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http://www.aimspress.com/journal/era

ERA, 32(3): 1873–1900.
DOI: 10.3934/era.2024086
Received: 31 August 2023
Revised: 09 February 2024
Accepted: 19 February 2024
Published: 04 March 2024

Research article

A stochastic Gilpin-Ayala nonautonomous competition model driven by mean-reverting OU process with finite Markov chain and Lévy jumps

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Abstract: The Ornstein-Uhlenbeck (OU) process was used to simulate random perturbations in the environment. Considering the influence of telegraph noise and jump noise, a stochastic Gilpin-Ayala nonautonomous competition model driven by the mean-reverting OU process with finite Markov chain and Lévy jumps was established, and the asymptotic behaviors of the stochastic Gilpin-Ayala nonautonomous competition model were studied. First, the existence of the global solution of the stochastic Gilpin-Ayala nonautonomous competition model was proven by the appropriate Lyapunov function. Second, the moment boundedness of the solution of the stochastic Gilpin-Ayala nonautonomous competition model was discussed. Third, the existence of the stationary distribution of the solution of the stochastic Gilpin-Ayala nonautonomous competition model was obtained. Finally, the extinction of the stochastic Gilpin-Ayala nonautonomous competition model was proved. The theoretical results were verified by numerical simulations.

Keywords: stochastic Gilpin-Ayala nonautonomous competition model; moment boundedness of solution; Ornstein-Uhlenbeck (OU) process; the existence of stationary distribution; extinction

1. Introduction

As a famous model in population dynamics, the classical logistic model proposed by Verhulst [1] in 1838 has attracted the attention and research of many experts and scholars. However, in the classical logistic model, the exponential growth for species is linear, and the results obtained by this linear hypothesis are quite different from the survival of species in real life. Therefore, in order to describe the real problems more accurately, Ayala and Gilpin [2] proposed the following model in 1973:

$$\mathrm{d}x(t) = x(t)\left(r - ax^{\theta}(t)\right)\mathrm{d}t,$$

where $x(t)$ is the population size at t moments, r is the intrinsic growth rate, $a > 0$ is the intraspecific competition coefficient, $\frac{a}{r}$ is the environmental carrying capacity, and θ is the positive parameter of the modified classical logistic model. In nature, any biological population will interact with other populations. According to the different interactions, the population system can be divided into three types, which include competition, predator-prey, and mutualism [3]. Therefore, we consider the following two-species Gilpin-Ayala competition model

$$\begin{cases} \mathrm{d}x(t) = x(t)\left(r_1 - a_{11}x^{\theta_1}(t) - a_{12}y(t)\right)\mathrm{d}t \\ \mathrm{d}y(t) = y(t)\left(r_2 - a_{21}x(t) - a_{22}y^{\theta_2}(t)\right)\mathrm{d}t, \end{cases} \tag{1.1}$$

where $a_{ii} > 0$ $(i = 1, 2)$ denotes intraspecific competition coefficients, $a_{ij} > 0$ $(i, j = 1, 2, i \neq j)$ denotes interspecific competition coefficients, and θ_i $(i = 1, 2)$ denotes the positive parameters of the modified classical logistic model.

On the other hand, the development of the population will be disturbed by various uncertain environmental factors, which will change its growth state in a short time. Therefore, for the study of the population, we also need to consider the influence of random factors. Random disturbances can be roughly divided into three categories: white noise, telegraph noise, and jump noise. Due to these different degrees of interference, the birth rate, death rate, competition coefficient, and other parameters of the population system also show a certain degree of random fluctuations [4] .

For the characterization of white noise, the accepted method [5–7] is to assume that the intrinsic growth rate r_1, r_2 in model (1.1) is linearly disturbed by Gaussian white noise, that is, $r_1(t) = r_1 + \sigma_1 \frac{\mathrm{d}B_1(t)}{\mathrm{d}t}, r_2(t) = r_2 + \sigma_2 \frac{\mathrm{d}B_2(t)}{\mathrm{d}t}$. However, in a randomly changing environment, it is unreasonable to use a linear function of Gaussian white noise to simulate parameter perturbations [8]. For any time interval $[0, t]$, let $\langle r_i(t)\rangle$ be the time average of $r_i(t)$ (i = 1,2). There is

$$\langle r_i(t)\rangle := \frac{1}{t}\int_0^t r_i(s)\mathrm{d}s = r_i + \frac{\alpha}{t}B(t) \sim N(r_i, \frac{\alpha^2}{t}), i = 1, 2,$$

where $N(\cdot,\cdot)$ is a one-dimensional normal distribution. Obviously, we get that the average growth rate $\langle r_i(t)\rangle, i = 1, 2$ has a variance $\frac{\alpha^2}{t}$ on $[0, t]$, which tends to infinity at $t \to 0^+$. This means that random fluctuations of the growth rate $r_i(t), i = 1, 2$ will become very large in a small time interval.

Therefore, we consider another method of simulating random perturbations, that is, the intrinsic growth rate r_1, r_2 of model (1.1) satisfies the mean-reverting Ornstein-Uhlenbeck(OU) process [9–11] in the form of

$$\mathrm{d}r_1(t) = \beta_1[\bar{r}_1 - r_1(t)]\mathrm{d}t + \sigma_1\mathrm{d}B_1(t), \mathrm{d}r_2(t) = \beta_2[\bar{r}_2 - r_2(t)]\mathrm{d}t + \sigma_2\mathrm{d}B_2(t), \tag{1.2}$$

where β_1,β_2 is the reversion speed, σ_1, σ_2 is the intensity of environmental fluctuation, $\bar{r}_1, \bar{r}_2$ is the mean reversion level, and $\beta_1, \beta_2, \sigma_1, \sigma_2 > 0$.

Integrating from 0 to t on the both sides of the OU process (1.2), we get

$$\begin{aligned} r_1(t) &= \bar{r}_1 + [r_1(0) - \bar{r}_1]e^{-\beta_1 t} + \sigma_1\int_0^t e^{-\beta_1(t-s)}\mathrm{d}B_1(s), \\ r_2(t) &= \bar{r}_2 + [r_2(0) - \bar{r}_2]e^{-\beta_2 t} + \sigma_2\int_0^t e^{-\beta_2(t-s)}\mathrm{d}B_2(s), \end{aligned} \tag{1.3}$$

where $r_1(0), r_2(0)$ are the initial values of the OU process $r_1(t), r_2(t)$. It is easy to have that the expectation and variance of $r_1(t), r_2(t)$ are

$$\mathbb{E}[r_1(t)] = \bar{r}_1 + [r_1(0) - \bar{r}_1] e^{-\beta_1 t}, Var[r_1(t)] = \frac{\sigma_1^2}{2\beta_1}\left(1 - e^{-2\beta_1 t}\right),$$
$$\mathbb{E}[r_2(t)] = \bar{r}_2 + [r_2(0) - \bar{r}_2] e^{-\beta_2 t}, Var[r_2(t)] = \frac{\sigma_2^2}{2\beta_2}\left(1 - e^{-2\beta_2 t}\right).$$

then $r_1(t)$ follows the Gaussian distribution $N\left(\bar{r}_1 + [r_1(0) - \bar{r}_1] e^{-\beta_1 t}, \frac{\sigma_1^2}{2\beta_1}\left(1 - e^{-2\beta_1 t}\right)\right)$. According to the property of Brownian motion, we obtain that $\sigma_1 \int_0^t e^{-\beta_1(t-s)} \mathrm{d}B_1(s)$ follows the Gaussian distribution $N\left(0, \frac{\sigma_1^2}{2\beta_1}\left(1 - e^{-2\beta_1 t}\right)\right)$. Similarly, $r_2(t) \sim N\left(\bar{r}_2 + [r_2(0) - \bar{r}_2] e^{-\beta_2 t}, \frac{\sigma_2^2}{2\beta_2}\left(1 - e^{-2\beta_2 t}\right)\right)$ and $\sigma_2 \int_0^t e^{-\beta_2(t-s)} \mathrm{d}B_2(s) \sim N\left(0, \frac{\sigma_2^2}{2\beta_2}\left(1 - e^{-2\beta_2 t}\right)\right)$.

However, in the real world, in addition to small disturbances in the environment, the population is also disturbed by telegraph noises and jump noises. The telegraph noise can be explained as a switching between two or more states of the environment [12–16], and the regime switching can be modeled by a right-continuous Markov chain $(\xi(t))_{t\geq 0}$ taking values in a finite state space $S = \{1, 2, \cdots, p\}$ [17, 18]. The jump noise can change the survival state of the population in an instant, for example, earthquakes, hurricanes, epidemics and so on [19, 20]. The introduction of levy jumps in the basic model is a reasonable way to describe these phenomena [19, 20]. At the same time, we also consider that some biological parameters change with time. Therefore, it is not reasonable to consider the autonomous system only [3], so we consider the following stochastic Gilpin-Ayala nonautonomous competition model driven by the mean-reverting OU process with finite Markov chain and Lévy jumps:

$$\begin{cases} \mathrm{d}x(t) = x(t^-)\left[\left(r_1(t) - a_{11}(t)x^{\theta_1}(t^-) - a_{12}(t)y(t^-)\right)\mathrm{d}t + \int_Z \gamma_1(\xi(t), z) N(\mathrm{d}t, \mathrm{d}z)\right] \\ \mathrm{d}y(t) = y(t^-)\left[\left(r_2(t) - a_{21}(t)x(t^-) - a_{22}(t)y^{\theta_2}(t^-)\right)\mathrm{d}t + \int_Z \gamma_2(\xi(t), z) N(\mathrm{d}t, \mathrm{d}z)\right] \\ \mathrm{d}r_1(t) = \beta_1[\bar{r}_1 - r_1(t)]\mathrm{d}t + \sigma_1 \mathrm{d}B_1(t) \\ \mathrm{d}r_2(t) = \beta_2[\bar{r}_2 - r_2(t)]\mathrm{d}t + \sigma_2 \mathrm{d}B_2(t), \end{cases} \tag{1.4}$$

where $x(t^-), y(t^-)$ are the left limit of $x(t), y(t)$, $B_i(t)(i = 1, 2)$ are independent standard Brownian motions defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$, and $a_{ij}(t)(i, j = 1, 2)$ are nonnegative continuous bounded functions defined on $[0, \infty)$. $\xi(t)$ is a continuous time Markov chain taking values in a finite state space $S = \{1, 2, \cdots, p\}$. N is a Poisson counting measure with characteristic measure v with $v(Z) < \infty$, and Z is a measurable subset of $(0, \infty)$. $\tilde{N}$ represents a compensating random measure of Poisson random measure N, defined as $\tilde{N}(\mathrm{d}t, \mathrm{d}z) = N(\mathrm{d}t, \mathrm{d}z) - v(\mathrm{d}z)\mathrm{d}t$. In order to adapt to the corresponding biological significance, we assume that for all $k \in S, z \in Z$, the jump diffusion coefficient satisfies $\gamma_1(k, z) > -1, \gamma_2(k, z) > -1$. The elements of the generator matrix $Q = \left(q_{ij}\right)_{p\times p}$ of the Markov chain $\xi(t)$ satisfy

$$\mathbb{P}(\xi(t + \triangle t) = j \mid \xi(t) = i) = \begin{cases} q_{ij} \triangle t + o(\triangle t), j \neq i \\ 1 + q_{ij} \triangle t + o(\triangle t), j = i, \end{cases}$$

where $\triangle t > 0$. When $i \neq j$, $q_{ij} > 0$ denotes the transition rate from state i to state j, and $\sum_{j=1}^{p} q_{ij} = 0$.

Suppose that the Markov chain $\xi(t)$ is irreducible, which means that the Markov chain $\xi(t)$ has a unique stationary distribution $\pi = \left(\pi_1, \pi_2, \cdots, \pi_p\right) \in \mathbb{R}^{1\times p}$ and satisfies $\pi Q = 0$, where $\sum_{i=1}^{p} \pi_i = 1, \pi_i > 0, i \in S$.

This paper mainly studies the dynamic behaviors of a stochastic Gilpin-Ayala nonautonomous competition model (1.4) driven by the mean-reverting OU process with finite Markov chain and Lévy jumps. In summary, the Gilpin Ayala model studied in this article assumes that the exponential growth of species in ecosystems is nonlinear first, which is a more realistic population model compared to the classical logistic model. Second, most of the current methods of characterizing white noise assume that the population parameters are linearly disturbed by Gaussian white noise, and the research on the properties of such random models is also very comprehensive [16–24]. The mean-reverting OU process used in this paper is a more reasonable improvement of the above method of white noise characterization, but there are few studies on the behavior of this kind of stochastic model [25–28]. In addition, we also consider the effects of telegraph noise and Lévy noises on the survival of the population, combined with the characteristics of some biological parameters changing with time, so we construct model (1.4) in this paper. As far as we know, no experts and scholars have studied the properties of this kind of model, so it is very meaningful to study the dynamic behaviors of this model.

For convenience, the following marks are taken in this article.

If $f(t)$ is a bounded continuous function on $[0, \infty)$, let

$$f^u = \sup_{t\in[0,\infty)} f(t), f^l = \inf_{t\in[0,\infty)} f(t).$$

For sequence c_{ij} $(1 \le i, j \le n)$, let

$$\check{c} = \max_{1\le i,j\le n} c_{ij}, \hat{c} = \min_{1\le i,j\le n} c_{ij}.$$

2. Existence and uniqueness of global solution

Assumption 2.1. For all state $i \in S, q > 0$, and $k \in \{1, 2, ..., n\}$, there exists a constant $c > 0$ and the following inequalities hold

$$(1) \int_Z \left[|\ln(1+\gamma_k(i,z))| \vee (\ln(1+\gamma_k(i,z)))^2\right] \nu(\mathrm{d}z) < c,$$

$$(2) \int_Z |\gamma_k(i,z)|^q \nu(\mathrm{d}z) < c,$$

$$(3) \int_Z |(1+\gamma_k(i,z))^q - 1| \nu(\mathrm{d}z) < c.$$

Theorem 2.1. If Assumption 2.1 holds, for any initial value $(x(0), y(0), r_1(0), r_2(0)) \in \mathbb{R}_+^2 \times \mathbb{R}^2$ and $\xi(0) \in S$, there exists a unique solution $(x(t), y(t), r_1(t), r_2(t))$ of model (1.4) on $t \ge 0$, and it will remain in $\mathbb{R}_+^2 \times \mathbb{R}^2$ with probability one.

