_2 + \epsilon_2 - \varepsilon_2\|\sigma_2\|)Z_1 + \max_{x\in[1,e],|z|\leq Z_0} |g(x,z)|$, so we have

$$f(x,z) \geq (\xi_1+\epsilon_1-\varepsilon_1\|\gamma_2\|)|z|-\gamma_1(x)-C_f, \ g(x,z) \geq (\xi_2+\epsilon_2-\varepsilon_2\|\sigma_2\|)|z|-\sigma_1(x)-C_g, \ z \in \mathbb{R}, x \in [1,e]. \tag{3.2}$$

Note that ε_i $(i = 1, 2)$ can be chosen arbitrarily small, and let a sufficiently large R_1 satisfy:

$$R_1 > \frac{\frac{\beta-1}{\beta}N_1 + (\beta-1)N_2}{\beta - 1 - \varepsilon_1\frac{\|\gamma_2\|}{\beta\Gamma(\beta)} - \varepsilon_2\|\sigma_2\|\int_1^e \frac{d\eta(x)}{x}},$$

$$R_1 > \frac{(\epsilon_1 - \varepsilon_1\|\gamma_2\|)\left(\frac{N_1}{\beta} + N_2\right) + (\xi_1 + \epsilon_1 - \varepsilon_1\|\gamma_2\|)(N_1 + N_2)}{(\epsilon_1 - \varepsilon_1\|\gamma_2\|)\left[1 - \varepsilon_1\frac{\|\gamma_2\|}{\beta(\beta-1)\Gamma(\beta)} - \varepsilon_2\frac{\|\sigma_2\|\int_1^e \frac{d\eta(x)}{x}}{\beta-1}\right] - (\xi_1 + \epsilon_1 - \varepsilon_1\|\gamma_2\|)\left[\varepsilon_1\frac{\|\gamma_2\|}{(\beta-1)\Gamma(\beta)} + \varepsilon_2\frac{\|\sigma_2\|\int_1^e \frac{d\eta(x)}{x}}{\beta-1}\right]},$$

and

$$R_1 > \frac{(\epsilon_2 - \varepsilon_2\|\sigma_2\|)\left(\frac{N_1}{\beta} + N_2\right) + (\xi_2 + \epsilon_2 - \varepsilon_2\|\sigma_2\|)(N_1 + N_2)}{(\epsilon_2 - \varepsilon_2\|\sigma_2\|)\left[1 - \varepsilon_1\frac{\|\gamma_2\|}{\beta(\beta-1)\Gamma(\beta)} - \varepsilon_2\frac{\|\sigma_2\|\int_1^e \frac{d\eta(x)}{x}}{\beta-1}\right] - (\xi_2 + \epsilon_2 - \varepsilon_2\|\sigma_2\|)\left[\varepsilon_1\frac{\|\gamma_2\|}{(\beta-1)\Gamma(\beta)} + \varepsilon_2\frac{\|\sigma_2\|\int_1^e \frac{d\eta(x)}{x}}{\beta-1}\right]}, \tag{3.3}$$

where

$$N_1 = \frac{\|\gamma_1\| + \|\gamma_2\| M_1^* + C_f}{(\beta-1)\Gamma(\beta)},\ N_2 = \frac{\int_1^e \frac{d\eta(x)}{x}}{\beta-1}[\|\sigma_1\| + \|\sigma_2\| M_2^* + C_g].$$

In the following, we shall prove

$$z - Tz \neq \mu\zeta_{\xi_1,\xi_2},\ z \in \partial B_{R_1}, \mu \geq 0, \tag{3.4}$$

where ζ_{ξ_1,ξ_2} is defined by (2.6) and B_{R_1} is an open ball: $\{z \in E : \|z\| < R_1\}$. If the claim (3.4) isn't satisfied, then there exist $z_1 \in \partial B_{R_1}$ and $\mu_1 \geq 0$ such that

$$z_1 - Tz_1 = \mu_1\zeta_{\xi_1,\xi_2}. \tag{3.5}$$

Note that $\mu_1 \neq 0$. If not, z_1 is a fixed point of T and the theorem is proved. In order to prove (3.5), let

$$\widetilde{z}_1(x) = \int_1^e G_\beta(x,y)[\gamma_1(y)+\gamma_2(y)M_1(z_1(y))+C_f]\frac{dy}{y}+\frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e[\sigma_1(x)+\sigma_2(x)M_2(z_1(x))+C_g]\frac{d\eta(x)}{x}, x \in [1,e]. \tag{3.6}$$

