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Research article

Eigenvalue properties of Sturm-Liouville problems with transmission conditions dependent on the eigenparameter

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Abstract: This paper studies a discontinuous Sturm-Liouville problem in which the spectral parameter appears not only in the differential equation but also in the transmission conditions. By constructing an appropriate Hilbert space and inner product, the eigenvalue and eigenfunction problems of the Sturm-Liouville problem are transformed into an eigenvalue problem of a certain self-adjoint operator. Next, the eigenfunctions of the problem and some properties of the eigenvalues are given via construction of the basic solution. The Green's function for the Sturm-Liouville problem is also given. Finally, the continuity of the eigenvalues and eigenfunctions of the problem is discussed. Especially, the differential expressions of the eigenvalues for some parameters have been obtained, including the parameters in the eigenparameter-dependent transmission conditions.

Keywords: Sturm-Liouville problems; eigenparameter-dependent transmission conditions; Green's function; dependence of eigenvalues

1. Introduction

Sturm's theory is one of the most practical and extensive theoretical studies in both theoretical and applied mathematics. Many problems in mathematics and physics need to be expressed as Sturm-Liouville (S-L) boundary–eigenvalue problems. Among the huge number of studies on S-L problems, the dependence of eigenvalues on the problems is one of the more important research branches, and it has contributed many important developments [1–5]. These investigations are crucial to the development of the basic theory of differential operators and the accompanying numerical calculations for the spectra, as well as the inverse spectral problems. For example, in the classical S-L problems, the eigenvalue dependence properties have been extensively studied by many authors [1–4]. In [6], the authors have studied similar problems on differential equations, and in [7] and [8] the authors extended the problem to discrete S-L problems and high-dimensional S-L problems. The dependence of eigenvalues of the Dirac equation on the problems has also been considered in a recent paper [9].

In recent years, the theory of differential equations with discontinuous properties, that is the S-L problems with transmission conditions have also attracted much attention, and the corresponding studies on eigenvalue dependence can also be found in [10]. And another research topic related to S-L problems is the so called S-L problem with eigenparameter-dependent boundary conditions(BCs). These topics have been triggered by physical issues like heat conduction problems and vibrating string problems, as well as magnetic fluid mechanics [11–13]. There are many studies on these problems, including self-adjoint realization, spectral properties, inverse spectral theory etc., see [14–20]. There are still several studies on the eigenvalue dependence on the problems for higher-order boundary value problems with transmission conditions or eigenparameter-dependent BCs [21–24].

As an organic combination of the above mentioned two problems, the S-L problems with eigenparameter-dependent transmission conditions have attracted some scholars attention [25–30]. In [25], the asymptotic expressions for eigenvalues of the S-L problem with the spectral parameter contained in the transmission conditions were studied by the authors. In [27,28], the asymptotic expressions for eigenvalues and the Green's functions for the S-L problem with Herglotz-type eigenparameter-dependent transmission conditions are given for an appropriate Hilbert space. The corresponding inverse spectral problems can be found in [29, 30]. However, corresponding studies of the eigenvalue dependence of such problems have not yet been given, especially for the spectral parameter that appears in both of the transmission conditions. Such problems appear in non-uniform vibrating strings, electronic signal amplifiers and other issues of sciences [27, 28, 31]. Motivated by this, in this paper, we will consider the S-L problems with eigenparameter-dependent transmission conditions and show some eigenvalue properties, especially the Green's function and eigenvalue dependence of such problems. We show the continuity and differential properties of the eigenvalues for the data, including the BCs, the coefficient functions and the eigenparameter-dependent transmission conditions.

This paper is divided into seven parts. Following this introduction, in Section 2, the basic notations and the operator theoretic formulation of the considered problems, as well as some properties are explained. In Section 3, several basic properties of eigenvalues and eigenfunctions are given. Section 4 shows the Green's function for the problem. The continuity of eigenvalues and eigenfunctions is proved in Section 5. In Section 6, the differential expressions for the eigenvalues for each parameter are derived. At last, the concluding remarks or this study are provided in Section 7.

2. Notation and basic properties

In this section we will describe the basic problem of the present paper and show some basic properties corresponding to this eigenvalue problem. To this end, we first convert the considered problem to a linear operator by constructing a Hilbert space associated with a new inner product that is based on the BCs and the eigenparameter-dependent transmission conditions. Then we prove the self-adjointness of the operator and show that the eigenvalues are real and the eigenfunctions corresponding to different eigenvalues are orthogonal to each other.

Consider the second-order S-L differential equation given by

$$-(pz')' + qz = \mu wz \quad on \;\; K = [a, e) \cup (e, b], \;\; -\infty < a < e < b < \infty, \tag{2.1}$$

with the BCs

$$\cos\gamma_1 z(a) - \sin\gamma_1 (pz')(a) = 0, \;\; \gamma_1 \in [0, \pi), \tag{2.2}$$

$$\cos\gamma_2 z(b) - \sin\gamma_2 (pz')(b) = 0, \quad \gamma_2 \in (0,\pi], \tag{2.3}$$

and eigenparameter-dependent transmission conditions given by

$$z(e-) + (\mu\eta_1 - \xi_1)(pz')(e-) + (pz')(e+) = 0, \tag{2.4}$$

$$(pz')(e-) - z(e+) + (\mu\eta_2 - \xi_2)(pz')(e+) = 0. \tag{2.5}$$

Here μ is the spectral parameter, $z(e-)$ denotes the left limit of the function z at point e, $z(e+)$ denotes the right limit of the function z at point e, and the coefficient functions satisfy the following conditions:

$$r = \frac{1}{p}, q, w \in L(K,\mathbb{R}), \textit{ and } p > 0, w > 0, \ \textit{ a.e. on } K, \tag{2.6}$$

where $L(K,\mathbb{R})$ represents the Lebesgue integrable real valued functions on K. And we assume that the parameters in transmission conditions satisfy the following conditions:

$$\eta_i, \ \xi_i \in \mathbb{R}, \ \eta_i > 0, \ i = 1, 2. \tag{2.7}$$

Let $\theta[z] = w^{-1}(-(pz')' + qz)$ *on* K and define a weighted space as follows:

$$H_w = L_w^2(K) = \left\{ z : \int_a^e |z(x)|^2 w(x)\mathrm{d}x + \int_e^b |z(x)|^2 w(x)\mathrm{d}x < \infty \right\},$$

together with the inner product $\langle f, g\rangle_{H_w} = \int_a^e f\bar{g}w\mathrm{d}x + \int_e^b f\bar{g}w\mathrm{d}x$ for any $f, g \in H_w$, where the overbar denotes the complex conjugate.

For any $z, \Lambda \in H_w$, the Lagrange bracket $[z, \Lambda]$ of the functions z and Λ can then be introduced as follows:

$$[z, \Lambda] = z(p\bar{\Lambda}') - (pz')\bar{\Lambda}. \tag{2.8}$$

Let us consider the set associated with the functions considered in the present paper as follows:

$$S = \left\{ z \in L_w^2(K) : z, (pz') \in AC_{loc}(K), \ \ \theta[z] \in L_w^2(K) \right\},$$

where $AC_{loc}(K)$ denotes the set of all local absolutely continuous functions on K; then, for two arbitrary functions $z, \Lambda \in S$, the following Lagrange identity holads:

$$\langle \theta[z], \Lambda\rangle_{H_w} - \langle z, \theta[\Lambda]\rangle_{H_w} = [z, \Lambda]_a^{e-} + [z, \Lambda]_{e+}^{b}$$

where $[z, \Lambda]_{t_1}^{t_2} = [z, \Lambda](t_2) - [z, \Lambda](t_1)$.

Define the direct sum space as follows:

$$\mathcal{H} = H_w \oplus \mathbb{C} \oplus \mathbb{C},$$

and the new inner product on this space as follows:

$$\langle F, G\rangle_{\mathcal{H}} = \int_a^e f\bar{g}w\mathrm{d}x + \int_e^b f\bar{g}w\mathrm{d}x + \eta_1 f_1\overline{g_1} + \eta_2 f_2\overline{g_2},$$

for any $F = (f(x), f_1, f_2)^T$, $G = (g(x), g_1, g_2)^T \in \mathcal{H}$. Then it is easy to verify that this direct sum space $\mathcal{H}$ is a Hilbert space.

