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Research article

The m-weak group inverse for rectangular matrices

D. Mosić[1], P. S. Stanimirović[1,2,*] and L. A. Kazakovtsev[2]

[1] University of Niš, Faculty of Sciences and Mathematics, Višegradska 33, 18000 Niš, Serbia

[2] Laboratory Hybrid Methods of Modelling and Optimization in Complex Systems, Siberian Federal University, Prosp. Svobodny 79, 660041 Krasnoyarsk, Russia

* **Correspondence:** Email: pecko@pmf.ni.ac.rs.

Abstract: An extension of the m-weak group inverse (or m-WGI) on the set of rectangular matrices is provided to solve some systems of matrix equations. The extension is termed as the W-weighted m-WGI (or W-m-WGI). The W-m-WGI presents a new, wider class of generalized inverses which involves some already defined generalized inverses, such as the m-WGI, W-weighted weak group, and W-weighted Drazin inverse. Basic properties and diverse characterizations are proved for W-m-WGI. Several expressions for computing W-m-WGI are proposed in terms of known generalized inverses and projectors, as well as its limit and integral representations. The W-m-WGI class is utilized to solve some linear matrix equations and express their general solutions. Some new properties of the weighted generalized group inverse and recognized properties of the W-weighted Drazin inverse are obtained as corollaries. Numerical and symbolic test examples are presented to verify the obtained results.

Keywords: m-weak group inverse; W-weighted weak group inverse; W-weighted core-EP inverse; W-weighted Drazin inverse

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1. Introduction

Throughout this work, $\mathbb{C}^{p\times n}$ denotes the set involving $p \times n$ matrices with complex entries, and, for $A \in \mathbb{C}^{p\times n}$, $\operatorname{rank}(A)$ is its rank, A^* is its conjugate-transpose matrix, $\mathcal{N}(A)$ is its null space, and $\mathcal{R}(A)$ is its range. The index $\operatorname{ind}(A)$ of $A \in \mathbb{C}^{p\times p}$ is the smallest nonnegative integer k for which the equality $\operatorname{rank}(A^k) = \operatorname{rank}(A^{k+1})$ is satisfied. The symbol I denotes the identity matrix of adequate size. Standard notations P_S and $P_{S,T}$ denote, respectively, the orthogonal projector onto a subspace S and a projector onto S along T when $\mathbb{C}^p$ is equal to the direct sum of the subspaces S and T.

Several definitions and properties of generalized inverses which are upgraded in this research are given. The Moore-Penrose inverse of $A \in \mathbb{C}^{p\times n}$ is uniquely determined $A^\dagger = X \in \mathbb{C}^{n\times p}$ as the solution to well-known Penrose equations [1]:

$$A = AXA, \qquad X = XAX, \qquad AX = (AX)^*, \qquad XA = (XA)^*.$$

If X satisfies only equation $XAX = X$, it is an outer inverse of A. The outer inverse of A which is uniquely determined by the null space S and the range T is labeled with $A^{(2)}_{T,S} = X \in \mathbb{C}^{n\times p}$ and satisfies

$$XAX = X, \quad \mathcal{N}(X) = S, \quad \mathcal{R}(X) = T,$$

where $s \le r = \operatorname{rank}(A)$ is the dimension of the subspace $T \subseteq \mathbb{C}^n$, and $p - s$ is the dimension of the subspace $S \subseteq \mathbb{C}^p$.

The following notation will be used:

$$\mathbb{C}^{p,n;k} := \{(A, W) : A \in \mathbb{C}^{p\times n},\ W \in \mathbb{C}^{n\times p}\backslash\{0\} \text{ and } k = \max\{\operatorname{ind}(AW), \operatorname{ind}(WA)\}\}.$$

The notion of the Drazin inverse was extended to rectangular matrices in [2]. For selected $(A, W) \in \mathbb{C}^{p,n;k}$, the W-weighted Drazin inverse $A^{\mathrm{D},W} = X \in \mathbb{C}^{p\times n}$ of A is uniquely determined by the matrix equations

$$XWAWX = X, \quad AWX = XWA, \quad (AW)^{k+1}XW = (AW)^k.$$

Especially, if $p = n$ and $W = I$, $A^{\mathrm{D},I} := A^{\mathrm{D}}$ reduces to the Drazin inverse of A. Further, for $\operatorname{ind}(A) = 1$, $A^{\mathrm{D}} := A^{\#}$ becomes the group inverse of A. Recall that [2]

$$A^{\mathrm{D},W} = A[(WA)^{\mathrm{D}}]^2 = [(AW)^{\mathrm{D}}]^2 A.$$

The notion of the core-EP inverse, proposed in [3] for a square matrix, was generalized to a rectangular matrix in [4]. If $(A, W) \in \mathbb{C}^{p,n;k}$, the W-weighted core-EP inverse of A is the unique solution $A^{\textcircled{\dagger},W} = X \in \mathbb{C}^{p\times n}$ to

$$WAWX = P_{\mathcal{R}(WA)^k}, \quad \mathcal{R}((AW)^k) = \mathcal{R}(X).$$

In a special case $p = n$ and $W = I$, $A^{\textcircled{\dagger},I}$ becomes the core-EP inverse $A^{\textcircled{\dagger}}$ of A. According to original definitions in [5] and [6–8], it is important to note

$$A^{\textcircled{\dagger},W} = A[(WA)^{\textcircled{\dagger}}]^2$$

and

$$A^{\textcircled{\dagger}} = A^{\mathrm{D}}A^k(A^k)^{\dagger} = A^k\left(A^{k+1}\right)^{\dagger}.$$

Some useful characterizations and representations of the core-EP inverse are presented in [3, 9–14]. In the case $\operatorname{ind}(A) = 1$, $A^{\textcircled{\dagger}}$ reduces to the core inverse $A^{\textcircled{\#}} = A^{\#}AA^{\dagger}$ of A [15].

The weak group inverse (WGI) was presented for a square matrix in [16] as an extension of the group inverse. The WGI is extended in [17] to a rectangular matrix and in [18] to Hilbert space operators. For $(A, W) \in \mathbb{C}^{p,n;k}$, the W-weighted WGI (W-WGI) of A is the unique solution $A^{\textcircled{w},W} = X \in \mathbb{C}^{p\times n}$ of the system [17, 18]

$$AWXWX = X, \quad AWX = A^{\textcircled{\dagger},W}WA,$$

and it is expressed by [17, 18]

$$A^{\textcircled{w},W} = (A^{\textcircled{\dagger},W}W)^2 A.$$

When $p = n$ and $W = I$, $A^{\circledR,I} := A^{\circledR}$ reduces to the WGI of A

$$A^{\circledR} = (A^{\textcircled{\dagger}})^2 A.$$

Remark that, for $1 = \operatorname{ind}(A)$, $A^{\circledR} = A^{\#}$. Useful results about WGI were given in [17–23].

The concept of the m-weak group inverse (m-WGI) was introduced in [24] as an extension of the WGI. Exactly, if $m \in \mathbb{N}$, the m-WGI of $A \in \mathbb{C}^{n\times n}$ is the unique matrix $A^{\circledR_m} = X \in \mathbb{C}^{n\times n}$ such that [25]

$$AX = (A^{\textcircled{\dagger}})^m A^m, \quad AX^2 = X. \tag{1.1}$$

Recall that

$$A^{\circledR_m} = (A^{\textcircled{\dagger}})^{m+1} A^m.$$

Clearly, $A^{\circledR_1} = A^{\circledR}$, and particularly $A^{\circledR_2} = (A^{\textcircled{\dagger}})^3 A^2$ becomes the generalized group (GG) inverse of A, established in [26]. It is interesting that, if $\operatorname{ind}(A) \le m$, $A^{\circledR_m} = A^{\mathrm{D}}$. Various properties of m-WGI were presented in [24,25,27,28].