By $\gamma_1 + \gamma_2 M_1(z_1) + C_f \in P$ and $\sigma_1 + \sigma_2 M_2(z_1) + C_g \in P$, Lemma 2.4 implies that

$$\widetilde{z}_1 \in P_0.$$

Moreover, we calculate $z_1 + \widetilde{z}_1$ as follows:

$$\begin{aligned} z_1(x) + \widetilde{z}_1(x) &= (Tz_1)(x) + \widetilde{z}_1(x) + \mu_1\zeta_{\xi_1,\xi_2}(x) \\ &= \int_1^e G_\beta(x,y)[f(y,z_1(y)) + \gamma_1(y) + \gamma_2(y)M_1(z_1(y)) + C_f]\frac{dy}{y} \\ &\quad + \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e [g(x,z_1(x)) + \sigma_1(x) + \sigma_2(x)M_2(z_1(x)) + C_g]\frac{d\eta(x)}{x} + \mu_1\zeta_{\xi_1,\xi_2}(x), x \in [1,e]. \end{aligned}$$

From (H2), (2.7), and Lemma 2.4 we know

$$z_1 + \widetilde{z}_1 \in P_0. \tag{3.7}$$

By (3.1), (3.3) and (3.6), note that $\|z_1\| = R_1$, and we have

$$\begin{aligned} \widetilde{z}_1(x) &\leq \int_1^e G_\beta(x,y)[\gamma_1(y) + \gamma_2(y)(\varepsilon_1|z_1(y)| + M_1^*) + C_f]\frac{dy}{y} \\ &\quad + \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e [\sigma_1(x) + \sigma_2(x)(\varepsilon_2|z_1(x)| + M_2^*) + C_g]\frac{d\eta(x)}{x} \\ &\leq \int_1^e G_\beta(e,y)[\|\gamma_1\| + \|\gamma_2\|(\varepsilon_1\|z_1\| + M_1^*) + C_f]\frac{dy}{y} \\ &\quad + \frac{1}{\beta-1}\int_1^e [\|\sigma_1\| + \|\sigma_2\|(\varepsilon_2\|z_1\| + M_2^*) + C_g]\frac{d\eta(x)}{x} \\ &= \frac{\|\gamma_1\| + \|\gamma_2\|(\varepsilon_1 R_1 + M_1^*) + C_f}{\beta(\beta-1)\Gamma(\beta)} + \frac{\int_1^e \frac{d\eta(x)}{x}}{\beta-1}[\|\sigma_1\| + \|\sigma_2\|(\varepsilon_2 R_1 + M_2^*) + C_g] \\ &< R_1. \end{aligned}$$

Note that $\widetilde{z}_1, z_1 + \widetilde{z}_1 \in P_0$. From (3.2) and (H2), we have

$$\begin{aligned}
&(Tz_1)(x) + \widetilde{z}_1(x) \\
&\quad \geq \int_1^e G_\beta(x,y)[(\xi_1 + \epsilon_1 - \varepsilon_1\|\gamma_2\|)|z_1(y)| - \gamma_1(y) - C_f + \gamma_1(y) + \gamma_2(y)M_1(z_1(y)) + C_f]\frac{dy}{y} \\
&\qquad + \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e [(\xi_2 + \epsilon_2 - \varepsilon_2\|\sigma_2\|)|z_1(x)| - \sigma_1(x) - C_g + \sigma_1(x) + \sigma_2(x)M_2(z_1(x)) + C_g]\frac{d\eta(x)}{x} \\
&\quad \geq \int_1^e G_\beta(x,y)(\xi_1 + \epsilon_1 - \varepsilon_1\|\gamma_2\|)z_1(y)\frac{dy}{y} + \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e (\xi_2 + \epsilon_2 - \varepsilon_2\|\sigma_2\|)z_1(x)\frac{d\eta(x)}{x} \\
&\quad = \int_1^e G_\beta(x,y)(\xi_1 + \epsilon_1 - \varepsilon_1\|\gamma_2\|)[z_1(y) + \widetilde{z}_1(y)]\frac{dy}{y} + \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e (\xi_2 + \epsilon_2 - \varepsilon_2\|\sigma_2\|)[z_1(x) + \widetilde{z}_1(x)]\frac{d\eta(x)}{x} \\
&\qquad - \int_1^e G_\beta(x,y)(\xi_1 + \epsilon_1 - \varepsilon_1\|\gamma_2\|)\widetilde{z}_1(y)\frac{dy}{y} - \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e (\xi_2 + \epsilon_2 - \varepsilon_2\|\sigma_2\|)\widetilde{z}_1(x)\frac{d\eta(x)}{x} \\
&\quad \geq \xi_1\int_1^e G_\beta(x,y)[z_1(y) + \widetilde{z}_1(y)]\frac{dy}{y} + \xi_2\frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e [z_1(x) + \widetilde{z}_1(x)]\frac{d\eta(x)}{x}
\end{aligned} \tag{3.8}$$