The following notations shall be utilized for a brief clarification

$$\hat{\mathcal{M}}_1(z) = \frac{1}{\eta_1}(\xi_1(pz')(e-) - z(e-) - (pz')(e+)), \quad \mathcal{M}_1(z) = (pz')(e-),$$

$$\hat{\mathcal{M}}_2(z) = \frac{1}{\eta_2}(\xi_2(pz')(e+) + z(e+) - (pz')(e-)), \quad \mathcal{M}_2(z) = (pz')(e+).$$

Then the eigenparameter-dependent transmission conditions given by (2.4) and (2.5) can be expressed as follow:

$$\mu\mathcal{M}_1(z) = \hat{\mathcal{M}}_1(z), \ \ \mu\mathcal{M}_2(z) = \hat{\mathcal{M}}_2(z).$$

Set

$$A_a = \begin{pmatrix} \cos\gamma_1 & -\sin\gamma_1 \\ 0 & 0 \end{pmatrix}, \quad B_b = \begin{pmatrix} 0 & 0 \\ \cos\gamma_2 & -\sin\gamma_2 \end{pmatrix},$$

$$C_\mu = \begin{pmatrix} 1 & \mu\eta_1 - \xi_1 \\ 0 & 1 \end{pmatrix}, \quad D_\mu = \begin{pmatrix} 0 & 1 \\ -1 & \mu\eta_2 - \xi_2 \end{pmatrix},$$

then the BCs (2.2) and (2.3) can be written as follows:

$$A_a\mathcal{Z}(a) + B_b\mathcal{Z}(b) = 0, \ \ \mathcal{Z} = \begin{pmatrix} z \\ pz' \end{pmatrix}, \tag{2.9}$$

and the eigenparameter-dependent transmission conditions given by (2.4) and (2.5) can be written as follows:

$$C_\mu\mathcal{Z}(e-) + D_\mu\mathcal{Z}(e+) = 0, \ \ \mathcal{Z} = \begin{pmatrix} z \\ pz' \end{pmatrix}. \tag{2.10}$$

Now we define a new operator and its domain as follows:

$$\mathcal{S}(\mathbf{T}) = \{Z = \begin{pmatrix} z \\ z_1 \\ z_2 \end{pmatrix} \in \mathcal{H} : z \in S, \ z(e \pm 0) = \lim_{x \to e\pm 0} z(x), \ (pz')(e \pm 0) = \lim_{x \to e\pm 0} (pz')(x)$$
$$exist, \ \ and \ z_1 = \mathcal{M}_1(z), \ z_2 = \mathcal{M}_2(z), \ A_a\mathcal{Z}(a) + B_b\mathcal{Z}(b) = 0\},$$

with the following rule

$$\mathbf{T}\begin{pmatrix} z \\ z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \theta[z] \\ \hat{\mathcal{M}}_1(z) \\ \hat{\mathcal{M}}_2(z) \end{pmatrix},$$

where $\theta[z] = \mu z$, $z \in S$, $x \in K$, satisfying (2.6) and (2.7). Thus, the problem given by (2.1)–(2.5) can be expressed in the following form

$$\mathbf{T}Z = \mu Z, \ Z = \begin{pmatrix} z \\ z_1 \\ z_2 \end{pmatrix} \in \mathcal{S}(\mathbf{T}). \tag{2.11}$$

Next we will discuss the self-adjointness of the operator $\mathbf{T}$.

Lemma 1. *$\mathcal{S}(\mathbf{T})$ is dense in $\mathcal{H}$.*

Proof. Suppose that $F = (f(x), f_1, f_2)^T \in \mathcal{H}$ and $F \perp \mathcal{S}(\mathbf{T})$; we will prove that $F = (0,0,0)^T$. Since $C_0^\infty \oplus \{0\} \oplus \{0\} \subset \mathcal{S}(\mathbf{T})$, for arbitrary $G = (g(x), 0, 0)^T \in C_0^\infty \oplus \{0\} \oplus \{0\}$, we have

$$\langle F, G\rangle = \int_a^e f\bar{g}w\mathrm{d}x + \int_e^b f\bar{g}w\mathrm{d}x = 0.$$

Because C_0^∞ is dense in $L_w^2[a,b]$, it follows that $f(x) = 0$, that is, $F = (0, f_1, f_2)^T$. For any $Z = (z(x), z_1, z_2)^T \in \mathcal{S}(\mathbf{T})$, we have

$$\langle F, Z\rangle = \eta_1 f_1\overline{z_1} + \eta_2 f_2\overline{z_2} = 0,$$

by the inner product in $\mathcal{H}$. Since z_1 and z_2 are arbitrary, we have that $f_1 = 0$ and $f_2 = 0$. Hence $F = (0,0,0)^T$, and the proof is completed.

Lemma 2. *The operator $\mathbf{T}$ is symmetric.*

Proof. Let $F = (f, f_1, f_2)^T$ and $G = (g, g_1, g_2)^T \in \mathcal{S}(\mathbf{T})$; then,

$$\begin{aligned}\langle \mathbf{T}F, G\rangle - \langle F, \mathbf{T}G\rangle =& \int_a^e [(-pf')'\bar{g} + qf\bar{g}]\mathrm{d}x + \int_e^b [(-pf')'\bar{g} + qf\bar{g}]\mathrm{d}x \\ &- [\int_a^e [(-p\bar{g}')'f + q\bar{g}f]\mathrm{d}x + \int_e^b [(-p\bar{g}')'f + q\bar{g}f]\mathrm{d}x] + \eta_1\hat{\mathcal{M}}_1(f)\mathcal{M}_1(\bar{g}) \\ &+ \eta_2\hat{\mathcal{M}}_2(f)\mathcal{M}_2(\bar{g}) - [\eta_1\hat{\mathcal{M}}_1(\bar{g})\mathcal{M}_1(f) + \eta_2\hat{\mathcal{M}}_2(\bar{g})\mathcal{M}_2(f)] \\ =& [f,g]_a^{e-} + [f,g]_{e+}^b + [f,g](e+) - [f,g](e-).\end{aligned} \tag{2.12}$$

So

$$\langle \mathbf{T}F, G\rangle - \langle F, \mathbf{T}G\rangle = [f,g](b) - [f,g](a). \tag{2.13}$$

By the BC (2.3), when $\gamma_2 \in (0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \pi]$, we have

$$[f,g](b) = \tan\gamma_2(pf')(b)(p\overline{g'})(b) - \tan\gamma_2(pf')(b)(p\overline{g'})(b) = 0, \tag{2.14}$$

and when $\gamma_2 = \frac{\pi}{2}$, we have that $(pz')(b) = 0$; thus, we can conclude that $[f,g](b) = 0$.

Similarly

$$[f,g](a) = 0. \tag{2.15}$$

Consequently, we have

$$\langle \mathbf{T}F, G\rangle - \langle F, \mathbf{T}G\rangle = 0.$$

Therefore, the operator $\mathbf{T}$ is symmetric.

Theorem 1. *$\mathbf{T}$ is a self-adjoint operator in $\mathcal{H}$.*

Proof. As $\mathbf{T}$ is symmetric, now we need to prove that for any $Z = (z(x), z_1, z_2)^T \in \mathcal{S}(\mathbf{T})$ and some $Y \in \mathcal{S}(\mathbf{T}^*)$, $U \in \mathcal{H}$ satisfying that $\langle \mathbf{T}Z, Y\rangle = \langle Z, U\rangle$, then $Y \in \mathcal{S}(\mathbf{T})$ and $\mathbf{T}Y = U$, where $Y = (y(x), y_1, y_2)^T$, $U = (u(x), u_1, u_2)^T$ and $\mathbf{T}^*$ is the adjoint operator of $\mathbf{T}$, i.e.,

(1) $y(x),\ (py')(x) \in AC(K),\ \theta[y] \in H_w$;

(2) $\cos\gamma_1 y(a) - \sin\gamma_1 (py')(a) = 0$, $\cos\gamma_2 y(b) - \sin\gamma_2 (py')(b) = 0$;

(3) $y_1 = \mathcal{M}_1(y) = (py')(e-)$, $y_2 = \mathcal{M}_2(y) = (py')(e+)$;

(4) $u_1 = \hat{\mathcal{M}}_1(y) = \frac{1}{\eta_1}(\xi_1 (py')(e-) - y(e-) - (py')(e+))$;

(5) $u_2 = \hat{\mathcal{M}}_2(y) = \frac{1}{\eta_2}(\xi_2 (py')(e+) + y(e+) - (py')(e-))$;

(6) $u(x) = \theta[y]$.

