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Research article

Pointwise Jacobson type necessary conditions for optimal control problems governed by impulsive differential systems

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Abstract: This work focuses on an exploration of the pointwise Jacobson-type necessary conditions for optimal control problems governed by differential systems with impulse at fixed times; the pointwise Jacobson-type necessary optimality conditions refer to a type of pointwise second-order necessary optimality conditions for optimal singular control in the classical sense. By introducing an impulsive linear matrix Riccati differential equation, we derive the integral representation of the functional second-order variational. Based on this, the integral form of the second-order necessary conditions and the pointwise Jacobson-type necessary conditions are obtained. Incidentally, we have established the Legendre-Clebsch condition and the pointwise Legendre-Clebsch condition. Finally, an example is provided to illustrate the effectiveness of the main result.

Keywords: pointwise Jacobson-type necessary conditions; Legendre-Clebsch condition; optimal singular control; impulsive differential systems; optimal control

1. Introduction

Impulsive differential equations are employed for the analysis of real-world phenomena that are characterized by the state of instantaneous system changes. It has been found to play a vital role in many areas, such as sampled-data control, communication networks, industrial robots, biology, and so on. Due to the widespread existence of impulsive perturbation, extensive research has been dedicated to the exploration of the stability of impulsive differential systems (see [1–4]). Meanwhile, some scholars have focused on studying the optimal impulsive control problem, which has resulted in numerous interesting findings (see [5–9]).

Finding the necessary optimality conditions is one of the central tasks in optimal control theory. Pontryagin and his co-authors have made milestone contributions (see [10]). As pointed out in [11]: "The mathematical significance of the maximum principle lies in that maximizing the Hamiltonian is much easier than the original control problem that is infinite-dimensional". Essentially, the Hamiltonian maximization occurs pointwise. Since Pontryagin's maximum principle was discovered, the first-order and second-order necessary conditions for optimal control problems in both finite- and infinite-dimensional spaces have been extensively researched (see [11, 12]).

However, it is not always possible to find the optimal control by pointwise maximizing the Hamiltonian, and this kind of optimal control problem is often referred to as the singular control problems. For singular control problems, the first task is to discover new necessary conditions to distinguish optimal singular controls from other singular controls; one way to do this is to look for second-order conditions. It is common to seek a second-order necessary condition that requires a quadratic functional to be non-negative. Ideally, we would prefer the second-order necessary conditions to have pointwise characteristics that are similar to Pontryagin's maximum principle, which means that the pointwise mode is preferable for maximizing a particular function. The pointwise necessary optimality conditions are reviewed in [13]; the Jacobson conditions and the Goh conditions are generally considered as two types of pointwise second-order necessary optimality conditions for the optimal singular control problem. In addition, references [14–16] are very comprehensive references on the singular control problem. Regarding the Goh conditions, the original contribution can be seen in [17]. In short, the Goh conditions are obtained by applying Goh's transformation. Goh's transformation approaches have been designed to transform the original singular problem into a new nonsingular one; then, for the nonsingular optimal control problem, the classical Legendre-Clebsch condition may be applied. In this process, Goh's transformation may be used several times; therefore, the Goh conditions are also called the generalized Legendre-Clebsch conditions; recent research results are very abundant, for example, references [18–21] and related references. Conversely, limited attention has been given to investigating the Jacobson-type conditions.

There is an interesting story about Jacobson-type necessary conditions. Until Jacobson discovered the Jacobson-type necessary conditions, it was thought that there was no Riccati-type matrix differential equation for singular control problems. In fact, Jacobson introduced a linear matrix Riccati differential equation that is similar to the nonlinear matrix Riccati differential equation in the standard LQ problem. Thus, a "new" necessary condition for singular control was obtained in [22]. After this, Jacobson found that this new condition is different from the generalized Legendre-Clebsch condition, and Jacobson has demonstrated that these two necessary conditions are generally insufficient for optimality in [23]. Therefore, the matrix Riccati differential equation can not only solve nonsingular problems, they can also solve important optimal singular control problems; singular control and nonsingular control can even be considered under a unified framework (see Theorem 4.2 in [15, 24]).

Recent studies [25, 26] have established the Jacobson-type pointwise second-order necessary optimality conditions for deterministic and stochastic optimal singular control problems, respectively. Particularly, the relaxation methods for control problems have been used to solve the singular control problem of a lack of a linear structure in control problems; also, the pointwise second-order necessary conditions have been obtained in [25]. When discussing the second-order necessary conditions, only the allowable set of singular control is considered, rather than the original allowable control set, which can greatly reduce the computational expense of singular control problems. References [25, 26] have also been applied to derive the pointwise second-order necessary optimality conditions for singular control problems with constraints in finite- or infinite-dimensional spaces. For further details, please refer to [13, 27–31]. Theorem 4.3 in [25] describes the Jacobson-type second-order necessary optimality condition of singular control problems governed by pulseless controlled systems according to Pontryagin.

For the definition of singular control in the classical sense and in the Pontryagin sense, see Definitions 1 and 2 in [14], as well as (4.4) and (4.5) in [25].

Regarding the importance of the impulsive system, the second-order necessary conditions for the optimal control problem governed by impulsive differential systems have been studied in [32], which focuses on impulsive systems with multi-point nonlocal and integral boundary conditions; the second-order necessary conditions of the integral form has been obtained by introducing the impulsive matrix function directly. When discussing the second-order necessary conditions for singular control, the perturbation control takes its value in full space, and not in the singular control region (see (3.13)); therefore, the conclusion is not reached by the pointwise Jacobson-type necessary conditions. In addition, references [33–35] consider second-order necessary conditions whereby the measure is impulsive control; the system can experience an infinite number of jumps in a finite amount of time, but this is different from what we are going to consider.

Inspired by the above discussions, we aimed to study the following optimal control problem, as governed by impulsive differential systems, which differs from the previous works.

Problem P: Let $T > 0$ and $\Lambda = \{ t_i \mid 0 < t_1 < t_2 < \cdots < t_k < T \} \subset (0, T)$ be given; $U \subseteq \mathbb{R}^m$ is a nonempty bounded convex open set.

$$\min J(u(\cdot)) = \int_0^T l(t, y(t), u(t))dt + G(y(T)), \tag{1.1}$$

subject to

$$\begin{cases} \dot{y}(t) = f(t, y(t), u(t)), & t \in [0, T] \setminus \Lambda, \\ y(t_i+) - y(t_i) = J_i(y(t_i)), & t_i \in \Lambda, \\ y(0) = y_0, \end{cases} \tag{1.2}$$

where $u(\cdot) \in \mathcal{U}_{ad} = \{ u(\cdot) \mid u(\cdot) \text{ measurable}, u(t) \in U\}$, and l, G, f, and J_i $(i = 1, 2, \cdots, k)$ are the given maps.

Here, we generalize the control model of [15, 25] to an impulsive controlled system. For Problem P, the difficulty lies in the introduction of the appropriate impulsive adjoint matrix differential equation, which we overcome by borrowing the method presented in [25]. The main conclusions include Theorems 3.1 and 4.6, where the former is a generalization of Theorem 4.2 in [15] to impulsive controlled systems; the latter is a similar conclusion of Theorem 4.3 in [25], but the difference is that Theorem 4.6 is a conclusion of the impulsive controlled systems, and in the classical sense, while Theorem 4.3 in [25] is a conclusion for the pulseless controlled systems, and in the Pontryagin sense.

The main novelties and contributions of this paper can be summarized as follows: (i) generalization of pulseless controlled systems to impulsive controlled systems, which is helpful to increase the applicability of the model; (ii) through the use of $C([0, T])$ as a dense subspace in $L^1([0, T])$, as per the results of functional analysis, the condition of conclusion is weakened; (iii) pointwise Jacobson-type necessary conditions have been obtained, which facilitates the calculations for and distinguishes the optimal singular control from other singular controls in the classical sense (see Definition 2.2).

The outline of this paper is as follows. Some preliminaries are proposed in Section 2. In Section 3, the integral-form second-order necessary conditions are derived. In Section 4, the pointwise Jacobson-type necessary conditions are given. An example is considered to elucidate the proposed main results in Section 5, and Section 6 concludes this paper.

2. Preliminaries

In this section, we will present some preliminaries, which includes basic assumptions, the definition of the singular control in the classical sense, the solvability of impulsive systems, and a lemma that has been obtained via functional analysis.

Let $B^\top$ denote the transposition of a matrix B. Define $C^1([0,T]\setminus\Lambda,\mathbb{R}^n)=\{y:[0,T]\longrightarrow\mathbb{R}^n \mid y$ is continuous differential at $t\in[0,T]\setminus\Lambda\}$ and $PC_l([0,T],\mathbb{R}^n)\,(PC_r([0,T],\ \mathbb{R}^n))=\{y:[0,T]\longrightarrow\mathbb{R}^n \mid y$ is continuous at $t\in[0,T]\setminus\Lambda,$ y is left (right) continuous and exists right (left) limit at $t\in\Lambda\}$, obviously endowed with the norm $\|y\|_{PC}=\sup\{\|y(t+)\|,\|y(t-)\| \mid\ t\in[0,T]\}$, $PC_l([0,T],\mathbb{R}^n)$; also, $PC_r([0,T],\mathbb{R}^n)$ denotes Banach spaces.

Let us assume the following:

(A1) $U\subseteq\mathbb{R}^m$ is a nonempty bounded convex open set.

(A2) The functions denoted by $F=\begin{pmatrix} f\\ l\end{pmatrix}:[0,T]\times\mathbb{R}^n\times U\longrightarrow\mathbb{R}^{n+1}$ are measurable in t and twice continuously differentiable in (y,u); for any $\rho>0$, there exists a constant $L(\rho)>0$ such that, for all $y,\hat{y}\in\mathbb{R}^n$ and $u,\hat{u}\in U$ with $\|y\|,\|\hat{y}\|,\|u\|,\|\hat{u}\|\le\rho$, and for all $t\in[0,T]$ such that

$$\begin{cases} \|F(t,y,u)-F(t,\hat{y},\hat{u})\|\le L(\rho)\left(\|y-\hat{y}\|+\|u-\hat{u}\|\right),\\ \|F_y(t,y,u)-F_y(t,\hat{y},\hat{u})\|\le L(\rho)\left(\|y-\hat{y}\|+\|u-\hat{u}\|\right),\\ \|F_u(t,y,u)-F_u(t,\hat{y},\hat{u})\|\le L(\rho)\left(\|y-\hat{y}\|+\|u-\hat{u}\|\right),\end{cases}$$

there is a constant $h>0$ such that

$$\|F(t,y,u)\|\le h\,(1+\|y\|),\ \text{for all } (t,u)\in[0,T]\times U, \tag{2.1}$$

where $F_y(t,y,u)^\top$ (or $F_u(t,y,u)^\top$) denotes the Jacobi matrix of F in y (or u).