Proof Noting that all the coefficients of model (1.4) satisfy the local Lipschitz condition, for any initial value $(x(0), y(0), r_1(0), r_2(0)) \in \mathbb{R}_+^2 \times \mathbb{R}^2$ and $\xi(0) \in S$, the system has a unique local solution $(x(t), y(t), r_1(t), r_2(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time of the solution [30]. Therefore, to prove the solution $(x(t), y(t), r_1(t), r_2(t))$ is global, it is needed to prove that $\tau_e = \infty$ with probability one only. Hence, we take a sufficiently large $p_0 > 0$ such that each component of $\left(x(0), y(0), e^{r_1(0)}, e^{r_2(0)}\right)$ falls within $\left[\frac{1}{p_0}, p_0\right]$. For each integer p_0 greater than p, we define the stopping time

$$\tau_p = \inf\left\{t \in (0, \tau_e) : x(t) \notin \left(\frac{1}{p}, p\right) \text{ or } y(t) \notin \left(\frac{1}{p}, p\right) \text{ or } e^{r_1(t)} \notin \left(\frac{1}{p}, p\right) \text{ or } e^{r_2(t)} \notin \left(\frac{1}{p}, p\right)\right\}, \tag{2.1}$$

Obviously, τ_p is monotonically increasing as p increases. For convenience, let $\tau_\infty = \lim_{p\to\infty} \tau_p$, then $\tau_\infty \leq \tau_e$ holds with probability one. Therefore, if $\tau_\infty = \infty$, then $\tau_e = \infty$ holds. In the following, we use the proof by contradiction to prove that $\tau_\infty = \infty$ is true. Suppose $\tau_\infty = \infty$ does not hold with probability one, then there exist constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}(\tau_\infty \leq T) > \varepsilon$, so there exists $p_1 \geq p_0$ such that

$$\mathbb{P}(\tau_p \leq T) \geq \varepsilon, \text{for all } p \geq p_1. \tag{2.2}$$

Define a C^2-function V on $\mathbb{R}_+^2 \times \mathbb{R}^2$

$$V(x(t), y(t), r_1(t), r_2(t)) = x(t) - 1 - \ln x(t) + y(t) - 1 - \ln y(t) + \frac{r_1^4(t)}{4} + \frac{r_2^4(t)}{4}.$$

When $x, y > 0$, there are inequalities $x - 1 \geq \ln x, y - 1 \geq \ln y$, so V is a nonnegative function. Using Itô formula, we can get

$$\begin{aligned} dV = {} & LV dt + \sigma_1 r_1^3 dB_1(t) + \sigma_2 r_2^3 dB_2(t) + \int_Z [x\gamma_1(\xi, z) - \ln(1 + \gamma_1(\xi, z))] \tilde{N}(dt, dz) \\ & + \int_Z [y\gamma_2(\xi, z) - \ln(1 + \gamma_2(\xi, z))] \tilde{N}(dt, dz), \end{aligned} \tag{2.3}$$

where

$$\begin{aligned} LV = {} & (x - 1)(r_1 - a_{11}(t)x^{\theta_1} - a_{12}(t)y) + (y - 1)(r_2 - a_{21}(t)x - a_{22}(t)y^{\theta_2}) + \beta_1 r_1^3(\bar{r}_1 - r_1) \\ & + \frac{3}{2}\sigma_1^2 r_1^2 + \beta_2 r_2^3(\bar{r}_2 - r_2) + \frac{3}{2}\sigma_2^2 r_2^2 + \int_Z [x\gamma_1(\xi, z) - \ln(1 + \gamma_1(\xi, z))] \nu(dz) \\ & + \int_Z [y\gamma_2(\xi, z) - \ln(1 + \gamma_2(\xi, z))] \nu(dz). \end{aligned} \tag{2.4}$$

Thus, there exists a constant $N > 0$ such that

$$LV \leq N. \tag{2.5}$$

Substituting Eq (2.5) into Eq (2.3), we have

$$\begin{aligned} dV \leq {} & N dt + \sigma_1 r_1^3 dB_1(t) + \sigma_2 r_2^3 dB_2(t) + \int_Z [x\gamma_1(\xi, z) - \ln(1 + \gamma_1(\xi, z))] \tilde{N}(dt, dz) \\ & + \int_Z [y\gamma_2(\xi, z) - \ln(1 + \gamma_2(\xi, z))] \tilde{N}(dt, dz). \end{aligned} \tag{2.6}$$

Taking the integral from 0 to $\tau_p \wedge T$ on both sides of Eq (2.6) and taking the expectation, we obtain

$$\begin{aligned} \mathbb{E}V\left(x(\tau_p \wedge T), y(\tau_p \wedge T), r_1(\tau_p \wedge T), r_2(\tau_p \wedge T)\right) & \leq V(x(0), y(0), r_1(0), r_2(0)) + N\mathbb{E}\left(\tau_p \wedge T\right) \\ & \leq V(x(0), y(0), r_1(0), r_2(0)) + NT. \end{aligned} \tag{2.7}$$

When $p \geq p_1$, let $\Omega_p = \left\{\tau_p \leq T\right\}$. From Eq (2.2), we can obtain $\mathbb{P}\left(\Omega_p\right) \geq \varepsilon$. From the definition of τ_p for each $\omega \in \Omega_p$, such that one of $x\left(\tau_p, \omega\right), y\left(\tau_p, \omega\right), e^{r_1\left(\tau_p,\omega\right)}, e^{r_2\left(\tau_p,\omega\right)}$ is equal to p or $\frac{1}{p}$ so that $V\left(x\left(\tau_p, \omega\right), y\left(\tau_p, \omega\right), r_1\left(\tau_p, \omega\right), r_2\left(\tau_p, \omega\right)\right)$ is not less than $(p - 1 - \ln p), \left(\frac{1}{p} - 1 + \ln p\right)$, or $\frac{1}{4}(\ln p)^4$, then

$$V\left(x\left(\tau_p, \omega\right), y\left(\tau_p, \omega\right), r_1\left(\tau_p, \omega\right), r_2\left(\tau_p, \omega\right)\right) \geq \min\left\{p - 1 - \ln p, \frac{1}{p} - 1 + \ln p, \frac{1}{4}(\ln p)^4\right\},$$

According to Eq (2.7), we can get

$$\begin{aligned} V(x(0), y(0), r_1(0), r_2(0)) + NT &\geq \mathbb{E}\left[I_{\Omega_p}(\omega) V\left(x\left(\tau_p, \omega\right), y\left(\tau_p, \omega\right), r_1\left(\tau_p, \omega\right), r_2\left(\tau_p, \omega\right)\right)\right] \\ &\geq \varepsilon \min\left\{p - 1 - \ln p, \frac{1}{p} - 1 + \ln p, \frac{1}{4}(\ln p)^4\right\}, \end{aligned}$$

where $I_{\Omega_p}(\omega)$ represents the indicator function of Ω_p. Let $p \to \infty$, then $\infty > V(x(0), y(0), r_1(0), r_2(0)) + NT = \infty$; we have a contradiction. Therefore, $\tau_\infty = \infty$ holds with probability one. Theorem 2.1 is proved.

3. Moment boundedness of solution

Theorem 3.1. If Assumption 2.1 holds, for any initial value $(x(0), y(0), r_1(0), r_2(0)) \in \mathbb{R}_+^2 \times \mathbb{R}^2$, and $\xi(0) \in S$, the solution $(x(t), y(t), r_1(t), r_2(t))$ of model (1.4) has the property that

$$\mathbb{E}[x(t)]^q \leq \kappa(q), \ \mathbb{E}[y(t)]^q \leq \kappa(q),$$

for any $q > 0$, where $\kappa(q)$ is a continuous function with respect to q. That is to say, the q th moment of the solution $(x(t), y(t), r_1(t), r_2(t))$ is bounded.

Proof For any $q \geq 2$, define a nonnegative C^2-function $V : \mathbb{R}_+^2 \times \mathbb{R}^2 \to \mathbb{R}_+$,

$$V(x(t), y(t), r_1(t), r_2(t)) = \frac{x^q(t)}{q} + \frac{y^q(t)}{q} + \frac{r_1^{2q}(t)}{2q} + \frac{r_2^{2q}(t)}{2q}.$$

Applying Itô formula to function V, we obtain

$$\begin{aligned} \mathrm{d}V = LV\mathrm{d}t + \sigma_1 r_1^{2q-1}\mathrm{d}B_1(t) + \sigma_2 r_2^{2q-1}\mathrm{d}B_2(t) + \int_Z \left(\frac{(x + x\gamma_1(\xi, z))^q}{q} - \frac{x^q}{q}\right)\tilde{N}(\mathrm{d}t, \mathrm{d}z) \\ + \int_Z \left(\frac{(y + y\gamma_2(\xi, z))^q}{q} - \frac{y^q}{q}\right)\tilde{N}(\mathrm{d}t, \mathrm{d}z), \end{aligned}$$

where

$$\begin{aligned} LV = {} & x^q(r_1 - a_{11}(t)x^{\theta_1} - a_{12}(t)y) + y^q(r_2 - a_{21}(t)x - a_{22}(t)y^{\theta_2}) \\ & + \beta_1 r_1^{2q-1}(\bar{r}_1 - r_1) + \frac{2q-1}{2}\sigma_1^2 r_1^{2q-2} + \beta_2 r_2^{2q-1}(\bar{r}_2 - r_2) + \frac{2q-1}{2}\sigma_2^2 r_2^{2q-2} \\ & + \int_Z \left(\frac{(x + x\gamma_1(\xi, z))^q}{q} - \frac{x^q}{q}\right)\nu(\mathrm{d}z) + \int_Z \left(\frac{(y + y\gamma_2(\xi, z))^q}{q} - \frac{y^q}{q}\right)\nu(\mathrm{d}z). \end{aligned}$$

Therefore,

$$\begin{aligned}LV \le& -a_{11}^{l}x^{\theta_1+q} - a_{22}^{l}y^{\theta_2+q} + |r_1|\,x^q + |r_2|\,y^q + \beta_1\bar{r}_1 r_1^{2q-1} + \beta_2\bar{r}_2 r_2^{2q-1} - \beta_1 r_1^{2q} - \beta_2 r_2^{2q} + \frac{2q-1}{2}\sigma_1^2 r_1^{2q-2}\\ &+ \frac{2q-1}{2}\sigma_2^2 r_2^{2q-2} + \frac{x^q}{q}\int_Z |(1+\gamma_1)^q - 1|\,\nu(\mathrm{d}z) + \frac{y^q}{q}\int_Z |(1+\gamma_2)^q - 1|\,\nu(\mathrm{d}z)\\ \le& -\hat{a}^l x^{\theta_1+q} - \hat{a}^l y^{\theta_2+q} + |r_1|\,x^q + |r_2|\,y^q + \beta_1\bar{r}_1 r_1^{2q-1} + \beta_2\bar{r}_2 r_2^{2q-1} - \beta_1 r_1^{2q} - \beta_2 r_2^{2q}\\ &+ \frac{2q-1}{2}\sigma_1^2 r_1^{2q-2} + \frac{2q-1}{2}\sigma_2^2 r_2^{2q-2} + \frac{cx^q}{q} + \frac{cy^q}{q}.\end{aligned} \tag{3.1}$$

Let $\eta = q\min\{\beta_1, \beta_2\}$. Using Itô formula again, we have

$$\begin{aligned}\mathrm{d}\left(e^{\eta t}V\right) =& \eta e^{\eta t}V\mathrm{d}t + e^{\eta t}\mathrm{d}V\\ =& \eta e^{\eta t}V\mathrm{d}t + e^{\eta t}\Bigg(LV\mathrm{d}t + \sigma_1 r_1^{2q-1}\mathrm{d}B_1(t) + \sigma_2 r_2^{2q-1}\mathrm{d}B_2(t) + \int_Z\left(\frac{(x + x\gamma_1(\xi,z))^q}{q} - \frac{x^q}{q}\right)\tilde{N}(\mathrm{d}t,\mathrm{d}z)\\ &+ \int_Z\left(\frac{(y + y\gamma_2(\xi,z))^q}{q} - \frac{y^q}{q}\right)\tilde{N}(\mathrm{d}t,\mathrm{d}z)\Bigg)\\ =& e^{\eta t}\left(\eta V + LV\right)\mathrm{d}t + e^{\eta t}\Bigg(\sigma_1 r_1^{2q-1}\mathrm{d}B_1(t) + \sigma_2 r_2^{2q-1}\mathrm{d}B_2(t)\int_Z\left(\frac{(x + x\gamma_1(\xi,z))^q}{q} - \frac{x^q}{q}\right)\tilde{N}(\mathrm{d}t,\mathrm{d}z)\\ &+ \int_Z\left(\frac{(y + y\gamma_2(\xi,z))^q}{q} - \frac{y^q}{q}\right)\tilde{N}(\mathrm{d}t,\mathrm{d}z)\Bigg).\end{aligned} \tag{3.2}$$

Integrating from 0 to t on both sides of Eq (3.2) and taking the expected value, we obtain

$$\mathbb{E}\left(e^{\eta t}V\right) = V(x(0), y(0), r_1(0), r_2(0)) + \mathbb{E}\int_0^t e^{\eta s}\left(\eta V + LV\right)\mathrm{d}s, \tag{3.3}$$

Combining with Eq (3.1), we have

$$\begin{aligned}\eta V + LV \le& \frac{\eta x^q}{q} + \frac{\eta y^q}{q} + \frac{\eta r_1^{2q}}{2q} + \frac{\eta r_2^{2q}}{2q} - \hat{a}^l x^{\theta_1+q} - \hat{a}^l y^{\theta_2+q} + |r_1|\,x^q + |r_2|\,y^q + \beta_1\bar{r}_1 r_1^{2q-1} + \beta_2\bar{r}_2 r_2^{2q-1}\\ &- \beta_1 r_1^{2q} - \beta_2 r_2^{2q} + \frac{2q-1}{2}\sigma_1^2 r_1^{2q-2} + \frac{2q-1}{2}\sigma_2^2 r_2^{2q-2} + \frac{cx^q}{q} + \frac{cy^q}{q}\\ \le& \sup_{(x,y,r_1,r_2)\in\mathbb{R}_+^2\times\mathbb{R}^2}\Bigg\{\frac{\eta x^q}{q} + \frac{\eta y^q}{q} + \frac{\eta r_1^{2q}}{2q} + \frac{\eta r_2^{2q}}{2q} - \hat{a}^l x^{\theta_1+q} - \hat{a}^l y^{\theta_2+q} + |r_1|\,x^q + |r_2|\,y^q + \beta_1\bar{r}_1 r_1^{2q-1}\\ &+\beta_2\bar{r}_2 r_2^{2q-1} - \beta_1 r_1^{2q} - \beta_2 r_2^{2q} + \frac{2q-1}{2}\sigma_1^2 r_1^{2q-2} + \frac{2q-1}{2}\sigma_2^2 r_2^{2q-2} + \frac{cx^q}{q} + \frac{cy^q}{q}\Bigg\} := \kappa_1(q).\end{aligned} \tag{3.4}$$