First, for any $V = (v(x), 0, 0)^T \in C_0^\infty \oplus \{0\} \oplus \{0\} \subset \mathcal{S}(\mathbf{T})$, we have that $\langle \mathbf{T}V, Y\rangle = \langle V, U\rangle$; hence,

$$\int_a^e (\theta[v])\bar{y}w\mathrm{d}x + \int_e^b (\theta[v])\bar{y}w\mathrm{d}x = \int_a^e v\bar{u}w\mathrm{d}x + \int_e^b v\bar{u}w\mathrm{d}x$$

holds, that is, $\langle \theta[v], y\rangle_{H_w} = \langle v, u\rangle_{H_w}$. By the classical theory of differential operators, it follows that (1) and (6) hold.

Next by (6) we get that for all $Z = (z(x), z_1, z_2)^T \in \mathcal{S}(\mathbf{T})$, $\langle \mathbf{T}Z, Y\rangle = \langle Z, U\rangle$ can be written as follows:

$$\langle \theta[z], y\rangle_{H_w} + \eta_1 \hat{\mathcal{M}}_1(z)\overline{y_1} + \eta_2 \hat{\mathcal{M}}_2(z)\overline{y_2} = \langle z, \theta[y]\rangle_{H_w} + \eta_1 \mathcal{M}_1(z)\overline{u_1} + \eta_2 \mathcal{M}_2(z)\overline{u_2}.$$

Given that

$$\langle \theta[z], y\rangle_{H_w} - \langle z, \theta[y]\rangle_{H_w} = [z, y](e-) - [z, y](e+),$$

we arrive at

$$\eta_1[\mathcal{M}_1(z)\overline{u_1} - \hat{\mathcal{M}}_1(z)\overline{y_1}] + \eta_2[\mathcal{M}_2(z)\overline{u_2} - \hat{\mathcal{M}}_2(z)\overline{y_2}] = [z, y](e-) - [z, y](e+). \tag{2.16}$$

Using Naimark's patching lemma [32], we choose $Z = (z(x), z_1, z_2)^T \in \mathcal{S}(\mathbf{T})$ such that

$$z(e-) = 1,\ (pz')(e-) = 0,\ z(e+) = 0,\ (pz')(e+) = 0,$$

this means that

$$\mathcal{M}_1(z) = 0,\ \mathcal{M}_2(z) = 0,\ \hat{\mathcal{M}}_1(z) = -\frac{1}{\eta_1},\ \hat{\mathcal{M}}_2(z) = 0,$$

then by (2.16), we have that $y_1 = \mathcal{M}_1(y) = (py')(e-)$. Using that similar method, one can prove a $y_2 = \mathcal{M}_2(y) = (py')(e+)$ is also true. Therefore, (3) holds.

We choose $Z = (z(x), z_1, z_2)^T \in \mathcal{S}(\mathbf{T})$ such that

$$z(e-) = 0,\ \ (pz')(e-) = 1,\ \ z(e+) = 0,\ \ (pz')(e+) = 0,$$

which imply that

$$\mathcal{M}_1(z) = 1,\ \mathcal{M}_2(z) = 0,\ \ \hat{\mathcal{M}}_1(z) = \frac{\xi_1}{\eta_1},\ \ \hat{\mathcal{M}}_2(z) = -\frac{1}{\eta_2},$$

then by (2.16), we have that $u_1 = \hat{\mathcal{M}}_1(y) = \frac{1}{\eta_1}(\xi_1 (py')(e-) - y(e-) - (py')(e+))$. Therefore, (4) holds. Using a similar method, one can prove that (5) is also true.

Choosing $Z = (z(x), z_1, z_2)^T \in \mathcal{S}(\mathbf{T})$ such that

$$z(a) = \sin\gamma_1,\ \ (pz')(a) = \cos\gamma_1,$$

and then combining the proof of symmetry and $[z, y](a) = 0$, we have that $\cos\gamma_1 y(a) - \sin\gamma_1 (py')(a) = 0$. Using a similar method, one can prove that $\cos\gamma_2 y(b) - \sin\gamma_2 (py')(b) = 0$ is also true. Therefore, (2) holds. Hence, the operator $\mathbf{T}$ is self-adjoint.

Corollary 1. *The eigenvalues of the operator* $\mathbf{T}$ *are all real.*

Corollary 2. *Let* $\mathcal{Y}_1(x)$ *and* $\mathcal{Y}_2(x)$ *be the eigenfunctions corresponding to two different eigenvalues of the problem given by (2.1)–(2.5). Then they are orthogonal to each other in the following sense:*

$$\int_a^e \mathcal{Y}_1(x)\overline{\mathcal{Y}_2}(x)w\mathrm{d}x + \int_e^b \mathcal{Y}_1(x)\overline{\mathcal{Y}_2}(x)w\mathrm{d}x + \eta_1\mathcal{M}_1(\mathcal{Y}_1)\overline{\mathcal{M}_1(\mathcal{Y}_2)} + \eta_2\mathcal{M}_2(\mathcal{Y}_1)\overline{\mathcal{M}_2(\mathcal{Y}_2)} = 0.$$

3. Eigenvalues and eigenfunctions

In this section we will introduce the Wronskian by constructing the fundamental solutions of the problem; we shall also that the zeros of the Wronskian constitute the eigenvalues of the S-L problem and that the eigenvalue problem is simple.

We construct the fundamental solutions of the differential Eq (2.1) as follows

$$\Upsilon(x,\mu) = \begin{cases} \Upsilon_1(x,\mu), & x \in [a,e), \\ \Upsilon_2(x,\mu), & x \in (e,b], \end{cases} \quad \Lambda(x,\mu) = \begin{cases} \Lambda_1(x,\mu), & x \in [a,e), \\ \Lambda_2(x,\mu), & x \in (e,b]. \end{cases}$$

Let $\Upsilon_{1\mu}(x) = \Upsilon_1(x,\mu)$ be the solution of Eq (2.1) on the interval $[a,e)$ satisfying the following initial conditions

$$z(a) = \sin\gamma_1, \quad (pz')(a) = \cos\gamma_1, \tag{3.1}$$

by virtue of [33], Eq (2.1) has a unique solution $\Upsilon_1(x,\mu)$ for each $\mu \in \mathbb{C}$, which is an entire function of μ for each fixed $x \in [a,e)$.

Now we can define the solution $\Upsilon_{2\mu}(x) = \Upsilon_2(x,\mu)$ of Eq (2.1) on the interval $(e,b]$ in terms of $\Upsilon_1(e-0,\mu)$ and $(p\Upsilon_1')(e-0,\mu)$ by applying the following initial conditions

$$\begin{aligned} z(e+) &= -(\mu\eta_2 - \xi_2)z(e-) - [(\mu\eta_2 - \xi_2)(\mu\eta_1 - \xi_1) - 1](pz')(e-), \\ (pz')(e+) &= -(z(e-) + (\mu\eta_1 - \xi_1)(pz')(e-)). \end{aligned} \tag{3.2}$$

For each $\mu \in \mathbb{C}$, Eq (2.1) has a unique solution $\Upsilon_2(x,\mu)$ on the interval $(e,b]$. Moreover, $\Upsilon_2(x,\mu)$ is an entire function of μ for each fixed $x \in (e,b]$.

Let $\Lambda_{2\mu}(x) = \Lambda_2(x,\mu)$ be the solution of Eq (2.1) on the interval $(e,b]$ satisfying the following initial conditions

$$z(b) = \sin\gamma_2, \quad (pz')(b) = \cos\gamma_2, \tag{3.3}$$

by virtue of [33], Eq (2.1) has a unique solution $\Lambda_2(x,\mu)$ for each $\mu \in \mathbb{C}$, which is an entire function of μ for each fixed $x \in (e,b]$.