Recent research about m-WGI as well as the fact that m-WGI is an important extension of the WGI, GG, Drazin inverse, and group inverse motivated us to further investigate this topic. The current popular trend in the research of generalized inverses consists in defining new generalized inverses that are based on suitable combinations of existing generalized inverses as well as in their application in solving appropriate systems of linear equations. Considering the system (1.1) for defining m-WGI, our first aim is to solve a system of matrix equations which is an extension of the system (1.1) from the square matrix case to an arbitrary case. Since the m-WGI is restricted to square matrices, our main goal is to extend this notion to W-m-WGI inverses on rectangular matrices. To solve a certain system of matrix equations on rectangular complex matrices, we extend the notions of m-WGI, W-WGI, and the W-weighted Drazin inverse by introducing a wider class of generalized inverses, termed as the W-weighted m-WGI (W-m-WGI) for a rectangular matrix. Particularly, an extension of the GG inverse on rectangular matrices is obtained. It is important to mention that we recover significant results for the W-weighted Drazin inverse in a particular case. A class of systems of linear equations is found that can be efficiently solved applying W-m-WGI. This results is an extension of known results about the W-weighted Drazin solution and the Drazin solution of exact linear systems.

The global structure of the work is based on sections with the following content. Several characterizations for the W-m-WGI are proved in Section 2 without and with projectors. We develop important expressions for the W-m-WGI based on core–EP, Drazin, and Moore-Penorse inverses of proper matrices. As a consequence, we introduce the weighted version of GG inverse and give its properties. Limit and integral formulae for computing the W-m-WGI are part of Section 3. Section 4 investigates applications of the W-m-WGI in solving specific matrix equations. Numerical experiments are presented in Section 5.

2. The W-weighted m-WGI

We introduce the W-weighted m-WGI on rectangular matrices as a class of generalized inverses that includes notions of the m-WGI and the W-weighted WGI.

Theorem 2.1. *If $(A, W) \in \mathbb{C}^{p,n;k}$ and $m \in \mathbb{N}$, then $X = (A^{\textcircled{\dagger},W} W)^{m+1}(AW)^{m-1}A$ is the unique solution to the matrix system*

$$AWX = (A^{\textcircled{\dagger},W} W)^m (AW)^{m-1} A \quad \text{and} \quad AWXWX = X. \tag{2.1}$$

Proof. Using the identity $AWA^{\textcircled{\dagger},W}WA^{\textcircled{\dagger},W} = A^{\textcircled{\dagger},W}$, the subsequent transformations are obtained for $X := (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m-1}A$:

$$AWX = AW(A^{\textcircled{\dagger},W}W)^{m+1}\,(AW)^{m-1}A = (A^{\textcircled{\dagger},W}W)^{m}\,(AW)^{m-1}A,$$

which further leads to

$$\begin{aligned} AWXWX &= (A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}AWX \\ &= (A^{\textcircled{\dagger},W}W)^m \left((AW)^m \left(A^{\textcircled{\dagger},W}W\right)^{m+1}\right)(AW)^{m-1}A \\ &= (A^{\textcircled{\dagger},W}W)^{m+1}\,(AW)^{m-1}A = X. \end{aligned}$$

Hence, $X = (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m-1}A$ is a solution to (2.1).

An arbitrary solution X to the system (2.1) satisfies

$$\begin{aligned} X &= (AWX)WX = (A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}(AWX) \\ &= (A^{\textcircled{\dagger},W}W)^m \left((AW)^{m-1} \left(A^{\textcircled{\dagger},W}W\right)^{m}\right)(AW)^{m-1}A \\ &= (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m-1}A, \end{aligned}$$

which leads to the conclusion that $X = (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m-1}A$ is the unique solution to (2.1). □

Definition 2.1. Under such conditions $(A, W) \in \mathbb{C}^{p,n;k}$ and $m \in \mathbb{N}$, the W-weighted m-WGI (shortly W-m-WGI) inverse of A is defined by the expression

$$\begin{aligned} A^{\textcircled{w}_m,W} &= (A^{\textcircled{\dagger},W}W)^{m+1}\,(AW)^{m-1}A \\ &= (A^{\textcircled{\dagger},W}W)^{m+1}\,A(WA)^{m-1}. \end{aligned}$$

Several special appearance forms of the W-m-WGI show its importance and are listed as follows:

- when $p = n$ and $W = I$, the I-m-WGI coincides with the m-WGI $A^{\textcircled{w}_m} = (A^{\textcircled{\dagger}})^{m+1}A^m$;
- if $m = 1$, then $(A^{\textcircled{\dagger},W}W)^2A = A^{\textcircled{w},W}$, that is, the W-1-WGI reduces to the W-WGI;
- for $m = 2$, the W-2-WGI is introduced as $A^{\textcircled{w}_2,W} = (A^{\textcircled{\dagger},W}W)^3AWA$ and presents an extension of the GG inverse;
- in the case $k \le m$, it follows $A^{\textcircled{w}_m,W} = A^{\mathrm{D},W}$ (see Lemma 2.1).

Some computationally useful representations of the W-m-WGI are developed in subsequent statements.

Lemma 2.1. *If $(A, W) \in \mathbb{C}^{p,n;k}$, $m \in \mathbb{N}$ and $l \ge k$, then*

$$\begin{aligned} A^{\textcircled{w}_m,W} &= A\left[(WA)^{\textcircled{\dagger}}\right]^{m+2}(WA)^m \\ &= A(WA)^{\textcircled{\dagger}}(WA)^{\textcircled{w}_m} \\ &= A\left[(WA)^{\mathrm{D}}\right]^{m+2}(WA)^l\left[(WA)^l\right]^{\dagger}(WA)^m = A\left[(WA)^{\mathrm{D}}\right]^{m+2}P_{\mathcal{R}((WA)^l)}(WA)^m \\ &= A(WA)^l\left[(WA)^{l+m+2}\right]^{\dagger}(WA)^m. \end{aligned}$$

Furthermore, for $m \ge k$, it follows that $A^{\textcircled{w}_m,W} = A^{\mathrm{D},W}$.

Proof. First, by induction on m, notice that $A^{\textcircled{\dagger},W} = A\left[(WA)^{\textcircled{\dagger}}\right]^2$ gives

$$\left(A^{\textcircled{\dagger},W}W\right)^{m+1} = \left(A\left[(WA)^{\textcircled{\dagger}}\right]^2 W\right)^{m+1} = A\left[(WA)^{\textcircled{\dagger}}\right]^{m+2} W.$$

Further, based on

$$\left[(WA)^{\textcircled{\dagger}}\right]^{m+2} = \left((WA)^{\mathrm{D}}(WA)^k\left[(WA)^k\right]^{\dagger}\right)^{m+2} = \left[(WA)^{\mathrm{D}}\right]^{m+2}(WA)^k\left[(WA)^k\right]^{\dagger}$$

and

$$(WA)^k[(WA)^k]^{\dagger} = P_{\mathcal{R}((WA)^k)} = P_{\mathcal{R}((WA)^l)} = (WA)^l\left[(WA)^l\right]^{\dagger},$$

we obtain

$$\begin{aligned} A^{\textcircled{w}_m,W} &= (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m-1}A = A\left[(WA)^{\textcircled{\dagger}}\right]^{m+2}(WA)^m \\ &= A\left[(WA)^{\mathrm{D}}\right]^{m+2}(WA)^k\left[(WA)^k\right]^{\dagger}(WA)^m \\ &= A\left[(WA)^{\mathrm{D}}\right]^{m+2}(WA)^{l+m+2}\left[(WA)^{l+m+2}\right]^{\dagger}(WA)^m \\ &= A(WA)^l\left[(WA)^{l+m+2}\right]^{\dagger}(WA)^m. \end{aligned}$$