Substituting Eq (3.4) into Eq (3.3), we get

$$\mathbb{E}\left(e^{\eta t}V\right) \le V(x(0), y(0), r_1(0), r_2(0)) + \mathbb{E}\int_0^t e^{\eta s}\kappa_1(q)\mathrm{d}s.$$

then

$$e^{\eta t}\mathbb{E}V \le V(x(0), y(0), r_1(0), r_2(0)) + \frac{e^{\eta t} - 1}{\eta}\kappa_1(q).$$

Further,

$$\begin{aligned}\limsup_{t\to\infty}\mathbb{E}\left[x^q(t)\right] &\le q\limsup_{t\to\infty}\mathbb{E}\left(V(x(t),y(t),r_1(t),r_2(t))\right)\\ &\le q\limsup_{t\to\infty}\left(\frac{V(x(0),y(0),r_1(0),r_2(0))}{e^{\eta t}}+\frac{e^{\eta t}-1}{\eta e^{\eta t}}\kappa_1(q)\right)\\ &=\frac{q\kappa_1(q)}{\eta}:=\kappa_2(q).\end{aligned}$$

Similarly , $\limsup_{t\to\infty}\mathbb{E}\left[y^q(t)\right]\le\kappa_2(q)$ holds, which then means $\mathbb{E}\left[x^q(t)\right]\le\kappa_2(q),\mathbb{E}\left[y^q(t)\right]\le\kappa_2(q),\forall t\ge 0,q\ge 2$. According to Hölder's inequality, for any $\tilde{q}\in(0,2)$, we obtain

$$\mathbb{E}\left[x^{\tilde{q}}(t)\right]\le\left(\mathbb{E}\left[x^2(t)\right]\right)^{\frac{\tilde{q}}{2}}\le(\kappa_2(2))^{\frac{\tilde{q}}{2}},\ \mathbb{E}\left[y^{\tilde{q}}(t)\right]\le\left(\mathbb{E}\left[y^2(t)\right]\right)^{\frac{\tilde{q}}{2}}\le(\kappa_2(2))^{\frac{\tilde{q}}{2}}.$$

Let $\kappa(q)=\max\left\{\kappa_2(q),(\kappa_2(2))^{\frac{\tilde{q}}{2}}\right\}$, then

$$\mathbb{E}\left[x^q(t)\right]\le\kappa(q),\mathbb{E}\left[y^q(t)\right]\le\kappa(q),\forall q>0.$$

Theorem 3.1 is proved.
Remark 3.1. Similar to the proof of Theorem 3.1, we can also get $\mathbb{E}\left[r_i(t)\right]^{2q}\le Q(q),i=1,2.$

4. Existence of a stationary distribution

Before giving the theorem of the existence of stationary distributions, we give several Lemmas as follows.
Assumption 4.1. $a_{11}^l-a_{21}^u>0,a_{22}^l-a_{12}^u>0,\beta_1>1,\beta_2>1.$
Lemma 4.1. Let $X^{a,j}(t)=(x(t),y(t),r_1(t),r_2(t))$ and $X^{\tilde{a},j}(t)=(\tilde{x}(t),\tilde{y}(t),\tilde{r}_1(t),\tilde{r}_2(t))$ be the solutions of model (1.4) with initial values of $(a,j)=((x(0),y(0),r_1(0),r_2(0)),\xi(0))\in D\times\mathbb{S}$ and $(\tilde{a},j)=((\tilde{x}(0),\tilde{y}(0),\tilde{r}_1(0),\tilde{r}_2(0)),\xi(0))\in D\times\mathbb{S}$, where D is any compact subset of $\mathbb{R}_+^2\times\mathbb{R}^2$. If Assumptions 2.1 and Assumption 4.1 hold, then the following equation holds

$$\lim_{t\to+\infty}\left(\mathbb{E}\left|x(t)-\tilde{x}(t)\right|+\mathbb{E}\left|y(t)-\tilde{y}(t)\right|+\mathbb{E}\left|r_1(t)-\tilde{r}_1(t)\right|+\mathbb{E}\left|r_2(t)-\tilde{r}_2(t)\right|\right)=0,a.s..$$

Proof Define a function W

$$W=\left|\ln x-\ln\tilde{x}\right|+\left|\ln y-\ln\tilde{y}\right|+\left|r_1-\tilde{r}_1\right|+\left|r_2-\tilde{r}_2\right|.$$

then we obtain

$$\begin{aligned}\mathrm{d}^+W&=sgn(x-\tilde{x})\mathrm{d}(\ln x-\ln\tilde{x})+sgn(y-\tilde{y})\mathrm{d}(\ln y-\ln\tilde{y})+sgn(r_1-\tilde{r}_1)\mathrm{d}(r_1-\tilde{r}_1)+sgn(r_2-\tilde{r}_2)\mathrm{d}(r_2-\tilde{r}_2)\\ &=sgn(x-\tilde{x})\left[(r_1-\tilde{r}_1)-a_{11}(t)(x^{\theta_1}-\tilde{x}^{\theta_1})-a_{12}(t)(y-\tilde{y})\right]\mathrm{d}t+sgn(y-\tilde{y})\left[(r_2-\tilde{r}_2)-a_{21}(t)(x-\tilde{x})\right.\\ &\quad\left.-a_{22}(t)(y^{\theta_2}-\tilde{y}^{\theta_2})\right]\mathrm{d}t+sgn(r_1-\tilde{r}_1)\left[-\beta_1(r_1-\tilde{r}_1)\right]\mathrm{d}t+sgn(r_2-\tilde{r}_2)\left[-\beta_2(r_2-\tilde{r}_2)\right]\mathrm{d}t\\ &\le -a_{11}^l\left|x^{\theta_1}-\tilde{x}^{\theta_1}\right|\mathrm{d}t-a_{22}^l\left|y^{\theta_2}-\tilde{y}^{\theta_2}\right|\mathrm{d}t+a_{21}^u\left|x-\tilde{x}\right|\mathrm{d}t+a_{12}^u\left|y-\tilde{y}\right|\mathrm{d}t-(\beta_1-1)\left|r_1-\tilde{r}_1\right|\mathrm{d}t\\ &\quad-(\beta_2-1)\left|r_2-\tilde{r}_2\right|\mathrm{d}t.\end{aligned}\tag{4.1}$$

Taking the integral on both sides of Eq (4.1), and taking the expectation, we obtain

$$\mathbb{E}W \leq W(0) - a_{11}^l \int_0^t \mathbb{E}\left|x^{\theta_1} - \tilde{x}^{\theta_1}\right| \mathrm{d}s - a_{22}^l \int_0^t \mathbb{E}\left|y^{\theta_2} - \tilde{y}^{\theta_2}\right| \mathrm{d}s + a_{21}^u \int_0^t \mathbb{E}\left|x - \tilde{x}\right| \mathrm{d}s + a_{12}^u \int_0^t \mathbb{E}\left|y - \tilde{y}\right| \mathrm{d}s$$
$$- (\beta_1 - 1) \int_0^t \mathbb{E}\left|r_1 - \tilde{r}_1\right| \mathrm{d}s - (\beta_2 - 1) \int_0^t \mathbb{E}\left|r_2 - \tilde{r}_2\right| \mathrm{d}s.$$

Noting $\mathbb{E}W(t) \geq 0$, we have

$$\begin{aligned} & a_{11}^l \int_0^t \mathbb{E}\left|x^{\theta_1} - \tilde{x}^{\theta_1}\right| \mathrm{d}s + a_{22}^l \int_0^t \mathbb{E}\left|y^{\theta_2} - \tilde{y}^{\theta_2}\right| \mathrm{d}s - a_{21}^u \int_0^t \mathbb{E}\left|x - \tilde{x}\right| \mathrm{d}s - a_{12}^u \int_0^t \mathbb{E}\left|y - \tilde{y}\right| \mathrm{d}s \\ & + (\beta_1 - 1) \int_0^t \mathbb{E}\left|r_1 - \tilde{r}_1\right| \mathrm{d}s + (\beta_2 - 1) \int_0^t \mathbb{E}\left|r_2 - \tilde{r}_2\right| \mathrm{d}s \\ \leq\ & W(0). \end{aligned} \tag{4.2}$$

Let $\theta_1 = \theta_2 = 1$, then

$$\begin{aligned} & (a_{11}^l - a_{21}^u) \int_0^t \mathbb{E}\left|x - \tilde{x}\right| \mathrm{d}s + (a_{22}^l - a_{12}^u) \int_0^t \mathbb{E}\left|y - \tilde{y}\right| \mathrm{d}s + (\beta_1 - 1) \int_0^t \mathbb{E}\left|r_1 - \tilde{r}_1\right| \mathrm{d}s \\ & + (\beta_2 - 1) \int_0^t \mathbb{E}\left|r_2 - \tilde{r}_2\right| \mathrm{d}s \\ \leq\ & W(0). \end{aligned}$$

Thus, according Assumption 4.1, we have

$$\mathbb{E}\left|x - \tilde{x}\right| \in L^1[0, +\infty), \mathbb{E}\left|y - \tilde{y}\right| \in L^1[0, +\infty).$$

Therefore, according (4.2), we get

$$\begin{aligned} & a_{11}^l \int_0^t \mathbb{E}\left|x^{\theta_1} - \tilde{x}^{\theta_1}\right| \mathrm{d}s + a_{22}^l \int_0^t \mathbb{E}\left|y^{\theta_2} - \tilde{y}^{\theta_2}\right| \mathrm{d}s + (\beta_1 - 1) \int_0^t \mathbb{E}\left|r_1 - \tilde{r}_1\right| \mathrm{d}s + (\beta_2 - 1) \int_0^t \mathbb{E}\left|r_2 - \tilde{r}_2\right| \mathrm{d}s \\ \leq\ & W(0) + a_{21}^u \int_0^t \mathbb{E}\left|x - \tilde{x}\right| \mathrm{d}s + a_{12}^u \int_0^t \mathbb{E}\left|y - \tilde{y}\right| \mathrm{d}s \\ \leq\ & +\infty. \end{aligned}$$

Thus, we have

$$\mathbb{E}\left|x^{\theta_1} - \tilde{x}^{\theta_1}\right| \in L^1[0, +\infty), \mathbb{E}\left|y^{\theta_2} - \tilde{y}^{\theta_2}\right| \in L^1[0, +\infty), \mathbb{E}\left|r - \tilde{r}_1\right| \in L^1[0, +\infty), \mathbb{E}\left|r - \tilde{r}_2\right| \in L^1[0, +\infty).$$

According to model (1.4), there is

$$\mathbb{E}(x(t)) = x(0) + \int_0^t \left[\mathbb{E}(r_1(s)x(s)) - \mathbb{E}(a_{11}(s)x^{\theta_1+1}(s)) - \mathbb{E}(a_{12}(s)x(s)y(s))\right] \mathrm{d}s + \mathbb{E}\left[\int_0^t \int_Z \gamma_1(\xi(s), z)x(s)\nu(\mathrm{d}z)\mathrm{d}s\right],$$
$$\mathbb{E}(y(t)) = y(0) + \int_0^t \left[\mathbb{E}(r_2(s)y(s)) - \mathbb{E}(a_{21}(s)x(s)y(s)) - \mathbb{E}(a_{22}(s)y^{\theta_2+1}(s))\right] \mathrm{d}s + \mathbb{E}\left[\int_0^t \int_Z \gamma_2(\xi(s), z)y(s)\nu(\mathrm{d}z)\mathrm{d}s\right],$$
$$\mathbb{E}(r_1(t)) = r_1(0) + \int_0^t \left[\mathbb{E}(\beta_1\bar{r}_1) - \mathbb{E}(\beta_1 r_1(s))\right] \mathrm{d}s,$$
$$\mathbb{E}(r_2(t)) = r_2(0) + \int_0^t \left[\mathbb{E}(\beta_2\bar{r}_2) - \mathbb{E}(\beta_2 r_2(s))\right] \mathrm{d}s.$$

Therefore, $\mathbb{E}(x(t)), \mathbb{E}(y(t)), \mathbb{E}(r_1(t))$, and $\mathbb{E}(r_2(t))$ are continuously differentiable. According to Theorem 3.1 and remark 3.1, we have

$$\frac{\mathrm{d}\mathbb{E}(x(t))}{\mathrm{d}t} \leq \mathbb{E}\left[x(t)\left|r_1(t)\right|\right] + c\mathbb{E}(x(t)) \leq \frac{1}{2}\mathbb{E}\left[x^2(t) + |r_1(t)|^2\right] + c\mathbb{E}(x(t)) \leq \frac{1}{2}(Q(1) + \kappa(2)) + c\kappa(1),$$
$$\frac{\mathrm{d}\mathbb{E}(y(t))}{\mathrm{d}t} \leq \mathbb{E}\left[y(t)\left|r_2(t)\right|\right] + c\mathbb{E}(y(t)) \leq \frac{1}{2}\mathbb{E}\left[y^2(t) + |r_2(t)|^2\right] + c\mathbb{E}(y(t)) \leq \frac{1}{2}(Q(1) + \kappa(2)) + c\kappa(1),$$
$$\frac{\mathrm{d}\mathbb{E}(r_1(t))}{\mathrm{d}t} \leq \beta_1\left|\bar{r}_1\right| + \beta_1\mathbb{E}\left|r_1(t)\right| \leq \beta_1\left|\bar{r}_1\right| + \beta_1 Q(1)^{\frac{1}{2}}, \quad \frac{\mathrm{d}\mathbb{E}(r_2(t))}{\mathrm{d}t} \leq \beta_2\left|\bar{r}_2\right| + \beta_2\mathbb{E}\left|r_2(t)\right| \leq \beta_2\left|\bar{r}_2\right| + \beta_2 Q(1)^{\frac{1}{2}}.$$

So, $\mathbb{E}(x(t)), \mathbb{E}(y(t)), \mathbb{E}(r_1(t))$, and $\mathbb{E}(r_2(t))$ are uniformly continuous. According to the Barbalat lemma, it can be concluded that $\lim_{t\to+\infty}\mathbb{E}|x-\tilde{x}| = 0$ *a.s.*, $\lim_{t\to+\infty}\mathbb{E}|y-\tilde{y}| = 0$ *a.s.*, $\lim_{t\to+\infty}\mathbb{E}|r_1-\tilde{r}_1| = 0$ *a.s.*, $\lim_{t\to+\infty}\mathbb{E}|r_2-\tilde{r}_2| = 0$ *a.s.*; therefore, Lemma 4.1 is proven.

