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Research article

Multiple positive solutions for a singular tempered fractional equation with lower order tempered fractional derivative

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Abstract: Let $\alpha \in (1,2], \beta \in (0,1)$ with $\alpha - \beta > 1$. This paper focused on the multiplicity of positive solutions for a singular tempered fractional boundary value problem

$$\begin{cases} -{}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t) = p(t)h\left(e^{\lambda t}u(t), {}_0^R\mathbb{D}_t^{\beta,\lambda}u(t)\right), t \in (0,1), \\ {}_0^R\mathbb{D}_t^{\beta,\lambda}u(0) = 0, \ \ {}_0^R\mathbb{D}_t^{\beta,\lambda}u(1) = 0, \end{cases}$$

where $h \in C([0,+\infty) \times [0,+\infty), [0,+\infty))$ and $p \in L^1([0,1],(0,+\infty))$. By applying reducing order technique and fixed point theorem, some new results of existence of the multiple positive solutions for the above equation were established. The interesting points were that the nonlinearity contained the lower order tempered fractional derivative and that the weight function can have infinite many singular points in $[0,1]$.

Keywords: singularity; multiplicity; tempered fractional equation; reducing order technique

1. Introduction

In this paper, we consider the multiplicity of positive solutions for the following singular tempered fractional equation with lower order tempered fractional derivative

$$\begin{cases} -{}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t) = p(t)h\left(e^{\lambda t}u(t), {}_0^R\mathbb{D}_t^{\beta,\lambda}u(t)\right), \ \ t \in (0,1), \\ {}_0^R\mathbb{D}_t^{\beta,\lambda}u(0) = 0, \ \ {}_0^R\mathbb{D}_t^{\beta,\lambda}u(1) = 0, \end{cases} \tag{1.1}$$

where $\alpha \in (1,2], \beta \in (0,1)$ with $\alpha - \beta > 1$, $h \in C([0,+\infty) \times [0,+\infty), [0,+\infty))$, $p \in L^1((0,1),(0,+\infty))$, which implies that the weight function can have infinite many singular points in $[0,1]$.

The equation (1.1) contains a tempered fractional derivative ${}_0^R\mathbb{D}_t^{\alpha,\lambda}$, which is actually obtained by multiplying an exponential factor in the Riemann-Liouville fractional derivative ${}_0^R\mathscr{D}_t^{\alpha}$, i.e., the following relationship exists between tempered fractional derivative and Riemann-Liouville fractional derivative

$$ {}_0^R\mathbb{D}_t^{\alpha,\lambda}u(t) = e^{-\lambda t}{}_0^R\mathscr{D}_t^{\alpha}(e^{\lambda t}u(t)). \tag{1.2}$$

For the definition of the standard Riemann-Liouville fractional derivative and integral, we refer the reader to [1–5].

As the optimization of the Riemann-Liouville fractional derivative, the tempered fractional derivative has many advantages, which not only overcomes the defect of using the power law of the classical fractional derivative in the mathematical sense, such as the Riemann-Liouville fractional derivatives [6, 7], the Caputo fractional derivatives [8], Hadamard fractional derivatives [9–11] etc, but also brings many practical applications. It especially describes the anomalous diffusion phenomena in Brownian motion with the semi-heavy tails or semi-long range dependence, such as the limits of random walk with an exponentially tempered jump distribution [12, 13], transient super-diffusion [14], anomalous diffusions in heterogeneous systems [15], exponentially tempering Lévy flights with both the α-stable and Gaussian trends [16], and pure jump Lévy process with the fractional derivatives in the risk management of financial derivatives traded over the counter [17].

In recent work [18], an upper and lower solutions technique has been employed to establish the existence of positive solutions for a singular tempered fractional turbulent flow model in a porous medium

$$\begin{cases} {}_0^R\mathbb{D}_t^{\alpha,\lambda}\left(\varphi_p\left({}_0^R\mathbb{D}_t^{\beta,\lambda}u(t)\right)\right) = f(t,u(t)), \quad t \in (0,1), \\ u(0) = 0, \ \ {}_0^R\mathbb{D}_t^{\beta,\lambda}u(0) = 0, \ \ u(1) = \displaystyle\int_0^1 e^{-\lambda(1-t)}u(t)dt, \end{cases} \tag{1.3}$$

with $\alpha \in (0,1]$, $\beta \in (1,2]$ and the nonlinearity f is decreasing in the second variable. In [19], in view of the monotone iterative method, the iterative properties of positive solutions for a tempered fractional equation

$$\begin{cases} {}_0^R\mathbb{D}_t^{\vartheta,\mu}u(t) = f(t, e^{\mu t}u(t), {}_0^R\mathbb{D}_t^{\delta,\mu}u(t)), \quad t \in (0,1), \\ {}_0^R\mathbb{D}_t^{\delta,\mu}u(0) = 0, \ \ {}_0^R\mathbb{D}_t^{\delta,\mu}u(1) = \displaystyle\int_0^1 e^{-\mu(1-t)}{}_0^R\mathbb{D}_t^{\delta,\mu}u(t)dt, \end{cases} \tag{1.4}$$

were established, where $\vartheta \in (1,2]$, $\delta \in (0,1)$ with $\vartheta - \delta > 1$, μ is a positive constant, and $f : (0,1) \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is a continuous and nondecreasing function with respect to the two space variables. Recently, by using spaces theories [20–24], smooth theories [25–27], operator method [28, 29], the method of moving sphere [30], critical point theories [31–34] and so on, some other types of fractional equations was also studied [35–46].

However, when the nonlinearity contains lower order tempered fractional derivative, the results of multiplicity of positive solutions have not yet been obtained. In order to fill this gap, by applying the reducing order technique and fixed point theorem, some new results of existence of the multiple positive solutions for the above equation are established in this paper. The interesting points are that the nonlinearity contains the lower order tempered fractional derivative and the weight function can have infinitely many singular points in $[0,1]$.

2. Preliminaries and lemmas

In this section, we first recall definitions and some useful properties of the Riemann-Liouville fractional derivative and integral.

Definition 2.1 ((2.1.1) on page 69 in [1]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1}u(s)ds$$

provided that the righthand side is pointwise and defined on $(0, +\infty)$.

Definition 2.2 ((2.1.5) on page 70 in [1]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a function $u : (0, +\infty) \to \mathbb{R}$ is given by

$$\mathscr{D}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t (t-s)^{n-\alpha-1}u(s)ds,$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the greatest integer part of the number α, provided that the righthand side is pointwise and defined on $(0, +\infty)$.

The following properties of the Riemann-Liouville fractional derivative and integral can be found on pages 73–75 (Lemmas 2.3–2.5) in [1].

Lemma 2.1. *Suppose $u(t) \in C[0,1] \cap L^1[0,1]$ and $\alpha > \beta > 0$. Let $n = [\alpha] + 1$, then*

(i)

$$I^{\alpha} {}_0^R\mathscr{D}_t^{\alpha}u(t) = u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2} + \cdots + c_nt^{\alpha-n}, \tag{2.1}$$

where $c_i \in \mathbb{R}, i = 1, 2, 3, ..., n$.