using the fact that

$$\begin{aligned}
&\int_1^e G_\beta(x,y)(\epsilon_1 - \varepsilon_1\|\gamma_2\|)[z_1(y) + \widetilde{z}_1(y)]\frac{dy}{y} + \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e (\epsilon_2 - \varepsilon_2\|\sigma_2\|)[z_1(x) + \widetilde{z}_1(x)]\frac{d\eta(x)}{x} \\
&\quad - \int_1^e G_\beta(x,y)(\xi_1 + \epsilon_1 - \varepsilon_1\|\gamma_2\|)\widetilde{z}_1(y)\frac{dy}{y} - \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e (\xi_2 + \epsilon_2 - \varepsilon_2\|\sigma_2\|)\widetilde{z}_1(x)\frac{d\eta(x)}{x} \\
&\quad \geq 0, x \in [1,e].
\end{aligned} \tag{3.9}$$

In what follows, we prove that (3.9) holds. Note that $\|z_1\| = R_1$, and by (3.7), we have

$$z_1(x) + \widetilde{z}_1(x) \geq (\ln x)^{\beta-1}\|z_1 + \widetilde{z}_1\| \geq (\ln x)^{\beta-1}(R_1 - \|\widetilde{z}_1\|).$$

Consequently, note that $G_\beta(x,y) \leq \frac{(\ln x)^{\beta-1}(1-\log y)^{\beta-2}}{\Gamma(\beta)}, x, y \in [1,e]$, and (3.9) is greater than

$$\begin{aligned}
&\int_1^e G_\beta(x,y)(\epsilon_1 - \varepsilon_1\|\gamma_2\|)(\ln y)^{\beta-1}(R_1 - \|\widetilde{z}_1\|)\frac{dy}{y} + \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e (\epsilon_2 - \varepsilon_2\|\sigma_2\|)(\ln x)^{\beta-1}(R_1 - \|\widetilde{z}_1\|)\frac{d\eta(x)}{x} \\
&\quad - \int_1^e G_\beta(x,y)(\xi_1 + \epsilon_1 - \varepsilon_1\|\gamma_2\|)\int_1^e (\ln y)^{\beta-1}\frac{(1-\ln\tau)^{\beta-2}}{\Gamma(\beta)}[\gamma_1(\tau) + \gamma_2(\tau)M_1(z_1(\tau)) + C_f]\frac{d\tau}{\tau}\frac{dy}{y} \\
&\quad - \int_1^e G_\beta(x,y)(\xi_1 + \epsilon_1 - \varepsilon_1\|\gamma_2\|)\frac{(\ln y)^{\beta-1}}{\beta-1}\int_1^e [\sigma_1(\tau) + \sigma_2(\tau)M_2(z_1(\tau)) + C_g]\frac{d\eta(\tau)}{\tau}\frac{dy}{y} \\
&\quad - \frac{(\ln x)^{\beta-1}}{(\beta-1)\Gamma(\beta)}\int_1^e (\xi_2 + \epsilon_2 - \varepsilon_2\|\sigma_2\|)\int_1^e (\ln x)^{\beta-1}(1-\ln y)^{\beta-2}[\gamma_1(y) + \gamma_2(y)M_1(z_1(y)) + C_f]\frac{dy}{y}\frac{d\eta(x)}{x} \\
&\quad - \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e (\xi_2 + \epsilon_2 - \varepsilon_2\|\sigma_2\|)\frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e [\sigma_1(y) + \sigma_2(y)M_2(z_1(y)) + C_g]\frac{d\eta(y)}{y}\frac{d\eta(x)}{x}.
\end{aligned}$$

(3.3) enables us to obtain

$$\begin{aligned}
&(\epsilon_1 - \varepsilon_1\|\gamma_2\|)(R_1 - \|\widetilde{z}_1\|)\\
&\quad - (\xi_1 + \epsilon_1 - \varepsilon_1\|\gamma_2\|)\int_1^e \frac{(1-\ln\tau)^{\beta-2}}{\Gamma(\beta)}[\gamma_1(\tau) + \gamma_2(\tau)M_1(z_1(\tau)) + C_f]\frac{d\tau}{\tau}\\
&\quad - \frac{\xi_1 + \epsilon_1 - \varepsilon_1\|\gamma_2\|}{\beta - 1}\int_1^e [\sigma_1(\tau) + \sigma_2(\tau)M_2(z_1(\tau)) + C_g]\frac{d\eta(\tau)}{\tau}\\
&\geq (\epsilon_1 - \varepsilon_1\|\gamma_2\|)(R_1 - \|\widetilde{z}_1\|)\\
&\quad - \frac{\xi_1 + \epsilon_1 - \varepsilon_1\|\gamma_2\|}{(\beta-1)\Gamma(\beta)}[\|\gamma_1\| + \|\gamma_2\|(\varepsilon_1 R_1 + M_1^*) + C_f]\\
&\quad - \frac{\xi_1 + \epsilon_1 - \varepsilon_1\|\gamma_2\|}{\beta - 1}\int_1^e \frac{d\eta(\tau)}{\tau}[\|\sigma_1\| + \|\sigma_2\|(\varepsilon_2 R_1 + M_2^*) + C_g]\\
&\geq 0
\end{aligned}$$