Lemma 4.2. For any $q > 0$ and any D on $\mathbb{R}^2_+ \times \mathbb{R}^2$, there is

$$\sup_{(a,j)\in D\times\mathbb{S}} \mathbb{E}\left[\sup_{0\leq s\leq t}\left|X^{a,j}(s)\right|^q\right] < +\infty, \forall t \geq 0.$$

Proof From model (1.4), it can be inferred that

$$x(t) = x(0) + \int_0^t \left[\left(r_1(s)x(s) - a_{11}(s)x^{\theta_1+1}(s) - a_{12}(s)x(s)y(s)\right)\right]\mathrm{d}s + \int_0^t\int_Z \gamma_1\left(\xi(s), z\right)x(s)N(\mathrm{d}s, \mathrm{d}z),$$
$$y(t) = y(0) + \int_0^t \left[\left(r_2(s)y(s) - a_{21}(s)x(s)y(s) - a_{22}(s)y^{\theta_2+1}(s)\right)\right]\mathrm{d}s + \int_0^t\int_Z \gamma_2\left(\xi(s), z\right)y(s)N(\mathrm{d}s, \mathrm{d}z).$$

By the Holder inequality and the moment inequality, there is $k = 1, 2, \cdots$, and we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{(k-1)\lambda\leq s\leq k\lambda}|x(s)|^q\right]\\
&\leq 3^{q-1}\mathbb{E}\left[\sup_{(k-1)\lambda\leq s\leq k\lambda}|x((k-1)\lambda)|^q\right] + 3^{q-1}\mathbb{E}\left[\sup_{(k-1)\lambda\leq s\leq k\lambda}\left|\int_{(k-1)\lambda}^{k\lambda}\int_Z \gamma_1\left(\xi(s), z\right)x(s)N(\mathrm{d}s, \mathrm{d}z)\right|^q\right]\\
&\quad + 3^{q-1}\mathbb{E}\left[\sup_{(k-1)\lambda\leq s\leq k\lambda}\left|\int_{(k-1)\lambda}^{k\lambda}\left[r_1(s)x(s) - a_{11}(s)x^{\theta_1+1}(s) - a_{12}(s)x(s)y(s)\right]\mathrm{d}s\right|^q\right].
\end{aligned} \tag{4.3}$$

According to Theorem 3.1 and remark 3.1, we obtain

$$\begin{aligned}
&\mathbb{E}\left[\sup_{(k-1)\lambda\leq s\leq k\lambda}\left|\int_{(k-1)\lambda}^{k\lambda}\left[r_1(s)x(s) - a_{11}(s)x^{\theta_1+1}(s) - a_{12}(s)x(s)y(s)\right]\mathrm{d}s\right|^q\right]\\
&\leq \lambda^q\mathbb{E}\left[\sup_{(k-1)\lambda\leq s\leq k\lambda}|x(s)|^q\left|r_1(s) - a_{11}(s)x(s)^{\theta_1} - a_{12}(s)y(s)\right|^q\right]\\
&\leq \frac{1}{2}\lambda^q\mathbb{E}\left[\sup_{(k-1)\lambda\leq s\leq k\lambda}|x(s)|^{2q}\right] + \frac{1}{2}\lambda^q\mathbb{E}\left[\sup_{(k-1)\lambda\leq s\leq k\lambda}\left|r_1(s) - a_{11}(s)x(s)^{\theta_1} - a_{12}(s)y(s)\right|^{2q}\right]\\
&\leq \frac{1}{2}\lambda^q\mathbb{E}\left[\sup_{(k-1)\lambda\leq s\leq k\lambda}|x(s)|^{2q}\right] + \frac{1}{2}\lambda^q 3^{(2q-1)}\left[\mathbb{E}\left[\sup_{(k-1)\lambda\leq s\leq k\lambda}|r_1(s)|^{2q}\right] + (a_{11}^u)^{2q}\mathbb{E}\left[\sup_{(k-1)\lambda\leq s\leq k\lambda}|x(s)|^{2q\theta_1}\right]\right.\\
&\quad \left.+(a_{12}^u)^{2q}\mathbb{E}\left[\sup_{(k-1)\lambda\leq s\leq k\lambda}|y(s)|^{2q}\right]\right]\\
&:= M_1(q)\lambda^q.
\end{aligned} \tag{4.4}$$

By Assumption 2.1 and the Kunita inequality, we get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{(k-1)\lambda\le s\le k\lambda}\left|\int_{(k-1)\lambda}^{k\lambda}\int_Z \gamma_1\left(\xi(s),z\right)x(s)N(\mathrm{d}s,\mathrm{d}z)\right|^q\right]\\
&\le 2^{q-1}D_q\left\{\mathbb{E}\left[\int_{(k-1)\lambda}^{k\lambda}\int_Z \left|\gamma_1\left(\xi(s),z\right)x(s)\right|^2 v(\mathrm{d}z)\mathrm{d}s\right]^{\frac{q}{2}}+\mathbb{E}\int_{(k-1)\lambda}^{k\lambda}\int_Z \left|\gamma_1\left(\xi(s),z\right)x(s)\right|^q v(\mathrm{d}z)\mathrm{d}s\right\}\\
&\quad+2^{q-1}\mathbb{E}\left[\sup_{(k-1)\lambda\le s\le k\lambda}\left|\int_{(k-1)\lambda}^{k\lambda}\int_Z \gamma_1\left(\xi(s),z\right)x(s)v(\mathrm{d}z)\mathrm{d}s\right|^q\right]\\
&\le 2^{q-1}D_q\lambda^{\frac{q}{2}}\mathbb{E}\left(c\left|x(s)\right|^2\right)^{\frac{q}{2}}+2^{q-1}D_q\lambda\mathbb{E}\left(c\left|x(s)\right|^q\right)+2^{q-1}\mathbb{E}\left[\sup_{(k-1)\lambda\le s\le k\lambda}\left|c\int_{(k-1)\lambda}^{k\lambda}x(s)\mathrm{d}s\right|^q\right]\\
&\le 2^{q-1}D_q\lambda^{\frac{q}{2}}\kappa(q)c^{\frac{q}{2}}+2^{q-1}D_q\lambda\kappa(q)c+2^{q-1}c^q\lambda^q\kappa(q).
\end{aligned}\tag{4.5}$$

According to model (1.4), it can also be concluded that

$$\begin{aligned}
r_1(t)&=r_1(0)+\int_0^t\beta_1\left[\bar{r}_1-r_1(s)\right]\mathrm{d}s+\int_0^t\sigma_1\mathrm{d}B_1(s),\\
r_2(t)&=r_2(0)+\int_0^t\beta_2\left[\bar{r}_2-r_2(s)\right]\mathrm{d}s+\int_0^t\sigma_2\mathrm{d}B_2(s).
\end{aligned}$$

then

$$\begin{aligned}
\mathbb{E}\left[\sup_{(k-1)\lambda\le s\le k\lambda}|r_1(s)|^q\right]&\le 3^{q-1}\mathbb{E}\left[\sup_{(k-1)\lambda\le s\le k\lambda}|r_1((k-1)\lambda)|^q\right]+3^{q-1}\mathbb{E}\left[\sup_{(k-1)\lambda\le s\le k\lambda}\left|\int_{(k-1)\lambda}^{k\lambda}\beta_1\left[\bar{r}_1-r_1(s)\right]\mathrm{d}s\right|^q\right]\\
&\quad+3^{q-1}\mathbb{E}\left[\sup_{(k-1)\lambda\le s\le k\lambda}\left|\int_{(k-1)\lambda}^{k\lambda}\sigma_1\mathrm{d}B_1(s)\right|^q\right].
\end{aligned}\tag{4.6}$$

According to remark 3.1, we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{(k-1)\lambda\le s\le k\lambda}\left|\int_{(k-1)\lambda}^{k\lambda}\beta_1\left[\bar{r}_1-r_1(s)\right]\mathrm{d}s\right|^q\right]&\le\lambda^q\mathbb{E}\left[\sup_{(k-1)\lambda\le s\le k\lambda}|\beta_1\bar{r}_1-\beta_1r_1(s)|^q\right]\\
&\le\lambda^q\mathbb{E}\left[\sup_{(k-1)\lambda\le s\le k\lambda}\left(2^{q-1}|\beta_1\bar{r}_1|^q+2^{q-1}|\beta_1r_1(s)|^q\right)\right]\\
&\le\lambda^q\mathbb{E}\left[2^{q-1}\beta_1^q|\bar{r}_1|^q\right]+\lambda^q\mathbb{E}\left[2^{q-1}\beta_1^q\sup_{(k-1)\lambda\le s\le k\lambda}|r_1(s)|^q\right]\\
&\le\lambda^q\left(2^{q-1}\beta_1^q|\bar{r}_1|^q+2^{q-1}\beta_1^qQ(\frac{q}{2})\right)\\
&:=M_2(q)\lambda^q.
\end{aligned}\tag{4.7}$$

According to the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}\mathbb{E}\left[\sup_{(k-1)\lambda\le s\le k\lambda}\left|\int_{(k-1)\lambda}^{k\lambda}\sigma_1 \mathrm{d}B_1(s)\right|^q\right] &\le C_q\mathbb{E}\left[\int_{(k-1)\lambda}^{k\lambda}|\sigma_1|^2\,\mathrm{d}s\right]^{\frac{q}{2}}\\ &\le C_q\mathbb{E}\left(|\sigma_1|^2\lambda\right)^{\frac{q}{2}}\\ &= C_q\,|\sigma_1|^q\,\lambda^{\frac{q}{2}}\\ &:= M_3(q)\lambda^{\frac{q}{2}}.\end{aligned} \tag{4.8}$$

According to Eqs (4.3)–(4.5), we have

$$\sup_{(a,j)\in D\times\mathbb{S}}\mathbb{E}\left[\sup_{0\le s\le t}\left|x^{a,j}(s)\right|^q\right] < +\infty, \forall s\in[0,t], \forall t\ge 0.$$

According to Eqs (4.6)–(4.8), we have

$$\sup_{(a,j)\in D\times\mathbb{S}}\mathbb{E}\left[\sup_{0\le s\le t}\left|r_1^{a,j}(s)\right|^q\right] < +\infty, \forall s\in[0,t], \forall t\ge 0.$$

Similarly, it can be inferred that

$$\sup_{(a,j)\in D\times\mathbb{S}}\mathbb{E}\left[\sup_{0\le s\le t}\left|y^{a,j}(s)\right|^q\right] < +\infty, \forall s\in[0,t], \forall t\ge 0,$$

$$\sup_{(a,j)\in D\times\mathbb{S}}\mathbb{E}\left[\sup_{0\le s\le t}\left|r_2^{a,j}(s)\right|^q\right] < +\infty, \forall s\in[0,t], \forall t\ge 0.$$

Thus, Lemma 4.2 is proved.

Here, in order to prove the following lemma, we introduce the following symbols. Define $B(\mathbb{R}_+^2\times\mathbb{R}^2\times\mathbb{S})$ as the set of all probability measures on $\mathbb{R}_+^2\times\mathbb{R}^2\times\mathbb{S}$, and for any two measures $p_1, p_2\in B$, define the metric d_H as follows:

$$d_H(p_1,p_2) = \sup_{h\in H}\left|\sum_{i=1}^{p}\int_{\mathbb{R}_+^2\times\mathbb{R}^2}h(x,i)p_1(\mathrm{d}x,i) - \sum_{i=1}^{p}\int_{\mathbb{R}_+^2\times\mathbb{R}^2}h(x,i)p_2(\mathrm{d}x,i)\right|,$$

where $H = \left\{h:\mathbb{R}_+^2\times\mathbb{R}^2\times\mathbb{S}\to R \mid |h(x,i)-h(y,j)|\le|x-y|+|i-j|, |h(\cdot\times\cdot)|\le 1\right\}$.

Lemma 4.3. If Assumption 2.1 and Assumption 4.1 hold, for any compact subset D over $\mathbb{R}_+^2\times\mathbb{R}^2$, the following formula holds

$$\lim_{t\to+\infty} d_H\left(p(t,a,i,\cdot\times\cdot), p(t,\tilde{a},j,\cdot\times\cdot)\right) = 0,$$

where $a,\tilde{a}\in D, i, j\in\mathbb{S}$.

Proof For any $i, j\in\mathbb{S}$, define the stopping time

$$\alpha_{ij} = \inf\left\{t\ge 0:\xi_i(t)=\xi_j(t)\right\}.$$

Because of the ergodic nature of Markov chains, $\alpha_{ij}<\infty$. So, for any $\varepsilon>0$, there exists a positive number T, such that

$$\mathbb{P}\left\{\alpha_{ij}\le T\right\} > 1-\frac{\varepsilon}{8}, \forall i, j\in\mathbb{S}.$$

For such T, by Lemma 4.2, there is a sufficiently large number $R > 0$ for

$$\mathbb{P}\{\Omega_{a,i}\} > 1 - \frac{\varepsilon}{16}, \forall (a,i) \in D \times \mathbb{S}, \tag{4.9}$$

where $\Omega_{a,i} = \left\{\left|X^{a,i}\right| \leq R, \forall t \in [0,T]\right\}$. Now, fix $a, \tilde{a} \in D, i, j \in \mathbb{S}$ and set $\Omega_1 = \Omega_{a,i} \cap \Omega_{\tilde{a},j}$. For any $h \in H$ and $t \geq T$, calculate

$$\begin{aligned}
\left|\mathbb{E}h(X^{a,i}, \xi_i(t)) - \mathbb{E}h(X^{\tilde{a},j}, \xi_j(t))\right| &\leq 2\mathbb{P}\left\{\alpha_{ij} > T\right\} + \mathbb{E}\left(I_{\{\alpha_{ij}\leq T\}} \mid h(X^{a,i}, \xi_i(t)) - h(X^{\tilde{a},j}, \xi_j(t)) \mid\right) \\
&\leq \frac{\varepsilon}{4} + \mathbb{E}\left[I_{\{\alpha_{ij}\leq T\}}\mathbb{E}\left(\left|h(X^{a,i}, \xi_i(t)) - h(X^{\tilde{a},j}, \xi_j(t))\right| \mid \mathcal{F}_{\alpha_{ij}}\right)\right] \\
&\leq \frac{\varepsilon}{4} + \mathbb{E}\left[I_{\{\alpha_{ij}\leq T\}}\mathbb{E}\left|h(X^{u,k}(t-\alpha_{ij}), \xi_k(t-\alpha_{ij})) - h(X^{v,k}(t-\alpha_{ij}), \xi_k(t-\alpha_{ij}))\right|\right] \\
&\leq \frac{\varepsilon}{4} + \mathbb{E}\left[I_{\{\alpha_{ij}\leq T\}}\mathbb{E}\left(2 \wedge \left|X^{u,k}(t-\alpha_{ij}) - X^{v,k}(t-\alpha_{ij})\right|\right)\right] \\
&\leq \frac{\varepsilon}{4} + 2\mathbb{P}(\Omega - \Omega_1) + \mathbb{E}\left[I_{\Omega_1\cap\{\alpha_{ij}\leq T\}}\mathbb{E}\left(2 \wedge \left|X^{u,k}(t-\alpha_{ij}) - X^{v,k}(t-\alpha_{ij})\right|\right)\right],
\end{aligned} \tag{4.10}$$

where $u = X^{a,i}(\alpha_{ij}), v = X^{\tilde{a},j}(\alpha_{ij}), k = \xi_i(\alpha_{ij}) = \xi_j(\alpha_{ij})$. According to Lemma 4.1, there is $T_1 > 0$ and $\mathbb{E}\left|X^{a,i} - X^{\tilde{a},i}\right| \leq \frac{\varepsilon}{2}, \forall t \geq T_1$. When $|a| \vee |\tilde{a}| \leq R, i \in \mathbb{S}$, there is

$$\mathbb{E}\left(2 \wedge \left|X^{a,i} - X^{\tilde{a},i}\right|\right) \leq \mathbb{E}\left|X^{a,i} - X^{\tilde{a},i}\right| \leq \frac{\varepsilon}{2}. \tag{4.11}$$