In the case $m \geq k$, it follows that

$$\begin{aligned} A^{\textcircled{w}_m,W} &= A\left[(WA)^{\mathrm{D}}\right]^{m+2}(WA)^k\left[(WA)^k\right]^{\dagger}(WA)^m = A\left[(WA)^{\mathrm{D}}\right]^{m+2}(WA)^m \\ &= A\left[(WA)^{\mathrm{D}}\right]^2 = A^{\mathrm{D},W}. \end{aligned}$$

□

Remark 2.1. *Note that* $A^{\textcircled{w}_m,W} = A(WA)^{\textcircled{\dagger}}(WA)^{\textcircled{w}_m}$ *implies the interesting identity* $WA^{\textcircled{w}_m,W} = P_{\mathcal{R}((WA)^l)}(WA)^{\textcircled{w}_m}$ *for* $l \geq k$. *This last identity is an extension of the classical property of the* W*-weighted Drazin inverse* $WA^{\mathrm{D},W} = (WA)^{\mathrm{D}}$. *About the dual property* $A^{\mathrm{D},W}W = (AW)^{\mathrm{D}}$, *if the equality* $A^{\textcircled{\dagger},W} = [(AW)^{\textcircled{\dagger}}]^2A$ *is satisfied (which is not true in general; for details see* [5]), *we can verify that* $A^{\textcircled{w}_m,W} = (AW)^{\textcircled{\dagger}}(AW)^{\textcircled{w}_m}A$ *and so* $A^{\textcircled{w}_m,W}W = (AW)^{\textcircled{\dagger}}(AW)^{\textcircled{w}_m}AW$.

Representations for the W-2-WGI and W-weighted Drazin inverse are obtained as consequences of Lemma 2.1 when $m = 2$ or $m = l \geq k$, respectively.

Corollary 2.1. *If* $(A, W) \in \mathbb{C}^{p,n;k}$ *and* $l \geq k$, *then*

$$\begin{aligned} A^{\textcircled{w}_2,W} &= A\left[(WA)^{\textcircled{\dagger}}\right]^4(WA)^2 \\ &= A\left[(WA)^{\mathrm{D}}\right]^4(WA)^l\left[(WA)^l\right]^{\dagger}(WA)^2 = A\left[(WA)^{\mathrm{D}}\right]^4 P_{\mathcal{R}((WA)^l)}(WA)^2 \\ &= A(WA)^l\left[(WA)^{l+4}\right]^{\dagger}(WA)^2 \end{aligned}$$

and

$$\begin{aligned} A^{\mathrm{D},W} &= A\left[(WA)^{\textcircled{\dagger}}\right]^{l+2}(WA)^l = A\left[(WA)^{\mathrm{D}}\right]^2 \\ &= A(WA)^l\left[(WA)^{2(l+1)}\right]^{\dagger}(WA)^l. \end{aligned}$$

Notice that Corollary 2.1 recovers the known expressions for the W-weighted Drazin inverse [29, 30].

In Lemma 2.2, we show that the W-m-WGI $A^{\circledR_m,W}$ is an outer inverse of WAW and find its range and null spaces.

Lemma 2.2. *If* $(A, W) \in \mathbb{C}^{p,n;k}$ *and* $m \in \mathbb{N}$*, the following representations are valid:*

(i) $A^{\circledR_m,W} = (WAW)^{(2)}_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*(WA)^m)}$;

(ii) $WAWA^{\circledR_m,W} = P_{\mathcal{R}((WA)^k),\mathcal{N}([(WA)^k]^*(WA)^m)}$;

(iii) $A^{\circledR_m,W}WAW = P_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*(WA)^{m+1}W)}$.

Proof. (i) Based on Lemma 2.1 it follows that $A^{\circledR_m,W} = A(WA)^k\left[(WA)^{k+m+2}\right]^\dagger(WA)^m$, which yields $\mathcal{R}(A^{\circledR_m,W}) \subseteq \mathcal{R}((AW)^k)$ and

$$\begin{aligned} A^{\circledR_m,W}WAWA^{\circledR_m,W} &= A(WA)^k\left[(WA)^{k+m+2}\right]^\dagger(WA)^{k+m+2}\left[(WA)^{k+m+2}\right]^\dagger(WA)^m \\ &= A(WA)^k\left[(WA)^{k+m+2}\right]^\dagger(WA)^m = A^{\circledR_m,W}. \end{aligned}$$

Another application of Lemma 2.1 yields

$$\begin{aligned} (AW)^k &= \left[(AW)^{\mathrm{D}}\right]^{m+2}(AW)^{k+m+2} = A\left[(WA)^{\mathrm{D}}\right]^{m+2}(WA)^{k+m+1}W \\ &= A\left[(WA)^{\mathrm{D}}\right]^{m+2}(WA)^k\left[(WA)^k\right]^\dagger(WA)^{k+m+1}W \\ &= A^{\circledR_m,W}(WA)^{k+1}W \end{aligned} \tag{2.2}$$

and so $\mathcal{R}((AW)^k) \subseteq \mathcal{R}(A^{\circledR_m,W})$. Thus, $\mathcal{R}(A^{\circledR_m,W}) = \mathcal{R}((AW)^k)$. Also,

$$\begin{aligned} \mathcal{N}\left(\left[(WA)^k\right]^*(WA)^m\right) &= \mathcal{N}\left(\left[(WA)^{k+m+2}\right]^*(WA)^m\right) = \mathcal{N}\left(\left[(WA)^{k+m+2}\right]^\dagger(WA)^m\right) \\ &= \mathcal{N}\left(A(WA)^k\left[(WA)^{k+m+2}\right]^\dagger(WA)^m\right) = \mathcal{N}\left(A^{\circledR_m,W}\right). \end{aligned}$$

(ii) By the part (i), $WAWA^{\circledR_m,W}$ is a projector, and

$$\mathcal{N}(WAWA^{\circledR_m,W}) = \mathcal{N}(A^{\circledR_m,W}) = \mathcal{N}([(WA)^k]^*(WA)^m).$$

The equalities $WAWA^{\circledR_m,W} = (WA)^{k+2}\left[(WA)^{k+m+2}\right]^\dagger(WA)^m$ and

$$\begin{aligned} (WA)^k &= (WA)^{k+2}\left[(WA)^{\mathrm{D}}\right]^2 = (WA)^{k+2}\left[(WA)^{k+2}\right]^\dagger(WA)^{k+2}\left[(WA)^{\mathrm{D}}\right]^2 \\ &= (WA)^{k+2}P_{\mathcal{R}([(WA)^{k+2}]^*)}\left[(WA)^{\mathrm{D}}\right]^2 = (WA)^{k+2}P_{\mathcal{R}([(WA)^{m+k+2}]^*)}\left[(WA)^{\mathrm{D}}\right]^2 \\ &= (WA)^{k+2}\left[(WA)^{m+k+2}\right]^\dagger(WA)^{m+k+2}\left[(WA)^{\mathrm{D}}\right]^2 \\ &= WAWA^{\circledR_m,W}(WA)^{k+2}\left[(WA)^{\mathrm{D}}\right]^2 \\ &= WAWA^{\circledR_m,W}(WA)^k \end{aligned}$$

imply $\mathcal{R}(WAWA^{\circledR_m,W}) = \mathcal{R}((WA)^k)$.