and

$$\begin{aligned}
&(\epsilon_2 - \varepsilon_2\|\sigma_2\|)(R_1 - \|\widetilde{z}_1\|)\\
&\quad - \frac{\xi_2 + \epsilon_2 - \varepsilon_2\|\sigma_2\|}{\Gamma(\beta)}\int_1^e (1-\ln y)^{\beta-2}[\gamma_1(y) + \gamma_2(y)M_1(z_1(y)) + C_f]\frac{dy}{y}\\
&\quad - \frac{\xi_2 + \epsilon_2 - \varepsilon_2\|\sigma_2\|}{\beta - 1}\int_1^e [\sigma_1(y) + \sigma_2(y)M_2(z_1(y)) + C_g]\frac{d\eta(y)}{y}\\
&\geq (\epsilon_2 - \varepsilon_2\|\sigma_2\|)(R_1 - \|\widetilde{z}_1\|)\\
&\quad - \frac{\xi_2 + \epsilon_2 - \varepsilon_2\|\sigma_2\|}{(\beta-1)\Gamma(\beta)}[\|\gamma_1\| + \|\gamma_2\|(\varepsilon_1 R_1 + M_1^*) + C_f]\\
&\quad - \frac{\xi_2 + \epsilon_2 - \varepsilon_2\|\sigma_2\|}{\beta - 1}\int_1^e \frac{d\eta(y)}{y}[\|\sigma_1\| + \|\sigma_2\|(\varepsilon_2 R_1 + M_2^*) + C_g]\\
&\geq 0.
\end{aligned}$$

As a result, (3.9) holds, as required. From (3.5) and (3.8), we have

$$\begin{aligned}
z_1(x) + \widetilde{z}_1(x) &= (Tz_1)(x) + \widetilde{z}_1(x) + \mu_1\zeta_{\xi_1,\xi_2}(x)\\
&\geq (L_{\xi_1,\xi_2}(z_1 + \widetilde{z}_1))(x) + \mu_1\zeta_{\xi_1,\xi_2}(x),\ x \in [1, e].
\end{aligned}$$

Define a set $W = \{\mu : z_1 + \widetilde{z}_1 \geq \mu\zeta_{\xi_1,\xi_2}\}$ and $\mu^* = \sup W$, then $\mu_1 \in W$ and $\mu^* \geq \mu_1$. Hence, note that $L_{\xi_1,\xi_2} : P \to P$, and we have

$$\begin{aligned}
z_1(x) + \widetilde{z}_1(x) &\geq \mu_1\zeta_{\xi_1,\xi_2}(x) + (L_{\xi_1,\xi_2}(z_1 + \widetilde{z}_1))(x)\\
&\geq \mu_1\zeta_{\xi_1,\xi_2}(x) + (L_{\xi_1,\xi_2}\mu^*\zeta_{\xi_1,\xi_2})(x) = \mu_1\zeta_{\xi_1,\xi_2}(x) + \mu^* r(L_{\xi_1,\xi_2})\zeta_{\xi_1,\xi_2}(x) \geq (\mu^* + \mu_1)\zeta_{\xi_1,\xi_2}(x),
\end{aligned}$$

which contradicts the definition of μ^*. As a result, (3.4) holds, and Lemma 2.6 implies that

$$\deg(I - T, B_{R_1}, 0) = 0. \tag{3.10}$$

By (H6), there exists $r_1 \in (0, R_1)$ such that

$$|f(x,z)| \leq \xi_3|z|,\ \ |g(x,z)| \leq \xi_4|z|,\ \text{ for } |z| \in [0, r_1], x \in [1, e]. \tag{3.11}$$

In what follows, we prove that

$$Tz \neq \mu z, z \in \partial B_{r_1}, \mu \geq 1, \tag{3.12}$$

where $B_{r_1} = \{z \in E : \|z\| < r_1\}$. If (3.12) doesn't hold, and there exist $z_2 \in \partial B_{r_1},\ \mu_2 \geq 1$, such that