Given $\omega \in \Omega_1 \cap \left\{\alpha_{ij} \leq T\right\}, |u| \vee |v| \leq R$, it is obtained from the above formula that

$$\mathbb{E}\left(2 \wedge \left|X^{u,k}(t-\alpha_{ij}) - X^{v,k}(t-\alpha_{ij})\right|\right) < \frac{\varepsilon}{2}, \forall t \geq T + T_1. \tag{4.12}$$

By formula (4.10)–(4.12), we can obtain

$$\left|\mathbb{E}h(X^{a,i}, \xi_i(t)) - \mathbb{E}h(X^{\tilde{a},j}, \xi_j(t))\right| \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon, \forall t \geq T + T_1.$$

Since h is arbitrary, then

$$\sup_{h\in H}\left|\mathbb{E}h(X^{a,i}(t), \xi_i(t)) - \mathbb{E}h(X^{\tilde{a},j}(t), \xi_j(t))\right| \leq \varepsilon, \forall t \geq T + T_1.$$

namely,

$$d_H\left(p(t, a, i, \cdot \times \cdot), p(t, \tilde{a}, j, \cdot \times \cdot)\right) \leq \varepsilon, \forall t \geq T + T_1.$$

for all $a, \tilde{a} \in D, i, j \in \mathbb{S}$. The proof is therefore complete.

Lemma 4.4. If Assumption 2.1 and Assumption 4.1 hold, for any $(a,i) \in \mathbb{R}_+^2 \times \mathbb{R}^2 \times \mathbb{S}$, $\{p(t, a, i, \cdot \times \cdot) \mid t \geq 0\}$ is the Cauchy in space $B(\mathbb{R}_+^2 \times \mathbb{R}^2 \times \mathbb{S})$ with metric d_H.

Proof Fix any $(a,i) \in \mathbb{R}_+^2 \times \mathbb{R}^2 \times \mathbb{S}$. We only need to prove that for any $\varepsilon > 0$, there is a $T > 0$ such that

$$d_H\left(p(t+s, a, i, \cdot \times \cdot), p(t, a, i, \cdot \times \cdot)\right) \leq \varepsilon, \forall t \geq T, s > 0,$$

which is equivalent to proof

$$\sup_{h\in H}\left|\mathbb{E}h(X^{a,i}(t+s), \xi_i(t+s)) - \mathbb{E}h(X^{a,i}(t), \xi_i(t))\right| \leq \varepsilon, \forall t \geq T, s > 0. \tag{4.13}$$

For any $h \in H, t, s > 0$, we have

$$
\begin{aligned}
&\left|\mathbb{E}h(X^{a,i}(t+s), \xi_i(t+s)) - \mathbb{E}h(X^{a,i}(t), \xi_i(t))\right| \\
&= \left|\mathbb{E}\left[\mathbb{E}(h(X^{a,i}(t+s), \xi_i(t+s)) \mid \mathcal{F}_s\right] - \mathbb{E}h(X^{a,i}(t), \xi_i(t))\right| \\
&= \left|\sum_{l=1}^{p} \int_{\mathbb{R}_+^2 \times \mathbb{R}^2} \mathbb{E}h(X^{z_0,l}(t), \xi_l(t)) p(s, a, i, \mathrm{d}z_0 \times \{l\}) - \mathbb{E}h(X^{a,i}(t), \xi_i(t))\right| \\
&\leq \sum_{l=1}^{p} \int_{\mathbb{R}_+^2 \times \mathbb{R}^2} \left|\mathbb{E}h(X^{z_0,l}(t), \xi_l(t)) - \mathbb{E}h(X^{a,i}(t), \xi_i(t))\right| p(s, a, i, \mathrm{d}z_0 \times \{l\}) \\
&\leq 2p(s, a, i, \bar{D}_{\mathbb{R}}^c \times \mathbb{S}) + \sum_{l=1}^{p} \int_{\bar{D}_{\mathbb{R}}} \left|\mathbb{E}h(X^{z_0,l}(t), \xi_l(t)) - \mathbb{E}h(X^{a,i}(t), \xi_i(t))\right| \times p(s, a, i, \mathrm{d}z_0 \times \{l\}),
\end{aligned}
\tag{4.14}
$$

where $\bar{D}_{\mathbb{R}} = \left\{a \in \mathbb{R}_+^2 \times \mathbb{R}^2 \mid |a| \leq R\right\}, \bar{D}_{\mathbb{R}}^c = (\mathbb{R}_+^2 \times \mathbb{R}^2) - \bar{D}_{\mathbb{R}}$. According to Chebyshev's inequality, the transition probability $\{p(t, a, i, \mathrm{d}z_0 \times \{l\} \mid t \geq 0)\}$ is compact, i.e., for any $\varepsilon > 0$, there exists a compact subset $D = D(\varepsilon, a, i)$ over $\mathbb{R}_+^2 \times \mathbb{R}^2$, such that $p(t, a, i, D \times \mathbb{S}) \geq 1 - \varepsilon, \forall t \geq 0$, where R is sufficiently large and we have

$$
p(s, a, i, \bar{D}_{\mathbb{R}}^c \times \mathbb{S}) < \frac{\varepsilon}{4}, \forall s \geq 0. \tag{4.15}
$$

According to Lemma 4.3, there exists $T > 0$ such that

$$
\sup_{h \in H} \left|\mathbb{E}h(X^{z_0,l}(t), \xi_l(t)) - \mathbb{E}h(X^{a,i}(t), \xi_i(t))\right| < \frac{\varepsilon}{2}, \forall t > T, (z_0, l) \in \bar{D}_{\mathbb{R}} \times \mathbb{S}. \tag{4.16}
$$

Substituting (4.15) and (4.16) into (4.14), we have

$$
\left|\mathbb{E}h(X^{a,i}(t+s), \xi_i(t+s)) - \mathbb{E}h(X^{a,i}(t), \xi_i(t))\right| < \varepsilon, \forall t \geq T, s > 0. \tag{4.17}
$$

Since h is arbitrary, then inequality (4.13) holds.

Lemma 4.5 [31]. Let $M(t), t \geq 0$ be a local martingale with initial value $M(0) = 0$. If $\lim_{t \to +\infty} \rho_M(t) < \infty$, then $\lim_{t \to +\infty} \frac{M(t)}{t} = 0$, where $\rho_M(t) = \int_0^t \frac{\mathrm{d}\langle M, M \rangle (s)}{(1+s)^2}, t \geq 0$, and $\langle M, M \rangle (t)$ is the quadratic variational process of $M(t)$.

Lemma 4.6 If Assumption 2.1 holds, the solutions of model (1.4) follow:

$$
\limsup_{t \to \infty} \frac{\ln x(t)}{t} \leq 0, \limsup_{t \to \infty} \frac{\ln y(t)}{t} \leq 0. \tag{4.18}
$$

Proof Define a founction $W(t) = (x(t) + y(t))^q = w(t)^q, q \geq 1$. Using Itô formula, we can get

$$
\begin{aligned}
LW = {} & q(x+y)^{q-1}\left[r_1 x - a_{11}x^{\theta_1+1} - a_{12}xy\right] + q(x+y)^{q-1}\left[r_2 y - a_{21}xy - a_{22}y^{\theta_2+1}\right] \\
& + x^q \int_Z \left[(1 + \gamma_1(\xi(s), z))^q - 1\right] \nu(\mathrm{d}z) + y^q \int_Z \left[(1 + \gamma_2(\xi(s), z))^q - 1\right] \nu(\mathrm{d}z) \\
\leq {} & q(x+y)^{q-1}\left(|r_1|\, x + |r_2|\, y\right) + cx^q + cy^q \\
\leq {} & q\,|r_1|\, w^q + q\,|r_2|\, w^q + 2cw^q \\
\leq {} & \frac{q}{2q+1}|r_1|^{2q+1} + \frac{q}{2q+1}|r_2|^{2q+1} + \frac{4q^2}{2q+1} w^{q+\frac{1}{2}} + 2cw^q.
\end{aligned}
$$

Let $\theta > 0$ be sufffciently small and satisfy $n\theta \le t \le (n+1)\theta, n = 1, 2, ...$. It follows that

$$\mathbb{E}\left[\sup_{n\theta\le t\le(n+1)\theta} w^q(t)\right] = \mathbb{E}\left[w^q(n\theta)\right] + I,$$

where

$$\begin{aligned} I &= \mathbb{E}\left[\sup_{n\theta\le t\le(n+1)\theta}\left|\int_{n\theta}^{t} LW\mathrm{d}s\right|\right] \\ &\le \mathbb{E}\left[\sup_{n\theta\le t\le(n+1)\theta}\left|\int_{n\theta}^{t}\left(\frac{q}{2q+1}\left|r_1(s)\right|^{2q+1} + \frac{q}{2q+1}\left|r_2(s)\right|^{2q+1} + \frac{4q^2}{2q+1}w(s)^{q+\frac{1}{2}} + 2cw(s)^q\right)\mathrm{d}s\right|\right] \\ &\le \frac{4q^2}{2q+1}\mathbb{E}\left[\int_{n\theta}^{(n+1)\theta} w(s)^{q+\frac{1}{2}}\mathrm{d}s\right] + 2c\mathbb{E}\left[\int_{n\theta}^{(n+1)\theta} w(s)^q\mathrm{d}s\right] + \frac{q}{2q+1}\mathbb{E}\left[\int_{n\theta}^{(n+1)\theta}\left(\left|r_1(s)\right|^{2q+1} + \left|r_2(s)\right|^{2q+1}\right)\mathrm{d}s\right] \\ &\le \frac{4q^2}{2q+1}\theta\mathbb{E}\left[\sup_{n\theta\le t\le(n+1)\theta} w(t)^{q+\frac{1}{2}}\right] + 2c\theta\mathbb{E}\left[\sup_{n\theta\le t\le(n+1)\theta} w(t)^q\right] + \frac{q}{2q+1}\theta\mathbb{E}\left[\sup_{n\theta\le t\le(n+1)\theta}\left(\left|r_1(t)\right|^{2q+1} + \left|r_2(t)\right|^{2q+1}\right)\right]. \end{aligned}$$

Choose θ sufficiently small such that $I < h(q)$, therefore,

$$\mathbb{E}\left[\sup_{n\theta\le t\le(n+1)\theta} w^q(t)\right] \le 2h(q).$$

Let ε be an arbitrary positive constant. Based on Chebyshev's inequality, it follows that

$$\mathbb{P}\left\{\sup_{n\theta\le t\le(n+1)\theta} w^q(t) > (n\theta)^{1+\varepsilon}\right\} \ge \frac{2h(q)}{(n\theta)^{1+\varepsilon}}, n = 1, 2, \cdots .$$

By the Borel–Cantelli Lemma, there exists an integer-valued random variable $n_0(\omega)$ such that for almost all $\omega \in \Omega$, when $n \ge n_0$, we have

$$\sup_{n\theta\le t\le(n+1)\theta} w^q(t) \le (n\theta)^{1+\varepsilon}.$$

Hence, for almost all $\omega \in \Omega$, if $n \ge n_0$ and $n\theta \le t \le (n+1)\theta$, we have

$$\limsup_{t\to\infty}\frac{\ln w^q(t)}{\ln t} \le \limsup_{t\to\infty}\frac{(1+\varepsilon)\ln(n\theta)}{\ln(n\theta)},$$

Let $\varepsilon \to 0$. We have

$$\limsup_{t\to\infty}\frac{\ln w^q(t)}{\ln t} \le 1, a.s.,$$

then

$$\limsup_{t\to\infty}\frac{\ln w(t)}{\ln t} \le \frac{1}{q}, a.s..$$

Thus,

$$\limsup_{t\to\infty}\frac{\ln w(t)}{t} \le \limsup_{t\to\infty}\frac{\ln w(t)}{\ln t} \times \limsup_{t\to\infty}\frac{\ln t}{t} \le 0,$$

and it follows that

$$\limsup_{t\to\infty}\frac{\ln x(t)}{t} \le 0, \limsup_{t\to\infty}\frac{\ln y(t)}{t} \le 0.$$