(ii)

$$I^{\alpha}I^{\beta}u(t) = I^{\alpha+\beta}u(t), \ \ {}_0^R\mathscr{D}_t^{\beta}I^{\alpha}u(t) = I^{\alpha-\beta}u(t), \ \ {}_0^R\mathscr{D}_t^{\beta}I^{\beta}u(t) = u(t).$$

Lemma 2.2. *Let $p \in L^1([0,1], (0, +\infty))$, then the singular linear tempered fractional equation*

$$\begin{cases} -{}_0^R\mathbb{D}_t^{\alpha-\beta,\lambda}u(t) = p(t), \\ u(0) = 0, \ \ u(1) = 0, \end{cases} \tag{2.2}$$

has a unique positive solution $u(t)$ provided that $1 < \alpha - \beta \leq 2$, which can be expressed by

$$u(t) = \int_0^1 H(t,s)p(s)ds, \tag{2.3}$$

where

$$H(t,s) = \begin{cases} \dfrac{t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}e^{-\lambda t}e^{\lambda s}, 0 \leq s \leq t \leq 1; \\ \dfrac{t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}e^{-\lambda t}e^{\lambda s}, \ \ 0 \leq t \leq s \leq 1. \end{cases} \tag{2.4}$$

is the Green function of (2.2).

Proof. In fact, it follows from $1 < \alpha - \beta \leq 2$, (1.2) and (2.1) that

$$e^{\lambda t}u(t) = -\frac{1}{\Gamma(\alpha-\beta)}\int_0^t (t-s)^{\alpha-\beta-1}e^{\lambda s}p(s)ds + b_1 t^{\alpha-\beta-1} + b_2 t^{\alpha-\beta-2}, t \in [0,1].$$

Since $u(0) = 0$ and $u(1) = 0$, we have $b_2 = 0$ and

$$b_1 = \frac{1}{\Gamma(\alpha-\beta)}\int_0^1 (1-s)^{\alpha-\beta-1}e^{\lambda s}p(s)ds.$$

Thus

$$\begin{aligned}
u(t) &= \frac{1}{\Gamma(\alpha-\beta)}\left[\int_0^1 (1-s)^{\alpha-\beta-1}e^{-\lambda t}e^{\lambda s}p(s)dst^{\alpha-\beta-1} - \int_0^t (t-s)^{\alpha-\beta-1}e^{-\lambda t}e^{\lambda s}p(s)ds\right] \\
&= \frac{1}{\Gamma(\alpha-\beta)}\int_0^t \left[t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1}\right]e^{-\lambda t}e^{\lambda s}p(s)ds \\
&+ \frac{1}{\Gamma(\alpha-\beta)}\int_t^1 t^{\alpha-\beta-1}(1-s)^{\alpha-\beta-1}e^{-\lambda t}e^{\lambda s}p(s)ds \\
&= \int_0^1 H(t,s)p(s)ds, \quad t \in [0,1].
\end{aligned}$$

The following Lemma has been proven (see Lemma 2.3 of [6] or Lemma 3 of [4]).

Lemma 2.3. *Suppose $\alpha \in (1,2], \beta \in (0,1)$ with $\alpha - \beta > 1$, then $H(t,s)$ is a nonnegative continuous function in $[0,1]\times[0,1]$ satisfying, for any $(t,s) \in [0,1]\times[0,1]$,*

$$\frac{t^{\alpha-\beta-1}(1-t)e^{-\lambda t}}{\Gamma(\alpha-\beta)}(1-s)^{\alpha-\beta-1}se^{\lambda s} \leq H(t,s) \leq \frac{t^{\alpha-\beta-1}(1-t)e^{-\lambda t}}{\Gamma(\alpha-\beta)} \; or \; \left(\frac{(1-s)^{\alpha-\beta-1}se^{\lambda s}}{\Gamma(\alpha-\beta)}\right). \tag{2.5}$$

Suppose $\alpha \in (1,2], \beta \in (0,1)$ with $\alpha - \beta > 1$. In order to use the reducing order technique, we introduce the following integral transformation

$$u(t) = e^{-\lambda t}I^{\beta}(e^{\lambda t}y(t)), \quad t \in [0,1] \tag{2.6}$$

and then consider the following reducing order problem

$$\begin{cases} -{}_0^R\mathbb{D}_t^{\alpha-\beta,\lambda}y(t) = p(t)h(I^{\beta}(e^{\lambda t}y(t)), y(t)), \quad t \in (0,1), \\ y(0) = 0, \quad y(1) = 0. \end{cases} \tag{2.7}$$

Lemma 2.4. *Suppose $\alpha \in (1,2], \beta \in (0,1)$ with $\alpha - \beta > 1$. The reducing order problem (2.7) is equivalent to the singular tempered fractional equation (1.1). In particular, if y is a positive solution of the problem (2.7), then $u(t) = e^{-\lambda t}I^{\beta}(e^{\lambda t}y(t))$ is a positive solution of the singular tempered fractional equation (1.1).*

Proof. We first suppose that u is a positive solution of the singular tempered fractional equation (1.1). Let

$$u(t) = e^{-\lambda t} I^{\beta}(e^{\lambda t} y(t)).$$

Noticing that $\alpha \in (1, 2], \beta \in (0, 1)$ with $\alpha - \beta > 1$, let $n = [\alpha] + 1$, i.e., n is the smallest integer greater than or equal to α, then it follows from (1.2) and Lemma 2.1 that

$$\begin{aligned}
{}_0^R\mathbb{D}_t^{\alpha,\lambda} u(t)) &= e^{-\lambda t} {}_0^R\mathscr{D}_t^{\alpha}(e^{\lambda t} u(t)) = e^{-\lambda t} \frac{d^n}{dt^n} I^{n-\alpha}(e^{\lambda t} u(t)) \\
&= e^{-\lambda t} \frac{d^n}{dt^n} I^{n-\alpha}(I^{\beta}(e^{\lambda t} y(t))) = e^{-\lambda t} \frac{d^n}{dt^n} I^{n-\alpha+\beta}(e^{\lambda t} y(t))) \\
&= e^{-\lambda t} {}_0^R\mathscr{D}_t^{\alpha-\beta}(e^{\lambda t} y(t))) = {}_0^R\mathbb{D}_t^{\alpha-\beta,\lambda} y(t),
\end{aligned} \tag{2.8}$$

and

$$ {}_0^R\mathbb{D}_t^{\beta,\lambda} u(t)) = e^{-\lambda t} {}_0^R\mathscr{D}_t^{\beta}(e^{\lambda t} u(t)) = e^{-\lambda t} {}_0^R\mathscr{D}_t^{\beta}(I^{\beta}(e^{\lambda t} y(t))) = y(t). \tag{2.9}$$

By (2.8) and (2.9), we have

$$\begin{cases} -{}_0^R\mathbb{D}_t^{\alpha-\beta,\lambda} y(t) = -{}_0^R\mathbb{D}_t^{\alpha,\lambda} u(t)) = p(t)h\left(e^{\lambda t} u(t), {}_0^R\mathbb{D}_t^{\beta,\lambda} u(t)\right) = p(t)h\left(I^{\beta}(e^{\lambda t} y(t)), y(t)\right), t \in (0, 1), \\ y(0) = {}_0^R\mathbb{D}_t^{\beta,\lambda} u(0) = 0, \quad y(1) = {}_0^R\mathbb{D}_t^{\beta,\lambda} u(1) = 0, \end{cases}$$

thus, y solves the equation (2.7).

Conversely, suppose that y is any solution of the reducing order problem (2.7), then we have

$$\begin{cases} -{}_0^R\mathbb{D}_t^{\alpha-\beta,\lambda} y(t) = p(t)h(I^{\beta}(e^{\lambda t} y(t)), y(t)), \quad t \in (0, 1), \\ y(0) = 0, \quad y(1) = 0. \end{cases}$$

Make integral transformation (2.6), similar to (2.8) and (2.9), we have

$$-{}_0^R\mathbb{D}_t^{\alpha,\lambda} u(t)) = -{}_0^R\mathbb{D}_t^{\alpha-\beta,\lambda} y(t), \quad {}_0^R\mathbb{D}_t^{\beta,\lambda} u(t)) = y(t).$$

Substituting the above formulas into (2.7), we get

$$-{}_0^R\mathbb{D}_t^{\alpha,\lambda} u(t)) = p(t)h\left(I^{\beta}(e^{\lambda t} y(t)), y(t)\right) = p(t)h\left(e^{\lambda t} u(t), {}_0^R\mathbb{D}_t^{\beta,\lambda} u(t)\right), \quad 0 < t < 1,$$

and

$${}_0^R\mathbb{D}_t^{\beta,\lambda} u(0) = 0, \quad {}_0^R\mathbb{D}_t^{\beta,\lambda} u(1) = 0,$$

which implies that $u(t) = e^{-\lambda t} I^{\beta}(e^{\lambda t} y(t))$ solves the singular tempered fractional equation (1.1).