(A3) the functions denoted by $\tilde{J}_i=\begin{pmatrix} J_i\\ G\end{pmatrix}:\mathbb{R}^n\longrightarrow\mathbb{R}^{n+1}$ $(i=1,2,\cdots,k)$ are twice continuously differentiable in x, and, for any $\rho>0$, there exists a constant $L(\rho)>0$ such that, for all $y,\hat{y}\in\mathbb{R}^n$ with $\|y\|,\|\hat{y}\|\le\rho$, we have

$$\begin{cases} \|\tilde{J}_i(y)-\tilde{J}_i(\hat{y})\|\le L(\rho)\|y-\hat{y}\|,\\ \|\tilde{J}_{ix}(y)-\tilde{J}_{iy}(\hat{y})\|\le L(\rho)\|y-\hat{y}\|,\end{cases}$$

where $\tilde{J}_{iy}(y)^\top$ denotes the Jacobi matrix of $\tilde{J}_i$ in y.

Denote $H(t)=l\,(t,y(t),u(t))+\langle f(t,y(t),u(t)),\varphi(t)\rangle$. To simplify the notation, $[t]$ is used to replace $(t,\bar{y}(t),\overline{\varphi}(t),\bar{u}(t))$ when evaluating the dynamics f and the Hamiltonian H; for example, $f[t]=f(t,\bar{y}(t),\bar{u}(t))$ and $H[t]=l\,(t,\bar{y}(t),\bar{u}(t))+\langle f(t,\bar{y}(t),\bar{u}(t)),\overline{\varphi}(t)\rangle$.

Remark 2.1. *$H(t)=l\,(t,y(t),u(t))+\langle f(t,y(t),u(t)),\varphi(t)\rangle$ is often referred to as the Hamiltonian function, or, simply, the Hamiltonian. The Hamiltonian function is represented in many reports as $\widetilde{H}(t)=-l\,(t,y(t),u(t))+\langle f(t,y(t),u(t)),\varphi(t)\rangle$; the difference between the two is that Pontryagin's maximum principle maximizes the Hamiltonian $\widetilde{H}$ or minimizes the Hamiltonian H. Accordingly, it also leads to a difference between positivity and negativity of the optimal inequality in the maximum principle. In this article, we use H to denote the Hamiltonian; in other words, we need to minimize the Hamiltonian H.*

Now, we will prove an elementary theorem that will be useful in the following sections.

Theorem 2.2. *Let (A1)–(A3) hold; for any fixed* $u \in \mathcal{U}_{ad}$*, the control system (1.2) has a unique solution* $y^u \in PC_l\left([0,T],\mathbb{R}^n\right)$ *given by*

$$y^u(t) = y_0 + \int_0^t f\left(s, y^u(s), u(s)\right) ds + \sum_{0<t_i<t} J_i\left(y^u(t_i)\right), \tag{2.2}$$

and there exists a constant $M = M\left(h, T, y_0, J_1(0), J_2(0), \cdots, J_k(0)\right)$ *such that*

$$\|y^u(t)\| \leq M \text{ for all } (t,u) \in [0,T]\times\mathcal{U}_{ad}. \tag{2.3}$$

Proof. By the qualitative theory of differential equations, it follows from (A1)–(A3) that the system of equations

$$\begin{cases} \dot{y}(t) = f(t, y(t), u(t)), & t \in [0, t_1], \\ y(0) = y_0, \end{cases}$$

has a unique solution $y^u \in C\left([0,t_1],\mathbb{R}^n\right)$ given by

$$y^u(t) = y_0 + \int_0^t f\left(s, y^u(s), u(s)\right) ds, \quad t \in [0, t_1],$$

and

$$\|y^u(t)\| \leq e^{ht}\left(ht_1 + \|y_0\|\right) \text{ for all } (t,u) \in [0,t_1]\times\mathcal{U}_{ad}.$$

Let

$$y_1 = y^u(t_1) + J_1\left(y^u(t_1)\right),$$

then, one also can infer that the system of equations

$$\begin{cases} \dot{y}(t) = f(t, y(t), u(t)), & t \in (t_1, t_2], \\ y(t_1+) = y_1, \end{cases}$$

has a unique solution $y^u \in C\left((t_1,t_2],\mathbb{R}^n\right)$ given by

$$y^u(t) = y_0 + \int_0^t f\left(s, y^u(s), u(s)\right) ds + J_1\left(y^u(t_1)\right), t \in (t_1, t_2],$$

and

$$\|y^u(t)\| \leq e^{h(t-t_1)}\left(h(t_2-t_1) + \|y_1\|\right) \text{ for all } (t,u) \in (t_1,t_2]\times\mathcal{U}_{ad}.$$

Using a step by step method, together with

$$\|J_i\left(y^u(t_i)\right)\| \leq L\left(\|y^u(t_i)\|\right)\|y^u(t_i)\| + \|J_i(0)\|, \; i = 1, 2, \cdots, k,$$

it is not difficult to claim that there exists a constant M such that (2.2) and (2.3) hold. Therefore, we have completed the proof of Theorem 2.2.

To establish the main conclusion of Theorem 4.6, we now introduce the definition of singular control in the classical sense (see Definition 2 in [14], as well as (4.4) in [25]).

Definition 2.3. *We refer to the elements in the following equation as singular controls in the classical sense:*

$$\overline{\mathcal{U}}_{ad} = \{v(\cdot) \in \mathcal{U}_{ad} \mid H_u(t, \bar{y}(t), v(t), \overline{\varphi}(t)) = 0, H_{uu}(t, y(t), v(t), \varphi(t)) \equiv 0\}.$$

Remark 2.4. *If the Hamiltonian H is linear in control u, then $\overline{\mathcal{U}}_{ad}$ is a singular control set in the classical sense.*

Now, we shall introduce a fundamental lemma, as derived from functional analysis, that will be utilized to establish the necessary condition for Problem P.

Lemma 2.5. *Let $h(t)$ be an n-dimensional piecewise continuous vector value function on $[0, T]$, and suppose that*

$$\int_0^T \langle h(t), a(t)\rangle dt = 0,$$

for all n-dimensional piecewise continuous vector value functions $a(t)$ on $[0, T]$; then, $h(t) = 0$ at all continuous moments of $h(t)$ on $[0, T]$.

Proof. Suppose that $h(t)$ at some continuous moments $\bar{t}$ when $h(\bar{t}) \neq 0$ because $h(t)$ is an n-dimensional piecewise continuous vector value function; therefore, there exists an interval $I_{\bar{t}}$ at $\bar{t}$ such that

$$h(t) \neq 0, t \in I_{\bar{t}}.$$

In this case, given the following:

$$a(t) = \begin{cases} h(t), & t \in I_{\bar{t}}, \\ 0, & t \in [0, T] \setminus I_{\bar{t}}, \end{cases}$$

then

$$\int_0^T h^\top(t)h(t)dt = \int_{I_{\bar{t}}} h^\top(t)h(t)dt = \int_{I_{\bar{t}}} \|h(t)\|^2 dt > 0.$$

This is a contradiction. Therefore, we have finished the proof of Lemma 2.5.

3. Integral-form second-order necessary condition

The purpose of this section is to prove Theorem 3.1, which establishes the integral form of the second-order necessary conditions for Problem P. The basic idea is that the condition of being second-order variational and non-negative is a necessary condition for an optimal control problem. We will borrow the method adopted in [25] and introduce a linear impulsive adjoint matrix to prove it.

Theorem 3.1. *Let (A1)–(A3) hold and $\bar{u}$ represent the optimal control of J over $\mathcal{U}_{ad}$; it is necessary that there exist functions $(\bar{y}, \overline{\varphi}, \overline{W}, \overline{\Phi}) \in PC_l([0, T], \mathbb{R}^n) \times PC_r([0, T], \mathbb{R}^n) \times PC_r([0, T], \mathbb{R}^{n\times n}) \times PC_l([0, T], \mathbb{R}^{n\times n})$ such that the following equations and inequality hold:*

$$\begin{cases} \dot{\bar{y}}(t) = f(t, \bar{y}(t), \bar{u}(t)), & t \in [0, T] \setminus \Lambda, \\ \bar{y}(t_i+) - \bar{y}(t_i) = J_i(\bar{y}(t_i)), & t_i \in \Lambda, \\ \bar{y}(0) = y_0; \end{cases} \tag{3.1}$$

$$\begin{cases} \dot{\overline{\varphi}}(t) = -f_y(t, \bar{y}(t), \bar{u}(t))\overline{\varphi}(t) - l_y(t, \bar{y}(t), \bar{u}(t)), & t \in [0, T] \setminus \Lambda, \\ \overline{\varphi}(t_i-) = \overline{\varphi}(t_i) + J_{iy}(\bar{y}(t_i))\overline{\varphi}(t_i), & t_i \in \Lambda, \\ \overline{\varphi}(T) = G_y(\bar{y}(T)); \end{cases} \tag{3.2}$$

$$\begin{cases} \dot{\overline{W}}(t) = -f_y(t, \bar{y}(t), \bar{u}(t))\overline{W}(t) - \overline{W}(t)f_y(t, \bar{y}(t), \bar{u}(t))^\top \\ \qquad\qquad -\overline{\varphi}(t)^\top f_{yy}(t, \bar{y}(t), \bar{u}(t)) - l_{yy}(t, \bar{y}(t), \bar{u}(t)), & t \in [0, T] \setminus \Lambda, \\ \overline{W}(t_i-) = \overline{W}(t_i) + J_{iy}(\bar{y}(t_i))\overline{W}(t_i) + \overline{W}(t_i+)J_{iy}(\bar{y}(t_i))^\top \\ \qquad\qquad +J_{iy}(\bar{y}(t_i))\overline{W}(t_i)J_{iy}(\bar{y}(t_i))^\top + \overline{\varphi}(t_i)^\top J_{iyy}(\bar{y}(t_i)), & t_i \in \Lambda, \\ \overline{W}(T) = G_{yy}(\bar{y}(T)); \end{cases} \tag{3.3}$$

$$\begin{cases} \dot{\overline{\Phi}}(t) = f_y(t, \bar{y}(t), \bar{u}(t))^\top \overline{\Phi}(t), & t \in [0, T] \setminus \Lambda, \\ \overline{\Phi}(t_i+) = \overline{\Phi}(t_i) + J_{iy}(\bar{y}(t_i))^\top \overline{\Phi}(t_i), & t_i \in \Lambda, \\ \overline{\Phi}(0) = I; \end{cases} \tag{3.4}$$

$$\begin{aligned} &\frac{1}{2}\int_0^T \langle H_{uu}[t][u(t) - \bar{u}(t)], u(t) - \bar{u}(t)\rangle\, dt \\ &+ \int_0^T \int_0^t \Big\langle \Big[H_{uy}[t] + \overline{W}(t)f_u[t]^\top\Big] [u(t) - \bar{u}(t)], \\ &\qquad \overline{\Phi}(t)\overline{\Phi}(s)^{-1} f_u[s]^T [u(s) - \bar{u}(s)]ds\Big\rangle\, dt \geq 0 \textit{ for all } u(\cdot) \in \mathcal{U}_{ad}. \end{aligned}$$