Lemma 4.7. If Assumption 2.1 holds, $\bar{r}_1 + \sum_{i=1}^{p} h_1(i)\pi_i > 0, \bar{r}_2 + \sum_{i=1}^{p} h_2(i)\pi_i > 0$, where $h_m(i) = \int_Z \ln(1+\gamma_m(i,z))\,\nu(\mathrm{d}z), m = 1,2$, then populations $x(t), y(t)$ are weakly persistent.
Proof According to the definition of weakly persistent, we need to prove $\limsup_{t\to\infty} x(t) > 0, \limsup_{t\to\infty} y(t) > 0$. If the conclusion is not true, then $\mathbb{P}(U) > 0$, where $U = \left\{\omega : \limsup_{t\to\infty} x(t,\omega) = 0, \limsup_{t\to\infty} y(t,\omega) = 0\right\}$. Applying Itô formula to $\ln x(t), \ln y(t)$, and integrating from 0 to t, we have

$$\begin{aligned}
\frac{\ln x(t)}{t} &= \frac{\ln x(0)}{t} + \frac{1}{t}\int_0^t \Big(r_1 - a_{11}(s)x^{\theta_1} - a_{12}(s)y\Big)\,\mathrm{d}s + \frac{1}{t}\int_0^t \int_Z \ln(1+\gamma_1(\xi(s),z))\,\nu(\mathrm{d}z)\mathrm{d}s + \frac{M_1(t)}{t},\\
\frac{\ln y(t)}{t} &= \frac{\ln y(0)}{t} + \frac{1}{t}\int_0^t \Big(r_2 - a_{21}(s)x - a_{22}(s)y^{\theta_2}\Big)\,\mathrm{d}s + \frac{1}{t}\int_0^t \int_Z \ln(1+\gamma_2(\xi(s),z))\,\nu(\mathrm{d}z)\mathrm{d}s + \frac{M_2(t)}{t},
\end{aligned} \tag{4.19}$$

where

$$M_1(t) = \int_0^t \int_Z \ln(1+\gamma_1(\xi(s),z))\,\tilde{N}(\mathrm{d}s,\mathrm{d}z), M_2(t) = \int_0^t \int_Z \ln(1+\gamma_2(\xi(s),z))\,\tilde{N}(\mathrm{d}s,\mathrm{d}z).$$

By Assumption 2.1,

$$\begin{aligned}
\langle M_1, M_1\rangle(t) &= \int_0^t \int_Z [\ln(1+\gamma_1(\xi(s),z))]^2\,\nu(\mathrm{d}z)\mathrm{d}s < ct,\\
\langle M_2, M_2\rangle(t) &= \int_0^t \int_Z [\ln(1+\gamma_2(\xi(s),z))]^2\,\nu(\mathrm{d}z)\mathrm{d}s < ct.
\end{aligned}$$

From Lemma 4.5, we achieve

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = 0, \lim_{t\to\infty} \frac{M_2(t)}{t} = 0.$$

On the one hand, combining the strong law of large numbers [31] and the definition of the OU process, we have

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t r_1(s)\mathrm{d}s = \bar{r}_1, \lim_{t\to\infty} \frac{1}{t}\int_0^t r_2(s)\mathrm{d}s = \bar{r}_2.$$

If for all $\omega \in U$, $\limsup_{t\to\infty} x(t,\omega) = 0, \limsup_{t\to\infty} y(t,\omega) = 0$. Combining formula (4.19), we have

$$0 \geq \limsup_{t\to\infty} \frac{\ln x(t,\omega)}{t} = \bar{r}_1 + \sum_{i=1}^{p} h_1(i)\pi_i > 0, 0 \geq \limsup_{t\to\infty} \frac{\ln y(t,\omega)}{t} = \bar{r}_2 + \sum_{i=1}^{p} h_2(i)\pi_i > 0.$$

This contradicts the assumption $\mathbb{P}(U) > 0$, then $\limsup_{t\to\infty} x(t) > 0, \limsup_{t\to\infty} y(t) > 0$.

Theorem 4.1. If Assumption 2.1 and Assumption 4.1 hold , and $\bar{r}_1 + \sum_{i=1}^{p} h_1(i)\pi_i > 0, \bar{r}_2 + \sum_{i=1}^{p} h_2(i)\pi_i > 0$, where $h_m(i) = \int_Z \ln(1+\gamma_m(i,z))\,\nu(\mathrm{d}z), m = 1,2$, then the model (1.4) has a unique ergodic stationary distribution.
Proof To prove Theorem 4.1, first prove that there is a probability measure $\eta(\cdot\times\cdot) \in B$, such that for any $(a, j) \in \mathbb{R}_+^2 \times \mathbb{R}^2 \times \mathbb{S}$, the transition probability $p(t, a, j, \cdot\times\cdot)$ for $X^{a,j}(t)$ converges weakly to $\eta(\cdot\times\cdot)$.

According to Proposition 2.5 [32], weak convergence of probability measures is the concept of a metric, i.e., $p(t, a, j, \cdot \times \cdot)$ weakly converges to $\eta(\cdot \times \cdot)$ is equivalent to the existence of metric d such that $\lim_{t\to+\infty} d\,(p(t, a, j, \cdot \times \cdot), \eta(\cdot \times \cdot)) = 0$.

So, we only need to prove that to any $(a, j) \in \mathbb{R}_+^2 \times \mathbb{R}^2 \times \mathbb{S}$, there is

$$\lim_{t\to+\infty} d_H\,(p(t, a, j, \cdot \times \cdot), \eta(\cdot \times \cdot)) = 0.$$

From Lemma 4.4, $\{p(t, 0, 1, \cdot \times \cdot \mid t \geq 0)\}$ is the Cauchy in the space $B(\mathbb{R}_+^2 \times \mathbb{R}^2 \times \mathbb{S})$ of metric d_H, so there is a unique $\eta(\cdot \times \cdot) \in B$ that makes

$$\lim_{t\to+\infty} d_H\,(p(t, 0, 1, \cdot \times \cdot), \eta(\cdot \times \cdot)) = 0.$$

By Lemma 4.3 and the triangle inequality, we have

$$\lim_{t\to+\infty} d_H\,(p(t, a, j, \cdot \times \cdot), \eta(\cdot \times \cdot)) \leq \lim_{t\to+\infty} \left[d_H\,(p(t, a, j, \cdot \times \cdot), p(t, 0, 1, \cdot \times \cdot)) + d_H\,(p(t, 0, 1, \cdot \times \cdot), \eta(\cdot \times \cdot))\right] = 0.$$

That is, the distribution of $(X(t), \xi(t))$ weakly converges to η.

By the Kolmogorov-Chapman equation, we know that η is constant. From Corollary 3.4.3 [33], it follows that η is strongly mixed. From Theorem 3.2.6 [33], we know that η is ergodic.

5. Extinction

Theorem 5.1. If Assumption 2.1 holds, for any initial value $(x(0), y(0), r_1(0), r_2(0)) \in \mathbb{R}_+^2 \times \mathbb{R}^2$ and $\xi(0) \in S$, the solution $(x(t), y(t), r_1(t), r_2(t))$ of system (1.4) has the property that

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} \leq \bar{r}_1 + \sum_{i=1}^{p} h_1(i)\pi_i, \quad \limsup_{t\to\infty} \frac{\ln y(t)}{t} \leq \bar{r}_2 + \sum_{i=1}^{p} h_2(i)\pi_i.$$

where $h_m(i) = \int_Z \ln\,(1 + \gamma_m(i, z))\,\nu(\mathrm{d}z), m = 1, 2$. In particular, if $\bar{r}_1 + \sum_{i=1}^{p} h_1(i)\pi_i < 0, \bar{r}_2 + \sum_{i=1}^{p} h_2(i)\pi_i < 0$, then $x(t), y(t)$ are extinct.

Proof Applying Itô formula to $\ln x(t), \ln y(t)$, we can get

$$\begin{aligned}
\mathrm{d}\ln x(t) &= \left(r_1 - a_{11}(t)x^{\theta_1} - a_{12}(t)y\right)\mathrm{d}t + \int_Z \ln\,(1 + \gamma_1(\xi(t), z))\,N(\mathrm{d}t, \mathrm{d}z) \\
&= \left(r_1 - a_{11}(t)x^{\theta_1} - a_{12}(t)y\right)\mathrm{d}t + \int_Z \ln\,(1 + \gamma_1(\xi(t), z))\,\nu(\mathrm{d}z)\mathrm{d}t + \int_Z \ln\,(1 + \gamma_1(\xi(t), z))\,\tilde{N}(\mathrm{d}t, \mathrm{d}z), \\
\mathrm{d}\ln y(t) &= \left(r_2 - a_{21}(t)x - a_{22}(t)y^{\theta_2}\right)\mathrm{d}t + \int_Z \ln\,(1 + \gamma_2(\xi(t), z))\,N(\mathrm{d}t, \mathrm{d}z) \\
&= \left(r_2 - a_{21}(t)x - a_{22}(t)y^{\theta_2}\right)\mathrm{d}t + \int_Z \ln\,(1 + \gamma_2(\xi(t), z))\,\nu(\mathrm{d}z)\mathrm{d}t + \int_Z \ln\,(1 + \gamma_2(\xi(t), z))\,\tilde{N}(\mathrm{d}t, \mathrm{d}z).
\end{aligned}$$

Integrating from 0 to t, we have

$$\ln x(t) = \ln x(0) + \int_0^t \left(r_1 - a_{11}(s)x^{\theta_1} - a_{12}(s)y\right)\mathrm{d}s + \int_0^t \int_Z \ln\,(1 + \gamma_1(\xi(s), z))\,\nu(\mathrm{d}z)\mathrm{d}s + M_1(t), \quad (5.1)$$

$$\ln y(t) = \ln y(0) + \int_0^t \left(r_2 - a_{21}(s)x - a_{22}(s)y^{\theta_2}\right) \mathrm{d}s + \int_0^t \int_Z \ln\left(1 + \gamma_2(\xi(s), z)\right) \nu(\mathrm{d}z)\mathrm{d}s + M_2(t), \quad (5.2)$$

where

$$M_1(t) = \int_0^t \int_Z \ln\left(1 + \gamma_1(\xi(s), z)\right) \tilde{N}(\mathrm{d}s, \mathrm{d}z), M_2(t) = \int_0^t \int_Z \ln\left(1 + \gamma_2(\xi(s), z)\right) \tilde{N}(\mathrm{d}s, \mathrm{d}z).$$

By Assumption 2.1,

$$\langle M_1, M_1 \rangle (t) = \int_0^t \int_Z \left[\ln\left(1 + \gamma_1(\xi(s), z)\right)\right]^2 \nu(\mathrm{d}z)\mathrm{d}s < ct,$$
$$\langle M_2, M_2 \rangle (t) = \int_0^t \int_Z \left[\ln\left(1 + \gamma_2(\xi(s), z)\right)\right]^2 \nu(\mathrm{d}z)\mathrm{d}s < ct.$$

From Lemma 4.5, we achieve

$$\lim_{t\to\infty} \frac{M_1(t)}{t} = 0, \lim_{t\to\infty} \frac{M_2(t)}{t} = 0.$$

On the one hand, combining the strong law of large numbers [31] and the definition of the OU process, we have

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t r_1(s)\mathrm{d}s = \bar{r}_1, \lim_{t\to\infty} \frac{1}{t} \int_0^t r_2(s)\mathrm{d}s = \bar{r}_2.$$

According to (5.1) and (5.2), we obtain

$$\ln x(t) \le \ln x(0) + \int_0^t r_1(s)\mathrm{d}s + M_1(t) + \int_0^t h_1(\xi(s))\mathrm{d}s,$$
$$\ln y(t) \le \ln y(0) + \int_0^t r_2(s)\mathrm{d}s + M_2(t) + \int_0^t h_2(\xi(s))\mathrm{d}s.$$

then

$$\frac{\ln x(t)}{t} \le \frac{\ln x(0)}{t} + \frac{\int_0^t r_1(s)\mathrm{d}s}{t} + \frac{M_1(t)}{t} + \frac{\int_0^t h_1(\xi(s))\mathrm{d}s}{t},$$
$$\frac{\ln y(t)}{t} \le \frac{\ln y(0)}{t} + \frac{\int_0^t r_2(s)\mathrm{d}s}{t} + \frac{M_2(t)}{t} + \frac{\int_0^t h_2(\xi(s))\mathrm{d}s}{t}.$$

According to the ergodicity of the Markov chain, we obtain

$$\limsup_{t\to\infty} \frac{\ln x(t)}{t} \le \limsup_{t\to\infty} \frac{\int_0^t r_1(s)\mathrm{d}s}{t} + \limsup_{t\to\infty} \frac{\int_0^t h_1(\xi(s))\mathrm{d}s}{t} = \bar{r}_1 + \sum_{i=1}^{p} h_1(i)\pi_i,$$
$$\limsup_{t\to\infty} \frac{\ln y(t)}{t} \le \limsup_{t\to\infty} \frac{\int_0^t r_2(s)\mathrm{d}s}{t} + \limsup_{t\to\infty} \frac{\int_0^t h_2(\xi(s))\mathrm{d}s}{t} = \bar{r}_2 + \sum_{i=1}^{p} h_2(i)\pi_i.$$

When $\bar{r}_1 + \sum_{i=1}^{p} h_1(i)\pi_i < 0, \bar{r}_2 + \sum_{i=1}^{p} h_2(i)\pi_i < 0$, it implies $\lim_{t\to\infty} x(t) = 0, \lim_{t\to\infty} y(t) = 0$, then $x(t), y(t)$ are extinct. Theorem 5.1 is proved.

Remark 5.1. Lemma 4.7 and Theorems 4.1 and 5.1 have very important biological explanations.

From the theoretical results obtained, it can be seen that when $\bar{r}_1 + \sum_{i=1}^{p} h_1(i)\pi_i > 0, \bar{r}_2 + \sum_{i=1}^{p} h_2(i)\pi_i > 0$, population $x(t), y(t)$ will be weakly persistent, and if the parameters of the model are controlled within a certain range, the system has a stationary distribution, which indicates the persistence of population growth. When $\bar{r}_1 + \sum_{i=1}^{p} h_1(i)\pi_i < 0, \bar{r}_2 + \sum_{i=1}^{p} h_2(i)\pi_i < 0$, population $x(t), y(t)$ will be extinct. That is, the survival and extinction of the biological population $x(t), y(t)$ simulated by model (1.4) completely depends on the symbol of $\bar{r}_1 + \sum_{i=1}^{p} h_1(i)\pi_i, \bar{r}_2 + \sum_{i=1}^{p} h_2(i)\pi_i$.

The following analyzes the effects of white noise simulated by the OU process on species survival and extinction. Since the OU process acts on the intrinsic growth rate r_1, r_2, if the model (1.4) is not affected by Markov switching and jump noise, then the model is in the following form:

$$\begin{cases} \mathrm{d}x(t) = x(t)\Big(r_1(t) - a_{11}(t)x^{\theta_1}(t) - a_{12}(t)y(t)\Big)\mathrm{d}t \\ \mathrm{d}y(t) = y(t)\Big(r_2(t) - a_{21}(t)x(t) - a_{22}(t)y^{\theta_2}(t)\Big)\mathrm{d}t \\ \mathrm{d}r_1(t) = \beta_1[\bar{r}_1 - r_1(t)]\mathrm{d}t + \sigma_1\mathrm{d}B_1(t) \\ \mathrm{d}r_2(t) = \beta_2[\bar{r}_2 - r_2(t)]\mathrm{d}t + \sigma_2\mathrm{d}B_2(t), \end{cases} \tag{5.3}$$

Using a similar method as above, it can be proven that when $\bar{r}_1 > 0, \bar{r}_2 > 0$, population $x(t), y(t)$ are weakly persistent; when $\bar{r}_1 < 0, \bar{r}_2 < 0$, population $x(t), y(t)$ are extinct. That is, when the system is only disturbed by white noise, the survival and extinction of the population is only related to the symbol of the average growth rate $\bar{r}_1, \bar{r}_2$ of the population.