(iii) It is clear, by (i), that $\mathcal{R}(A^{\circledR_m,W}WAW) = \mathcal{R}(A^{\circledR_m,W}) = \mathcal{R}((AW)^k)$. The identity $\mathcal{N}(A^{\circledR_m,W}WAW) = \mathcal{N}\left(\left[(WA)^k\right]^*(WA)^{m+1}W\right)$ is verified in a similar manner as in (i). □

Remark 2.2. *For $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{q\times t}$, $C \in \mathbb{C}^{s\times p}$, $M \in \mathbb{C}^{p\times m}$, and $N \in \mathbb{C}^{n\times q}$, by* [31, 32]*, the* (M,N)*-weighted* (B,C)*-inverse of* A *is represented by* $A^{(2,M,N)}_{(B,C)} = (MAN)^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$. *By Lemma 2.2,* $A^{\circledW_m,W} = (WAW)^{(2)}_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*(WA)^m)}$*, and thus* $A^{\circledW_m,W}$ *is the* (W,W)*-weighted* $((AW)^k), [(WA)^k]^*(WA)^m)$*-inverse of* A. *Since the* (B,C)*-inverse of* A *is given as* $A^{(2)}_{(B,C)} = A^{(2)}_{\mathcal{R}(B),\mathcal{N}(C)}$ [33]*, it follows that* $A^{\circledW_m,W} = (WAW)^{(2)}_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*(WA)^m)}$ *is the* $\left((AW)^k, [(WA)^k]^*(WA)^m\right)$*-inverse of* WAW.

Lemma 2.2 and the Urquhart formula [1] give the next representations for $A^{\circledW_m,W}$.

Corollary 2.2. *If* $(A,W) \in \mathbb{C}^{p,n;k}$ *and* $m \in \mathbb{N}$*, the* W*-*m*-WGI of* A *is represented as*

$$A^{\circledW_m,W} = (AW)^k \left(\left[(WA)^k\right]^* (WA)^{m+k+1} W\right)^\dagger [(WA)^k]^*(WA)^m.$$

If $m = 2$ or $m = k$ in Lemma 2.2 and Corollary 2.2, we obtain the next properties related to the W-2-WGI and W-weighted Drazin inverse.

Corollary 2.3. *If* $(A,W) \in \mathbb{C}^{p,n;k}$*, the following statements hold:*

(i) $A^{\circledW_2,W} = (WAW)^{(2)}_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*(WA)^2)} = (AW)^k \left(\left[(WA)^k\right]^* (WA)^{k+3} W\right)^\dagger \left[(WA)^k\right]^* (WA)^2$;
(ii) $WAWA^{\circledW_2,W} = P_{\mathcal{R}((WA)^k),\mathcal{N}([(WA)^k]^*(WA)^2)}$;
(iii) $A^{\circledW_2,W}WAW = P_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*(WA)^3 W)}$;
(iv) $A^{\mathrm{D},W} = (WAW)^{(2)}_{\mathcal{R}((AW)^k),\mathcal{N}((WA)^k)} = (AW)^k \left([(WA)^k]^*(WA)^{2k+1} W\right)^\dagger [(WA)^k]^*(WA)^k$;
(v) $WAWA^{\mathrm{D},W} = P_{\mathcal{R}((WA)^k),\mathcal{N}((WA)^k)}$;
(vi) $A^{\mathrm{D},W}WAW = P_{\mathcal{R}((AW)^k),\mathcal{N}((AW)^k)}$.

Some necessary and sufficient conditions for a rectangular matrix to be the W-m-WGI are considered.

Theorem 2.2. *If* $(A,W) \in \mathbb{C}^{p,n;k}$*,* $X \in \mathbb{C}^{p\times n}$*, and* $m \in \mathbb{N}$*, the subsequent statements are equivalent:*

(i) $X = A^{\circledW_m,W}$;
(ii) $AWX = (A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$ *and* $\mathcal{R}(X) = \mathcal{R}((AW)^k)$;
(iii) $AWX = (A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}((AW)^k)$;
(iv) $AWXWX = X$, $X(WA)^{k+1}W = (AW)^k$ *and* $[(WA)^k]^*(WA)^{m+1}WX = [(WA)^k]^*(WA)^m$;
(v) $XWAWX = X$, $\mathcal{R}(X) = \mathcal{R}((AW)^k)$ *and* $[(WA)^k]^*(WA)^{m+1}WX = [(WA)^k]^*(WA)^m$;
(vi) $XWAWX = X$, $AWX = (A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$ *and* $XWA = (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^mA$;
(vii) $XWAWX = X$, $WAWX = W(A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$ *and* $XWAW = (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m+1}$;
(viii) $XWAWX = X$, $AWXWA = (A^{\textcircled{\dagger},W}W)^m(AW)^mA$, $AWX = (A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$ *and* $XWA = (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^mA$;
(ix) $XWAWX = X$, $WAWXWAW = W(A^{\textcircled{\dagger},W}W)^m(AW)^{m+1}$, $WAWX = W(A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$ *and* $XWAW = (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m+1}$;
(x) $X = A^{\textcircled{\dagger},W}WAWX$ *and* $AWX = (A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$ *(or* $WAWX = W(A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$*)*;
(xi) $X = AWA^{\textcircled{\dagger},W}WX$ *and* $A^{\textcircled{\dagger},W}WX = (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m-1}A$;
(xii) $X = A^{\mathrm{D},W}WAWX$ *and* $AWX = (A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$ *(or* $WAWX = W(A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$*)*;
(xiii) $X = (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m+1}X$ *and* $AWX = (A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$ *(or* $WAWX = W(A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$*)*;

(xiv) $X = XW(A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$ *and* $XWA = (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^m A$ *(or* $XWAW = (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m+1}$*).*

Proof. (i) ⇒ (ii): It follows from Theorem 2.1 and Lemma 2.2.

(ii) ⇒ (iii): This implication is obvious.

(iii) ⇒ (i): Because $\mathcal{R}(X) \subseteq \mathcal{R}((AW)^k)$, we have

$$X = (AW)^k U = (AW)^k[(AW)^k]^\dagger (AW)^k U = (AW)^k[(AW)^k]^\dagger X,$$

for some $U \in \mathbb{C}^{p\times n}$. Notice that, by $AWX = (A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$,

$$\begin{aligned}
AWXWX &= (A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}AWX \\
&= \left(A\left[(WA)^{\textcircled{\dagger}}\right]^2 W\right)^m (AW)^m X \\
&= A\left[(WA)^{\textcircled{\dagger}}\right]^{m+1} W(AW)^m X \\
&= A\left[(WA)^{\mathrm{D}}\right]^{m+1} (WA)^k[(WA)^k]^\dagger W(AW)^m X \\
&= A\left[(WA)^{\mathrm{D}}\right]^{m+1} (WA)^k[(WA)^k]^\dagger W(AW)^m(AW)^k[(AW)^k]^\dagger X \\
&= A\left[(WA)^{\mathrm{D}}\right]^{m+1} (WA)^{m+k}W[(AW)^k]^\dagger X \\
&= \left[(AW)^{\mathrm{D}}\right]^{m+1} (AW)^{m+k+1}[(AW)^k]^\dagger X \\
&= (AW)^k\left[(AW)^k\right]^\dagger X \\
&= X.
\end{aligned}$$

An application of Theorem 2.1 leads to the conclusion $X = A^{\textcircled{w}_m,W}$.