Proof. Now, let $(\bar{y}(\cdot), \bar{u}(\cdot))$ be the given optimal pair and $\varepsilon \in (0, 1]$. For an arbitrary but fixed $u(\cdot) \in \mathcal{U}_{ad}$, let $u^\varepsilon(\cdot) = \bar{u}(\cdot) + \varepsilon(u(\cdot) - \bar{u}(\cdot))$. It follows from the assumption (A1) that $u^\varepsilon(\cdot) \in \mathcal{U}_{ad}$; according to Theorem 2.2, u^ε determines the unique allowed state $y^\varepsilon(\cdot)$; then, we can get from (A2), (A3) and Theorem 2.2 (see (2.2) and (2.3)) that

$$\begin{aligned} & \|y^\varepsilon(t) - \bar{y}(t)\| \\ \leq\; & \int_0^t \|f(s, y^\varepsilon(s), u^\varepsilon(s)) - f(s, \bar{y}(s), \bar{u}(s))\|\, ds + \sum_{0<t_i<t} \|J_i(y^\varepsilon(t_i)) - J_i(\bar{y}(t_i))\| \\ \leq\; & L(M)\int_0^t (\|y^\varepsilon(t) - \bar{y}(t)\| + \varepsilon\,\|u(t) - \bar{u}(t)\|)\, ds + L(M)\sum_{0<t_i<t} \|y^\varepsilon(t_i) - \bar{y}(t_i)\|. \end{aligned}$$

Using the impulse integral inequality (see [1]), we have

$$\lim_{\varepsilon \to 0} \|y^\varepsilon - \bar{y}\|_{PC} = 0. \tag{3.5}$$

Let

$$Y(t) = \lim_{\varepsilon \to 0} Y^\varepsilon(t) = \lim_{\varepsilon \to 0} \frac{y^\varepsilon(t) - y(t)}{\varepsilon}.$$

In the same way as for (3.5), it is easy to show that

$$\lim_{\varepsilon\to 0}\|Y^{\varepsilon}-Y\|_{PC}=0, \tag{3.6}$$

and Y solves the following system of variational equations:

$$\begin{cases} \dot{Y}(t)=f_y[t]^{\top}Y(t)+f_u[t]^{\top}(u(t)-\bar{u}(t)), & t\in[0,T]\setminus\Lambda,\\ Y(t_i+)=Y(t_i)+J_{iy}\left(\bar{y}(t_i)\right)^{\top}Y(t_i), & t_i\in\Lambda,\\ Y(0)=0. \end{cases} \tag{3.7}$$

To obtain the first-order necessary condition for Problem P, the following proposition will be used.

Proposition 3.2. *Let (A2) and (A3) hold and $\overline{\varphi}\in PC_r([0,T],\mathbb{R}^n)$ be the solution of the impulsive adjoint equation given by (3.2). Then*

$$\begin{aligned} &\int_0^T \langle f_u(t,\bar{y}(t),\bar{u}(t))\,\overline{\varphi}(t),u(t)-\bar{u}(t)\rangle\,dt\\ =\;&\int_0^T \Big\langle l_y(s,\bar{y}(s),\bar{u}(s)),Y(s)\Big\rangle\,ds+\Big\langle G_y(\bar{y}(T)),Y(T)\Big\rangle. \end{aligned} \tag{3.8}$$

Proof. Since $C([0,T])$ is a dense subspace in $L^1([0,T])$, there exist function sequences $\left\{f_y^{\alpha}\right\}$, $\left\{l_y^{\alpha}\right\}\subseteq C([0,T]$ such that

$$f_y^{\alpha}(\cdot)\longrightarrow f_y(\cdot,\bar{y}(\cdot),\bar{u}(\cdot))\text{ and }l_y^{\alpha}(\cdot)\longrightarrow l_y(\cdot,\bar{y}(\cdot),\bar{u}(\cdot))\text{ in }L^1([0,T])\text{ as }\alpha\to\infty. \tag{3.9}$$

Moreover, it follows from (A3) that the system of linear impulsive differential equations given by

$$\begin{cases} \dot{\varphi}_{\alpha}(t)=-f_y^{\alpha}(t)\varphi_{\alpha}(t)-l_y^{\alpha}(t), & t\in[0,T]\setminus\Lambda,\\ \varphi_{\alpha}(t_i-)=\varphi_{\alpha}(t_i)+J_{iy}(\bar{y}(t_i))\varphi_{\alpha}(t_i), & t_i\in\Lambda,\\ \varphi_{\alpha}(T)=G_y(\bar{y}(T)), \end{cases} \tag{3.10}$$

has a unique solution $\varphi_{\alpha}\in PC_r([0,T],\mathbb{R}^n)\bigcap C^1([0,T]\setminus\Lambda,\mathbb{R}^n)$, and that there is a constant $\beta>0$ such that

$$\|\varphi_{\alpha}\|_{PC}\leq\beta\text{ for all }\alpha.$$

Hence, we have

$$\begin{aligned} &\|\varphi_{\alpha}(t)-\overline{\varphi}(t)\|\\ \leq\;&\int_t^T\left\|l_y^{\alpha}(s)-l_y(s,\bar{y}(s),\bar{u}(s))\right\|ds+\int_t^T\left\|f_y^{\alpha}(s)-f_y(s,\bar{y}(s),\bar{u}(s))\right\|\|\varphi_{\alpha}(s)\|\,ds\\ &+\int_t^T\left\|f_y(s,\bar{y}(s),\bar{u}(s))\right\|\|\varphi_{\alpha}(s)-\overline{\varphi}(s)\|\,ds+\sum_{t<t_i<T}\left\|J_{iy}(\bar{y}(t_i))\right\|\|\varphi_{\alpha}(t_i)-\overline{\varphi}(t_i)\|. \end{aligned}$$

Therefore, using the same method as for (3.5), it is not difficult to show that

$$\lim_{\alpha\to\infty}\|\varphi_{\alpha}-\overline{\varphi}\|_{PC}=0. \tag{3.11}$$

Consequently, we can infer from (3.7) and (3.10) that

$$\begin{aligned}
&\int_0^T \langle f_u(t,\bar{y}(t),\bar{u}(t))\,\varphi_\alpha(t), u(t)-\bar{u}(t)\rangle\,dt\\
=&\int_0^T \langle \varphi_\alpha(t), f_u(t,\bar{y}(t),\bar{u}(t))^\top (u(t)-\bar{u}(t))\rangle\,dt\\
=&\int_0^T \left\langle \varphi_\alpha(t), \dot{Y}(t) - f_y^\top(t,\bar{y}(t),\bar{u}(t))\,Y(t)\right\rangle dt\\
=&\ \langle \varphi_\alpha(T), Y(T)\rangle - \sum_{i=1}^{k}\left[\langle \varphi_\alpha(t_i), Y(t_i+)\rangle - \langle \varphi_\alpha(t_i-), Y(t_i)\rangle\right]\\
&-\int_0^T \left\langle \dot{\varphi}_\alpha(t) + f_y^\alpha(t)\varphi_\alpha(t), Y(t)\right\rangle dt\\
=&\left\langle G_y(\bar{y}(T)), Y(T)\right\rangle - \int_0^T \left\langle \dot{\varphi}_\alpha(t) + f_y^\alpha(t)\varphi_\alpha(t), Y(t)\right\rangle dt\\
=&\left\langle G_y(\bar{y}(T)), Y(T)\right\rangle + \int_0^T \left\langle l_y^\alpha(t), Y(t)\right\rangle dt.
\end{aligned}$$

Let $\alpha \longrightarrow \infty$ in the above expression; using (3.9) and (3.11), we have (3.8). Therefore, we have finished the proof of Proposition 3.2.

Based on the above proposition, we now continue to prove Theorem 3.1.

By the optimality of $\bar{u}$, one can ascertain from the assumptions (A2) and (A3), (3.2), (3.6), (3.7), and Proposition 3.2 (see (3.8)) that

$$\begin{aligned}
0 \leq\ & \lim_{\varepsilon\to 0}\frac{J(u^\varepsilon)-J(\bar{u})}{\varepsilon}\\
=\ & \lim_{\varepsilon\to 0}\int_0^T \left\langle \int_0^1 l_y(t,\bar{y}(t)+\tau(y^\varepsilon(t)-\bar{y}(t)),\bar{u}(t))\,d\tau, Y^\varepsilon(t)\right\rangle dt\\
&+\lim_{\varepsilon\to 0}\int_0^T \left\langle \int_0^1 l_u(t,y^\varepsilon(t),\bar{u}(t)+\tau\varepsilon(u(t)-\bar{u}(t)))\,d\tau, u(t)-\bar{u}(t)\right\rangle dt\\
&+\lim_{\varepsilon\to 0}\left\langle \int_0^1 G_y(\bar{y}(T)+\tau(y^\varepsilon(T)-\bar{y}(T)))\,d\tau, Y^\varepsilon(T)\right\rangle\\
=\ & \int_0^T \left\langle l_y(t,\bar{y}(t),\bar{u}(t)), Y(t)\right\rangle dt + \int_0^T \langle l_u(t,\bar{y}(t),\bar{u}(t)), u(t)-\bar{u}(t)\rangle\,dt\\
&+\left\langle G_y(\bar{y}(T)), Y(T)\right\rangle\\
=\ & \int_0^T \langle l_u(t,\bar{y}(t),\bar{u}(t)) + f_u(t,\bar{y}(t),\bar{u}(t))\,\overline{\varphi}(t), u(t)-\bar{u}(t)\rangle\,dt,
\end{aligned}$$

which leads to the following optimal inequality:

$$\int_0^T \langle l_u(t,\bar{y}(t),\bar{u}(t)) + f_u(t,\bar{y}(t),\bar{u}(t))\overline{\varphi}(t), u(t)-\bar{u}(t)\rangle\,dt \geq 0.$$

Because of the arbitrariness of $u \in \mathcal{U}_{ad}$, we get

$$\int_0^T \langle l_u(t, \bar{y}(t), \bar{u}(t)) + f_u(t, \bar{y}(t), \bar{u}(t))\overline{\varphi}(t), u(t) - \bar{u}(t)\rangle \, dt = 0. \tag{3.12}$$

Moreover, combining this with (A1) and Proposition 3.2, it follows that, for all continuous time points $t \in [0, T]$ of $\bar{u}(t)$, we have

$$H_u[t] = l_u(t, \bar{y}(t), \bar{u}(t)) + \langle f_u(t, \bar{y}(t), \bar{u}(t)), \overline{\varphi}(t)\rangle = 0.$$

We define

$$\overline{U}(t) = \{v \in U \mid l_u(t, \bar{y}(t), v) + \langle f_u(t, \bar{y}(t), v), \overline{\varphi}(t)\rangle = 0\}. \tag{3.13}$$

We refer to this as the singular control region in the classical sense, which will be used later.