When $\bar{r}_1 > 0, \bar{r}_2 > 0$, the species only disturbed by white noise are weakly persistent. If the system is affected by jump noise and Markov switching again, and satisfies $\bar{r}_1 + \sum_{i=1}^{p} h_1(i)\pi_i < 0, \bar{r}_2 + \sum_{i=1}^{p} h_2(i)\pi_i < 0$, the species are extinct. When $\bar{r}_1 < 0, \bar{r}_2 < 0$, the species that are only disturbed by white noise are extinct, but if there is jump noise and Markov switching such that $\bar{r}_1 + \sum_{i=1}^{p} h_1(i)\pi_i > 0, \bar{r}_2 + \sum_{i=1}^{p} h_2(i)\pi_i > 0$, species are weakly persistent. Therefore, it can be obtained that jump noise and Markov switching can make the survival system extinct and the extinction system survive.

Remark 5.2. The following is to analyze the effect of jump diffusion coefficient $\gamma_1(i,z), \gamma_2(i,z)$ on population survival and extinction. If $\gamma_1(k,z) < 0, \gamma_2(k,z) < 0$, then $h_1(i) < 0, h_2(i) < 0$, which means that jumping noise is detrimental to the survival of the population. If $\gamma_1(i,z) > 0, \gamma_2(i,z) > 0$, then $h_1(i) > 0, h_2(i) > 0$, which means that jumping noise is beneficial to the survival of the population.

Remark 5.3. Now, consider the subsystem of the system in a certain state i

$$\begin{cases} \mathrm{d}x(t) = x(t^-)\Big[\Big(r_1(t) - a_{11}(t)x^{\theta_1}(t^-) - a_{12}(t)y(t^-)\Big)\mathrm{d}t + \int_Z \gamma_1(i,z)N(\mathrm{d}t,\mathrm{d}z)\Big] \\ \mathrm{d}y(t) = y(t^-)\Big[\Big(r_2(t) - a_{21}(t)x(t^-) - a_{22}(t)y^{\theta_2}(t^-)\Big)\mathrm{d}t + \int_Z \gamma_2(i,z)N(\mathrm{d}t,\mathrm{d}z)\Big] \\ \mathrm{d}r_1(t) = \beta_1[\bar{r}_1 - r_1(t)]\mathrm{d}t + \sigma_1\mathrm{d}B_1(t) \\ \mathrm{d}r_2(t) = \beta_2[\bar{r}_2 - r_2(t)]\mathrm{d}t + \sigma_2\mathrm{d}B_2(t), \end{cases} \tag{5.4}$$

Using the above similar proof method, it can be obtained that if $\bar{r}_1 + h_1(i) > 0, \bar{r}_2 + h_2(i) > 0$, the population $x(t), y(t)$ described by the system (5.4) will be weakly persistent, and if $\bar{r}_1 + h_1(i) < 0, \bar{r}_2 + h_2(i) < 0$, then the population $x(t), y(t)$ will be extinct.

Remark 5.4. Now, we analyze the impact of Markov switching on population survival and extinction. If for a certain state $i \in S$, there is $\bar{r}_1 + h_1(i) < 0, \bar{r}_2 + h_2(i) < 0$, then the corresponding subsystem

(5.4) is extinct. It can be seen from Theorem 5.1 that if every subsystem of system (1.4) is extinct, then the result of Markov switching is that system (1.4) is still extinct. However, Lemma 4.7 and Theorems 4.1 and 5.1 reveal an interesting phenomenon: If some subsystems are extinct and some are weakly persistent, the value of the total system after Markov mixing may be greater than zero. At this time, the whole system is weakly persistent, so we can see the adjustment effect of Markov switching on the survival condition of the whole population survival system.

6. Computer simulations

In order to verify the above theoretical results on the stochastic Gilpin-Ayala nonautonomous competition model (1.4), we use the Euler-Maruyama method [34] and the R language and select the appropriate parameters for numerical verification. Given that the state space of Markov chain $\xi(t)$ is $S = \{1, 2, 3\}$ and the generated matrix is $Q = \begin{pmatrix} -4 & 2 & 2 \\ 1 & -2 & 1 \\ 2 & 3 & -5 \end{pmatrix}$, the only stationary distribution of the Markov chain $\xi(t)$ is $\pi = (\pi_1, \pi_2, \pi_3) = \left(\frac{7}{29}, \frac{16}{29}, \frac{6}{29}\right)$. The combination of parameters is shown in Table 1, and the data is from [27, 35–38].

According to literature [38], in this section, we consider the survival conditions of the competition system (1.4) of two grassland populations and discuss the effect of stochastic interference on two grassland species competition in system (1.4). In model (1.4), $x(t)$ represents the numbers of *Phleum pratense* and $y(t)$ represents the numbers of *Trifolium repens*.

Table 1. Several combinations of biological parameters of model (1.4).

Combinations	Value
$\mathcal{A}_1$	$a_{11}(t) = 0.4 + 0.1\sin t, a_{12}(t) = 0.2\sin^2 t, a_{21}(t) = 0.07\cos^2 t, a_{22}(t) = 1.5(1 - 0.7\cos t), \theta_1 = 2, \theta_2 = 1, \gamma_1(1) = 0.1, \gamma_1(2) = 0.4, \gamma_1(3) = 0.2, \gamma_2(1) = 0.2, \gamma_2(2) = 0.3, \gamma_2(3) = 0.1, \beta_1 = 0.4, \beta_2 = 0.2, \bar{r}_1 = 0.103, \bar{r}_2 = 0.076, \sigma_1 = 0.3, \sigma_2 = 0.1$
$\mathcal{A}_2$	$a_{11}(t) = 0.4 + 0.1\sin t, a_{12}(t) = 0.2\sin^2 t, a_{21}(t) = 0.5\cos^2 t, a_{22}(t) = 0.5(1 - 0.4\cos t), \theta_1 = 2, \theta_2 = 2, \gamma_1(1) = 0.3, \gamma_1(2) = 0.2, \gamma_1(3) = 0.1, \gamma_2(1) = 0.2, \gamma_2(2) = 0.3, \gamma_2(3) = 0.4, \beta_1 = 0.4, \beta_2 = 0.2, \bar{r}_1 = 0.103, \bar{r}_2 = 0.076, \sigma_1 = 0.3, \sigma_2 = 0.2, q = 2$
$\mathcal{A}_3$	$a_{11}(t) = 0.4 + 0.1\sin t, a_{12}(t) = 0.2\sin^2 t, a_{21}(t) = 0.07\cos^2 t, a_{22}(t) = 1.5(1 - 0.7\cos t), \theta_1 = 2, \theta_2 = 1.5, \gamma_1(1) = 0.1, \gamma_1(2) = 0.2, \gamma_1(3) = 0.3, \gamma_2(1) = 0.4, \gamma_2(2) = 0.3, \gamma_2(3) = 0.2, \beta_1 = 1.3, \beta_2 = 1.1, \bar{r}_1 = 0.103, \bar{r}_2 = 0.076, \sigma_1 = 0.2, \sigma_2 = 0.1$
$\mathcal{A}_4$	$a_{11}(t) = 0.4 + 0.1\sin t, a_{12}(t) = 0.2\sin^2 t, a_{21}(t) = 0.07\cos^2 t, a_{22}(t) = 1.5(1 - 0.7\cos t), \theta_1 = 0.3, \theta_2 = 0.5, \gamma_1(1) = -0.11, \gamma_1(2) = -0.15, \gamma_1(3) = -0.3, \gamma_2(1) = -0.13, \gamma_2(2) = -0.19, \gamma_2(3) = -0.3, \beta_1 = 0.4, \beta_2 = 0.2, \bar{r}_1 = -0.13, \bar{r}_2 = -0.1, \sigma_1 = 0.02, \sigma_2 = 0.05$

Example 6.1. Let $v(Z) = 1$. Take the initial value of model (1.4) as $x(0) = 0.3, y(0) = 0.2, r_1(0) =$

$0.12, r_2(0) = 0.21$, choose the combination $\mathcal{A}_1$ as the parameter value of model (1.4), and use the R language for numerical simulation, and Figure 1 is obtained. The numerical simulation results show that the global solution of the stochastic Gilpin-Ayala population model (1.4) exists, and Theorem 2.1 is verified.

The blue lines in Figure 1(a),(b) show the trend of $x(t), y(t)$, whose growth rate is disturbed by the OU process. The green lines in Figure 1(a),(b) represent the global solution of $x(t), y(t)$ under the disturbance of the OU process, Markov chains, and Lévy noise. Combined with Theorem 2.1, it can be seen that *Phleum pratense* and *Trifolium repens* will continue to grow no matter what environmental disturbance populations *Phleum pratense* and *Trifolium repens* are subjected to. The red lines in Figure 1(c),(d) represent intrinsic growth rates r_1, r_2 of *Phleum pratense* and *Trifolium repens*, while the blue lines in Figure 1(c),(d) represent population growth rates disturbed by the OU process, indicating that the interference of random environmental factors will make the growth rate $r_1(t), r_2(t)$ fluctuate randomly under the interference of the OU process.

In the two grassland species competition system (1.4), we assume that Lévy jumps are affected by the Markov chains and the Lévy jump values are both positive. According to remark 5.2, it indicates that the jump noise plays a role in promoting the population growth. It can also be seen from Figure 1(a),(b) that under the action of positive Lévy jumps, the population number represented by the green lines are larger than the blue lines at the same time. We compared Figures 1(a),(b) in the two grassland species competition system (1.4), the numbers of *Phleum pratense* (Figure 1(a)) is more than the numbers of *Trifolium repens* (Figure 1(b)), which indicates that *Phleum pratense* has a more competitive advantage in the resource competition, and the number of *Trifolium repens* will remain lower than that of *Phleum pratense*.

Example 6.2. Let $\nu(Z) = 1$. Take the initial value of model (1.4) as $x(0) = 0.1, y(0) = 0.1, r_1(0) = 0.12, r_2(0) = 0.13$, choose the combination $\mathcal{A}_2$ as the parameter value of model (1.4), and use the R language for numerical simulation, and Figure 2 is obtained. In Figure 2(a), $\kappa(q)$ denotes a continuous function with respect to q, and $\mathbb{E}(x^q), \mathbb{E}(y^q)$ denote the q-th moment of $x(t), y(t)$. The numerical simulation results show that $\mathbb{E}(x^q), \mathbb{E}(y^q)$ are less than $\kappa(q)$, so $\mathbb{E}(x^q) \leq \kappa(q), \mathbb{E}(y^q) \leq \kappa(q), q > 0$ hold and Theorem 3.1 is verified.

On the other hand, from the biological point of view, due to the limited resources in the natural environment, no biological population can grow without limit, so we hope that the solution of the two grassland species competition system (1.4) is ultimately bounded. Therefore, in Figure 2, we set $q = 2$ to get $\mathbb{E}(x^2) \leq \kappa(2), \mathbb{E}(y^2) \leq \kappa(2)$, which indicates that the system (1.4) is bounded by second order moments, and population *Trifolium repens* and *Phleum pratense* will not grow wildly and maintain a healthy growth, which obeys the significance of biology.

Example 6.3. Let $\nu(Z) = \frac{1}{2}$. Take the initial value of model (1.4) as $x(0) = 0.3, y(0) = 0.2, r_1(0) = 0.12, r_2(0) = 0.21$, choose the combination $\mathcal{A}_3$ as the parameter value of model (1.4), and use the R language for numerical simulation, and Figure 3 is obtained. Figure 3(a),(c) represent the solution of $x(t), y(t)$, and Figure 3(b),(d) represent the histogram of the solution of $x(t), y(t)$. The numerical simulation results show that when $\bar{r}_1 + \sum_{i=1}^{3} h_1(i)\pi_i > 0, \bar{r}_2 + \sum_{i=1}^{3} h_2(i)\pi_i > 0$, $x(t), y(t)$ obey normal distribution approximately, it means model (1.4) has a stationary distribution η, and Theorem 4.1 holds.

From Figure 3(a),(c), it can be seen that the population size of *Phleum pratense* $x(t)$ is mostly between 0.75–1.0 and the population size of *Trifolium repens* $y(t)$ is mostly between 0.3–0.4, mainly

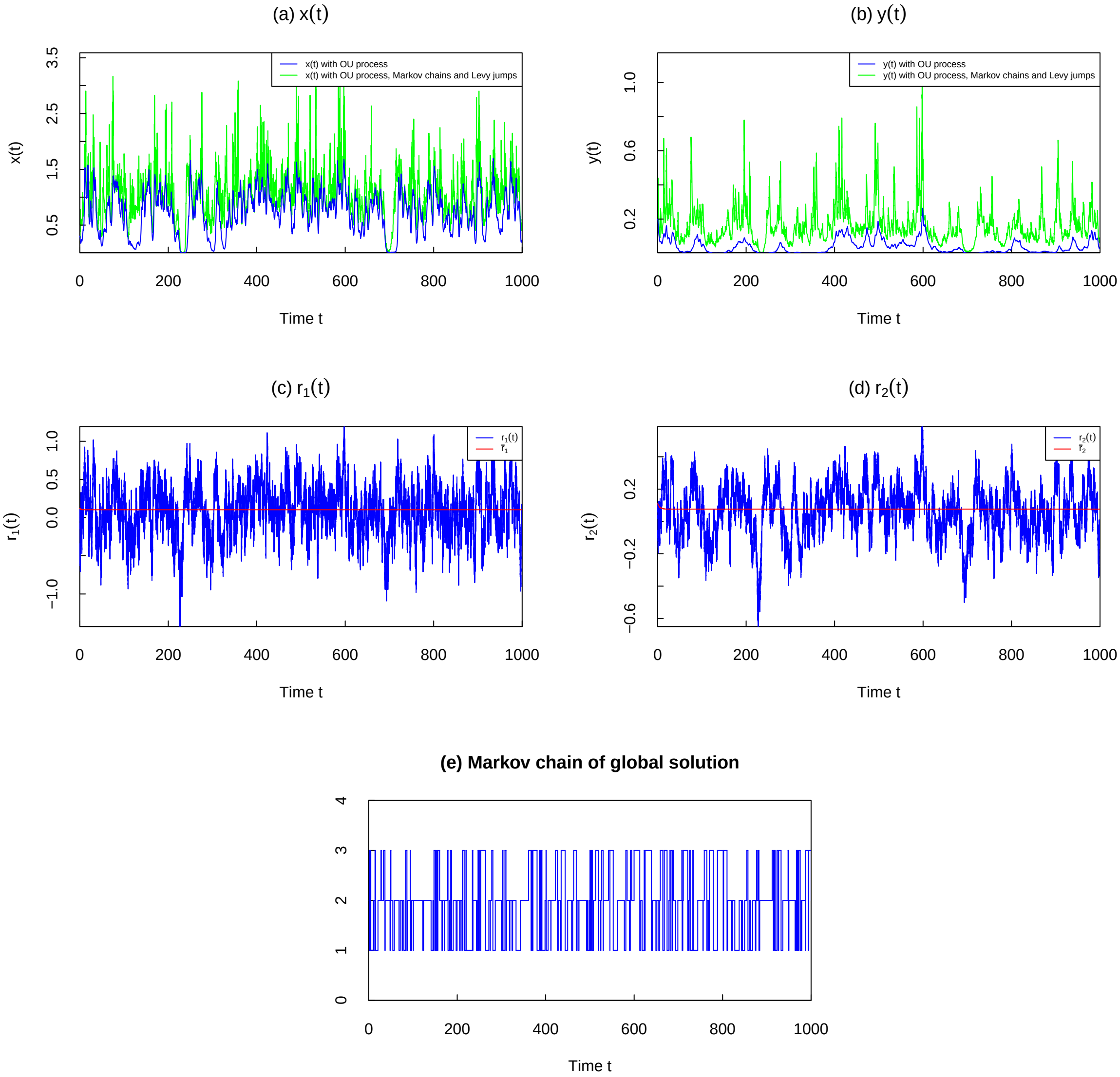

Figure 1. Global solution.