(i) ⇒ (iv): For $X = A^{\textcircled{w}_m,W}$, Theorem 2.1 implies $AWXWX = X$. The equality (2.2) gives $X(WA)^{k+1}W = (AW)^k$. Using Lemma 2.1, we get $X = A(WA)^k[(WA)^{k+m+2}]^\dagger(WA)^m$, which implies

$$\begin{aligned}
[(WA)^k]^*(WA)^{m+1}WX &= \left[(WA)^k\right]^*(WA)^{m+1}WA(WA)^k\left[(WA)^{k+m+2}\right]^\dagger(WA)^m \\
&= \left[(WA)^k\right]^*(WA)^{k+m+2}\left[(WA)^{k+m+2}\right]^\dagger(WA)^m \\
&= \left[(WA)^k\right]^*(WA)^k\left[(WA)^k\right]^\dagger(WA)^m \\
&= \left[(WA)^k\right]^*(WA)^m.
\end{aligned}$$

(iv) ⇒ (v): Notice, by $X(WA)^{k+1}W = (AW)^k$ and

$$X = AWXWX = (AW)^2(XW)^2X = \cdots = (AW)^r(XW)^rX, \tag{2.3}$$

for arbitrary $r \in \mathbb{N}$, that $\mathcal{R}(X) = \mathcal{R}((AW)^k)$. Hence, $X = (AW)^kU$, for some $U \in \mathbb{C}^{p\times n}$, and

$$XWAWX = XWAW(AW)^kU = (X(WA)^{k+1}W)U = (AW)^kU = X.$$

(v) ⇒ (i): The assumptions $\mathcal{R}(X) = \mathcal{R}((AW)^k)$ and $XWAWX = X$ give $(AW)^k = XV = XWAW(XV) = XW(AW)^{k+1}$, for some $V \in \mathbb{C}^{n\times p}$. Since $X = (AW)^kU$, for some $U \in \mathbb{C}^{p\times n}$, we get

$$AWXWX = AWXW(AW)^kU = AW\left(XW(AW)^{k+1}\right)(AW)^{\mathrm{D}}U$$

$$= AW(AW)^k(AW)^{\mathrm{D}}U = (AW)^kU = X.$$

The assumption $\left[(WA)^k\right]^*(WA)^{m+1}WX = \left[(WA)^k\right]^*(WA)^m$ yields

$$\begin{aligned}
(WA)^k\left[(WA)^k\right]^\dagger(WA)^{m+1}WX &= \left(\left[(WA)^k\right]^\dagger\right)^*\left[(WA)^k\right]^*(WA)^{m+1}WX\\
&= \left(\left[(WA)^k\right]^\dagger\right)^*\left[(WA)^k\right]^*(WA)^m\\
&= (WA)^k\left[(WA)^k\right]^\dagger(WA)^m.
\end{aligned}$$

Because (2.3) holds, we obtain

$$\begin{aligned}
(A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A &= A[(WA)^{\textcircled{\dagger}}]^{m+1}(WA)^m\\
&= A\left[(WA)^{\mathrm{D}}\right]^{m+1}(WA)^k\left[(WA)^k\right]^\dagger(WA)^m\\
&= A\left[(WA)^{\mathrm{D}}\right]^{m+1}(WA)^k\left[(WA)^k\right]^\dagger(WA)^{m+1}WX\\
&= A\left[(WA)^{\mathrm{D}}\right]^{m+1}(WA)^k\left[(WA)^k\right]^\dagger(WA)^{m+1}W(AW)^k(XW)^kX\\
&= A\left[(WA)^{\mathrm{D}}\right]^{m+1}(WA)^{k+m+1}W(XW)^kX\\
&= AW\left((AW)^k(XW)^kX\right)\\
&= AWX.
\end{aligned}$$

Theorem 2.1 gives $X = A^{\textcircled{w}_m,W}$.

(i) $\Rightarrow$ (vi) $\Rightarrow$ (vii): These implications are clear.

(vii) $\Rightarrow$ (i): Using $XWAWX = X$, $WAWX = W(A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$ and $XWAW = (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m+1}$, we get

$$\begin{aligned}
X &= X(WAWX) = XW(A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A\\
&= (XWAW)(A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m-1}A\\
&= (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m+1}(A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m-1}A\\
&= (A^{\textcircled{\dagger},W}W)^{m+1}AWA^{\textcircled{\dagger},W}W(AW)^{m-1}A\\
&= (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m-1}A.
\end{aligned}$$

(vi) $\Leftrightarrow$ (viii) and (vii) $\Leftrightarrow$ (ix): These equivalences are evident.

(i) $\Rightarrow$ (x): From $X = (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m-1}A$, we observe

$$A^{\textcircled{\dagger},W}WAWX = A^{\textcircled{\dagger},W}WAW(A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m-1}A = (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m-1}A = X.$$

(x) $\Rightarrow$ (i): Applying $X = A^{\textcircled{\dagger},W}WAWX$ and $AWX = (A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A$, we obtain $X = A^{\textcircled{\dagger},W}W(AWX) = A^{\textcircled{\dagger},W}W(A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A = (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^{m-1}A$.

(i) $\Leftrightarrow$ (xiv): This equivalence follows as (i) $\Leftrightarrow$ (vii). □

We also characterize the W-m-WGI in the following two ways.

Theorem 2.3. *If* $(A, W) \in \mathbb{C}^{p,n;k}$ *and* $m \in \mathbb{N}$*, then*

(i) $A^{\circledS_m,W}$ *is the unique solution to*

$$WAWX = P_{\mathcal{R}((WA)^k),\mathcal{N}([(WA)^k]^*(WA)^m)} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}((AW)^k); \tag{2.4}$$

(ii) $A^{\circledS_m,W}$ *is the unique solution to*

$$XWAW = P_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*(WA)^{m+1}W)} \quad \textit{and} \quad \mathcal{R}(X^*) \subseteq \mathcal{R}([(WA)^m]^*(WA)^k). \tag{2.5}$$

Proof. (i) By Lemma 2.2, $X = A^{\circledS_m,W}$ is a solution to (2.4). If the system (2.4) has two solutions X and X_1, notice

$$WAW(X - X_1) = P_{\mathcal{R}((WA)^k),\mathcal{N}([(WA)^k]^*(WA)^m)} - P_{\mathcal{R}((WA)^k),\mathcal{N}([(WA)^k]^*(WA)^m)} = 0$$

and $\mathcal{R}(X - X_1) \subseteq \mathcal{R}((AW)^k)$. Therefore,

$$\mathcal{R}(X - X_1) \subseteq \mathcal{N}(WAW) \cap \mathcal{R}((AW)^k) \subseteq \mathcal{N}((AW)^k) \cap \mathcal{R}((AW)^k) = \{0\},$$

i.e., $X = X_1$ is the unique solution of the system of Eqs (2.4).
(ii) Lemmas 2.1 and 2.2 imply validity of (2.5) for $X = A^{\circledS_m,W} = A(WA)^k[(WA)^{k+m+2}]^\dagger(WA)^m$.

The assumption that two solutions X and X_1 satisfy (2.5) leads to the conclusion

$$\begin{aligned} \mathcal{R}(X^* - X_1^*) &\subseteq \mathcal{R}([(WA)^m]^*(WA)^k) \cap \mathcal{N}((WAW)^*) \\ &\subseteq \mathcal{R}((WAWA^{\circledS_m,W})^*) \cap \mathcal{N}((WAWA^{\circledS_m,W})^*) = \{0\}, \end{aligned}$$

that is, $X = X_1$. □

Corresponding characterizations of the W-2-WGI and W-weighted Drazin inverse are derived as particular cases $m = 2$ and $m = k$ of Theorem 2.3, respectively.