Let

$$Z^\varepsilon(\cdot) \equiv \begin{pmatrix} Z_1^\varepsilon(\cdot) & Z_2^\varepsilon(\cdot) & \cdots & Z_n^\varepsilon(\cdot)\end{pmatrix}^\top = \frac{Y^\varepsilon(t) - Y(t)}{\varepsilon},$$

and

$$Z_k(t) = \lim_{\varepsilon \to 0} Z_k^\varepsilon(t).$$

In the same way as for (3.5), one can also claim from (A2) and (A3) that

$$\lim_{\varepsilon \to 0} \|Z^\varepsilon - Z\|_{PC([0,T])} = 0, \tag{3.14}$$

and $Z(\cdot)$ denotes the solution of the following system of equations:

$$\begin{cases} \dot{Z}(t) = f_y(t, \bar{y}(t), \bar{u}(t))^\top Z(t) \\ \qquad + \frac{1}{2}\langle f_{uu}(t, \bar{y}(t), \bar{u}(t))[u(t) - \bar{u}(t)], [u(t) - \bar{u}(t)]\rangle \\ \qquad + \left\langle f_{uy}(t, \bar{y}(t), \bar{u}(t))[u(t) - \bar{u}(t)], Y(t)\right\rangle \\ \qquad + \frac{1}{2}\left\langle f_{yy}(t, \bar{y}(t), \bar{u}(t))Y(t), Y(t)\right\rangle, & t \in (0, T] \setminus \Lambda, \\ Z(t_i+) = Z(t_i) + J_{iy}^\top(\bar{y}(t_i))Z(t_i) \\ \qquad + \frac{1}{2}\left\langle J_{iyy}(\bar{y}(t_i))Y(t_i), Y(t_i)\right\rangle, & t_i \in \Lambda, \\ Z(0) = 0. \end{cases} \tag{3.15}$$

where

$$\begin{aligned} &\frac{1}{2}\langle f_{uu}(t, \bar{y}(t), \bar{u}(t))[u(t) - \bar{u}(t)], [u(t) - \bar{u}(t)]\rangle \\ = \; &\frac{1}{2}\begin{Bmatrix} \left\langle f_{uu}^1(t, \bar{y}(t), \bar{u}(t))[u(t) - \bar{u}(t)], [u(t) - \bar{u}(t)]\right\rangle \\ \left\langle f_{uu}^2(t, \bar{y}(t), \bar{u}(t))[u(t) - \bar{u}(t)], [u(t) - \bar{u}(t)]\right\rangle \\ \vdots \\ \left\langle f_{uu}^n(t, \bar{y}(t), \bar{u}(t))[u(t) - \bar{u}(t)], [u(t) - \bar{u}(t)]\right\rangle \end{Bmatrix}, \end{aligned}$$

$$
\begin{aligned}
&\frac{1}{2}\left\langle f_{yy}(t,\bar{y}(t),\bar{u}(t))Y(t),Y(t)\right\rangle\\
=\;&\frac{1}{2}\left\{\begin{array}{c}
\left\langle f^1_{yy}(t,\bar{y}(t),\bar{u}(t))Y(t),Y(t)\right\rangle\\
\left\langle f^2_{yy}(t,\bar{y}(t),\bar{u}(t))Y(t),Y(t)\right\rangle\\
\vdots\\
\left\langle f^n_{yy}(t,\bar{y}(t),\bar{u}(t))Y(t),Y(t)\right\rangle
\end{array}\right\},
\end{aligned}
$$

and

$$
\begin{aligned}
&\left\langle f_{uy}(t,\bar{y}(t),\bar{u}(t))[u(t)-\bar{u}(t)],Y(t)\right\rangle\\
=\;&\left\{\begin{array}{c}
\left\langle f^1_{uy}(t,\bar{y}(t),\bar{u}(t))[u(t)-\bar{u}(t)]),Y(t)\right\rangle\\
\left\langle f^2_{uy}(t,\bar{y}(t),\bar{u}(t))[u(t)-\bar{u}(t)],Y(t)\right\rangle\\
\vdots\\
\left\langle f^n_{uy}(t,\bar{y}(t),\bar{u}(t))[u(t)-\bar{u}(t)],Y(t)\right\rangle
\end{array}\right\}.
\end{aligned}
$$

Meanwhile, one can infer from (3.7) that $X := YY^\top$ is the solution to the following system of equations:

$$
\begin{cases}
\dot{X}(t) = f_y(t,\bar{y}(t),\bar{u}(t))^\top X(t) & \\
\qquad +X(t)f_y(t,\bar{y}(t),\bar{u}(t)) & \\
\qquad +f_u(t,\bar{y}(t),\bar{u}(t))^\top(u(t)-\bar{u}(t))Y^\top(t) & \\
\qquad +Y(t)\left(f_u(t,\bar{y}(t),\bar{u}(t))^\top(u(t)-\bar{u}(t))\right)^\top, & t\in[0,T]\setminus\Lambda,\\
X(t_i+) = X(t_i) + J^\top_{iy}(\bar{y}(t_i))X(t_i) + X(t_i)J_{iy}(\bar{y}(t_i)) & \\
\qquad +J^\top_{iy}(\bar{y}(t_i))X(t_i)J_{iy}(\bar{y}(t_i)), & t_i\in\Lambda,\\
X(0) = 0.
\end{cases}
\tag{3.16}
$$

The subsequent proposition plays a crucial role in obtaining the integral form of the second-order necessary conditions.

Proposition 3.3. *Let (A2) and (A3) hold and $W \in PC_r([0,T],\mathbb{R}^{n\times n})$, $\overline{\Phi} \in PC_l([0,T],\mathbb{R}^{n\times n})$ be the solution of (3.3) and (3.4). Then,*

$$
\begin{aligned}
&\int_0^T \left\langle \overline{W}(t)f_u(t,\bar{y}(t),\bar{u}(t))^\top(u(t)-\bar{u}(t)),Y(t)\right\rangle dt\\
=\;&\frac{1}{2}\int_0^T \left\langle\left[l_{yy}(t,\bar{y}(t),\bar{u}(t)) + \overline{\varphi}(t)^\top f_{yy}(t,\bar{y}(t),\bar{u}(t))\right]Y(t),Y(t)\right\rangle dt\\
&+\frac{1}{2}\left\langle G_{yy}(\bar{y}(T))\,Y(T),Y(T)\right\rangle + \frac{1}{2}\sum_{i=1}^k\left\langle \overline{\varphi}(t_i)^\top J_{iyy}(\bar{y}(t_i))Y(t_i)),Y(t_i))\right\rangle,
\end{aligned}
\tag{3.17}
$$

and

$$\int_0^T \left\langle \left(l_{uy}\left(t,\bar{y}(t),\bar{u}(t)\right) + \overline{\varphi}(t)^\top f_{uy}\left(t,\bar{y}(t),\bar{u}(t)\right) + \overline{W}(t) f_u\left(t,\bar{y}(t),\bar{u}(t)\right)^T \right) [u(t)-\bar{u}(t)], Y(t) \right\rangle dt$$
$$= \int_0^T dt \int_0^t \left\langle \left(l_{uy}\left(t,\bar{y}(t),\bar{u}(t)\right) + \overline{\varphi}(t)^\top f_{uy}\left(t,\bar{y}(t),\bar{u}(t)\right) + \overline{W}(t) f_u\left(t,\bar{y}(t),\bar{u}(t)\right)^T \right) [u(t)-\bar{u}(t)], \right.$$
$$\left. \overline{\Phi}(t)\overline{\Phi}(s)^{-1} f_u(s,\bar{y}(s),\bar{u}(s))^T [u(s)-\bar{u}(s)] \right\rangle ds. \tag{3.18}$$

Proof. Since $C([0,T])$ is a dense subspace in $L^1([0,T])$, there exist function sequences $\left\{f_y^\alpha\right\}$, $\left\{f_{yy}^\alpha\right\}$, $\left\{f_{uy}^\alpha\right\}$, $\{f_u^\alpha\}$, $\left\{l_y^\alpha\right\}$, $\left\{l_{yy}^\alpha\right\}$, $\left\{l_{uy}^\alpha\right\} \subseteq C([0,T]$ such that

$$\begin{cases} f_y^\alpha(\cdot) \longrightarrow f_y(\cdot,\bar{y}(\cdot),\bar{u}(\cdot)), & l_y^\alpha(\cdot) \longrightarrow l_y(\cdot,\bar{y}(\cdot),\bar{u}(\cdot)), \\ f_{yy}^\alpha(\cdot) \longrightarrow f_{yy}(\cdot,\bar{y}(\cdot),\bar{u}(\cdot)), & l_{yy}^\alpha(\cdot) \longrightarrow l_{yy}(\cdot,\bar{y}(\cdot),\bar{u}(\cdot)), \\ f_{uy}^\alpha(\cdot) \longrightarrow f_{uy}(\cdot,\bar{y}(\cdot),\bar{u}(\cdot)), & l_{uy}^\alpha(\cdot) \longrightarrow l_{uy}(\cdot,\bar{y}(\cdot),\bar{u}(\cdot)), \\ f_u^\alpha \longrightarrow f_u, \end{cases} \quad \text{in } L^1([0,T]) \text{ as } \alpha \to \infty. \tag{3.19}$$