Suppose $E = C([0, 1]; \mathbb{R})$ with the norm

$$\|y\| = \max_{t \in [0,1]} |y(t)|.$$

Define a cone of E and an operator T, respectively:

$$P = \{y \in E : y(t) \geq t^{\alpha-\beta-1}(1 - t)e^{-\lambda t}\|y\|\},$$

and

$$(Ty)(t) = \int_0^1 H(t,s)p(s)h(I^\beta(e^{\lambda t}y(s)), y(s))ds. \tag{2.10}$$

In order to obtain the existence of a positive solution of the Eq (1.1), Lemma 2.4 indicates that we only consider the fixed points of the operator T.

Now, we list the following hypotheses, which are used in the rest of this paper.

(**C1**) $h \in C([0,+\infty) \times [0,+\infty), [0,+\infty))$ and $p \in L^1((0,1),(0,+\infty))$.

(**C2**) There is a constant $n > 0$ such that for any $0 \leq u + v \leq \left(1 + \frac{e^\lambda}{\Gamma(\beta+1)}\right)n$,

$$h(u,v) < \mu n,$$

where

$$\mu = \left[\frac{ne^\lambda}{\Gamma(\alpha-\beta)}\int_0^1 p(s)ds\right]^{-1}.$$

(**C3**) There is a constant $\rho > 0$ such that for any

$$\left(\frac{\Gamma(\alpha-\beta)\left(\frac{1}{4}\right)^{\alpha-1}}{\Gamma(\alpha)} + \left(\frac{1}{4}\right)^{\alpha-\beta} e^{-\frac{3}{4}\lambda}\right)\rho \leq u + v \leq \left(1 + \frac{e^\lambda}{\Gamma(\beta+1)}\right)\rho$$

it implies

$$h(u,v) \geq \varrho\rho,$$

where

$$\varrho = 8^{\alpha-\beta}e^{\frac{1}{4}\lambda}\Gamma(\alpha-\beta)\left[\int_{\frac{1}{4}}^{\frac{3}{4}} p(s)ds\right]^{-1}.$$

Lemma 2.5. *Assume $\alpha \in (1,2], \beta \in (0,1)$ with $\alpha - \beta > 1$ and (**C1**) holds, then the operator $T: P \to P$ is completely continuous.*

Proof. First by (**C1**), T is continuous on $[0,1]$. For any $y \in P$, there exists a constant $M > 0$ such that $\|y\| \leq M$, then

$$0 \leq I^\beta(e^{\lambda s}y(s)) = \int_0^t \frac{(t-s)^{\beta-1}e^{\lambda s}y(s)}{\Gamma(\beta)}ds \leq \frac{Me^\lambda}{\Gamma(\beta)}. \tag{2.11}$$

Let

$$\aleph = \max_{(u,v)\in[0,\frac{Me^\lambda}{\Gamma(\beta)}]\times[0,M]} h(u,v),$$

then by (2.11), for any $y \in P$, we have

$$\begin{aligned}
\|Ty\| &= \max_{t\in[0,1]}\int_0^1 H(t,s)p(s)h(I^\beta(e^{\lambda t}y(s)), y(s))ds \\
&\leq \int_0^1 \frac{(1-s)^{\alpha-\beta-1}se^{\lambda s}}{\Gamma(\alpha-\beta)}p(s)h(I^\beta(e^{\lambda t}y(s)), y(s))ds \\
&\leq \aleph\int_0^1 \frac{(1-s)^{\alpha-\beta-1}se^{\lambda s}}{\Gamma(\alpha-\beta)}p(s)ds \\
&\leq \frac{\aleph e^\lambda}{\Gamma(\alpha-\beta)}\int_0^1 p(s)ds < +\infty,
\end{aligned} \tag{2.12}$$

which implies that $T : P \to E$ is well-defined. In addition, by (2.5) and (2.12), we have

$$(Ty)(t) \geq \int_0^1 \frac{(1-s)^{\alpha-\beta-1} s e^{\lambda s}}{\Gamma(\alpha-\beta)} p(s) h(I^{\beta}(e^{\lambda t} y(s)), y(s)) ds$$
$$\times t^{\alpha-\beta-1}(1-t)e^{-\lambda t} \geq \|Ty\| t^{\alpha-\beta-1}(1-t)e^{-\lambda t},$$

which implies that $T(P) \subset P$.

In the end, by using the standard arguments and combining the Ascoli-Arzela theorem, we know $T(P) \subset P$ is completely continuous.

Our proof of main results depends on the fixed point theorem of cone expansion and compression (see Theorem 2.3.3 on page 93 of [47]).

Lemma 2.6. *[47] Suppose P is a cone of real Banach space E, the bounded open subsets Ω_1, Ω_2 of E satisfy $\theta \in \Omega_1, \overline{\Omega_1} \subset \Omega_2$. Let $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ be a completely continuous operator such that either*

(1) $\|Tz\| \leq \|z\|, z \in P \cap \partial\Omega_1$ and $\|Tz\| \geq \|z\|, z \in P \cap \partial\Omega_2$, or

(2) $\|Tz\| \geq \|z\|, z \in P \cap \partial\Omega_1$ and $\|Tz\| \leq \|z\|, z \in P \cap \partial\Omega_2$;

then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

3. Main results

For the convenience of the proof, we first introduce the following notations whenever the limits exist or not:

$$h_0 = \lim_{u+v \to 0} \frac{h(u,v)}{u+v}, \quad h_\infty = \lim_{u+v \to +\infty} \frac{h(u,v)}{u+v},$$

and then state our main results as follows.

Theorem 3.1. *Assume that* (**C1**) *and* (**C2**) *hold, and*

$$h_0 = +\infty, h_\infty = +\infty,$$

then the tempered fractional equation (1.1) *has at least two positive solutions u_1, u_2; moreover, there exist four constants $A_1, B_1, A_2, B_2 > 0$ such that*

$$A_1 t^{\alpha-1} e^{-\lambda t} \leq u_1(t) \leq B_1 e^{\lambda(1-t)}, \quad A_2 t^{\alpha-1} e^{-\lambda t} \leq u_2(t) \leq B_2 e^{\lambda(1-t)}.$$

Proof. It follows from $h_0 = +\infty$ that there exist $0 < m < n$ and a sufficiently large constant

$$N > \frac{32^{\alpha-\beta} e^{\lambda} \Gamma(\alpha-\beta)}{\int_{\frac{1}{4}}^{\frac{3}{4}} p(s) ds}$$

such that for any $0 < u + v \leq \left(1 + \frac{e^{\lambda}}{\Gamma(\beta+1)}\right) m$,

$$h(u,v) \geq N(u+v).$$

Take $\Omega_m = \{y \in E : \|y\| < m\}$, and $\partial\Omega_m = \{y \in E : \|y\| = m\}$, then for any $y \in P \cap \partial\Omega_m$, one has

$$I^\beta(e^{\lambda s}y(s)) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} e^{\lambda s}y(s)ds \le \frac{e^\lambda m t^\beta}{\Gamma(\beta+1)} \le \frac{e^\lambda m}{\Gamma(\beta+1)}, \tag{3.1}$$

then

$$I^\beta(e^{\lambda s}y(s)) + y(s) \le \left(1 + \frac{e^\lambda}{\Gamma(\beta+1)}\right)m. \tag{3.2}$$