Corollary 2.4. *The following statements are valid for* $(A, W) \in \mathbb{C}^{p,n;k}$*:*

(i) $A^{\circledS_2,W}$ *is the unique solution to*

$$WAWX = P_{\mathcal{R}((WA)^k),\mathcal{N}([(WA)^k]^*(WA)^2)} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}((AW)^k);$$

(ii) $A^{\circledS_2,W}$ *is the unique solution to*

$$XWAW = P_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*(WA)^3W)} \quad \textit{and} \quad \mathcal{R}(X^*) \subseteq \mathcal{R}\left([(WA)^2]^*(WA)^k\right);$$

(iii) $A^{\mathrm{D},W}$ *is the unique solution to*

$$WAWX = P_{\mathcal{R}((WA)^k),\mathcal{N}((WA)^k)} \quad \textit{and} \quad \mathcal{R}(X) \subseteq \mathcal{R}((AW)^k);$$

(iv) $A^{\mathrm{D},W}$ *is the unique solution to*

$$XWAW = P_{\mathcal{R}((AW)^k),\mathcal{N}((AW)^k)} \quad \textit{and} \quad \mathcal{R}(X^*) \subseteq \mathcal{R}([(WA)^k]^*).$$

3. Further results on W-m-WGI

Some formulae for the W-m-WGI are given in this section.

We present a relation between a nonsingular bordered matrix and the W-m-WGI. Precisely, by Theorem 3.1, when the inverse of a proper bordered matrix is known, then the corresponding position of that inverse gives the W-m-WGI.

Theorem 3.1. *Let $(A, W) \in \mathbb{C}^{p,n;k}$ and $m \in \mathbb{N}$. Assume that full-column rank matrices G and H^* fulfill*

$$\mathcal{N}\left(\left[(WA)^k\right]^* (WA)^m\right) = \mathcal{R}(G) \qquad \text{and} \qquad \mathcal{R}((AW)^k) = \mathcal{N}(H).$$

Then,

$$N = \begin{bmatrix} WAW & G \\ H & 0 \end{bmatrix}$$

is nonsingular, and

$$N^{-1} = \begin{bmatrix} A^{\circledR_m, W} & (I - A^{\circledR_m, W} WAW)H^\dagger \\ G^\dagger(I - WAWA^{\circledR_m, W}) & -G^\dagger(WAW - WAWA^{\circledR_m, W} WAW)H^\dagger \end{bmatrix}. \tag{3.1}$$

Proof. Lemma 2.2 gives $A^{\circledR_m, W} = (WAW)^{(2)}_{\mathcal{R}((AW)^k), \mathcal{N}([(WA)^k]^*(WA)^m)}$. Since

$$\begin{aligned}\mathcal{R}\left(I - WAWA^{\circledR_m, W}\right) &= \mathcal{N}\left(WAWA^{\circledR_m, W}\right) \\ &= \mathcal{N}\left(A^{\circledR_m, W}\right) = \mathcal{N}\left(\left[(WA)^k\right]^* (WA)^m\right) \\ &= \mathcal{R}(G) = \mathcal{R}(GG^\dagger) = \mathcal{N}(I - GG^\dagger),\end{aligned}$$

we have $(I - GG^\dagger)(I - WAWA^{\circledR_m, W}) = 0$, that is, $GG^\dagger(I - WAWA^{\circledR_m, W}) = (I - WAWA^{\circledR_m, W})$. From $\mathcal{R}(A^{\circledR_m, W}) = \mathcal{R}((AW)^k) = \mathcal{N}(H)$, we get $HA^{\circledR_m, W} = 0$. Set Y for the right hand side of (3.1). Then,

$$\begin{aligned} NY &= \left[\begin{array}{c|c} WAWA^{\circledR_m, W} + GG^\dagger(I - WAWA^{\circledR_m, W}) & WAW(I - A^{\circledR_m, W} WAW)H^\dagger - GG^\dagger(I - WAWA^{\circledR_m, W})WAWH^\dagger \\ HA^{\circledR_m} & H(I - A^{\circledR_m} WAW)H^\dagger \end{array}\right] \\ &= \left[\begin{array}{c|c} WAWA^{\circledR_m, W} + I - WAWA^{\circledR_m, W} & (I - WAWA^{\circledR_m, W})WAWH^\dagger - (I - WAWA^{\circledR_m, W})WAWH^\dagger \\ 0 & HH^\dagger \end{array}\right] \\ &= \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} = I. \end{aligned}$$

and similarly $YN = I$. So, N is nonsingular with $N^{-1} = Y$. □

We investigate limit and integral expressions for W-m-WGI motivated by limit and integral formulae of known generalized inverses [34–37].

Theorem 3.2. *If $(A, W) \in \mathbb{C}^{p,n;k}$, $m \in \mathbb{N}$, and $l \geq k$, then*

$$\begin{aligned} A^{\circledR_m, W} &= \lim_{\lambda \to 0} A(WA)^l \left[(WA)^{l+m+2}\right]^* \left((WA)^{l+m+2} \left[(WA)^{l+m+2}\right]^* + \lambda I\right)^{-1} (WA)^m \\ &= \lim_{\lambda \to 0} A(WA)^l \left(\left[(WA)^{l+m+2}\right]^* (WA)^{l+m+2} + \lambda I\right)^{-1} \left[(WA)^{l+m+2}\right]^* (WA)^m. \end{aligned}$$

Proof. Lemma 2.1 gives $A^{ⓜ_m,W} = A(WA)^l\left[(WA)^{l+m+2}\right]^{\dagger}(WA)^m$. According to the limit representation for the Moore-Penrose inverse given in [36], we derive

$$\begin{aligned}[(WA)^{l+m+2}]^{\dagger} &= \lim_{\lambda\to 0}\left[(WA)^{l+m+2}\right]^*\left((WA)^{l+m+2}\left[(WA)^{l+m+2}\right]^*+\lambda I\right)^{-1}\\ &= \lim_{\lambda\to 0}\left(\left[(WA)^{l+m+2}\right]^*(WA)^{l+m+2}+\lambda I\right)^{-1}[(WA)^{l+m+2}]^*,\end{aligned}$$

which implies the rest. □

Since W-m-WGI belongs to outer inverses, the limit representation of the outer inverse proposed in [35] implies the limit representation of the W-m-WGI.

Theorem 3.3. *Let $(A, W) \in \mathbb{C}^{p,n;k}$ and $m \in \mathbb{N}$. Suppose that $H_1 \in \mathbb{C}_s^{p\times s}$, $\mathcal{R}(H_1) = \mathcal{R}((AW)^k)$, $H_2 \in \mathbb{C}_s^{s\times n}$, and $n-s$ is the dimension of the subspace $\mathcal{N}(H_2) = \mathcal{N}\left(\left[(WA)^k\right]^*(WA)^m\right)$ in $\mathbb{C}^n$. Then,*

$$\begin{aligned}A^{ⓜ_m} &= \lim_{v\to 0}H_1(vI+H_2WAWH_1)^{-1}H_2\\ &= \lim_{u\to 0}(uI+H_1H_2WAW)^{-1}H_1H_2 = \lim_{v\to 0}H_1H_2(vI+WAWH_1H_2)^{-1}.\end{aligned}$$

Proof. Since $A^{ⓜ_m,W} = (WAW)^{(2)}_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*(WA)^m)}$, by [35, Theorem 7], it follows that

$$A^{ⓜ_m,W} = \lim_{u\to 0}H_1(uI+H_2WAWH_1)^{-1}H_2.$$

□

Some integral formulae are established for the W-m-WGI.