Define the solution $\Lambda_{1\mu}(x) = \Lambda_1(x,\mu)$ of Eq (2.1) on the interval $[a,e)$ in terms of $\Lambda_2(e+0,\mu)$ and $(p\Lambda_2')(e+0,\mu)$ by applying the following initial conditions

$$\begin{aligned} z(e-) &= -(\mu\eta_1 - \xi_1)z(e+) + [(\mu\eta_1 - \xi_1)(\mu\eta_2 - \xi_2) - 1](pz')(e+), \\ (pz')(e-) &= z(e+) - (\mu\eta_2 - \xi_2)(pz')(e+). \end{aligned} \tag{3.4}$$

For each $\mu \in \mathbb{C}$, Eq (2.1) has a unique solution $\Lambda_1(x,\mu)$ on the interval $[a,e)$. Moreover, $\Lambda_1(x,\mu)$ is an entire function of μ for each fixed $x \in [a,e)$.

From the theory of linear ordinary differential equations, the Wronskians denoted by

$$\omega_j(\mu) := W(\Upsilon_j(x,\mu), \Lambda_j(x,\mu)), \quad j = 1,2$$

are independent of $x \in K$.

Lemma 3. *For each $\mu \in \mathbb{C}$*

$$\omega_2(\mu) = \omega_1(\mu).$$

Proof. Due to the Wronskians being independent of x, then by (2.2) and (2.4), it follows that

$$\begin{aligned}
\omega_2(\mu) = \omega_2(\mu)|_{x=e} &= \det\begin{pmatrix} \Upsilon_2(e+,\mu) & \Lambda_2(e+,\mu) \\ (p\Upsilon_2')(e+,\mu) & (p\Lambda_2')(e+,\mu) \end{pmatrix} \\
&= \det\begin{pmatrix} f_2(\mu)\Upsilon_1(e-,\mu) + h(\mu)(p\Upsilon_1')(e-,\mu) & \Upsilon_1(e-,\mu) + f_1(\mu)(p\Upsilon_1')(e-,\mu) \\ f_2(\mu)\Lambda_1(e-,\mu) + h(\mu)(p\Lambda_1')(e-,\mu) & \Lambda_1(e-,\mu) + f_1(\mu)(p\Lambda_1')(e-,\mu) \end{pmatrix} \\
&= \det\begin{pmatrix} f_2(\mu) & h(\mu) \\ 1 & f_1(\mu) \end{pmatrix}\omega_1(\mu) = \omega_1(\mu),
\end{aligned}$$

where $f_1(\mu) = \mu\eta_1 - \xi_1$, $f_2(\mu) = \mu\eta_2 - \xi_2$, $h(\mu) = f_1(\mu)f_2(\mu) - 1$.

The proof is completed.

Let

$$\omega(\mu) = \omega_1(\mu),$$

then the following lemma holds.

Lemma 4. *The complex number μ is an eigenvalue of the S-L problem given by (2.1)–(2.5) if and only if μ is the zero point of $\omega(\mu)$, that is, if $\omega(\mu) = 0$ holds.*

Proof. Let μ_0 be the eigenvalue of the S-L problem given by (2.1)–(2.5) and $z(x,\mu_0)$ be the eigenfunction corresponding to μ_0. Then we have that $\omega(\mu_0) = 0$. In fact, if we assume that $\omega(\mu_0) \neq 0$, this implies that $W(\Upsilon_j(x,\mu), \Lambda_j(x,\mu)) \neq 0 (j = 1,2)$. Then, given (3.1)–(3.4), the functions $\Upsilon_1(x,\mu_0)$, $\Lambda_1(x,\mu_0)$ and $\Upsilon_2(x,\mu_0)$, $\Lambda_2(x,\mu_0)$ are linearly independent in $[a,e)$ and $(e,b]$ respectively. Therefore, the solution $z(x,\mu_0)$ of Eq (2.1) can be expressed as follows:

$$z(x,\mu_0) = \begin{cases} c_{11}\Upsilon_1(x,\mu_0) + c_{12}\Lambda_1(x,\mu_0), & x \in [a,e), \\ d_{11}\Upsilon_2(x,\mu_0) + d_{12}\Lambda_2(x,\mu_0), & x \in (e,b], \end{cases}$$

where at least one constant among c_{11}, c_{12}, d_{11} and d_{12} is not zero. However, by incorporating this representation into the BCs (2.2) and (2.3), we obtain that $c_{12} = 0$ and $d_{11} = 0$. By incorporating $z(x,\mu_0)$ into the transmission conditions given by (2.4) and (2.5), we obtain that $c_{11} = d_{11} = 0$ and $d_{12} = c_{12} = 0$. This leads to a contradiction; thus, the eigenvalues of the S-L problem given by (2.1)–(2.5) are all zero points of $\omega(\mu)$.

Conversely, let $\omega(\mu_0) = 0$; then, $W(\Upsilon_1(x,\mu_0), \Lambda_1(x,\mu_0)) = 0$, and consequently the functions $\Upsilon_1(x,\mu_0)$, $\Lambda_1(x,\mu_0)$ are linearly dependent solutions of Eq (2.1) in the interval $[a,e)$, i.e.,

$$\Upsilon_1(x,\mu_0) = k_1\Lambda_1(x,\mu_0),$$

for some $k_1 \neq 0$. In this case, we have

$$\begin{aligned}
\cos\gamma_1\Lambda(a) - \sin\gamma_1(p\Lambda')(a) &= \cos\gamma_1\Lambda_1(a,\mu_0) - \sin\gamma_1(p\Lambda_1')(a,\mu_0) \\
&= k_1(\cos\gamma_1\Upsilon_1(a,\mu_0) - \sin\gamma_1(p\Upsilon_1')(a,\mu_0)) = 0.
\end{aligned} \tag{3.5}$$

Therefore, $\Lambda_1(x,\mu_0)$ is the solution satisfying the BC (2.2) on the interval $[a,e)$. And $\Lambda_2(x,\mu_0)$ is the solution satisfying the BC (2.3) on the interval $(e,b]$. Thus, $\Lambda(x,\mu_0)$ is the solution satisfying the

BCs (2.2) and (2.3) and the transmission conditions given by (2.4) and (2.5); hence, $\Lambda(x,\mu_0)$ is the eigenfunction corresponding to the eigenvalue μ_0 of the problem given by (2.1)–(2.5). This completes the proof.

Lemma 5. *The eigenvalues of the S-L problem given by (2.1)–(2.5) are simple.*

Proof. By Corollary 1, since the eigenvalues of the S-L problem given by (2.1)–(2.5) are all real, we let $\mu = m,\ m \in \mathbb{R}$, differentiating the equation $\theta[\Lambda(x,\mu)] = \mu\Lambda(x,\mu)$ with respect to μ; then, we have

$$\theta[\Lambda_\mu(x,\mu)] = \mu\Lambda_\mu(x,\mu) + \Lambda(x,\mu),$$

where $\Lambda_\mu(x,\mu)$ is the partial derivative of $\Lambda(x,\mu)$ with respect to μ. By $\mu = m$, then

$$\langle\theta[\Lambda_\mu],\Upsilon\rangle - \langle\Lambda_\mu,\theta[\Upsilon]\rangle = \langle\mu\Lambda_\mu + \Lambda,\Upsilon\rangle - \langle\Lambda_\mu,\mu\Upsilon\rangle = \langle m\Lambda_\mu + \Lambda,\Upsilon\rangle - \langle\Lambda_\mu, m\Upsilon\rangle = \langle\Lambda,\Upsilon\rangle, \tag{3.6}$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $L_w^2(K)$ defined above.