Thus,

$$\begin{aligned}
\|Ty\| \ge (Ty)\left(\frac{1}{2}\right) &= \int_0^1 H\left(\frac{1}{2}, s\right)p(s)h(I^\beta(e^{\lambda s}y(s)), y(s))ds \\
&\ge \frac{2^{-\alpha+\beta}e^{-\frac{1}{2}\lambda}}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} s e^{\lambda s} p(s) N\left(I^\beta(e^{\lambda s}y(s)) + y(s)\right)ds \\
&\ge \frac{2^{-\alpha+\beta}e^{-\frac{1}{2}\lambda}}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} s e^{\lambda s} p(s) N y(s)ds \\
&\ge \frac{2^{-\alpha+\beta}e^{-\frac{1}{2}\lambda}}{\Gamma(\alpha-\beta)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\alpha-\beta-1} s e^{\lambda s} p(s) N y(s)ds \\
&\ge \frac{N}{8^{\alpha-\beta}e^{\frac{5}{4}\lambda}\Gamma(\alpha-\beta)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\alpha-\beta-1} s e^{\lambda s} p(s)ds\|y\| \\
&\ge \frac{N}{32^{\alpha-\beta}e^{\lambda}\Gamma(\alpha-\beta)} \int_{\frac{1}{4}}^{\frac{3}{4}} p(s)ds\|y\| \ge \|y\|.
\end{aligned} \tag{3.3}$$

So, for any $y \in P \cap \partial\Omega_m$, we have $\|Ty\| \ge \|y\|$.

Next, let $\Omega_n = \{y \in E : \|y\| < n\}$ and $\partial\Omega_n = \{y \in E : \|y\| = n\}$, then it follows from (3.2) and (**C2**) that for any $y \in P \cap \partial\Omega_n$, one has

$$\begin{aligned}
(Ty)(t) &= \int_0^1 H(t,s)p(s)h(I^\beta(e^{\lambda t}y(s), y(s))ds \\
&\le \int_0^1 \frac{(1-s)^{\alpha-\beta-1}se^{\lambda s}}{\Gamma(\alpha-\beta)} p(s)h(I^\beta(e^{\lambda t}y(s), y(s))ds \\
&\le \frac{e^\lambda}{\Gamma(\alpha-\beta)} \int_0^1 p(s)h(I^\beta(e^{\lambda t}y(s), y(s))ds \\
&\le \frac{\mu n e^\lambda}{\Gamma(\alpha-\beta)} \int_0^1 p(s)ds \le n.
\end{aligned} \tag{3.4}$$

Therefore for any $y \in P \cap \partial\Omega_n$, we have $\|Ty\| \le \|y\|$.

On the other hand, it follows from $h_\infty = +\infty$ that there exists $M > 0$ and

$$\eta \ge 32^{\alpha-\beta}e^\lambda\Gamma(\alpha-\beta)\left(\int_{\frac{1}{4}}^{\frac{3}{4}} p(s)ds\right)^{-1}$$

such that for any $u + v > M$, we have

$$h(u,v) \ge \eta(u+v).$$

Take $R = n + 4^{\alpha-\beta}e^{\frac{3}{4}\lambda}M$ and let $\Omega_R = \{y \in E : \|y\| < R\}$ and $\partial\Omega_R = \{y \in E : \|y\| = R\}$, then for any $y \in P \cap \partial\Omega_R$ and $[\frac{1}{4}, \frac{3}{4}]$, one has

$$y(t) \geq \left(\frac{1}{4}\right)^{\alpha-\beta} e^{-\frac{3}{4}\lambda}\|y\| \geq \left(\frac{1}{4}\right)^{\alpha-\beta} e^{-\frac{3}{4}\lambda}(n + 4^{\alpha-\beta}e^{\frac{3}{4}\lambda}M) \geq M.$$

Thus, for any $y \in P \cap \partial\Omega_R$, we have

$$\begin{aligned}
\|Ty\| \geq (Ty)\left(\frac{1}{2}\right) &= \int_0^1 H\left(\frac{1}{2}, s\right) p(s)h(I^\beta(e^{\lambda s}y(s)), y(s))ds \\
&\geq \frac{2^{-\alpha+\beta}e^{-\frac{1}{2}\lambda}}{\Gamma(\alpha-\beta)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\alpha-\beta-1} se^{\lambda s}p(s)\eta\left(I^\beta(e^{\lambda s}y(s)) + y(s)\right)ds \\
&\geq \frac{2^{-\alpha+\beta}e^{-\frac{1}{2}\lambda}}{\Gamma(\alpha-\beta)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\alpha-\beta-1} se^{\lambda s}p(s)\eta y(s)ds \\
&\geq \frac{2^{-\alpha+\beta}e^{-\frac{1}{2}\lambda}}{\Gamma(\alpha-\beta)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\alpha-\beta-1} se^{\lambda s}p(s)\eta y(s)ds \\
&\geq \frac{\eta}{8^{\alpha-\beta}e^{\frac{5}{4}\lambda}\Gamma(\alpha-\beta)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\alpha-\beta-1} se^{\lambda s}p(s)ds\|y\| \\
&\geq \frac{\eta}{32^{\alpha-\beta}e^{\lambda}\Gamma(\alpha-\beta)} \int_{\frac{1}{4}}^{\frac{3}{4}} p(s)ds\|y\| \geq \|y\|.
\end{aligned} \tag{3.5}$$

Therefore, for any $y \in P \cap \partial\Omega_R$, we have $\|Ty\| \leq \|y\|$.

According to Lemma 2.6, T has two fixed points $y_1 \in P \cap (\overline{\Omega}_m\backslash\Omega_n)$ and $y_2 \in P \cap (\overline{\Omega}_R\backslash\Omega_n)$ with $m \leq \|y_1\| \leq n \leq \|y_2\| \leq R$. Thus, it follows from Lemma 2.4 that the tempered fractional equation (1.1) has at least two positive solutions satisfying

$$\frac{m\Gamma(\alpha-\beta)t^{\alpha-1}e^{-\lambda t}}{\Gamma(\alpha)} \leq u_1(t) = e^{-\lambda t}I^\beta(e^{\lambda t}y_1(t)) \leq \frac{ne^{\lambda(1-t)}}{\Gamma(\beta+1)},$$

and

$$\frac{n\Gamma(\alpha-\beta)t^{\alpha-1}e^{-\lambda t}}{\Gamma(\alpha)} \leq u_2(t) = e^{-\lambda t}I^\beta(e^{\lambda t}y_2(t)) \leq \frac{Re^{\lambda(1-t)}}{\Gamma(\beta+1)}.$$

Theorem 3.2. *Assume that* (**C1**) *and* (**C3**) *hold and*

$$h_0 = 0, h_\infty = 0,$$

then the tempered fractional equation (1.1) *has at least two positive solutions* u_3, u_4*; moreover, there exist four constants* $A_3, B_3, A_4, B_4 > 0$ *such that*

$$A_3t^{\alpha-1}e^{-\lambda t} \leq u_3(t) \leq B_3e^{\lambda(1-t)}, \quad A_4t^{\alpha-1}e^{-\lambda t} \leq u_4(t) \leq B_4e^{\lambda(1-t)}.$$