$$Tz_2 = \mu_2 z_2,$$

then this, along with (3.11), implies that

$$\begin{aligned}
|z_2(x)| &= \frac{1}{\mu_2}|(Tz_2)(x)| \\
&\leq \left| \int_1^e G_\beta(x,y) f(y, z_2(y)) \frac{dy}{y} + \frac{(\ln x)^{\beta-1}}{\beta - 1} \int_1^e g(x, z_2(x)) \frac{d\eta(x)}{x} \right| \\
&\leq \int_1^e G_\beta(x,y) |f(y, z_2(y))| \frac{dy}{y} + \frac{(\ln x)^{\beta-1}}{\beta - 1} \int_1^e |g(x, z_2(x))| \frac{d\eta(x)}{x} \\
&\leq \xi_3 \int_1^e G_\beta(x,y) |z_2(y)| \frac{dy}{y} + \xi_4 \frac{(\ln x)^{\beta-1}}{\beta - 1} \int_1^e |z_2(x)| \frac{d\eta(x)}{x} \\
&= (L_{\xi_3,\xi_4}|z_2|)(x), x \in [1, e].
\end{aligned}$$

Note that $r(L_{\xi_3,\xi_4}) < 1$. This means that $\left(I - L_{\xi_3,\xi_4}\right)^{-1}$ exists and

$$\left(I - L_{\xi_3,\xi_4}\right)^{-1} = I + L_{\xi_3,\xi_4} + L^2_{\xi_3,\xi_4} + \cdots + L^n_{\xi_3,\xi_4} + \cdots .$$

Consequently, note that $\left(I - L_{\xi_3,\xi_4}\right)^{-1} : P \to P$, and we have

$$((I - L_{\xi_3,\xi_4})|z_2|)(x) \leq 0 \Rightarrow |z_2(x)| \leq \left(I - L_{\xi_3,\xi_4}\right)^{-1} 0 = 0,$$

which implies that $z_2(x) \equiv 0, x \in [1, e]$ and contradicts $z_2 \in \partial B_{r_1}$. Therefore, Lemma 2.7 implies that

$$\deg (I - T, B_{r_1}, 0) = 1.$$

Using this with (3.10), we have

$$\deg \left(I - T, B_{R_1} \backslash \overline{B}_{r_1}, 0\right) = \deg (I - T, B_{R_1}, 0) - \deg (I - T, B_{r_1}, 0) = -1.$$

Hence, T has at least one fixed point in $B_{R_1} \backslash \overline{B}_{r_1}$, i.e., (1.1) has at least one nontrivial solution. This completes the proof.

Now, we consider the case that our nonlinearities are suberlinear. From [8], we know that E's conjugate space $E^* := \{\rho : \rho \text{ has bounded variation on } [1, e]\}$. Moreover, the dual cone of P and the bounded linear functional on E are

$$P^* := \{\rho \in E^* : \rho \text{ is non-decreasing on } [1, e]\} \text{ and } \langle \rho, z \rangle = \int_1^e z(x) d\rho(x), z \in E, \rho \in E^*.$$

Note that $r(L_{\xi_5,\xi_6}) > 1$ in (H7), and from the similar method in [23], there exists $\psi_{\xi_5,\xi_6} \in P^* \backslash \{0\}$ such that

$$L^*_{\xi_5,\xi_6} \psi_{\xi_5,\xi_6} = r\left(L_{\xi_5,\xi_6}\right) \psi_{\xi_5,\xi_6}, \tag{3.13}$$

where $L^*_{\xi_5,\xi_6} : E^* \to E^*$ is the conjugate operator of L_{ξ_5,ξ_6}, denoted by

$$\left(L^*_{\xi_5,\xi_6}\rho\right)(y) := \xi_5 \int_1^y d\tau \int_1^e G_\beta(x,\tau)\frac{d\rho(x)}{x} + \xi_6\eta(y)\int_1^e \frac{(\ln x)^{\beta-1}}{\beta-1}\frac{d\rho(x)}{x}, \rho \in P^*.$$

Theorem 3.2. Let (H1)–(H4), (H7), and (H8) hold, then (1.1) has a nontrivial solution.
Proof. From (H7), there exists $r_2 > 0$ such that

$$f(x,z) \geq \xi_5|z|,\ g(x,z) \geq \xi_6|z|,\ \text{for } x \in [1,e], |z| \leq r_2. \tag{3.14}$$

For this r_2, we prove that

$$z - Tz \neq \mu\omega_0,\ z \in \partial B_{r_2}, \mu \geq 0, \tag{3.15}$$

where $\omega_0 \in P$ is a fixed element and $B_{r_2} = \{z \in E : \|z\| < r_2\}$. We use an argument by indirection, then there exist $z_3 \in \partial B_{r_2}, \mu_3 \geq 0$ such that

$$z_3 - Tz_3 = \mu_3\omega_0.$$

Note that by (3.14), we have

$$(Tz_3)(x) = \int_1^e G_\beta(x,y)f(y,z_3(y))\frac{dy}{y} + \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e g(x,z_3(x))\frac{d\eta(x)}{x} \geq 0, z_3 \in \partial B_{r_2}, x \in [1,e].$$