By the Lagrange identity, and by making use of (3.1)–(3.4), we get

$$\begin{aligned}
&\langle\theta[\Lambda_\mu],\Upsilon\rangle - \langle\Lambda_\mu,\theta[\Upsilon]\rangle\\
=&[\Lambda_{1\mu}(x,\mu)p(x)\overline{\Upsilon_1'(x,\mu)} - p(x)\Lambda_{1\mu}'(x,\mu)\overline{\Upsilon_1(x,\mu)}]_a^{e-}\\
&+ [\Lambda_{2\mu}(x,\mu)p(x)\overline{\Upsilon_2'(x,\mu)} - p(x)\Lambda_{2\mu}'(x,\mu)\overline{\Upsilon_2(x,\mu)}]_{e+}^b\\
=&\sin\gamma_1(p\Lambda_{1\mu}')(a) - \cos\gamma_1\Lambda_{1\mu}(a),
\end{aligned} \tag{3.7}$$

where $\Lambda_{j\mu}(x,\mu)$, $\Lambda_{j\mu}'(x,\mu)$, $\Upsilon_{j\mu}(x,\mu)$, $\Upsilon_{j\mu}'(x,\mu)$, $\Upsilon_\mu(x,\mu)$ are the respective partial derivatives of $\Lambda_j(x,\mu)$, $\Lambda_j'(x,\mu)$, $\Upsilon_j(x,\mu)$, $\Upsilon_j'(x,\mu)$, $\Upsilon(x,\mu)$ with respect to μ. By the definition of $\omega(\mu)$ and (3.6) and (3.7), we get

$$\omega'(\mu)|_{x=a} = \frac{d\omega_1(\mu)}{d\mu}|_{x=a} = \sin\gamma_1(p\Lambda_{1\mu}')(a) - \cos\gamma_1\Lambda_{1\mu}(a) = \langle\Lambda,\Upsilon\rangle. \tag{3.8}$$

Suppose that μ_0 is an eigenvalue of the S-L problem given by (2.1)–(2.5). Then $\omega(\mu_0) = 0$. Thus, there exist constants $c_j \neq 0$ ($j = 1,2$) such that

$$\Lambda_j(x,\mu_0) = c_j\Upsilon_j(x,\mu_0), \quad j = 1,2.$$

By (3.2) and (3.4), we obtain

$$\begin{aligned}
\Lambda_2(e+,\mu_0) &= c_1(-(\mu\eta_2 - \xi_2)\Upsilon_1(e-,\mu_0) - ((\mu\eta_2 - \xi_2)(\mu\eta_1 - \xi_1) - 1)(p\Upsilon_1')(e-,\mu_0))\\
&= c_1\Upsilon_2(e+,\mu_0),
\end{aligned}$$

$$(p\Lambda_2')(e+,\mu_0) = c_1(\Upsilon_1(e-,\mu_0) + (\mu\eta_1 - \xi_1)(p\Upsilon_1')(e-,\mu_0)) = c_1(p\Upsilon_2')(e+,\mu_0).$$

Thus, $c_1 = c_2 \neq 0$ and $\Lambda(x,\mu) = c_1\Upsilon(x,\mu)$. So, Eq (3.8) becomes

$$\omega'(\mu_0) = c_1\langle\Upsilon,\Upsilon\rangle = c_1(\int_a^e |\Upsilon_1(x,\mu_0)|^2 w\mathrm{d}x + \int_e^b |\Upsilon_2(x,\mu_0)|^2 w\mathrm{d}x) \neq 0. \tag{3.9}$$

Hence, the eigenvalue of the S-L problem given by (2.1)–(2.5); μ is simple.

4. Green's function

In this section, we show the Green's function for the S-L problem given by (2.1)–(2.5).

Let $\mu \in \Gamma = \{\mu \in \mathbb{C} | \omega(\mu) \neq 0\}$, and $F = (f(x), f_1, f_2)^T \in \mathcal{H}$; we define

$$\kappa(z) = -(pz')' + qz \quad on \quad K = [a, e) \cup (e, b].$$

Next, we focus on the nonhomogeneous differential equation given by

$$\kappa(z) - \mu wz = f(x), \quad x \in K, \tag{4.1}$$

together with the BCs and transmission conditions given by (2.2)–(2.5); we can represent the general solution of the differential equation $\kappa(z) - \mu wz = f_1(x)$ $(x \in [a, e))$ in the following form:

$$z_1 = c_{21}\Upsilon_1(x,\mu) + c_{22}\Lambda_1(x,\mu) + \frac{\Upsilon_1(x,\mu)}{\omega(\mu)} \int_a^x \Lambda_1(\xi,\mu) f_1(\xi) \mathrm{d}\xi - \frac{\Lambda_1(x,\mu)}{\omega(\mu)} \int_a^x \Upsilon_1(\xi,\mu) f_1(\xi) \mathrm{d}\xi, \tag{4.2}$$

where $f_1 = f(x)|_{[a,e)}$ and $c_{21}, c_{22} \in \mathbb{C}$. The general solution of the differential equation $\kappa(z) - \mu wz = f_2(x)$, $(x \in (e, b])$ can be represented as follows:

$$z_2 = d_{21}\Upsilon_2(x,\mu) + d_{22}\Lambda_2(x,\mu) + \frac{\Upsilon_2(x,\mu)}{\omega(\mu)} \int_e^x \Lambda_2(\xi,\mu) f_2(\xi) \mathrm{d}\xi - \frac{\Lambda_2(x,\mu)}{\omega(\mu)} \int_e^x \Upsilon_2(\xi,\mu) f_2(\xi) \mathrm{d}\xi, \tag{4.3}$$

where $f_2 = f(x)|_{(e,b]}$ and $d_{21}, d_{22} \in \mathbb{C}$. Taking the transmission conditions given by (2.4) and (2.5) into account along with (3.2) and (3.4), we obtain

$$\begin{aligned}
&z(e-) + (\mu\eta_1 - \xi_1)(pz')(e-) + (pz')(e+) \\
&= - [c_{21}(p\Upsilon_2')(e) + c_{22}(p\Lambda_2')(e) + \frac{(p\Upsilon_2')(e,\mu)}{\omega(\mu)} \int_a^e \Lambda_1(\xi,\mu) f_1(\xi) \mathrm{d}\xi \\
&\quad - \frac{(p\Lambda_2')(e,\mu)}{\omega(\mu)} \int_a^e \Upsilon_1(\xi,\mu) f_1(\xi) \mathrm{d}\xi] + d_{21}(p\Upsilon_2')(e) + d_{22}(p\Lambda_2')(e) = 0,
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&(pz')(e-) - z(e+) + (\mu\eta_2 - \xi_2)(pz')(e+) \\
&= - [d_{21}(p\Upsilon_1')(e) + d_{22}(p\Lambda_1')(e)] + c_{21}(p\Upsilon_1')(e) + c_{22}(p\Lambda_1')(e) \\
&\quad + \frac{(p\Upsilon_1')(e,\mu)}{\omega(\mu)} \int_a^e \Lambda_1(\xi,\mu) f_1(\xi) \mathrm{d}\xi - \frac{(p\Lambda_1')(e,\mu)}{\omega(\mu)} \int_a^e \Upsilon_1(\xi,\mu) f_1(\xi) \mathrm{d}\xi = 0.
\end{aligned} \tag{4.5}$$

From (4.4) and (4.5), we have

$$d_{21} = c_{21} + \frac{1}{\omega(\mu)} \int_a^e \Lambda_1(\xi,\mu) f_1(\xi) \mathrm{d}\xi, \ d_{22} = c_{22} - \frac{1}{\omega(\mu)} \int_a^e \Upsilon_1(\xi,\mu) f_1(\xi) \mathrm{d}\xi. \tag{4.6}$$

By the BC (2.2), we have

$$c_{21}(\cos\gamma_1 \Upsilon_1(a,\mu) - \sin\gamma_1 (p\Upsilon_1')(a,\mu)) + c_{22}(\cos\gamma_1 \Lambda_1(a,\mu) - \sin\gamma_1 (p\Lambda_1')(a,\mu)) = 0,$$

then we can obtain that $c_{22} = 0$.