Consequently, by (A3) and (3.19), one can infer that the systems of linear impulsive matrix differential equations given by

$$\begin{cases} \dot{W}_\alpha(t) = -f_y^\alpha(t)W_\alpha(t) - W_\alpha(t)f_y^\alpha(t)^\top - \overline{\varphi}(t)^\top f_{yy}^\alpha(t) - l_{yy}^\alpha(t), & t \in [0,T]\setminus\Lambda, \\ W_\alpha(t_i-) = W_\alpha(t_i+) + J_{iy}(\bar{y}(t_i))W_\alpha(t_i+) + W_\alpha(t_i+)J_{iy}(\bar{y}(t_i))^\top \\ \qquad\qquad +J_{iy}(\bar{y}(t_i))W_\alpha(t_i+)J_{iy}(\bar{y}(t_i))^\top + \overline{\varphi}(t_i)^\top J_{iyy}(\bar{y}(t_i)), & t_i \in \Lambda, \\ W_\alpha(T) = G_{yy}(\bar{y}(T)), \end{cases} \tag{3.20}$$

and

$$\begin{cases} \dot{\Phi}_\alpha(t) = f_y^\alpha(t)^\top \Phi_\alpha(t), & t \in [0,T]\setminus\Lambda, \\ \Phi_\alpha(t_i+) = \Phi_\alpha(t_i) + J_{iy}(\bar{y}(t_i))^\top \Phi_\alpha(t_i), & t_i \in \Lambda, \\ \Phi_\alpha(0) = I, \end{cases} \tag{3.21}$$

each have a unique solution $W_\alpha \in PC_r\left([0,T],\mathbb{R}^{n\times n}\right) \bigcap C^1\left([0,T]\setminus\Lambda,\mathbb{R}^{n\times n}\right)$ and $\Phi_\alpha \in PC_l([0,T], \mathbb{R}^{n\times n}) \bigcap C^1\left([0,T]\setminus\Lambda,\mathbb{R}^{n\times n}\right)$, respectively. Not only that, there exists a constant $\gamma > 0$ such that

$$\|W_\alpha\|_{PC} \le \gamma \text{ and } \|\Phi_\alpha\|_{PC} \le \gamma \text{ for all } \alpha.$$

Moreover, we have

$$\begin{aligned} &\left\|W_\alpha(t) - \overline{W}(t)\right\| \\ \le\ & \int_t^T \left\|l_{yy}^\alpha(s) - l_{yy}(s,\bar{y}(s),\bar{u}(s))\right\| ds + \int_t^T \|\overline{\varphi}(s)^\top\| \left\|f_{yy}^\alpha(s) - f_{yy}(s,\bar{y}(s),\bar{u}(s))\right\| ds \\ & +2\gamma \int_t^T \left\|f_y^\alpha(s) - f_y(s,\bar{y}(s),\bar{u}(s))\right\| ds + 2\int_t^T \left\|W_\alpha(s) - \overline{W}(s)\right\| \left\|f_y^\top(s,\bar{y}(s),\bar{u}(s))\right\| ds \\ & +2\sum_{t<t_i<T} \left(\left\|J_{iy}(\bar{y}(t_i))\right\| + \left\|J_{iy}(\bar{y}(t_i))\right\|^2\right) \left\|W_\alpha(t_i) - \overline{W}(t_i)\right\|, \end{aligned}$$

and

$$
\begin{aligned}
& \left\|\Phi_\alpha(t) - \overline{\Phi}(t)\right\| \\
\le\ & \gamma \int_0^t \left\| f_y^\alpha(s) - f_y(s, \bar{y}(s), \bar{u}(s)) \right\| ds + \int_0^t \left\| f_y(s, \bar{y}(s), \bar{u}(s)) \right\| \left\| \Phi_\alpha(s) - \overline{\Phi}(s) \right\| ds \\
+\ & \sum_{0<t_i<t} \left\| J_{iy}(\bar{y}(t_i)) \right\| \left\| \Phi_\alpha(t_i) - \overline{\Phi}(t_i) \right\|.
\end{aligned}
$$

In the same way as for (3.5), we obtain

$$
\lim_{\alpha\to\infty} \left\| W_\alpha - \overline{W} \right\|_{PC} = 0 \text{ and } \lim_{\alpha\to\infty} \left\| \Phi_\alpha - \overline{\Phi} \right\|_{PC} = 0. \tag{3.22}
$$

In addition, it is obvious from (3.20) that $W_\alpha^\top$ is also a solution of (3.20). This means that

$$
W_\alpha^\top(t) = W_\alpha(t) \text{ for all } t \in [0, T]. \tag{3.23}
$$

Since $tr(AB) = tr(BA)$ for all $k \times j$ matrix A and $j \times k$ matrix B, we can get from (3.2), (3.7), (3.16), (3.19), (3.20), (3.22), and (3.23) that

$$
\begin{aligned}
& 2\int_0^T \left\langle \overline{W}(t) f_u(t, \bar{y}(t), \bar{u}(t))^\top (u(t) - \bar{u}(t)), Y(t) \right\rangle dt \\
=\ & 2 \lim_{\alpha\to\infty} \int_0^T \left\langle W_\alpha(t) \left(\dot{Y}(t) - f_y(t, \bar{y}(t), \bar{u}(t))^\top Y(t) \right), Y(t) \right\rangle dt \\
=\ & \lim_{\alpha\to\infty} tr \int_0^T \Big[W_\alpha(t) \left(\dot{Y}(t) - f_y(t, \bar{y}(t), \bar{u}(t))^\top Y(t) \right) Y^\top(t) \\
& + W_\alpha(t) Y(t) \left(\dot{Y}(t)^\top - Y(t)^\top f_y(t, \bar{y}(t), \bar{u}(t)) \right) \Big] dt \\
=\ & \lim_{\alpha\to\infty} tr \int_0^T W_\alpha(t) \left[\dot{X}(t) - f_y(t, \bar{y}(t), \bar{u}(t))^\top X(t) - X(t) f_y(t, \bar{y}(t), \bar{u}(t)) \right] dt \\
=\ & \lim_{\alpha\to\infty} \Big\{ - \int_0^T \left\langle \left[\dot{W}_\alpha(t) + W_\alpha(t) f_y(t, \bar{y}(t), \bar{u}(t))^\top + f_y(t, \bar{y}(t), \bar{u}(t)) W_\alpha(t) \right] Y(t), Y(t) \right\rangle dt \\
& + \langle W_\alpha(T) Y(T), Y(T) \rangle + \sum_{i=1}^k \left[\langle W_\alpha(t_i-) Y(t_i), Y(t_i) \rangle - \langle W_\alpha(t_i) Y(t_i+), Y(t_i+) \rangle \right] \Big\} \\
=\ & \lim_{\alpha\to\infty} \Big\{ - \int_0^T \left\langle \left[\dot{W}_\alpha(t) + W_\alpha(t) f_y^\alpha(t)^\top + f_y^\alpha(t) W_\alpha(t) \right] Y(t), Y(t) \right\rangle dt \\
& + \langle W_\alpha(T) Y(T), Y(T) \rangle + \sum_{i=1}^k \Big[\langle (W_\alpha(t_i-) - W_\alpha(t_i)) Y(t_i), Y(t_i) \rangle \\
& - \left\langle \left(J_{iy}(\bar{y}(t_i)) W_\alpha(t_i) + W_\alpha(t_i) J_{iy}(\bar{y}(t_i))^\top \right) Y(t_i), Y(t_i) \right\rangle \\
& - \left\langle J_{iy}(\bar{y}(t_i)) W_\alpha(t_i) J_{iy}(\bar{y}(t_i))^\top Y(t_i), Y(t_i) \right\rangle \Big] \Big\} \\
=\ & \lim_{\alpha\to\infty} \Big\{ \int_0^T \left\langle \left[l_{yy}^\alpha(t) + \overline{\varphi}(t)^\top f_{yy}^\alpha(t) \right] Y(t), Y(t) \right\rangle dt + \langle W_\alpha(T) Y(T), Y(T) \rangle
\end{aligned}
$$

$$
\begin{aligned}
& + \sum_{i=1}^{k} \Big[\langle (W_\alpha(t_i-) - W_\alpha(t_i)) Y(t_i), Y(t_i) \rangle \\
& - \Big\langle \Big(J_{iy}\left(\bar{y}(t_i)\right) W_\alpha(t_i) + W_\alpha(t_i) J_{iy}\left(\bar{y}(t_i)\right)^\top \Big) Y(t_i), Y(t_i) \Big\rangle \\
& - \Big\langle J_{iy}\left(\bar{y}(t_i)\right) W_\alpha(t_i) J_{iy}\left(\bar{y}(t_i)\right)^\top Y(t_i), Y(t_i) \Big\rangle \Big] \Big\} \\
= \; & \int_0^T \Big\langle \Big[l_{yy}(t, \bar{y}(t), \bar{u}(t)) + \overline{\varphi}(t)^\top f_{yy}(t, \bar{y}(t), \bar{u}(t)) \Big] Y(t), Y(t) \Big\rangle \, dt \\
& + \Big\langle G_{yy}(\bar{y}(T)) Y(T), Y(T) \Big\rangle + \sum_{i=1}^{k} \Big\langle \overline{\varphi}(t_i)^\top J_{iyy}(\bar{y}(t_i)) Y(t_i), Y(t_i) \Big\rangle,
\end{aligned}
$$

i.e., (3.17) holds.

Now, let us prove (3.18). By (3.4), (3.7), (3.21), and (3.22), we have

$$
\begin{aligned}
Y(t) &= \overline{\Phi}(t) \int_0^t \overline{\Phi}(s)^{-1} f_u[s]^\top (u(s) - \bar{u}(s)) ds \\
&= \lim_{\alpha \to 0} \Phi_\alpha(t) \int_0^t \Phi_\alpha(s)^{-1} f_u[s]^\top (u(s) - \bar{u}(s)) ds,
\end{aligned}
$$

which means that (3.18) holds. Therefore, we have finished the proof of Proposition 3.3.

Based on the above propositions, we now continue to prove Theorem 3.1.