Proof. First, notice that $h_0 = 0$, for any $\epsilon > 0$. Let us select $0 < \kappa < \rho$ such that for any $0 < u + v < \left(1 + \frac{e^\lambda}{\Gamma(\beta+1)}\right)\kappa$,

$$h(u, v) \leq \epsilon(u + v).$$

Choose ϵ such that

$$\frac{\epsilon e^\lambda}{\Gamma(\alpha-\beta)}\left(1 + \frac{e^\lambda}{\Gamma(\beta+1)}\right)\int_0^1 p(s)ds \leq 1.$$

Now, let $\Omega_\kappa = \{y \in E : \|y\| < \kappa\}$ and $\partial\Omega_\kappa = \{y \in E : \|y\| = \kappa\}$, then for any $y \in P \cap \partial\Omega_\kappa$, the same as (3.2), we also have

$$I^\beta(e^{\lambda s}y(s)) + y(s) \leq \left(1 + \frac{e^\lambda}{\Gamma(\beta+1)}\right)\kappa, \tag{3.6}$$

which implies that for any $y \in P \cap \partial\Omega_\kappa$, the following estimation is valid:

$$h(I^\beta(e^{\lambda t}y(s)), y(s) \leq \epsilon(I^\beta(e^{\lambda t}y(s)) + y(s)).$$

Consequently, for any $y \in P \cap \partial\Omega_\kappa$, one gets

$$\begin{aligned}
(Ty)(t) &= \int_0^1 H(t, s)p(s)h(I^\beta(e^{\lambda t}y(s)), y(s))ds \\
&\leq \int_0^1 \frac{(1-s)^{\alpha-\beta-1}se^{\lambda s}}{\Gamma(\alpha-\beta)}p(s)h(I^\beta(e^{\lambda t}y(s)), y(s))ds \\
&\leq \frac{e^\lambda}{\Gamma(\alpha-\beta)}\int_0^1 p(s)h(I^\beta(e^{\lambda t}y(s)), y(s))ds \\
&\leq \frac{e^\lambda}{\Gamma(\alpha-\beta)}\int_0^1 p(s)\epsilon(I^\beta(e^{\lambda t}y(s)) + y(s))ds \\
&\leq \frac{e^\lambda}{\Gamma(\alpha-\beta)}\int_0^1 p(s)\epsilon\left(1 + \frac{e^\lambda}{\Gamma(\beta+1)}\right)\kappa ds \\
&\leq \frac{\epsilon e^\lambda}{\Gamma(\alpha-\beta)}\left(1 + \frac{e^\lambda}{\Gamma(\beta+1)}\right)\int_0^1 p(s)ds\kappa \leq \kappa.
\end{aligned} \tag{3.7}$$

(3.7) implies that $\|Ty\| \leq \|y\|, \quad y \in P \cap \partial\Omega_\kappa$.

Next, let $\Omega_\rho = \{y \in E : \|y\| < \rho\}$, and $\partial\Omega_\rho = \{y \in E : \|y\| = \rho\}$, then for any $y \in P \cap \partial\Omega_\rho$, we have

$$I^\beta(e^{\lambda s}s^{\alpha-\beta-1}y(s)) = \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}s^{\alpha-\beta-1}e^{\lambda s}y(s)ds \geq \frac{\Gamma(\alpha-\beta)t^{\alpha-1}\rho}{\Gamma(\alpha)}. \tag{3.8}$$

Thus, it follows from (3.8) that for any $y \in P \cap \partial\Omega_\rho$ and $[\frac{1}{4}, \frac{3}{4}]$, one has

$$\begin{aligned}
&\left(\frac{\Gamma(\alpha-\beta)\left(\frac{1}{4}\right)^{\alpha-1}}{\Gamma(\alpha)} + \left(\frac{1}{4}\right)^{\alpha-\beta}e^{-\frac{3}{4}\lambda}\right)\rho = \frac{\Gamma(\alpha-\beta)\left(\frac{1}{4}\right)^{\alpha-1}}{\Gamma(\alpha)}\rho + \left(\frac{1}{4}\right)^{\alpha-\beta}e^{-\frac{3}{4}\lambda}\rho \\
&= I^\beta(e^{\lambda s}s^{\alpha-\beta-1}y(s)) + y(t) \leq \left(1 + \frac{e^\lambda}{\Gamma(\beta+1)}\right)\rho.
\end{aligned} \tag{3.9}$$

So, for any $y \in P \cap \partial\Omega_\rho$, we have

$$
\begin{aligned}
\|Ty\| &\geq (Ty)\left(\frac{1}{2}\right) = \int_0^1 H\left(\frac{1}{2}, s\right) p(s)h(I^\beta(e^{\lambda s}y(s)), y(s))ds \\
&\geq \frac{2^{-\alpha+\beta}e^{-\frac{1}{2}\lambda}}{\Gamma(\alpha-\beta)} \int_0^1 (1-s)^{\alpha-\beta-1} s e^{\lambda s} p(s)h(I^\beta(e^{\lambda s}y(s)), y(s))ds \\
&\geq \frac{2^{-\alpha+\beta}e^{-\frac{1}{2}\lambda}}{\Gamma(\alpha-\beta)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\alpha-\beta-1} s e^{\lambda s} p(s)h(I^\beta(e^{\lambda s}y(s)), y(s))ds \\
&\geq \frac{2^{-\alpha+\beta}e^{-\frac{1}{2}\lambda}}{\Gamma(\alpha-\beta)} \int_{\frac{1}{4}}^{\frac{3}{4}} (1-s)^{\alpha-\beta-1} s e^{\lambda s} p(s)\varrho\rho ds \geq \frac{\varrho}{8^{\alpha-\beta}e^{\frac{1}{4}\lambda}\Gamma(\alpha-\beta)} \int_{\frac{1}{4}}^{\frac{3}{4}} p(s)ds\rho \\
&\geq \|y\|,
\end{aligned}
\tag{3.10}
$$

which implies that $\|Ty\| \geq \|y\|$ holds and $y \in P \cap \partial\Omega_\rho$.

On the other hand, since $h_\infty = 0$, for any $\epsilon > 0$, there exists $M^* > 0$ such that for any $u + v > M^*$

$$h(u, v) \leq \epsilon(u + v).$$

For the above $\epsilon > 0$, choose a sufficiently small one such that

$$\frac{\epsilon e^\lambda}{\Gamma(\alpha-\beta)}\left(1 + \frac{e^\lambda}{\Gamma(\beta+1)}\right)\int_0^1 p(s)ds < 1,$$

and take

$$R^* = \max\left\{\frac{\frac{e^\lambda}{\Gamma(\alpha-\beta)}\max_{0\leq u+v\leq M^*} h(u,v)\int_0^1 p(s)ds}{1 - \frac{\epsilon e^\lambda}{\Gamma(\alpha-\beta)}\left(1+\frac{e^\lambda}{\Gamma(\beta+1)}\right)\int_0^1 p(s)ds}, \rho + M^*\right\}.$$