Theorem 3.4. *If $(A, W) \in \mathbb{C}^{p,n;k}$, $m \in \mathbb{N}$, and $l \geq k$, then*

$$A^{ⓜ_m,W} = \int_0^{\infty} A(WA)^l\left[(WA)^{l+m+2}\right]^*\exp\left(-(WA)^{l+m+2}\left[(WA)^{l+m+2}\right]^* v\right)(WA)^m \mathrm{d}v.$$

Proof. According to [34],

$$\left[(WA)^{l+m+2}\right]^{\dagger} = \int_0^{\infty}\left[(WA)^{l+m+2}\right]^*\exp\left(-(WA)^{l+m+2}\left[(WA)^{l+m+2}\right]^* v\right)\mathrm{d}v.$$

The proof is completed utilizing $A^{ⓜ_m,W} = A(WA)^l\left[(WA)^{l+m+2}\right]^{\dagger}(WA)^m$. □

Theorem 3.5. *Let $(A, W) \in \mathbb{C}^{p,n;k}$ and $m \in \mathbb{N}$. If $H \in \mathbb{C}^{p\times n}$, $\mathcal{R}(H) = \mathcal{R}((AW)^k)$, and $\mathcal{N}(H) = \mathcal{N}(\left[(WA)^k\right]^*(WA)^m)$, then*

$$A^{ⓜ_m} = \int_0^{\infty}\exp\left[-H(HWAWH)^*HWAWv\right]H(HWAWH)^*H\,\mathrm{d}v.$$

Proof. Applying [37, Theorem 2.2], it follows that

$$\begin{aligned}A^{ⓜ_m,W} &= (WAW)^{(2)}_{\mathcal{R}((AW)^k),\mathcal{N}([(WA)^k]^*(WA)^m)}\\ &= \int_0^{\infty}\exp\left[-H(HWAWH)^*HWAWv\right]H(HWAWH)^*H\,\mathrm{d}v,\end{aligned}$$

which completes the proof. □

4. Applications of the W-m-WGI

The W-m-WGI is applicable in studying solvability of some matrix and vector equations.

In the case that $A \in \mathbb{C}^{m\times n}$, $x \in \mathbb{C}^n$, and $b \in \mathbb{C}^m$, to find approximation solution to inconsistent system of linear equations $Ax = b$, a classical approach is to ask for, so called, generalized solutions, defined as solutions to $GAx = Gb$ with respect to an appropriate matrix $G \in \mathbb{C}^{n\times m}$ [38]. It is important to mention that the system $GAx = Gb$ is consistent in the case $\mathrm{rank}(GA) = \mathrm{rank}(G)$. Such approach has been exploited extensively. One particular choice is $G = A^*$, which leads to widely used least-squares solutions obtained as solutions to the normal equation $A^*Ax = A^*b$. Another important choice is $m = n$, $G = A^k$, and $k = \mathrm{ind}(A)$, which leads to the so called Drazin normal equation $A^{k+1}x = A^kb$ and usage of the Drazin inverse solution $A^{\mathrm{D}}b$.

Starting from the known equation $WAWx = b$, we use $G = \left[(WA)^k\right]^* (WA)^m$ to obtain the following equation (4.1).

Theorem 4.1. *If $m \in \mathbb{N}$ and $(A, W) \in \mathbb{C}^{p,n;k}$, the general solution to*

$$\left[(WA)^k\right]^* (WA)^{m+1}Wx = \left[(WA)^k\right]^* (WA)^m b, \quad b \in \mathbb{C}^n, \tag{4.1}$$

is of the form

$$x = A^{\circledR_m,W}b + \left(I - A^{\circledR_m,W}WAW\right)u, \tag{4.2}$$

for arbitrary vector $u \in \mathbb{C}^n$.

Proof. Let x be represented as in (4.2). Theorem 2.2 gives

$$\left[(WA)^k\right]^* (WA)^{m+1}WA^{\circledR_m,W} = \left[(WA)^k\right]^* (WA)^m.$$

So, x is a solution to (4.1) by

$$\begin{aligned}\left[(WA)^k\right]^* (WA)^{m+1}Wx &= \left[(WA)^k\right]^* (WA)^{m+1}WA^{\circledR_m,W}b\\ &\quad + \left[(WA)^k\right]^* (WA)^{m+1}W(I - A^{\circledR_m,W}WAW)u\\ &= \left[(WA)^k\right]^* (WA)^m b.\end{aligned}$$

If Eq (4.1) has a solution x, based on

$$A^{\circledR_m,W} = A\left[(WA)^{\mathrm{D}}\right]^{m+2}(WA)^k\left[(WA)^k\right]^\dagger (WA)^m,$$

one concludes

$$\begin{aligned}A^{\circledR_m,W}b &= A[(WA)^{\mathrm{D}}]^{m+2}(WA)^k[(WA)^k]^\dagger(WA)^m b\\ &= A\left[(WA)^{\mathrm{D}}\right]^{m+2}([(WA)^k]^\dagger)^*\left[(WA)^k\right]^*(WA)^m b\\ &= A\left[(WA)^{\mathrm{D}}\right]^{m+2}([(WA)^k]^\dagger)^*\left[(WA)^k\right]^*(WA)^{m+1}Wx\\ &= A\left[(WA)^{\mathrm{D}}\right]^{m+2}(WA)^k\left[(WA)^k\right]^\dagger(WA)^{m+1}Wx\\ &= A^{\circledR_m,W}WAWx,\end{aligned}$$

which yields

$$x = A^{\circledR_m,W}b + x - A^{\circledR_m,W}WAWx = A^{\circledR_m,W}b + (I - A^{\circledR_m,W}WAW)x.$$

Hence, x possesses the pattern (4.2). $\square$

Choosing $m = 2$ or $m \geq k$ in Theorem 4.1, we obtain the next result.

Corollary 4.1. *Let $b \in \mathbb{C}^n$ and $(A, W) \in \mathbb{C}^{p,n;k}$.*

(i) *The general solution to*

$$\left[(WA)^k\right]^* (WA)^3 Wx = \left[(WA)^k\right]^* (WA)^2 b \tag{4.3}$$

possesses the form

$$x = A^{\circledast_2,W} b + (I - A^{\circledast_2,W} WAW)u,$$

for arbitrary $u \in \mathbb{C}^n$.

(ii) *If $m \geq k$, the general solution to*

$$(WA)^{m+1} Wx = (WA)^m b \tag{4.4}$$

(or equivalently $\left[(WA)^k\right]^* (WA)^{m+1} Wx = \left[(WA)^k\right]^* (WA)^m b$)
possesses the form

$$x = A^{\mathrm{D},W} b + (I - A^{\mathrm{D},W} WAW)u,$$

for arbitrary $u \in \mathbb{C}^n$.

We study assumptions which ensure the uniqueness of the solution to Eq (4.1).

Theorem 4.2. *If $m \in \mathbb{N}$ and $(A, W) \in \mathbb{C}^{p,n;k}$, $x = A^{\circledast_m,W} b$ is the unique solution to* (4.1) *in the space $\mathcal{R}((AW)^k)$.*

Proof. Theorem 4.1 implies that (4.1) has a solution $x = A^{\circledast_m,W} b \in \mathcal{R}(A^{\circledast_m,W}) = \mathcal{R}((AW)^k)$.