Since $\bar{u}$ represents optimal control of J over $\mathcal{U}_{ad}$, together with Proposition 3.2 (see (3.8)) and (3.12), we have

$$
\begin{aligned}
& \int_0^T \Big\langle l_y\left(t, \bar{y}(t), \bar{u}(t)\right), Y(t) \Big\rangle \, ds + \int_0^T \langle l_u\left(t, \bar{y}(t), \bar{u}(t)\right), u(t) - \bar{u}(t) \rangle \, dt + \Big\langle G_y\left(\bar{y}(T)\right), Y(T) \Big\rangle \\
= \; & \int_0^T \langle l_u\left(t, \bar{y}(t), \bar{u}(t)\right) + f_u\left(t, \bar{y}(t), \bar{u}(t)\right) \varphi(t), u(t) - \bar{u}(t) \rangle \, dt \\
= \; & 0 \text{ for all } u \in \mathcal{U}_{ad}.
\end{aligned} \tag{3.24}
$$

Taken together with (A2), (A3), and (3.24), one can get

$$
\begin{aligned}
& \frac{J\left(u^\varepsilon(\cdot)\right) - J\left(\bar{u}(\cdot)\right)}{\varepsilon} \\
= \; & \int_0^T \Big\langle \int_0^1 \left(l_u\left(t, y^\varepsilon(t), \bar{u}(t) + \tau\varepsilon\left(u(t) - \bar{u}(t)\right)\right) - l_u\left(t, \bar{y}(t), \bar{u}(t)\right) \right) d\tau, u(t) - \bar{u}(t) \Big\rangle \, dt \\
& + \int_0^T \Big\langle \int_0^1 \Big(l_y\left(t, \bar{y}(t) + \tau\left(y^\varepsilon(t) - \bar{y}(t)\right), \bar{u}(t)\right) - l_y\left(t, \bar{y}(t), \bar{u}(t)\right) \Big) d\tau, Y^\varepsilon(t) \Big\rangle \, dt \\
& + \int_0^T \Big\langle l_y\left(t, \bar{y}(t), \bar{u}(t)\right), Y^\varepsilon(t) - Y(t) \Big\rangle \, dt + \Big\langle G_y\left(\bar{y}(T)\right), Y^\varepsilon(T) - Y(T) \Big\rangle \\
& + \Big\langle \int_0^1 \Big(G_y\left(\bar{y}(T) + \tau\left(y^\varepsilon(T) - \bar{y}(T)\right)\right) - G_y\left(\bar{y}(T)\right) \Big) d\tau, Y^\varepsilon(T) \Big\rangle
\end{aligned}
$$

$$
\begin{aligned}
= \;& \varepsilon \int_0^T \left\langle \int_0^1 \tau \int_0^1 l_{uu}\left(t, y^\varepsilon(t), \bar{u}(t) + \nu\tau\varepsilon\left(u(t) - \bar{u}(t)\right)\right) d\nu d\tau [u(t) - \bar{u}(t)], u(t) - \bar{u}(t) \right\rangle dt \\
& + \varepsilon \int_0^T \left\langle \int_0^1 l_{uy}\left(t, \bar{y}(t) + \tau\left(y^\varepsilon(t) - \bar{y}(t)\right), \bar{u}(t)\right)(u(t) - \bar{u}(t)) d\tau, Y^\varepsilon(t) \right\rangle dt \\
& + \varepsilon \int_0^T \left\langle \int_0^1 \tau \int_0^1 l_{yy}\left(t, \bar{y}(t) + \nu\tau\left(y^\varepsilon(t) - \bar{y}(t)\right), \bar{u}(t)\right) d\nu d\tau Y^\varepsilon(t), Y^\varepsilon(t) \right\rangle dt \\
& + \varepsilon \int_0^T \left\langle l_y\left(t, \bar{y}(t), \bar{u}(t)\right), Z^\varepsilon(t) \right\rangle dt + \varepsilon \left\langle G_y\left(\bar{y}(T)\right), Z^\varepsilon(T) \right\rangle \\
& + \varepsilon \left\langle \int_0^1 \tau \int_0^1 G_{yy}\left(\bar{y}(T) + \nu\tau\left(y^\varepsilon(T) - \bar{y}(T)\right)\right) d\nu d\tau Y^\varepsilon(T), Y^\varepsilon(T) \right\rangle .
\end{aligned}
$$

Then, combining this with (3.6) and (3.14), the above expression, (A2), and (A3) leads to the following:

$$
\begin{aligned}
& \frac{1}{2} \int_0^T \langle l_{uu}\left(t, \bar{y}(t), \bar{u}(t)\right)[u(t) - \bar{u}(t)], u(t) - \bar{u}(t) \rangle dt \\
& + \int_0^T \left\langle l_{uy}\left(t, \bar{y}(t), \bar{u}(t)\right)(u(t) - \bar{u}(t)), Y(t) \right\rangle ds \\
& + \frac{1}{2} \int_0^T \left\langle l_{yy}\left(t, \bar{y}(t), \bar{u}(t)\right) Y(t), Y(t) \right\rangle dt + \frac{1}{2} \left\langle G_{yy}\left(\bar{y}(T)\right) Y(T), Y(T) \right\rangle \\
& + \int_0^T \left\langle l_y(t, \bar{y}(t), \bar{u}(t)), Z(t) \right\rangle ds + \left\langle G_y\left(\bar{y}(T)\right), Z(T) \right\rangle \geq 0 \text{ for all } u \in \mathcal{U}_{ad}.
\end{aligned}
$$

By (3.2), we have

$$
\begin{aligned}
& \frac{1}{2} \int_0^T \langle l_{uu}\left(t, \bar{y}(t), \bar{u}(t)\right)[u(t) - \bar{u}(t)], u(t) - \bar{u}(t) \rangle dt \\
& + \int_0^T \left\langle l_{uy}\left(t, \bar{y}(t), \bar{u}(t)\right)(u(t) - \bar{u}(t)), Y(t) \right\rangle ds \\
& + \frac{1}{2} \int_0^T \left\langle l_{yy}\left(t, \bar{y}(t), \bar{u}(t)\right) Y(t), Y(t) \right\rangle dt + \frac{1}{2} \left\langle G_{yy}\left(\bar{y}(T)\right) Y(T), Y(T) \right\rangle \\
& - \int_0^T \left\langle \dot{\bar{\varphi}}(t) + f_y(t, \bar{y}(t), \bar{u}(t))\bar{\varphi}(t), Z(t) \right\rangle ds + \left\langle G_y\left(\bar{y}(T)\right), Z(T) \right\rangle \geq 0 \text{ for all } u \in \mathcal{U}_{ad}.
\end{aligned}
\tag{3.25}
$$

Since $C([0,T])$ is a dense subspace in $L^1([0,T])$, there exist function sequences $\{u^\alpha\}, \{f^\alpha_{uu}\} \subseteq C([0,T])$ such that

$$
\lim_{\alpha\to\infty} \|u^\alpha - [u - \bar{u}]\|_{L^1} = 0 \text{ and } \lim_{\alpha\to\infty} \left\| f^\alpha_{uu}(\cdot) - f_{uu}(\cdot, \bar{y}(\cdot), \bar{u}(\cdot)) \right\|_{L^1} = 0. \tag{3.26}
$$

It follows immediately from (3.15), (3.19), and (3.26) that the system of equations given by

$$
\begin{cases}
\dot{Z}^\alpha(t) = f^\alpha_y(t)^\top Z^\alpha(t) + \frac{1}{2} u^\alpha(t)^\top f^\alpha_{uu}(t) u^\alpha(t) \\
\qquad\qquad + u^\alpha(t)^\top f^\alpha_{yu}(t) Y(t) + \frac{1}{2} Y(t)^\top f^\alpha_{yy}(t) Y(t), & t \in [0,T] \setminus \Lambda, \\
Z^\alpha(t_i+) = Z^\alpha(t_i) + \frac{1}{2} Y(t_i))^\top J_{iyy}(\bar{y}(t_i)) Y(t_i)) + J_{iy}(\bar{y}(t_i))^\top Z^\alpha(t_i), & t_i \in \Lambda, \\
Z^\alpha(0) = 0,
\end{cases}
\tag{3.27}
$$

has a unique solution $Z^{\alpha} \in PC_l([0,T],\mathbb{R}^n) \bigcap C^1([0,T]\setminus\Lambda,\mathbb{R}^n)$ and

$$\lim_{\alpha\to\infty} \|Z^{\alpha} - Z\|_{PC} = 0, \tag{3.28}$$

where $Z(\cdot)$ is the solution to (3.15).

Moreover, we can infer from (3.19) and (3.26)–(3.28) that

$$\begin{aligned}
& -\int_0^T \left\langle \dot{\overline{\varphi}}(t) + f_y(t,\bar{y}(t),\bar{u}(t))\overline{\varphi}(t), Z(t)\right\rangle dt + \left\langle G_y(\bar{y}(T)), Z(T)\right\rangle \\
= & -\lim_{\alpha\to\infty}\int_0^T \left\langle \dot{\overline{\varphi}}(t) + f_y^{\alpha}(t)\overline{\varphi}(t), Z^{\alpha}(t)\right\rangle dt + \left\langle G_y(\bar{y}(T)), Z(T)\right\rangle \\
= & \lim_{\alpha\to\infty}\left\{\int_0^T \left\langle \overline{\varphi}(t), \dot{Z}^{\alpha}(t) - f_y^{\alpha\top}(t)Z^{\alpha}(t)\right\rangle dt - \sum_{i=1}^{k}\left[\langle\overline{\varphi}(t_i-), Z^{\alpha}(t_i)\rangle - \langle\overline{\varphi}(t_i), Z^{\alpha}(t_i+)\rangle\right]\right\} \\
= & \frac{1}{2}\lim_{\alpha\to\infty}\int_0^T \left\langle \overline{\varphi}(t), u^{\alpha}(t)^{\top} f_{uu}^{\alpha}(t)u^{\alpha}(t) + 2u^{\alpha}(t)^{\top} f_{yu}^{\alpha}(t)Y(t) + Y(t)^{\top} f_{yy}^{\alpha}(t)Y(t)\right\rangle dt \\
& +\frac{1}{2}\sum_{i=1}^{k}\left\langle \overline{\varphi}(t_i), Y(t_i))^{\top} J_{iyy}(\bar{y}(t_i))Y(t_i))\right\rangle \\
= & \frac{1}{2}\int_0^T \langle \overline{\varphi}(t), [u(t)-\bar{u}(t)]^{\top} f_{uu}(t,\bar{y}(t),\bar{u}(t))[u(t)-\bar{u}(t)]\rangle\, dt \\
& + \int_0^T \left\langle \overline{\varphi}(t), (u(t)-\bar{u}(t))^{\top}(t) f_{uy}(t,\bar{y}(t),\bar{u}(t))Y(t)\right\rangle dt \\
& +\frac{1}{2}\int_0^T \left\langle \overline{\varphi}(t), Y(t)^{\top} f_{yy}(t,\bar{y}(t),\bar{u}(t))Y(t)\right\rangle dt \\
& +\frac{1}{2}\sum_{i=1}^{k}\left\langle \overline{\varphi}(t_i), Y(t_i)^{\top} J_{iyy}(\bar{y}(t_i))Y(t_i)))\right\rangle .
\end{aligned} \tag{3.29}$$