Next, let $\Omega_{R^*} = \{y \in E : \|y\| < R^*\}$, and $\partial\Omega_{R^*} = \{y \in E : \|y\| = R^*\}$, then for any $y \in P \cap \partial\Omega_{R^*}$, one has

$$
\begin{aligned}
(Ty)(t) &= \int_0^1 H(t,s)p(s)h(I^\beta(e^{\lambda t}y(s), y(s))ds \leq \int_0^1 \frac{(1-s)^{\alpha-\beta-1}se^{\lambda s}}{\Gamma(\alpha-\beta)} p(s)h(I^\beta(e^{\lambda t}y(s), y(s))ds \\
&\leq \frac{e^\lambda}{\Gamma(\alpha-\beta)}\left\{\int_{0\leq I^\beta(e^{\lambda t}y(s)+y(s)\leq M^*} p(s)h(I^\beta(e^{\lambda t}y(s), y(s))ds\right. \\
&\quad \left. + \int_{M^*\leq I^\beta(e^{\lambda t}y(s)+y(s)\leq\left(1+\frac{e^\lambda}{\Gamma(\beta+1)}\right)R^*} p(s)h(I^\beta(e^{\lambda t}y(s), y(s))ds\right\} \\
&\leq \frac{e^\lambda}{\Gamma(\alpha-\beta)}\left\{\max_{0\leq u+v\leq M^*} h(u,v)\int_0^1 p(s)ds\right. \\
&\quad \left. + \int_{M^*\leq I^\beta(e^{\lambda t}y(s)+y(s)\leq\left(1+\frac{e^\lambda}{\Gamma(\beta+1)}\right)R^*} p(s)\epsilon(I^\beta(e^{\lambda t}y(s) + y(s))ds\right\} \\
&\leq \frac{e^\lambda}{\Gamma(\alpha-\beta)}\left\{\max_{0\leq u+v\leq M^*} h(u,v) + \epsilon\left(1 + \frac{e^\lambda}{\Gamma(\beta+1)}\right)R^*\right)\int_0^1 p(s)ds \leq R^*.
\end{aligned}
\tag{3.11}
$$

Therefore, for any $y \in P \cap \partial\Omega_R$, we have $\|Ty\| \geq \|y\|$.

According to Lemma 2.6, T has two fixed points $y_3 \in P \cap (\overline{\Omega}_\rho \backslash \Omega_\kappa)$ and $y_4 \in P \cap (\overline{\Omega}_{R^*} \backslash \Omega_\rho)$ with $\kappa \le \|y_3\| \le \rho \le \|y_4\| \le R^*$. Thus, it follows from Lemma 2.4 that the tempered fractional equation (1.1) has at least two positive solutions satisfying

$$\frac{\kappa\Gamma(\alpha-\beta)t^{\alpha-1}e^{-\lambda t}}{\Gamma(\alpha)} \le u_3(t) = e^{-\lambda t}I^\beta(e^{\lambda t}y_3(t)) \le \frac{\rho e^{\lambda(1-t)}}{\Gamma(\beta+1)},$$

and

$$\frac{\rho\Gamma(\alpha-\beta)t^{\alpha-1}e^{-\lambda t}}{\Gamma(\alpha)} \le u_4(t) = e^{-\lambda t}I^\beta(e^{\lambda t}y_4(t)) \le \frac{R^* e^{\lambda(1-t)}}{\Gamma(\beta+1)}.$$

4. Examples

The tempered fractional diffusion equation has many important applications, including the tempered fractional Langevin and Vasicek differential equations [48] and the space-time tempered fractional diffusion-wave equation [49]. The new theorems established in the present paper are very useful in the area of tempered fractional calculus. We can find the multiple positive solutions for the singular tempered fractional equations with the lower order tempered fractional derivatives using the proposed theorems.

Let us apply the main results to solve two singular tempered fractional equations with lower order tempered fractional derivative.

Example 4.1. Let $\alpha = \frac{5}{3}$, $\beta = \frac{1}{3}$, $\lambda = 3$, and

$$h(u,v) = \begin{cases} (u+v)^{\frac{1}{2}}, & 0 \le u+v \le 47, \\ \dfrac{(u+v)^2}{47^{\frac{3}{2}}}, & u+v > 47. \end{cases}$$

We consider the multiplicity of positive solutions for the following singular tempered fractional equation with the lower order tempered fractional derivative:

$$\begin{cases} -{}_0^R\mathbb{D}_t^{\frac{5}{3},3}u(t) = \left|\dfrac{1}{2}-t\right|^{\frac{1}{2}} h\left(e^{3t}u(t), {}_0^R\mathbb{D}_t^{\frac{1}{3},3}u(t)\right), 0 \le t \le 1, \\ {}_0^R\mathbb{D}_t^{\frac{1}{3},3}u(0) = 0, \ {}_0^R\mathbb{D}_t^{\frac{1}{3},3}u(1) = 0. \end{cases} \tag{4.1}$$

Conclusion. The singular tempered fractional equation (4.1) has at least two positive solutions u_1, u_2: moreover, there exist four constants $A_1, B_1, A_2, B_2 > 0$ such that

$$A_1 t^{\frac{2}{3}} e^{-3t} \le u_1(t) \le B_1 e^{3(1-t)}, \ A_2 t^{\frac{2}{3}} e^{-3t} \le u_2(t) \le B_2 e^{3(1-t)}.$$

Proof. Here,

$$p(t) = \left|\frac{1}{2}-t\right|^{\frac{1}{2}}, \ h(u,v) = \begin{cases} \dfrac{(u+v)^{\frac{1}{2}}}{600}, & 0 \le u+v \le 47, \\ \dfrac{(u+v)^2}{600 \times 47^{\frac{3}{2}}}, & u+v > 47. \end{cases}$$

Clearly, (**C1**) holds and

$$h_0 = \lim_{u+v\to 0} \frac{h(u,v)}{u+v} = +\infty, \quad h_\infty = \lim_{u+v\to +\infty} \frac{h(u,v)}{u+v} = +\infty.$$

In the following, we verify the condition (**C2**). In fact, take $n = 2$, then

$$\mu = \left[\frac{ne^\lambda}{\Gamma(\alpha-\beta)} \int_0^1 p(s)ds\right]^{-1} = \left[\frac{2e^3}{\Gamma(\frac{4}{3})} \int_0^1 \left|\frac{1}{2} - s\right|^{\frac{1}{2}} ds\right]^{-1} = 0.0079,$$

and for any $0 \le u + v \le \left(1 + \frac{e^\lambda}{\Gamma(\beta+1)}\right)n = 2\left(1 + \frac{e^3}{\Gamma(\frac{4}{3})}\right) = 46.9853$, we have

$$h(u,v) < \frac{47^{\frac{1}{2}}}{600} = 0.0114 < \mu n = 0.0158,$$

which implies that (**C2**) holds.

Consequently, it follows from Theorem 3.1 that the tempered fractional equation (4.1) has at least two positive solutions u_1, u_2; moreover, there exist four constants $A_1, B_1, A_2, B_2 > 0$ such that

$$A_1 t^{\frac{2}{3}} e^{-3t} \le u_1(t) \le B_1 e^{3(1-t)}, \;\; A_2 t^{\frac{2}{3}} e^{-3t} \le u_2(t) \le B_2 e^{3(1-t)}.$$

Example 4.2. Let $\alpha = \frac{5}{3}$, $\beta = \frac{1}{3}$, $\lambda = 3$, and

$$h(u,v) = \begin{cases} 100\sqrt{2}(u+v)^2, & 0 \le u+v \le \dfrac{1}{2}, \\ 50(u+v)^{\frac{1}{2}}, & u+v > \dfrac{1}{2}. \end{cases}$$

We consider the multiplicity of positive solutions for the following singular tempered fractional equation with the lower order tempered fractional derivative

$$\begin{cases} -{}_0^R\mathbb{D}_t^{\frac{5}{3},3} u(t) = \left|\dfrac{1}{2} - t\right|^{\frac{1}{2}} h\left(e^{3t}u(t), {}_0^R\mathbb{D}_t^{\frac{1}{3},3}u(t)\right), 0 \le t \le 1, \\ {}_0^R\mathbb{D}_t^{\frac{1}{3},3}u(0) = 0, \;\; {}_0^R\mathbb{D}_t^{\frac{1}{3},3}u(1) = 0. \end{cases} \tag{4.2}$$