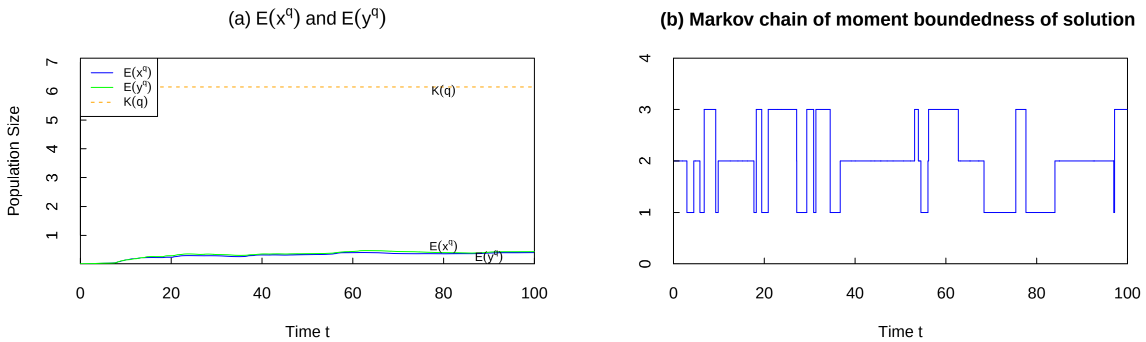

Figure 2. Moment boundedness of solution.

concentrated in the middle region. Figure 3(b),(d) of the frequency histogram of populations $x(t), y(t)$ also shows a trend of high in the middle and low at both ends and obeys normal distribution approximately.

In a biological sense, this indicates the long-term development trend of the two grassland species competition system (1.4) and represents the persistence of species *Phleum pratense* and *Trifolium repens*; that is, although the model (1.4) is a competitive system, if Assumption 4.1 and $\bar{r}_1 + \sum_{i=1}^{3} h_1(i)\pi_i > 0, \bar{r}_2 + \sum_{i=1}^{3} h_2(i)\pi_i > 0$ hold, then the population *Phleum pratense* and *Trifolium repens* will continue to grow and eventually reach a stable state.

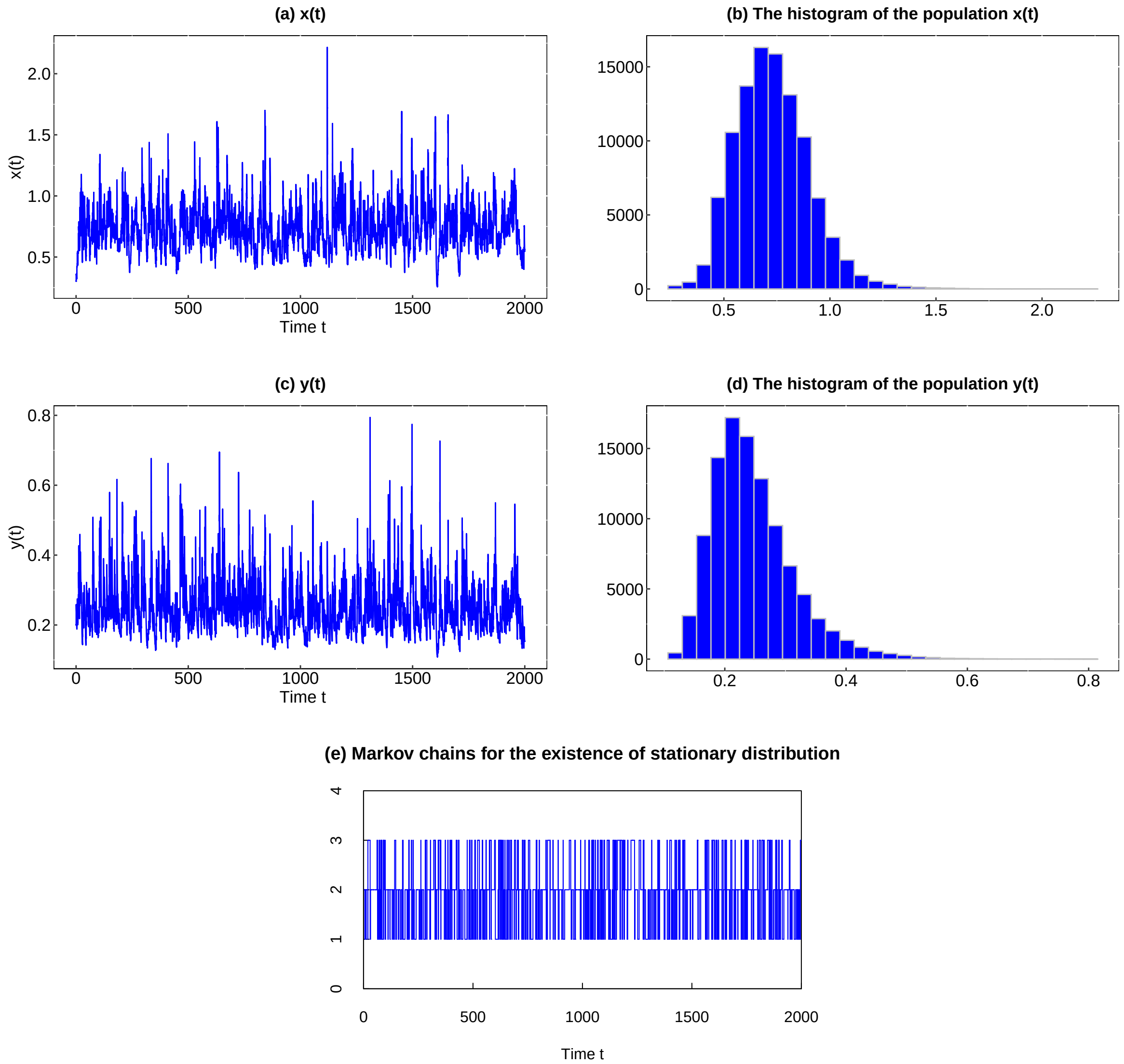

Figure 3. Existence of stationary distribution.

Example 6.4. Let $v(Z) = 1$. Take the initial value of model (1.4) as $x(0) = 0.5, y(0) = 0.5, r_1(0) = 0.1, r_2(0) = 0.1$, choose the combination $\mathcal{A}_4$ as the parameter value of model (1.4), and use the R language for numerical simulation, and Figure 4 is obtained. Calculate $\bar{r}_1 + \sum_{i=1}^{3} h_1(i)\pi_i \approx -0.21 < 0, \bar{r}_2 + \sum_{i=1}^{3} h_2(i)\pi_i \approx -0.25 < 0$. According Theorem 5.1, population $x(t), y(t)$ will be extinct.

The blue lines in Figure 4 (a) and (b) show the populations *Phleum pratense* and *Trifolium repens*, whose growth rate is disturbed by the OU process. This indicated that if populations *Phleum pratense* and *Trifolium repens* are affected by some subtle disturbance in the environment, such as changes in soil pH, nutrient density, etc., and $\bar{r}_1 < 0, \bar{r}_2 < 0$, then population *Phleum pratense* and *Trifolium repens* extinct at $t = 25$.

The green lines in Figure 4(a),(b) represent the global solution of the population under the disturbance of the OU process, Markov chains, and Lévy noise. *Phleum pratense* extinct at $t = 15$ and *Trifolium repens* extinct at $t = 10$. In this example, we let $\gamma_1(i,z) < 0, \gamma_2(i,z) < 0, i = 1,2,3$, which means populations Phleum pratense and Trifolium repens are subject to some adverse external disturbances, such as fires, floods, etc., then according remark 5.2, these adverse random disturbances in the environment will accelerate the extinction of populations *Phleum pratense* and *Trifolium repens*. This is also consistent with the phenomenon described in Figure 4(a),(b).

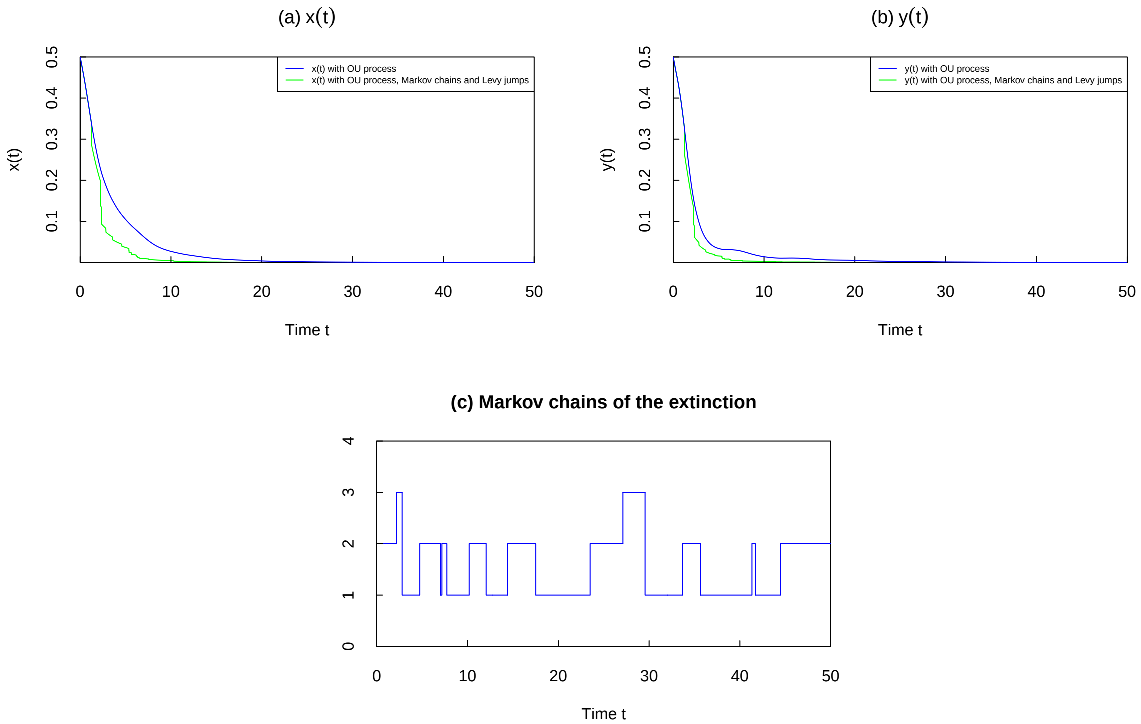

Figure 4. Extinction.

7. Conclusions

In this paper, we study the dynamic behaviors of a stochastic Gilpin-Ayala nonautonomous competition model (1.4) driven by the mean-reverting OU process with finite Markov chain and Lévy jumps. The existence and uniqueness of the global solution, the moment boundedness of the solution, the existence of the stationary distribution, and extinction of the stochastic Gilpin-Ayala nonautonomous competition model (1.4) are proved. The properties of the stochastic Gilpin-Ayala nonautonomous competition model (1.4) are verified by numerical examples. The existence and uniqueness of the global solution and the moment boundedness of the solution shows that under the interference of various random factors and the change of biological parameters with time, the population shows a fluctuating growth trend, and the intrinsic growth rate also fluctuates around the mean level; for any $q > 0$, the populations $x(t)$ and $y(t)$ have bounded q-th moments. The existence of stationary distribution and the extinction indicate that Markov chain and Lévy jump have a crucial impact on population growth. Combined with numerical examples of population parameters, we find that: The existence of stationary distribution indicates that when the jumping noise coefficient of the population is positive under different environmental states, combined with the average growth rate of the population, that is, $\bar{r}_1 + \sum_{i=1}^{3} h_1(i)\pi_i > 0, \bar{r}_2 + \sum_{i=1}^{3} h_2(i)\pi_i > 0$, it can be seen that the impact of jumping noise on the population under different states is beneficial to the population. The system solution has a stationary distribution, which indicates the persistence of population growth; extinction indicates that when the jumping noise coefficient of the population is negative under different environmental states, combined with the average growth rate of the population, that is, $\bar{r}_1 + \sum_{i=1}^{3} h_1(i)\pi_i < 0, \bar{r}_2 + \sum_{i=1}^{3} h_2(i)\pi_i < 0$, the impact of jumping noise under different states of the population is adverse to the population, and the population will be extinct.

However, in model (1.4), we only considered the effects of the OU process, finite Markov chain, and Lévy jumps on the survival of the population, and there were many factors that we did not consider, such as seasonal changes bring periodic changes, time delays, etc.. Therefore, in future work, we will consider more complex and realistic models to study population ecology.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This study was supported by National Natural Science Foundation of China (11401085), Heilongjiang Province Postdoctoral Funding Program (LBH-Q21059), Fundamental Research Projects of Chinese Central Universities (2572021DJ04).

Conflict of interest

The authors declare there is no conflicts of interest.

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