Taken together with (3.29), we deduce from (3.25) that

$$\begin{aligned}
& \frac{1}{2}\int_0^T \langle [l_{uu}(t,\bar{y}(t),\bar{u}(t)) + \overline{\varphi}(t)^{\top} f_{uu}(t,\bar{y}(t),\bar{u}(t))]\,[u(t)-\bar{u}(t)], u(t)-\bar{u}(t)\rangle\, ds \\
& + \int_0^T \left\langle \left[l_{uy}(t,\bar{y}(t),\bar{u}(t)) + \overline{\varphi}(t)^{\top} f_{uy}(t,\bar{y}(t),\bar{u}(t))\right][u(t)-\bar{u}(t)], Y(t)\right\rangle dt \\
& +\frac{1}{2}\int_0^T \left\langle \left[l_{yy}(t,\bar{y}(t),\bar{u}(t)) + \overline{\varphi}(t)^{\top} f_{yy}(t,\bar{y}(t),\bar{u}(t))\right] Y(t), Y(t)\right\rangle dt \\
& +\frac{1}{2}\left\langle G_{yy}(\bar{y}(T))\, Y(T), Y(T)\right\rangle + \frac{1}{2}\sum_{i=1}^{k}\left\langle \overline{\varphi}(t_i)^{\top} J_{iyy}(\bar{y}(t_i))Y(t_i)), Y(t_i))\right\rangle \geq 0 \text{ for all } u \in \mathcal{U}_{ad}.
\end{aligned} \tag{3.30}$$

The following inequality follows from (3.30) and (3.17):

$$\frac{1}{2}\int_0^T \langle [l_{uu}(t,\bar{y}(t),\bar{u}(t)) + \overline{\varphi}(t)^{\top} f_{uu}(t,\bar{y}(t),\bar{u}(t))]\,[u(t)-\bar{u}(t)], u(t)-\bar{u}(t)\rangle\, dt$$

$$+\int_0^T \Big\langle \Big[l_{uy}(t,\bar{y}(t),\bar{u}(t)) + \overline{\varphi}(t)^\top f_{uy}(t,\bar{y}(t),\bar{u}(t)) + \overline{W}(t) f_u(t,\bar{y}(t),\bar{u}(t))^\top\Big][u(t)-\bar{u}(t)], \\ Y(t)\Big\rangle\, dt \geq 0 \text{ for all } u \in \mathcal{U}_{ad}. \quad (3.31)$$

Then, by (3.18) and (3.31), we can show that

$$\frac{1}{2}\int_0^T \langle [l_{uu}(t,\bar{y}(t),\bar{u}(t)) + \overline{\varphi}(t)^\top f_{uu}(t,\bar{y}(t),\bar{u}(t))][u(t)-\bar{u}(t)], u(t)-\bar{u}(t)\rangle\, dt \\ + \int_0^T \int_0^t \Big\langle \Big[l_{uy}(t,\bar{y}(t),\bar{u}(t)) + \overline{\varphi}(t)^\top f_{uy}(t,\bar{y}(t),\bar{u}(t)) + \overline{W}(t) f_u(t,\bar{y}(t),\bar{u}(t))^\top\Big][u(t)-\bar{u}(t)], \\ \overline{\Phi}(t)\overline{\Phi}(s)^{-1} f_u(s,\bar{y}(s),\bar{u}(s))^T[u(s)-\bar{u}(s)]ds\Big\rangle\, dt \geq 0 \text{ for all } u \in \mathcal{U}_{ad}, \quad (3.32)$$

Thus, we have finished the proof of Theorem 3.1.

Remark 3.4. *Theorem 3.1 does not establish whether the optimal control of Problem P is singular or nonsingular; it is a unified conclusion, similar to the equations in (4.5.2) of Theorem 4.2 in [15]. Therefore, Theorem 3.1 is a generalization of Theorem 4.2 in [15] to impulsive controlled systems. Based on Theorem 3.1, singular control and nonsingular control in the classical sense can be considered under a unified framework for Problem P.*

4. Pointwise Jacobson type necessary conditions

In this section, on the basis of Theorem 3.1 in the previous section, we first obtain the Legendre-Clebsch condition; then, we give a corollary for the integral form of the second-order necessary optimality conditions for optimal singular control; finally, we give the pointwise Jacobson type necessary conditions and the pointwise Legendre-Clebsch condition.

Corollary 4.1. *Let (A1)–(A3) hold and $\bar{u}$ denote the optimal control of J over $\mathcal{U}_{ad}$; it is necessary that there exist a pair of functions $(\bar{y},\overline{\varphi}) \in PC_l([0,T],\mathbb{R}^n) \times PC_r([0,T],\mathbb{R}^n)$ such that (3.1), (3.2), and*

$$H_{uu}[t] \geq 0, \text{ for all the continuous time of } \bar{u}(t), t \in [0,T], \quad (4.1)$$

hold.

Proof. To prove Corollary 4.1, let $\bar{u}(\cdot) \in \mathcal{U}_{ad}$; take the special control variational problem as follows:

$$u(t)-\bar{u}(t) = \begin{cases} 0, & t \in [t_0,\bar{t}), \\ h, & t \in [\bar{t},\bar{t}+\varepsilon), \\ 0, & t \in [\bar{t}+\varepsilon, T], \end{cases} \quad (4.2)$$

where $h \in \mathbb{R}^r$ is a constant vector, ε is a sufficiently small positive number, $\bar{t}$ is any continuous time of $\bar{u}(t)$. For $u(t)-\bar{u}(t)$, $\overline{\Phi}(t)$ satisfies (3.4); then, the solution $Y(t)$ of the variational problem (3.7) is given by

$$Y(t) = \begin{cases} 0, & t \in [t_0,\bar{t}), \\ \int_{\bar{t}}^t \overline{\Phi}(t)\overline{\Phi}(s)^{-1} f_u[s]^\top h ds, & t \in [\bar{t},\bar{t}+\varepsilon), \\ \int_{\bar{t}}^{\bar{t}+\varepsilon} \overline{\Phi}(t)\overline{\Phi}(s)^{-1} f_u[s]^\top h ds, & t \in [\bar{t}+\varepsilon, T]. \end{cases}$$

Then

$$\|Y(t)\| = \begin{cases} 0, & t \in [t_0, \bar{t}), \\ O(\varepsilon), & t \in [\bar{t}, T], \end{cases}$$

where $\lim_{\varepsilon\to 0} \frac{O(\varepsilon)}{\varepsilon} = C$ (nonzero constant). Utilizing the continuity of the mean value theorem for integrals, we have

$$\begin{aligned}
\int_0^T Y(t)^\top H_{yy}[t]Y(t)dt &= \int_{\bar{t}}^{\bar{t}+\varepsilon} Y(t)^\top H_{yy}[t]Y(t)dt = O(\varepsilon^2) = o(\varepsilon), \\
\int_0^T Y(t)^\top H_{uy}[t][u(t) - \bar{u}(t)]dt &= \int_{\bar{t}}^{\bar{t}+\varepsilon} Y(t)^\top H_{uy}[t]hdt \\
&= \varepsilon Y(\bar{t})^\top H_{uy}[\bar{t}]h + o(\varepsilon), \\
\int_0^T [u(t) - \bar{u}(t)]^\top H_{uu}[t][u(t) - \bar{u}(t)]dt &= \int_{\bar{t}}^{\bar{t}+\varepsilon} h^\top H_{uu}[t]hdt \\
&= \varepsilon h^\top H_{uu}[\bar{t}]h + o(\varepsilon).
\end{aligned} \tag{4.3}$$

Substituting (4.3) into (3.32), we have

$$\varepsilon h^\top H_{uu}[t]h + o(\varepsilon) \geq 0.$$

Observe that $h^\top H_{uu}[t]h$ is independent of ε and ε can be arbitrarily small; we have

$$h^\top H_{uu}[t]h \geq 0.$$

Since $h \in \mathbb{R}^r$ is an arbitrary vector and t denotes arbitrary continuous time, (4.1) holds.

Remark 4.2. *Condition (4.1) represents the Legendre-Clebsch condition for the optimal control problem Problem P. At the same time, it also shows the rationality of the conventional hypothesis* $H_{uu}[t] \geq 0, t \in [t_0, T]$.

Remark 4.3. *For the LQ problem, where* $R[t] = H_{uu}[t]$*, when* $R[t] > 0, \forall t \in [t_0, T]$*, the problem is called a nonsingular problem; when* $R[t] = 0, \forall t \in [t_0, T]$*, the problem is called the totally singular case; when* $R[t] \geq 0, \forall t \in [t_0, T]$*, the problem is called the partially singular case (see Definitions (4.4)–(4.6) in [15]). Whether or not the optimal control problem is singular, and regardless of the kind of singularity, this classification standard is often adopted.*

The following is a corollary of Theorem 3.1 in the case in which $H_{uu}[t] \equiv 0$, that is, Problem P is a totally singular problem according to the definitions in [15].