Conclusion. The tempered fractional equation (4.2) has at least two positive solutions u_1, u_2; moreover, there exist four constants $A_1, B_1, A_2, B_2 > 0$ such that

$$A_1 t^{\frac{2}{3}} e^{-3t} \le u_1(t) \le B_1 e^{3(1-t)}, \;\; A_2 t^{\frac{2}{3}} e^{-3t} \le u_2(t) \le B_2 e^{3(1-t)}.$$

Proof. Here

$$p(t) = \left|\frac{1}{2} - t\right|^{\frac{1}{2}}, \;\; h(u,v) = \begin{cases} 100\sqrt{2}(u+v)^2, & 0 \le u+v \le \dfrac{1}{2}, \\ 50(u+v)^{\frac{1}{2}}, & u+v > \dfrac{1}{2}. \end{cases}$$

Clearly, (**C1**) holds and

$$h_0=\lim_{u+v\to 0}\frac{h(u,v)}{u+v}=0,\quad h_\infty=\lim_{u+v\to+\infty}\frac{h(u,v)}{u+v}=0.$$

Take $\rho=2$, and we have

$$\varrho=8^{\alpha-\beta}e^{\frac{1}{4}\lambda}\Gamma(\alpha-\beta)\left[\int_{\frac{1}{4}}^{\frac{3}{4}}p(s)ds\right]^{-1}=8^{\frac{4}{3}}e^{\frac{3}{4}}\Gamma\left(\frac{4}{3}\right)\left[\int_{\frac{1}{4}}^{\frac{3}{4}}=\left|\frac{1}{2}-s\right|^{\frac{1}{2}}ds\right]^{-1}=15.1238.$$

For any

$$0.8184=2\left(\frac{\Gamma(\frac{4}{3})\left(\frac{1}{4}\right)^{\frac{2}{3}}}{\Gamma(\frac{5}{3})}+\left(\frac{1}{4}\right)^{\frac{4}{3}}e^{-\frac{9}{4}}\right)\le u+v\le 2\left(1+\frac{e^3}{\Gamma(\frac{4}{3})}\right)=46.9844,$$

one has

$$h(u,v)\ge 50\times 0.8184^{\frac{1}{2}}=45.232>\varrho\rho=30.2476,$$

which implies that (**C3**) holds.

Consequently, it follows from Theorem 3.2 that the tempered fractional equation (4.2) has at least two positive solutions u_1,u_2; moreover, there exist four constants $A_1,B_1,A_2,B_2>0$ such that

$$A_1t^{\frac{2}{3}}e^{-3t}\le u_1(t)\le B_1e^{3(1-t)},\ \ A_2t^{\frac{2}{3}}e^{-3t}\le u_2(t)\le B_2e^{3(1-t)}.$$

5. Conclusions

This work studies the multiplicity of positive solutions for a class of singular tempered fractional equations with the lower order tempered fractional derivative. By applying reducing order technique and fixed point theorem, some new results of existence of the multiple positive solutions for the equation are established. The interesting points are the nonlinearity contains the lower order tempered fractional derivative and the weight function can have infinitely many singular points in $[0,1]$. However, in this study, the conditions $\alpha\in(1,2],\beta\in(0,1)$ with $\alpha-\beta>1$ are required; if $0<\alpha-\beta<1$ or h has singularity at space variables, these interesting problems are still worth future studying.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Xinguang Zhang is a guest editor of the special issue for ERA and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

References

1. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations in North-Holland Mathematics Studies*, Elsevier, **204** (2006).
2. J. He, X. Zhang, L. Liu, Y. Wu, Y. Cui, A singular fractional Kelvin-Voigt model involving a nonlinear operator and their convergence properties, *Boundary Value Probl.*, **2019** (2019), 112. https://doi.org/10.1186/s13661-019-1228-7
3. T. Ren, S. Li, X. Zhang, L. Liu,Maximum and minimum solutions for a nonlocal p-Laplacian fractional differential system from eco-economical processes, *Boundary Value Probl.*, **2017** (2017), 118. https://doi.org/10.1186/s13661-017-0849-y
4. J. Wu, X. Zhang, L. Liu, Y. Wu, Y. Cui, Convergence analysis of iterative scheme and error estimation of positive solution for a fractional differential equation, *Math. Modell. Anal.*, **23** (2018), 611–626. https://doi.org/10.3846/mma.2018.037
5. X. Zhang, Y. Wu, L. Caccetta, Nonlocal fractional order differential equations with changing-sign singular perturbation, *Appl. Math. Modell.*, **39** (2015), 6543–6552.https://doi.org/10.1016/j.apm.2015.02.005
6. J. Wu, X. Zhang, L. Liu, Y. Wu, Y. Cui, The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity, *Boundary Value Probl.*, **2018** (2018), 82. https://doi.org/10.1186/s13661-018-1003-1
7. K. Owolabi, Riemann-Liouville fractional derivative and application to model chaotic differential equations, *Progr. Fract. Differ. Appl.*, **4** (2018), 99–110. https://doi.org/10.18576/pfda/040204
8. R. Almeida,A Caputo fractional derivative of a function with respect to another function, *Commun. Nonlinear Sci. Numer. Simul.*, **44** (2017), 460–481. https://doi.org/10.1016/j.cnsns.2016.09.006
9. X. Zhang, D. Kong, H. Tian, Y. Wu, B. Wiwatanapataphee, An upper-lower solution method for the eigenvalue problem of Hadamard-type singular fractional differential equation, *Nonlinear Anal.-Model. Control*, **27** (2022), 789–802. https://doi.org/10.15388/namc.2022.27.27491
10. X. Zhang, L. Yu, J. Jiang, Y. Wu, Y. Cui, Positive solutions for a weakly singular Hadamard-type fractional differential equation with changing-sign nonlinearity, *J. Funct. Spaces*, **2020** (2020), 5623589. https://doi.org/10.1155/2020/5623589
11. X. Zhang, P. Xu , Y. Wu, B. Wiwatanapataphe, The uniqueness and iterative properties of solutions for a general Hadamard-type singular fractional turbulent flow model, *Nonlinear Anal.-Model. Control*, **27** (2022), 428–444. https://doi.org/10.15388/namc.2022.27.25473
12. Á. Cartea, D. Negrete, Fluid limit of the continuous-time random walk with general Lévy jump distribution functions, *Phys. Rev. E*, **76** (2007), 041105. https://doi.org/10.1103/PhysRevE.76.041105