Similarly, by the BC (2.3), we have

$$\cos\gamma_2[d_{21}\Upsilon_2(b)+d_{22}\Lambda_2(b)+\frac{\Upsilon_2(b,\mu)}{\omega(\mu)}\int_e^b\Lambda_2(\xi,\mu)f_2(\xi)\mathrm{d}\xi$$
$$-\frac{\Lambda_2(b,\mu)}{\omega(\mu)}\int_e^b\Upsilon_2(\xi,\mu)f_2(\xi)\mathrm{d}\xi]-\sin\gamma_2[d_{21}(p\Upsilon_2')(b)+d_{22}(p\Lambda_2')(b)$$
$$+\frac{(p\Upsilon_2')(b,\mu)}{\omega(\mu)}\int_e^b\Lambda_2(\xi,\mu)f_2(\xi)\mathrm{d}\xi-\frac{(p\Lambda_2')(b,\mu)}{\omega(\mu)}\int_e^b\Upsilon_2(\xi,\mu)f_2(\xi)\mathrm{d}\xi]=0.$$

Incorporating (4.6), we obtain that $d_{21}=-\frac{1}{\omega(\mu)}\int_e^b\Lambda_2(\xi,\mu)f_2(\xi)\mathrm{d}\xi$.
By (4.6), we get

$$c_{21}=-\frac{1}{\omega(\mu)}\int_a^e\Lambda_1(\xi,\mu)f_1(\xi)\mathrm{d}\xi-\frac{1}{\omega(\mu)}\int_e^b\Lambda_2(\xi,\mu)f_2(\xi)\mathrm{d}\xi,\ \ d_{22}=-\frac{1}{\omega(\mu)}\int_a^e\Upsilon_1(\xi,\mu)f_1(\xi)\mathrm{d}\xi.$$

Applying (4.2) and (4.3), we obtain

$$z_1(x,\xi)=-\frac{\Upsilon_1(x,\mu)}{\omega(\mu)}\int_x^e\Lambda_1(\xi,\mu)f_1(\xi)\mathrm{d}\xi-\frac{\Lambda_1(x,\mu)}{\omega(\mu)}\int_a^x\Upsilon_1(\xi,\mu)f_1(\xi)\mathrm{d}\xi$$
$$-\frac{\Upsilon_1(x,\mu)}{\omega(\mu)}\int_e^b\Lambda_2(\xi,\mu)f_2(\xi)\mathrm{d}\xi,\quad x\in[a,e),$$

$$z_2(x,\xi)=-\frac{\Upsilon_2(x,\mu)}{\omega(\mu)}\int_x^b\Lambda_2(\xi,\mu)f_2(\xi)\mathrm{d}\xi-\frac{\Lambda_2(x,\mu)}{\omega(\mu)}\int_e^x\Upsilon_2(\xi,\mu)f_2(\xi)\mathrm{d}\xi$$
$$-\frac{\Lambda_2(x,\mu)}{\omega(\mu)}\int_a^e\Upsilon_1(\xi,\mu)f_1(\xi)\mathrm{d}\xi,\quad x\in(e,b].$$

Denoting the Green's function of the problem as $G(x,\xi,\mu)$, then $z(x,\mu)$ can be represented as follows:

$$z(x,\mu)=\int_a^e G(x,\xi,\mu)f(\xi)\mathrm{d}\xi+\int_e^b G(x,\xi,\mu)f(\xi)\mathrm{d}\xi,$$

where

$$G(x,\xi,\mu)=\begin{cases}-\frac{\Upsilon_1(x,\mu)\Lambda_1(\xi,\mu)}{\omega(\mu)}, & a<x<\xi<e,\\ -\frac{\Lambda_1(x,\mu)\Upsilon_1(\xi,\mu)}{\omega(\mu)}, & a<\xi<x<e,\\ -\frac{\Upsilon_1(x,\mu)\Lambda_2(\xi,\mu)}{\omega(\mu)}, & a<x<e,e<\xi<b,\\ -\frac{\Lambda_2(x,\mu)\Upsilon_1(\xi,\mu)}{\omega(\mu)}, & a<\xi<e,e<x<b,\\ -\frac{\Upsilon_2(x,\mu)\Lambda_2(\xi,\mu)}{\omega(\mu)}, & e<x<\xi<b,\\ -\frac{\Lambda_2(x,\mu)\Upsilon_2(\xi,\mu)}{\omega(\mu)}, & e<\xi<x<b.\end{cases}$$

5. Continuous dependence of eigenvalues and eigenfunctions

In this section, the dependence of the eigenvalues on the given data will be presented; to this end let us consider the following set

$$\Omega=\left\{\vartheta=(\frac{1}{p},q,w,\gamma_1,\gamma_2,\xi_1,\xi_2,\eta_1,\eta_2):(2.6)-(2.7)\ hold\right\}.$$

The Banach space can then be introduced as follows:

$$\mathcal{B} := L(K) \oplus L(K) \oplus L(K) \oplus \mathbb{R}^6,$$

equipped with the following norm:

$$\begin{aligned}\| \vartheta \| = & \int_a^e (|\frac{1}{p}| + |q| + |w|)dx + \int_e^b (|\frac{1}{p}| + |q| + |w|)dx \\ & + |\gamma_1| + |\gamma_2| + |\xi_1| + |\xi_2| + |\eta_1| + |\eta_2|.\end{aligned}$$

Next, the continuous dependence of the eigenvalues on the parameters can be discussed.

Theorem 2. *Let* $\breve{\vartheta} = \left(\frac{1}{\breve{p}}, \breve{q}, \breve{w}, \breve{\gamma}_1, \breve{\gamma}_2, \breve{\xi}_1, \breve{\xi}_2, \breve{\eta}_1, \breve{\eta}_2\right) \in \Omega$. *Suppose that* $\mu = \mu(\vartheta)$ *is an eigenvalue of the S-L problem given by (2.1)–(2.5). Then* μ *is continuous for* $\breve{\vartheta}$, *that is, for any* $\varepsilon > 0$ *sufficiently small, there exists a* $\delta > 0$ *such that, for any* $\vartheta = (\frac{1}{p}, q, w, \gamma_1, \gamma_2, \xi_1, \xi_2, \eta_1, \eta_2) \in \Omega$ *satisfying*

$$\begin{aligned}\| \vartheta - \breve{\vartheta} \| = & \int_a^e (|\frac{1}{p} - \frac{1}{\breve{p}}| + |q - \breve{q}| + |w - \breve{w}|)dx + \int_e^b (|\frac{1}{p} - \frac{1}{\breve{p}}| + |q - \breve{q}| + |w - \breve{w}|)dx \\ & + |\gamma_1 - \breve{\gamma}_1| + |\gamma_2 - \breve{\gamma}_2| + |\xi_1 - \breve{\xi}_1| + |\xi_2 - \breve{\xi}_2| + |\eta_1 - \breve{\eta}_1| + |\eta_2 - \breve{\eta}_2| < \delta,\end{aligned}$$

the eigenvalue $\mu(\vartheta)$ *of the S-L problem given by (2.1)–(2.5) satisfies*

$$|\mu(\vartheta) - \mu(\breve{\vartheta})| < \varepsilon.$$

Proof. The proof is similar to that in [21], so we omit the details here.

Definition 1. *An eigenvector* $Z = (z, z_1, z_2)^T \in \mathcal{H}$ *of the S-L problem given by (2.1)–(2.5) is called a normalized eigenvector if Z satisfies*

$$\| (z, z_1, z_2)^T \|^2 = \langle (z, z_1, z_2)^T, (z, z_1, z_2)^T \rangle = \int_a^e z\bar{z}w\mathrm{d}x + \int_e^b z\bar{z}w\mathrm{d}x + \eta_1 z_1 \overline{z_1} + \eta_2 z_2 \overline{z_2} = 1.$$

Then by the above definition a continuity property of the corresponding eigenvector can be expressed as follows.

Theorem 3. *Let* $\mu(\breve{\vartheta})$, $\breve{\vartheta} \in \Omega$ *be an eigenvalue of the S-L problem given by (2.1)–(2.5) and* $U = (u, u_1, u_2)^T(\cdot, \breve{\vartheta}) \in \mathcal{H}$ *be a normalized eigenvector for* $\mu(\breve{\vartheta})$. *Then there exist normalized eigenvectors* $V = (v, v_1, v_2)^T(\cdot, \vartheta) \in \mathcal{H}$ *of* $\mu(\vartheta)$ *for* $\vartheta \in \Omega$ *such that when* $\vartheta \to \breve{\vartheta}$ *in* Ω, *it follows that*

$$v(x) \to u(x),\ (pv')(x) \to (pu')(x), \tag{5.1}$$

all uniformly on $[a, e) \cup (e, b]$, *and* $v_1 \to u_1$, $v_2 \to u_2$.