Corollary 4.4. *Let (A1)–(A3) hold and* $\bar{u}(\cdot) \in \overline{\mathcal{U}}_{ad}$ *denotes the optimal singular control of J over* $\mathcal{U}_{ad}$*; it is necessary that there exist functions* $(\overline{y}, \overline{\varphi}, \overline{W}, \overline{\Phi}) \in PC_l([0,T], \mathbb{R}^n) \times PC_r([0,T], \mathbb{R}^n) \times PC_r([0,T], \mathbb{R}^{n\times n}) \times PC_l([0,T], \mathbb{R}^{n\times n})$ *that satisfy (3.1)–(3.4) and*

$$\begin{aligned}
&\int_0^T dt \int_0^t \left\langle \left(\overline{W}(t) f_u[t]^\top + H_{uy}[t]\right)[u(t) - \bar{u}(t)], \overline{\Phi}(t)\overline{\Phi}(s)^{-1} f_u[s]^\top [u(s) - \bar{u}(s)] \right\rangle ds \\
&\geq 0, \quad \forall u \in \overline{\mathcal{U}}_{ad}.
\end{aligned}$$

Remark 4.5. *Corollary 4.4 is a similar conclusion of Theorem 4.3 in [25] for Problem P.*

The following is the pointwise Jacobson-type second-order necessary optimality condition for Problem P. The conclusion is similar to that of Theorem 4.3 in [25]. Note that the set $\overline{U}(t)$ (see (3.13)) of values for v differs from the set $\mathbb{R}^m$ of values that is described in [32]; thus, it essentially confirms the pointwise characteristic. The author of [25] has proven similar conclusions under the weaker condition, suggesting that the control region U is a Polish space. In fact, under our basic assumption (A1), we can use the method on page 93 in [15] to prove it.

Theorem 4.6. *Let (A1)–(A3) hold and $\bar{u}(\cdot) \in \overline{\mathcal{U}}_{ad}$ denote the optimal singular control of J over $\mathcal{U}_{ad}$; it is necessary that there exist functions $(\bar{y}, \overline{\varphi}, \overline{W}, \overline{\Phi}) \in PC_l([0,T],\mathbb{R}^n) \times PC_r([0,T],\mathbb{R}^n) \times PC_r([0,T],\mathbb{R}^{n\times n}) \times PC_l([0,T],\mathbb{R}^{n\times n})$ that satisfy (3.1)–(3.4), and, for all continuous time of $\bar{u}(t), t \in [0,T]$, we have*

$$\Big\langle \Big(\overline{W}(t) f_u[t]^\top + H_{uy}[t]\Big)[v - \bar{u}(t)], f_u[t]^\top [v - \bar{u}(t)] \Big\rangle \geq 0, \quad \forall v \in \overline{U}(t). \tag{4.4}$$

Proof. Note that Definition 2.3 implies that $H_{uu}[t] \equiv 0$ for all $t \in [0,T]$, apply the same control perturbation as for (4.2); we have

$$\begin{aligned} &\int_0^T \Big\langle \Big(\overline{W}(t) f_u[t]^\top + H_{yu}[t]\Big) h, Y(t) \Big\rangle dt \\ = &\int_{\bar{t}}^{\bar{t}+\varepsilon} \Big\langle \Big(\overline{W}(t) f_u[t]^\top + H_{yu}[t]\Big) h, Y(t) \Big\rangle dt, \end{aligned} \tag{4.5}$$

and the dominant term in the expansion of (4.5) for sufficiently small ε is given by

$$(\varepsilon)^2 \Big\langle \Big(\overline{W}(t) f_u[t]^\top + H_{yu}[t]\Big) h, f_u[t]^\top h \Big\rangle \Big|_{\bar{t}}.$$

Since $\bar{t}$ can be chosen as any continuous time of $\bar{u}(t), t \in [0,T]$, let $h = v - \bar{u}(t), \forall v \in \overline{U}(t)$; thus, (4.4) holds and we have finished the proof of Theorem 4.6.

Using the same idea as in Theorem 4.6, we can also obtain the pointwise Legendre-Clebsch necessary optimality condition corresponding to Corollary 4.1.

Corollary 4.7. *Let (A1)–(A3) hold and $\bar{u}$ denote the optimal control of J over $\mathcal{U}_{ad}$; it is necessary that there exist a pair of functions $(\bar{y}, \overline{\varphi}) \in PC_l([0,T],\mathbb{R}^n) \times PC_r([0,T],\mathbb{R}^n)$ such that (3.1), (3.2), and, for all continuous time of $\bar{u}(t), t \in [0,T]$*

$$(v - \bar{u}(t))^\top H_{uu}[t](v - \bar{u}(t)) \geq 0, \forall v \in \overline{U}(t), \tag{4.6}$$

hold.

Remark 4.8. *Comparing Corollaries 4.7 and 4.1, it can be found that if the pointwise condition is satisfied, $H_{uu} \geq 0$ is not required, as only (4.6) needs to be satisfied.*

5. Example

In this section, we will give an example to illustrate the effectiveness of Theorem 4.6.

Let

$$\min J(u(\cdot)) = y_2(1),$$

subject to

$$\begin{cases} \dot{y}_1(t) = u, & t \in [0,1] \setminus 0.5, \\ \dot{y}_2(t) = -y_1^2, & t \in [0,1] \setminus 0.5, \\ y_1(0.5+) = y_1(0.5) + y_1(0.5), \\ y_2(0.5+) = y_2(0.5), \\ y_1(0) = 0, \\ y_2(0) = 0. \end{cases}$$

Obviously, the Hamiltonian $H(t,y,u,\varphi) = \varphi_1 u - \varphi_2 y_1^2$ satisfies the conditions for linear control, as denoted by u and $u \in U = \mathbb{R}$. According to Remark (2.4), the problem is singular, i.e., $\overline{U}(t) = \mathbb{R}$. It is not difficult to assert that $\bar{u} \equiv 0$ denotes singular control; this is because, by $\bar{u} \equiv 0$, we can get that $\bar{y}_1 \equiv 0, \bar{y}_2 \equiv 0$, and $\overline{\varphi}_1 = 0, t \in [0,1]$; consequently, $H(t,y,u,\varphi) = \overline{\varphi}_1 \bar{u} - \overline{\varphi}_2 \bar{y}_1^2 \equiv 0$, $H_u \equiv 0$, and $H_{uu} \equiv 0$; by Definition 2.3, $\bar{u} \equiv 0, t \in [0,1]$ denotes singular control. The question is whether it is optimal singular control. Now, let us use Theorem 4.6 to determine that it must not be optimal singular control.

By (3.2), we have

$$\begin{cases} \dot{\overline{\varphi}}_1(t) = 2\overline{\varphi}_2 \bar{y}_1, & t \in [0,1] \setminus 0.5, \\ \dot{\overline{\varphi}}_2(t) = 0, & t \in [0,1] \setminus 0.5, \\ \overline{\varphi}_1(0.5-) = \overline{\varphi}_1(0.5), \\ \overline{\varphi}_2(0.5-) = \overline{\varphi}_1(0.5) + \overline{\varphi}_2(0.5), \\ \overline{\varphi}_1(1) = 0, \\ \overline{\varphi}_2(1) = 1. \end{cases}$$

Using (3.3), we have

$$\begin{aligned} \dot{\overline{W}}(t) &= \begin{bmatrix} \dot{\overline{w}}_{11}(t) & \dot{\overline{w}}_{12}(t) \\ \dot{\overline{w}}_{21}(t) & \dot{\overline{w}}_{22}(t) \end{bmatrix} = -\begin{bmatrix} 0 & -2\bar{y}_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \overline{w}_{11}(t) & \overline{w}_{12}(t) \\ \overline{w}_{21}(t) & \overline{w}_{22}(t) \end{bmatrix} \\ &\quad - \begin{bmatrix} \overline{w}_{11}(t) & \overline{w}_{12}(t) \\ \overline{w}_{21}(t) & \overline{w}_{22}(t) \end{bmatrix} \begin{bmatrix} 0 & 0 \\ -2\bar{y}_1 & 0 \end{bmatrix} + \begin{bmatrix} \overline{\varphi}_2(t) & 0 \\ 0 & 0 \end{bmatrix}, \end{aligned}$$

and

$$\begin{bmatrix} \overline{w}_{11}(1) & \overline{w}_{12}(1) \\ \overline{w}_{21}(1) & \overline{w}_{22}(1) \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},$$

and

$$\overline{W}(0.5-) = \begin{bmatrix} \overline{w}_{11}(0.5-) & \overline{w}_{12}(0.5-) \\ \overline{w}_{21}(0.5-) & \overline{w}_{22}(0.5-) \end{bmatrix}$$

$$
\begin{aligned}
&= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \overline{w}_{11}(0.5) & \overline{w}_{12}(0.5) \\ \overline{w}_{21}(0.5) & \overline{w}_{22}(0.5) \end{bmatrix} \\
&+ \begin{bmatrix} \overline{w}_{11}(0.5) & \overline{w}_{12}(0.5) \\ \overline{w}_{21}(0.5) & \overline{w}_{22}(0.5) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \overline{w}_{11}(0.5) & \overline{w}_{12}(0.5) \\ \overline{w}_{21}(0.5) & \overline{w}_{22}(0.5) \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.
\end{aligned}
$$

Substituting $\bar{u} \equiv 0, \bar{y}_1 \equiv 0$, and $\bar{y}_2 \equiv 0$ directly into the above equations, the following results can be obtained directly

$$
\begin{cases} \overline{\varphi}_1(t) \equiv 0, \\ \overline{\varphi}_2(t) \equiv 1, \end{cases} \quad t \in [0,1],
$$

and

$$
\begin{cases} \overline{w}_{11} = t - 1, t \in [0,1], \\ \overline{w}_{12} = \overline{w}_{21} = \overline{w}_{22} = \begin{cases} 0, & t \in [0.5, 1], \\ -0.5, & t \in [0, 0.5). \end{cases} \end{cases}
$$

By (4.4), the necessary condition for the singular control $\bar{u} \equiv 0$ to be the optimal control scheme is given by

$$
(t-1)v^2 \geq 0, \forall v \ \in \overline{U}(t). \tag{5.1}
$$

But, because of the above equations, (5.1) cannot be true for arbitrary but fixed $t \in (0,1)$. Therefore, regarding singular control $\bar{u} = 0$, according to Theorem 4.6, it must not be optimal singular control.

6. Conclusions

In this paper, we have investigated the pointwise Jacobson type necessary conditions for Problem P. By introducing an impulsive linear matrix Riccati differential equation, we have derived the integral representation of the functional second-order variational equation. On this basis, we obtained the integral form of the second-order necessary conditions and the pointwise Jacobson type necessary conditions for optimal singular control in the classical sense. Incidentally, the Legendre-Clebsch condition and the pointwise Legendre-Clebsch condition were also obtained. These conclusions have been derived under weaker conditions, thereby enriching existing conclusions. In the future, we will continue to research the pointwise Jacobson-type second-order necessary optimality conditions in the Pontryagin sense.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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