13. F. Sabzikar, M. M. Meerschaert, J. Chen, Tempered fractional calculus, *J. Comput. Phys.*, **293** (2015), 14–28. https://doi.org/10.1016/j.jcp.2014.04.024
14. B. Baeumera, M. M. Meerschaert, Tempered stable Lévy motion and transient super-diffusion, *J. Comput. Appl. Math.*, **233** (2010), 2438–2448. https://doi.org/10.1016/j.cam.2009.10.027
15. M. Meerschaert, Y. Zhang, B. Baeumer, Tempered anomalous diffusions in heterogeneous systems, *Geophys. Res. Lett.*, **35** (2008), 17403–17407. https://doi.org/10.1029/2008GL034899
16. C. Li, W. Deng, High order schemes for the tempered fractional diffusion equations, *Adv. Comput. Math.*, **42** (2016), 543–572. https://doi.org/10.1007/s10444-015-9434-z
17. G. Yuan, D. Ding, J. Duan, W. Lu, F. Wu, Total value adjustment of Bermudan option valuation under pure jump Lévy fluctuations, *Chaos*, **32** (2022), 023127. https://doi.org/10.1063/5.0072500
18. X. Zhang, P. Chen, H. Tian, Y. Wu, Upper and lower solution method for a singular tempered fractional equation with a p-Laplacian operator, *Fractal Fract.*, **7** (2023), 522. https://doi.org/10.3390/fractalfract7070522
19. X. Zhang, P. Chen, H. Tian,Y. Wu, The iterative properties for positive solutions of a tempered fractional equation, *Fractal Fract.*, **7** (2023), 761. https://doi.org/10.3390/fractalfract7100761
20. D. Chang, X. Duong, J. Li, W. Wang, Q. Wu, An explicit formula of Cauchy-Szegö kernel for quaternionic Siegel upper half space and applications, *Indiana Univ. Math. J.*, **70** (2021), 2451–2477. https://doi.org/10.1512/iumj.2021.70.8732
21. M. Yang, Z. Fu, S. Liu, Analyticity and existence of the Keller-Segel-Navier-Stokes equations in critical Besov spaces, *Adv. Nonlinear Stud.*, **18** (2018), 517–535. https://doi.org/10.1515/ans-2017-6046
22. M. Yang, Z. Fu, J. Sun, Existence and Gevrey regularity for a two-species chemotaxis system in homogeneous Besov spaces, *Sci. China Math.*, **60** (2017), 1837–1856. https://doi.org/10.1007/s11425-016-0490-y
23. J. Cao, D.Chang, Z. Fu, D. Yang, Real interpolation of weighted tent spaces, *Appl. Anal.*, **59** (2016), 2415–2443. https://doi.org/10.1080/00036811.2015.1091924
24. D. Chang, Z. Fu , D. Yang, S. Yang, Real-variable characterizations of Musielak-Orlicz-Hardy spaces associated with Schrödinger operators on domains, *Math. Methods Appl. Sci.*, **39** (2016), 533–569. https://doi.org/10.1002/mma.3501
25. P. Chen, X. Duong, J. Li, Q. Wu, Compactness of Riesz transform commutator on stratified Lie groups, *J. Funct. Anal.*, **277** (2019), 1639–1676. https://doi.org/10.1016/j.jfa.2019.05.008
26. S. Shi, Z. Fu, S. Lu, On the compactness of commutators of Hardy operators, *Pac. J. Math.*, **307** (2020), 239–256. https://doi.org/10.2140/pjm.2020.307.239
27. X. Duong, M. Lacey, J. Li, B. Wick, Q. Wu, Commutators of Cauchy-Szego type integrals for domains in C^n with minimal smoothness, *Indiana Univ. Math. J.*, **70** (2021), 1505–1541. https://doi.org/10.48550/arXiv.1809.08335
28. Z. Fu, S. Gong, S. Lu, W. Yuan, Weighted multilinear Hardy operators and commutators, *Forum Math.*, **27** (2015), 2825–2852. https://doi.org/10.1515/forum-2013-0064
29. L. Gu, Z. Zhang, Riemann boundary value problem for harmonic functions in Clifford analysis, *Math. Nachr.*, **287** (2014), 1001–1012. https://doi.org/10.1002/mana.201100302

30. G. Wang, Z. Liu, L. Chen, Classification of solutions for an integral system with negative exponents, *Complex Var. Elliptic Equations*, **64** (2019), 204–222. https://doi.org/10.1155/2020/8392397

31. Y. Wu, W. Chen, On strong indefinite Schrödinger equations with non-periodic potential, *J. Appl. Anal. Comput.*, **13** (2023), 1–10. https://doi.org/10.11948/20210036

32. M. Yang, Z. Fu, S. Liu, Analyticity and existence of the Keller-Segel-Navier-Stokes equations in critical Besov spaces, *Adv. Nonlinear Stud.*, **18** (2018), 517–535. https://doi.org/10.1515/ans-2017-6046

33. M. Yang, Z. Fu, J. Sun, Existence and large time behavior to coupled chemotaxis-fluid equations in Besov-Morrey spaces, *J. Differ. Equations*, **266** (2019), 5867–5894. https://doi.org/10.1016/j.jde.2018.10.050

34. S. Yang, D. Chang, D. Yang, Z. Fu, Gradient estimates via rearrangements for solutions of some Schrödinger equations, *Anal. Appl.*, **16** (2018), 339–361. https://doi.org/10.1142/S0219530517500142

35. W. Chen, Z. Fu, L. Grafakos, Y. Wu, Fractional Fourier transforms on L^p and applications, *Appl. Comput. Harmon. Anal.*, **55** (2021), 71–96. https://doi.org/10.1016/j.acha.2021.04.004

36. X. Zhang, J. Jiang, L. Liu, Y. Wu, Extremal solutions for a class of tempered fractional turbulent flow equations in a porous medium, *Math. Probl. Eng.*, **2020** (2020), 2492193. https://doi.org/10.1155/2020/2492193

37. B. Dong, Z. Fu, J. Xu, Riesz-Kolmogorov theorem in variable exponent Lebesgue spaces and its applications to Riemann-Liouville fractional differential equations, *Sci. China Math.*, **61** (2018), 1807–1824. https://doi.org/10.1007/s11425-017-9274-0

38. S. Shi, L. Zhang, G. Wang, Fractional non-linear regularity, potential and balayage, *J. Geom. Anal.*, **32** (2022), 221. https://doi.org/10.1007/s12220-022-00956-6

39. X. Guo, Z.Fu, An initial and boundary value problem of fractional Jeffreys' fluid in a porous half spaces, *Comput. Math. Appl.*, **78** (2019), 1801–1810. https://doi.org/10.1016/j.camwa.2015.11.020

40. S. Shi, Some notes on supersolutions of fractional p-Laplace equation, *J. Math. Anal. Appl.*, **463** (2018), 10521074. https://doi.org/10.1016/j.jmaa.2018.03.064

41. S. Shi, L. Zhang, Dual characterization of fractional capacity via solution of fractional p-Laplace equation, *Math. Nachr.*, **2020** (2020), 2233–2247. https://doi.org/10.1002/mana.201800438

42. S. Shi, Z. Zhai, L. Zhang, Characterizations of the viscosity solution of a nonlocal and nonlinear equation induced by the fractional p-Laplace and the fractional p-convexity, *Adv. Calc. Var.*, **17** (2023), 195–207. https://doi.org/10.1515/acv-2021-0110

43. H. Tang, G. Wang, Limiting weak type behavior for multilinear fractional integrals, *Nonlinear Anal.*, **2020** (2020), 197. https://doi.org/10.1016/j.na.2020.111858

44. H. Xu, L. Zhang, G. Wang, Some new inequalities and extremal solutions of a Caputo-Fabrizio fractional Bagley-Torvik differential equation, *Fractal Fract.*, **6** (2022), 488. https://doi.org/10.3390/fractalfract6090488

45. X. Zhang, L. Yu, J. Jiang, Y. Wu, Y. Cui, Solutions for a singular Hadamard-type fractional differential equation by the spectral construct analysis, *J. Funct. Spaces*, **2020** (2020), 8392397. https://doi.org/10.1155/2020/8392397

46. Y. Yang, Q. Wu, S. Jhang, Q. Kang, Approximation theorems associated with multidimensional fractional fouried reansform and applications in Laplace and heat equations, *Fractal. Fract.*, **6** (2022), 625. https://doi.org/10.3390/fractalfract6110625

47. D. J. Guo, V. Lakshmikantham, *Nonlinear Problems in Abstract Cone*, Academic Press Inc, New York, 1988.

48. N. A. Obeidat, M. S. Rawashdeh, Theories of tempered fractional calculus applied to tempered fractional Langevin and Vasicek equations, *Math. Methods Appl. Sci.*, **46** (2023), 8582–8595. https://doi.org/10.1002/mma.9003

49. Z. Qiu, X. Cao, Second-order numerical methods for the tempered fractional diffusion equations, *Adv. Differ. Equations*, **2019** (2019), 1–23. https://doi.org/10.1186/s13662-019-2417-5