Proof. Assume that $(z(x, \breve{\vartheta}), z_1(\breve{\vartheta}), z_2(\breve{\vartheta}))^T$ is an eigenvector for $\mu(\breve{\vartheta})$ with

$$\| z(x, \breve{\vartheta}) \| = \int_a^e z(x, \breve{\vartheta})\bar{z}(x, \breve{\vartheta})w(x)\mathrm{d}x + \int_e^b z(x, \breve{\vartheta})\bar{z}(x, \breve{\vartheta})w(x)\mathrm{d}x = 1.$$

There exists $\mu(\vartheta)$ such that

$$\mu(\vartheta) \to \mu(\breve{\vartheta}), \quad as \quad \vartheta \to \breve{\vartheta}.$$

Let the BC and the eigenparameter-dependent transmission condition matrix be denoted by

$$\begin{pmatrix} A_a & B_b \\ C_\mu & D_\mu \end{pmatrix}(\vartheta) = \begin{pmatrix} \cos\gamma_1 & -\sin\gamma_1 & 0 & 0 \\ 0 & 0 & \cos\gamma_2 & -\sin\gamma_2 \\ 1 & \mu(\vartheta)\eta_1 - \xi_1 & 0 & 1 \\ 0 & 1 & -1 & \mu(\vartheta)\eta_2 - \xi_2 \end{pmatrix},$$

then

$$\begin{pmatrix} A_a & B_b \\ C_\mu & D_\mu \end{pmatrix}(\vartheta) \to \begin{pmatrix} A_a & B_b \\ C_\mu & D_\mu \end{pmatrix}(\breve{\vartheta}), \quad as \quad \vartheta \to \breve{\vartheta}.$$

It follows from Theorem 3.2 of [1] that there exist eigenfunctions denoted by $z(x,\vartheta)$ for $\mu(\vartheta)$ such that $\| z(x,\vartheta) \| = 1$ and

$$z(x,\vartheta) \to z(x,\breve{\vartheta}),\ (pz')(x,\vartheta) \to (pz')(x,\breve{\vartheta}), \quad as \quad \vartheta \to \breve{\vartheta}\ in\ \Omega, \tag{5.2}$$

both uniformly on $[a,e) \cup (e,b]$, and

$$z_1(\vartheta) \to z_1(\breve{\vartheta}),\ z_2(\vartheta) \to z_2(\breve{\vartheta}), \quad as \quad \vartheta \to \breve{\vartheta}\ in\ \Omega. \tag{5.3}$$

Let

$$(v, v_1, v_2)^T = \frac{(z(x,\vartheta), z_1(\vartheta), z_2(\vartheta))^T}{\|(z(x,\vartheta), z_1(\vartheta), z_2(\vartheta))^T\|},\ (u, u_1, u_2)^T = \frac{(z(x,\breve{\vartheta}), z_1(\breve{\vartheta}), z_2(\breve{\vartheta}))^T}{\|(z(x,\breve{\vartheta}), z_1(\breve{\vartheta}), z_2(\breve{\vartheta}))^T\|},$$

$$pv' = \frac{(pz')(x,\vartheta)}{\|(z(x,\vartheta), z_1(\vartheta), z_2(\vartheta))^T\|},\ pu' = \frac{(pz')(x,\breve{\vartheta})}{\|(z(x,\breve{\vartheta}), z_1(\vartheta), z_2(\vartheta))^T\|}.$$

Then (5.1) holds by (5.2) and (5.3).

6. Differential equations for eigenvalues

The differentiability and the derivative formulas for the eigenvalues for each parameter in Theorem 2 are detailed in this section. The derivative formulas will be given in the form of a classical derivative or Frechet derivative, respectively, for different parameters. For the definition of the Frechet derivative the readers may refer to [1,9].

Theorem 4. *Let $\mu(\vartheta)$, $\vartheta \in \Omega$ be an eigenvalue of the S-L problem given by (2.1)–(2.5) and $U = (u, u_1, u_2)^T$ be a normalized eigenvector for $\mu(\vartheta)$; then, for each parameter in ϑ, μ is differentiable, moreover, the derivative formulas for μ can be deduced as follows:*

1) If we fix all parameters of ϑ except γ_1, then one has

$$\mu'(\gamma_1) = -\csc^2\gamma_1 |u(a)|^2.$$

2) If we fix all parameters of ϑ except γ_2, then one has

$$\mu'(\gamma_2) = \csc^2\gamma_2 |u(b)|^2.$$

3) If we fix all parameters of ϑ except ξ_1, then one has

$$\mu'(\xi_1) = |(pu')(e-)|^2.$$

4) If we fix all parameters of ϑ except ξ_2, then one has

$$\mu'(\xi_2) = |(pu')(e+)|^2.$$

5) If we fix all parameters of ϑ except η_1, then one has

$$\mu'(\eta_1) = -\mu|(pu')(e-)|^2.$$

6) If we fix all parameters of ϑ except η_2, then one has

$$\mu'(\eta_2) = -\mu|(pu')(e+)|^2.$$

7) If we fix all parameters of ϑ except w, then one has

$$d\mu_w(h) = -\mu[\int_a^e h|u|^2 + \int_e^b h|u|^2], \quad h \in L(J,\mathbb{R}), \quad h \to 0.$$

8) If we fix all parameters of ϑ except $\frac{1}{p}$, then one has

$$d\mu_{\frac{1}{p}}(h) = -[\int_a^e h|pu'|^2 + \int_e^b h|pu'|^2], \quad h \in L(J,\mathbb{R}), \quad h \to 0.$$

9) If we fix all parameters of ϑ except q, then one has

$$d\mu_q(h) = \int_a^e h|u|^2 + \int_e^b h|u|^2, \quad h \in L(J,\mathbb{R}), \quad h \to 0.$$

Proof. (a) With the exception of γ_1, let us fix the parameters of ϑ, and let $u = u(\cdot,\gamma_1)$ and $v = u(\cdot,\gamma_1 + h)$. Then

$$\begin{aligned}
&(\mu(\gamma_1 + h) - \mu(\gamma_1))\,[\int_a^e u\bar{v}w\mathrm{d}x + \int_e^b u\bar{v}w\mathrm{d}x] \\
=& - [u,v]_a^{e-} - [u,v]_{e+}^b \\
=&[u,v](a) - [u,v](b) + [u,v](e+) - [u,v](e-),
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
&(\mu(\gamma_1 + h) - \mu(\gamma_1))\,\eta_1 u_1 \overline{v_1} \\
=&\eta_1 u_1 \mu(\gamma_1 + h)\overline{v_1} - \eta_1 \mu(\gamma_1) u_1 \overline{v_1} \\
=&(pu')(e-)(\xi_1 (p\bar{v}')(e-) - \bar{v}(e-) - (p\bar{v}')(e+)) \\
&- (p\bar{v}')(e-)(\xi_1 (pu')(e-) - u(e-) - (pu')(e+)) \\
=&2[u,v](e-),
\end{aligned} \tag{6.2}$$

$$\begin{aligned}
&(\mu(\gamma_1 + h) - \mu(\gamma_1))\, \eta_2 u_2 \overline{v_2} \\
=&\eta_2 u_2 \mu(\gamma_1 + h)\overline{v_2} - \eta_2 \mu(\gamma_1) u_2 \overline{v_2} \\
=&(pu')(e+)(\xi_2(p\bar{v}')(e+) + \bar{v}(e+) - (p\bar{v}')(e-)) \\
&- (p\bar{v}')(e+)(\xi_2(pu')(e+) + u(e+) - (pu')(e-)) \\
=& - [u, v](e+) - [u, v](e-).
\end{aligned} \tag{6.3}$$

By the BC (2.3), we obtain

$$[u, v](b) = 0.$$

Thus,

$$\begin{aligned}
&(\mu(\gamma_1 + h) - \mu(\gamma_1))\, [\int_a^e u\bar{v}w\mathrm{d}x + \int_e^b u\bar{v}w\mathrm{d}x + \eta_1 u_1 \overline{v_1} + \eta_2 u_2 \overline{v_2}] \\
=&[u, v](a) \\
=&[\cot(\gamma_1 + h) - \cot\gamma_1] u(a)\bar{v}(a).
\end{aligned} \tag{6.4}$$

Then by dividing h and letting $h \to 0$, we arrive at

$$\mu'(\gamma_1) = - \csc^2 \gamma_1 |u(a)|^2. \tag{6.5}$$

Using a similar method, we can get 2).