Hence, $\mu_3 \geq 0, \omega_0 \in P$ enable us to find

$$z_3 = Tz_3 + \mu_3\omega_0 \geq 0,$$

and

$$\begin{aligned} z_3(x) &\geq (Tz_3)(x) \\ &\geq \xi_5 \int_1^e G_\beta(x,y)z_3(y)\frac{dy}{y} + \xi_6\frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e z_3(x)\frac{d\eta(x)}{x}. \end{aligned}$$

Multiplying $\frac{d\psi_{\xi_5,\xi_6}(x)}{x}$ on both sides of the above inequalities and integrating over $[1,e]$, from (3.13), we obtain

$$\begin{aligned} \int_1^e z_3(x)\frac{d\psi_{\xi_5,\xi_6}(x)}{x} &\geq \int_1^e \frac{d\psi_{\xi_5,\xi_6}(x)}{x}\left(\xi_5\int_1^e G_\beta(x,y)z_3(y)\frac{dy}{y} + \xi_6\frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e z_3(x)\frac{d\eta(x)}{x}\right) \\ &= \int_1^e \frac{z_3(y)}{y}d\left(\xi_5\int_1^y d\tau\int_1^e G_\beta(x,\tau)\frac{d\psi_{\xi_5,\xi_6}(x)}{x} + \xi_6\eta(y)\int_1^e \frac{(\ln x)^{\beta-1}}{\beta-1}\frac{d\psi_{\xi_5,\xi_6}(x)}{x}\right) \\ &= \int_1^e \frac{z_3(y)}{y}d(L^*_{\xi_5,\xi_6}\psi_{\xi_5,\xi_6})(y) \\ &= r\left(L_{\xi_5,\xi_6}\right)\int_1^e \frac{z_3(y)}{y}d\psi_{\xi_5,\xi_6}(y). \end{aligned} \tag{3.16}$$

Note that $\psi_{\xi_5,\xi_6} \in P^*\backslash\{0\}$, then from the definition of the Riemann-Stieltjes integral, we have

$$\int_1^e z_3(x)\frac{d\psi_{\xi_5,\xi_6}(x)}{x} = \lim_{\lambda\to 0}\sum_{i=1}^n \frac{z_3(\xi_i)}{\xi_i}\left[\psi_{\xi_5,\xi_6}(x_i) - \psi_{\xi_5,\xi_6}(x_{i-1})\right] \geq 0, \tag{3.17}$$

where $0 = x_0 < x_1 < \cdots < x_{n-1} < x_n < x_{n+1} = 1, \lambda = \max_{1\le i\le n}(x_i - x_{i-1}), \forall \xi_i \in [x_{i-1}, x_i], i = 1, 2, \cdots, n$. Therefore, $r(L_{\xi_5,\xi_6}) > 1$ and (3.16) enable us to obtain

$$\int_1^e z_3(x)\frac{d\psi_{\xi_5,\xi_6}(x)}{x} = 0. \tag{3.18}$$

Note that for all divisions x_i, (3.18) holds, and we only obtain $z_3(x) \equiv 0, x \in [1, e]$. Therefore, this contradicts $z_3 \in \partial B_{r_2}$ and (3.15) holds. Lemma 2.6 implies that

$$\deg(I - T, B_{r_2}, 0) = 0. \tag{3.19}$$

By (H8), there exist $d_1, d_2 > 0$ such that

$$|f(x,z)| \le \xi_7|z| + d_1,\ |g(x,z)| \le \xi_8|z| + d_2,\ \text{for } |z| \in \mathbb{R}, x \in [1, e]. \tag{3.20}$$

In what follows, we prove that $S = \{z \in E : Tz = \mu z, \mu \ge 1\}$ is a bounded set. If there exist $z_4 \in S, \mu_4 \ge 1$ such that

$$Tz_4 = \mu_4 z_4,$$

then (3.20) implies that

$$\begin{aligned}
|z_4(x)| &= \frac{1}{\mu_4}|(Tz_4)(x)| \\
&\le \int_1^e G_\beta(x,y)|f(y,z_4(y))|\frac{dy}{y} + \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e |g(x,z_4(x))|\frac{d\eta(x)}{x} \\
&\le \int_1^e G_\beta(x,y)(\xi_7|z_4(y)| + d_1)\frac{dy}{y} + \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e (\xi_8|z_4(x)| + d_2)\frac{d\eta(x)}{x} \\
&\le (L_{\xi_7,\xi_8}|z_4|)(x) + d_1\int_1^e G_\beta(e,y)\frac{dy}{y} + \frac{d_2\int_1^e \frac{d\eta(x)}{x}}{\beta-1},\ x \in [1, e].
\end{aligned}$$

Note that $r(L_{\xi_7,\xi_8}) < 1$. Thus $\left(I - L_{\xi_7,\xi_8}\right)^{-1}$ exists, and

$$\left(I - L_{\xi_7,\xi_8}\right)^{-1} = I + L_{\xi_7,\xi_8} + L^2_{\xi_7,\xi_8} + \cdots + L^n_{\xi_7,\xi_8} + \cdots.$$

Consequently, note that $\left(I - L_{\xi_7,\xi_8}\right)^{-1} : P \to P$, and we have

$$((I - L_{\xi_7,\xi_8})|z_4|)(x) \le \left(\frac{d_1}{\beta(\beta-1)\Gamma(\beta)} + \frac{d_2\int_1^e \frac{d\eta(x)}{x}}{\beta-1}\right)$$

and, thus,

$$|z_4(x)| \le \left(I - L_{\xi_7,\xi_8}\right)^{-1}\left(\frac{d_1}{\beta(\beta-1)\Gamma(\beta)} + \frac{d_2\int_1^e \frac{d\eta(x)}{x}}{\beta-1}\right),\ x \in [1, e],$$

which implies that S is a bounded set, as required. Let $R_2 > \sup S$ and $R_2 > r_2$, then we have

$$Tz \ne \mu z,\ z \in \partial B_{R_2}, \mu \ge 1.$$

Therefore, Lemma 2.7 implies that

$$\deg(I-T, B_{R_2}, 0) = 1.$$

Using this with (3.19), we have

$$\deg\left(I-T, B_{R_2}\backslash\overline{B}_{r_2}, 0\right) = \deg(I-T, B_{R_2}, 0) - \deg(I-T, B_{r_2}, 0) = -1.$$

Hence, T has at least one fixed point in $B_{R_2}\backslash\overline{B}_{r_2}$, i.e., (1.1) has at least one nontrivial solution. This completes the proof.

Theorem 3.3. Let (H1), (H4), and (H9) hold, then (1.1) has a nontrivial solution.

Proof. Define a linear operator $L_{\xi_9,\xi_{10}} : E \to E$ as follows:

$$(L_{\xi_9,\xi_{10}}z)(x) = \xi_9\int_1^e G_\beta(x,y)z(y)\frac{dy}{y} + \xi_{10}\frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e z(x)\frac{d\eta(x)}{x},\ x\in[1,e],\ z\in E.$$

Now, we prove that 1 isn't an eigenvalue for $L_{\xi_9,\xi_{10}}$. On the contrary, then there exists $z^* \in E\backslash\{0\}$ such that

$$L_{\xi_9,\xi_{10}}z^* = z^*. \tag{3.21}$$

By Lemma 2.2, we obtain

$$\begin{cases} -D^\beta z^*(x) = \xi_9 z^*(x),\ x\in(1,e),\\ z^*(1) = \delta z^*(1) = 0, \delta z^*(e) = \xi_{10}\displaystyle\int_1^e z^*(x)\frac{d\eta(x)}{x}. \end{cases}$$

Next, we will consider the following two cases.

Case 1. $\xi_9^2 + \xi_{10}^2 = 0$. By Lemma 2.2, we have

$$z^*(x) = \widetilde{d_1}(\ln x)^{\beta-1} + \widetilde{d_2}(\ln x)^{\beta-2} + \widetilde{d_3}(\ln x)^{\beta-3},$$

where $\widetilde{d_i} \in \mathbb{R}$, $i = 1,2,3$. Note that $\beta\in(2,3]$, the boundary condition $z^*(1) = \delta z^*(1) = \delta z^*(e) = 0$ implies that $\widetilde{d_i} = 0$, and $z^*(x) \equiv 0,\ x\in[1,e]$. This contradicts $z^* \in E\backslash\{0\}$.

Case 2. $\xi_9^2 + \xi_{10}^2 \neq 0$. From (3.21), we have

$$\begin{aligned} z^*(x) &= \xi_9\int_1^e G_\beta(x,y)z^*(y)\frac{dy}{y} + \xi_{10}\frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e z^*(x)\frac{d\eta(x)}{x}\\ &\le \|z^*\|\left(|\xi_9|\int_1^e G_\beta(e,y)\frac{dy}{y} + |\xi_{10}|\frac{\int_1^e \frac{d\eta(x)}{x}}{\beta-1}\right),\ x\in[1,e]. \end{aligned}$$

By (H9), we obtain that

$$\|z^*\| \le \|z^*\|\left(\frac{|\xi_9|}{\beta(\beta-1)\Gamma(\beta)} + |\xi_{10}|\frac{\int_1^e \frac{d\eta(x)}{x}}{\beta-1}\right) < \|z^*\|.$$

This is impossible.

As a result, we claim that 1 isn't an eigenvalue for $L_{\xi_9,\xi_{10}}$, as required.