(b) With the exception of ξ_1, let us fix the parameters of ϑ, and let $u = u(\cdot, \xi_1)$ and $v = u(\cdot, \xi_1 + h)$. Then

$$(\mu(\xi_1 + h) - \mu(\xi_1))[\int_a^e u\bar{v}w\mathrm{d}x + \int_e^b u\bar{v}w\mathrm{d}x] = -[u, v]_a^{e-} - [u, v]_{e+}^b, \tag{6.6}$$

$$\begin{aligned}
&(\mu(\xi_1 + h) - \mu(\xi_1))\, \eta_1 u_1 \overline{v_1} \\
=&\eta_1 u_1 \mu(\xi_1 + h)\overline{v_1} - \eta_1 \mu(\xi_1) u_1 \overline{v_1} \\
=&(pu')(e-)((\xi_1 + h)(p\bar{v}')(e-) - \bar{v}(e-) - (p\bar{v}')(e+)) \\
&- (p\bar{v}')(e-)(\xi_1(pu')(e-) - u(e-) - (pu')(e+)) \\
=&2[u, v](e-) + h(pu')(e-)(p\bar{v}')(e-),
\end{aligned} \tag{6.7}$$

and

$$(\mu(\xi_1 + h) - \mu(\xi_1))\, \eta_2 u_2 \overline{v_2} = -[u, v](e+) - [u, v](e-). \tag{6.8}$$

By the BCs (2.3) and (2.4) we obtain

$$[u, v](a) = [u, v](b) = 0.$$

Thus,

$$\begin{aligned}
&(\mu(\xi_1 + h) - \mu(\xi_1))[\int_a^e u\bar{v}w\mathrm{d}x + \int_e^b u\bar{v}w\mathrm{d}x + \eta_1 u_1 \overline{v_1} + \eta_2 u_2 \overline{v_2}] \\
=& h(pu')(e-)(p\bar{v}')(e-).
\end{aligned} \tag{6.9}$$

Then by dividing h and letting $h \to 0$, we arrive at

$$\mu'(\xi_1) = |(pu')(e-)|^2. \tag{6.10}$$

This is the result for 3). And using a similar method, we can get 4).

(c) With the exception of η_2, let us fix the parameters of ϑ, and let $u = u(\cdot, \eta_2)$ and $v = u(\cdot, \eta_2 + h)$. Then

$$(\mu(\eta_2 + h) - \mu(\eta_2))[\int_a^e u\bar{v}w\mathrm{d}x + \int_e^b u\bar{v}w\mathrm{d}x] = -[u, v]_a^{e-} - [u, v]_{e+}^b, \tag{6.11}$$

$$(\mu(\eta_2 + h) - \mu(\eta_2))\,\eta_1 u_1 \overline{v_1} = 2[u, v](e-), \tag{6.12}$$

and

$$\begin{aligned}
&(\mu(\eta_2 + h) - \mu(\eta_2))\,\eta_2 u_2 \overline{v_2} \\
=&\eta_2 u_2 \mu(\eta_2 + h)\overline{v_2} - \eta_2 \mu(\eta_2) u_2 \overline{v_2} \\
=&(pu')(e+)\frac{\eta_2}{\eta_2 + h}(\xi_2(p\bar{v}')(e+) + \bar{v}(e+) - (p\bar{v}')(e-)) \\
&- (p\bar{v}')(e+)(\xi_2(pu')(e+) + u(e+) - (pu')(e-)) \\
=&-[u, v](e+) - [u, v](e-) - \frac{h}{\eta_2 + h}(\xi_2(p\bar{v}')(e+) + \bar{v}(e+) - (p\bar{v}')(e-))(pu')(e+).
\end{aligned} \tag{6.13}$$

Combining (6.11)–(6.13) and the BCs (2.3) and (2.4), we obtain

$$\begin{aligned}
&(\mu(\eta_2 + h) - \mu(\eta_2))[\int_a^e u\bar{v}w\mathrm{d}x + \int_e^b u\bar{v}w\mathrm{d}x + \eta_1 u_1 \overline{v_1} + \eta_2 u_2 \overline{v_2}] \\
=&-\frac{h}{\eta_2 + h}(\xi_2(p\bar{v}')(e+) + \bar{v}(e+) - (p\bar{v}')(e-))(pu')(e+).
\end{aligned} \tag{6.14}$$

Then by dividing h and letting $h \to 0$, we arrive at

$$\mu'(\eta_2) = -\mu|(pu')(e+)|^2. \tag{6.15}$$

Using a similar method, we can get that 5) holds.

(d) With the exception of w, let us fix the parameters of ϑ, and let $u = u(\cdot, w)$ and $v = u(\cdot, w + h)$. Then

$$(\mu(w + h) - \mu(w))\,\eta_1 u_1 \overline{v_1} = 2[u, v](e-), \tag{6.16}$$

$$(\mu(w + h) - \mu(w))\,\eta_2 u_2 \overline{v_2} = -[u, v](e+) - [u, v](e-), \tag{6.17}$$

$$\begin{aligned}
&(\mu(w + h) - \mu(w))\,[\int_a^e u\bar{v}w\mathrm{d}x + \int_e^b u\bar{v}w\mathrm{d}x] \\
=&[u, v](a) - [u, v](b) + [u, v](e+) - [u, v](e-) \\
&- (\int_a^e hu\mu(w + h)\bar{v}dx + \int_e^b hu\mu(w + h)\bar{v}dx).
\end{aligned} \tag{6.18}$$

Combining (6.16)–(6.18) and the BCs (2.3) and (2.4), we obtain

$$\begin{aligned}
&(\mu(w + h) - \mu(w))[\int_a^e u\bar{v}wdx + \int_e^b u\bar{v}wdx + \eta_1 u_1 \overline{v_1} + \eta_2 u_2 \overline{v_2}] \\
=&-(\int_a^e hu\mu(w + h)\bar{v}dx + \int_e^b hu\mu(w + h)\bar{v}dx).
\end{aligned} \tag{6.19}$$

Let $h \to 0$, we arrive at

$$d\mu_w(h) = -\mu[\int_a^e h|u|^2 + \int_e^b h|u|^2]. \tag{6.20}$$

Using a similar method, we can get 8) and 9).

7. Conclusions

This paper represents the study of a new class of discontinuous S-L problems in which the spectral parameter appears in the differential equation and the transmission conditions. The eigenvalue and eigenfunction problems of the S-L problem have been converted into an eigenvalue problem for a specific self-adjoint operator by building an appropriate Hilbert space and inner product, and the self-adjointness of the operator in this case is provided. Next, some basic properties of eigenvalues were given via the construction of the fundamental solutions. The Green's function for this new class of S-L problem has also been derived. Then, the continuity of the eigenvalues and eigenfunctions of the problem was discussed. We obtained that the eigenvalues of the problem are continuously and smoothly dependent on the parameters which define the problem. Finally, the differential equations for the eigenvalues associated with the coefficient functions, the endpoints, the BCs, and transmission conditions were obtained. The results obtained here are further generalizations of eigenvalue dependence of the boundary value problems. As far as we know, there is no such eigenvalue dependence results for S-L problems with eigenparameter-dependent transmission conditions.

The eigenvalue problems and eigenvalue dependence problems of differential operators play an important role in mathematics and other fields of sciences. Such problems can be viewed as the theoretical basis of the ordinary differential equations and enable effective numerical computation of the eigenvalues, estimates of eigenvalues, and the inverse spectral theory of differential operators. For example, the sharp estimates of eigenvalues for the corresponding differential operator may require the use of our basic eigenvalue results here.

Use of AI tools declaration

The authors declare that they have not used artificial intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that they have no conflict of interest.

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