By (H9), for all $\varepsilon > 0$, there exists $\Lambda > 0$ such that

$$|f(x,z) - \xi_9 z| \le \varepsilon|z|,\ |g(x,z) - \xi_{10} z| \le \varepsilon|z|,\ \text{for } |z| > \Lambda,\ x \in [1,e].$$

Note that when $|z| \le \Lambda$ and $x \in [1,e]$, $|f(x,z) - \xi_9 z|$ and $|g(x,z) - \xi_{10} z|$ are bounded. Therefore, there exist $\varrho_i > 0$ $(i = 1,2)$ such that

$$|f(x,z) - \xi_9 z| \le \varepsilon|z| + \varrho_1,\ |g(x,z) - \xi_{10} z| \le \varepsilon|z| + \varrho_2,\ z \in \mathbb{R},\ x \in [1,e]. \tag{3.22}$$

Hence, (3.22) enables us to obtain

$$\begin{aligned}
\|Tz - L_{\xi_9,\xi_{10}} z\| &= \max_{x\in[1,e]} \left| \int_1^e G_\beta(x,y)[f(y,z(y)) - \xi_9 z(y)]\frac{dy}{y} + \frac{(\ln x)^{\beta-1}}{\beta-1}\int_1^e [g(x,z(x)) - \xi_{10} z(x)]\frac{d\eta(x)}{x} \right| \\
&\le \int_1^e G_\beta(e,y)|f(y,z(y)) - \xi_9 z(y)|\frac{dy}{y} + \frac{1}{\beta-1}\int_1^e |g(x,z(x)) - \xi_{10} z(x)|\frac{d\eta(x)}{x} \\
&\le \frac{\varepsilon\|z\| + \varrho_1}{\beta(\beta-1)\Gamma(\beta)} + \frac{\int_1^e \frac{d\eta(x)}{x}}{\beta-1}(\varepsilon\|z\| + \varrho_2).
\end{aligned}$$

This implies that

$$\lim_{\|z\|\to\infty} \frac{\|Tz - L_{\xi_9,\xi_{10}} z\|}{\|z\|} \le \varepsilon\left(\frac{1}{\beta(\beta-1)\Gamma(\beta)} + \frac{\int_1^e \frac{d\eta(x)}{x}}{\beta-1}\right).$$

Note that from the arbitrariness of ε, we know that

$$\lim_{\|z\|\to\infty} \frac{\|Tz - L_{\xi_9,\xi_{10}} z\|}{\|z\|} = 0.$$

Lemma 2.8 indicates that T has at least one fixed point, and note that $T0 \ne 0$ by $f, g(t,0) \not\equiv 0, t \in [1,e]$. Therefore, (1.1) has at least one nontrivial solution. This completes the proof.

4. Conclusions

We note that the spectral theory of linear operators can be used to study differential equations, see for example, [2,6,9,10]. In [9], the authors studied the following fractional boundary value problem

$$\begin{cases} D_{0+}^{\alpha} z(t) + q(t)f(t,z(t)) = 0, \quad 0 < t < 1, n-1 < \alpha \le n, \\ z(0) = z'(0) = \cdots = z^{(n-2)}(0) = 0, \quad z(1) = \int_0^1 z(s)dA(s), \end{cases}$$

where $\alpha \ge 2$, D_{0+}^{α} is the Riemann-Liouville derivative and $f \in C([0,1] \times (0,+\infty), \mathbb{R}^+)$. They obtained two existence theorems of positive solutions for their problem when the nonlinearity f satisfies one of the following growth conditions
sublinear condition: $\limsup_{z\to+\infty} \sup_{t\in[0,1]} \frac{f(t,z)}{z} < \lambda_1$, and $\liminf_{z\to0^+} \inf_{t\in[0,1]} \frac{f(t,z)}{z} > \lambda_1$;
superlinear condition: $\limsup_{z\to0^+} \sup_{t\in[0,1]} \frac{f(t,z)}{z} < \lambda_1$, and $\liminf_{z\to+\infty} \inf_{t\in[0,1]} \frac{f(t,z)}{z} > \lambda_1$, where $\lambda_1 = (r(L))^{-1}$, $r(L)$ is the spectral radius of the linear operator $(Lz)(t) = \int_0^1 G_{\text{Wang}}(t,s)q(s)z(s)ds$, G_{Wang} is their Green's function.

Compared with our problem, we don't incorporate the integral boundary conditions into the Green's function, and consider a new linear operator (2.5). Moreover, we obtain several existence theorems of nontrivial solutions under some conditions regarding the spectral radius of the linear operator. On the other hand, our nonlinearities f, g can be sign-changing, in contrast to the nonlinearities in [1–10], which are assumed to be nonnegative. These imply that our main results generalize and improve the corresponding ones in the works cited above.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflict of interest.

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