For two solutions $x, x_1 \in \mathcal{R}((AW)^k)$ to (4.1), by Lemma 2.2, we obtain

$$x - x_1 \in \mathcal{R}((AW)^k) \cap \mathcal{N}\left(\left[(WA)^k\right]^* (WA)^{m+1} W\right) = \mathcal{R}\left(A^{\circledast_m,W} WAW\right) \cap \mathcal{N}(A^{\circledast_m,W} WAW) = \{0\}.$$

Hence, the Eq (4.1) has uniquely determined solution $x = A^{\circledast_m,W} b$ in $\mathcal{R}((AW)^k)$. □

Theorem 4.2 gives the next particular results.

Corollary 4.2. *Let $b \in \mathbb{C}^n$ and $(A, W) \in \mathbb{C}^{p,n;k}$.*

(i) *$x = A^{\circledast_2,W} b$ is the unique solution in $\mathcal{R}((AW)^k)$ to* (4.3).
(ii) *$x = A^{\mathrm{D},W} b$ is the unique solution in $\mathcal{R}((AW)^k)$ to* (4.4).

Recall that, by [39], for $W \in \mathbb{C}^{n\times p} \backslash \{0\}$, $A \in \mathbb{C}^{p\times n}$, $\mathrm{ind}(AW) = k_1$, $\mathrm{ind}(WA) = k_2$, and $b \in \mathcal{R}((WA)^{k_2})$, $x = A^{\mathrm{D},W} b$ is the uniquely determined solution to

$$WAWx = b, \qquad x \in \mathcal{R}((AW)^{k_1}).$$

Specifically, if $A \in \mathbb{C}^{n\times n}$, $W = I$, $\mathrm{ind}(A) = k$, and $b \in \mathcal{R}(A^k)$, $x = A^{\mathrm{D}} b$ is the unique solution to [40]

$$Ax = b, \qquad x \in \mathcal{R}(A^k).$$

For $1 = \mathrm{ind}(A)$ and $b \in \mathcal{R}(A)$, $x = A^{\#} b$ is the uniquely determined solution to $Ax = b$. Notice that Theorem 4.2 and Corollary 4.2 recover the above mentioned results from [39] and [40].

5. Numerical examples

The identity (resp., zero) $\ell\times\ell$ matrix will be denoted by $\mathbf{I}_\ell$ (resp., $\mathbf{0}_\ell$). Denote by $\mathbf{D}_\ell^p$, $p \geq 1$, the $\ell\times\ell$ matrix with its pth leading diagonal parallel filled by the entries of the vector $\mathbf{1} = \{1,\dots,1\} \in \mathbb{C}^{\ell-p}$ and 0 in all other positions.

We perform numerical tests on the class of test matrices of index ℓ, given by

$$\left\{\left(\begin{array}{c|c} C\mathbf{I}_\ell & C_1\mathbf{I}_\ell \\ \hline \mathbf{0}_\ell & C_2\mathbf{D}_\ell^p \end{array}\right),\quad \ell > 0\right\},\quad C, C_1, C_2 \in \mathbb{C}. \tag{5.1}$$

Example 5.1. The test matrix A in this example is derived using $\ell = 4$ and $C = 2, C_1 = 3, C_2 = 1$ from the test set (5.1), and W is derived using $\ell = 4$ and $C = 1, C_1 = 3/2, C_2 = 4$ from the test set (5.1). Our intention is to perform numerical experiments on integer matrices using exact computation. Appropriate matrices are

$$A = \left(\begin{array}{c|c} 2\mathbf{I}_4 & 3\mathbf{I}_4 \\ \hline \mathbf{0}_4 & \mathbf{D}_4^1 \end{array}\right) = \left(\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 3 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$W = \left(\begin{array}{c|c} \mathbf{I}_4 & 3/2\mathbf{I}_4 \\ \hline \mathbf{0}_4 & 4\mathbf{D}_4^1 \end{array}\right) = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{3}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{3}{2} \\ \hline 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The matrices WA and AW fulfill $k = \operatorname{ind}(WA) = \operatorname{ind}(AW) = 2$.

(a) In the first part of this example, we calculate the Drazin inverse, the core-EP inverse, and W-m-WGI class of inverses based on their definitions. The Drazin inverse of WA is computed using

$$(WA)^{\mathrm{D}} = (WA)^2\left(WA^5\right)^\dagger (WA)^2 = \left(\begin{array}{c|c} 1/2\mathbf{I}_4 & 3/4\mathbf{I}_4 + 3/8\mathbf{D}_4^1 + 3/2\mathbf{D}_4^2 + 3/4\mathbf{D}_4^3 \\ \hline \mathbf{0}_4 & \mathbf{0}_4 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} \frac{1}{2} & 0 & 0 & 0 & \frac{3}{4} & \frac{3}{8} & \frac{3}{2} & \frac{3}{4} \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{3}{4} & \frac{3}{8} & \frac{3}{2} \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{3}{4} & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{3}{4} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

and the core-EP inverse of WA is equal to

$$(WA)^{\textcircled{\dagger}} = (WA)^k\left((WA)^{k+1}\right)^{\dagger} = \left(\begin{array}{c|c} 1/2\mathbf{I}_4 & \mathbf{0}_4 \\ \hline \mathbf{0}_4 & \mathbf{0}_4 \end{array}\right) = \left(\begin{array}{cccc|cccc} \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

The W-weighted Drazin inverse of A is equal to

$$A^{\mathrm{D},W} = A[(WA)^{\mathrm{D}}]^2 = (WA)^{\mathrm{D}}$$

and the W-weighted core-EP inverse of A is equal to

$$A^{\textcircled{\dagger},W} = A\left((WA)^{\textcircled{\dagger}}\right)^2 = (WA)^{\textcircled{\dagger}}.$$

The W-WGI (or W-1-WGI) inverse of A is given by

$$A^{\textcircled{w},W} = (A^{\textcircled{\dagger},W}W)^2A = \left(\begin{array}{c|c} 1/2\mathbf{I}_4 & 3/4\mathbf{I}_4 + 3/8\mathbf{D}_4^1 \\ \hline \mathbf{0}_4 & \mathbf{0}_4 \end{array}\right) = \left(\begin{array}{cccc|cccc} \frac{1}{2} & 0 & 0 & 0 & \frac{3}{4} & \frac{3}{8} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{3}{4} & \frac{3}{8} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{3}{4} & \frac{3}{8} \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{3}{4} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

the W-2-WGI inverse of A is equal to

$$A^{\textcircled{w}_2,W} = (A^{\textcircled{\dagger},W}W)^3\,AWA = A^{\mathrm{D},W},$$

and for each $m \geq k$ the W-m-WGI inverse of A satisfies

$$A^{\textcircled{w}_m,W} = (A^{\textcircled{\dagger},W}W)^{m+1}\,(AW)^{m-1}A = A^{\mathrm{D},W}.$$

Additionally, $A^{\textcircled{w}_m} = A^{\mathrm{D}}$ is checked for each $m \geq \mathrm{ind}(A)$.

(b) Representations involved in Lemma 2.1 are verified using

$$\begin{aligned} A\left[(WA)^{\textcircled{\dagger}}\right]^3 WA &= A\left[(WA)^{\mathrm{D}}\right]^3(WA)^2\left[(WA)^2\right]^{\dagger} WA \\ &= A(WA)^2\left[(WA)^5\right]^{\dagger} WA = A^{\textcircled{w},W}; \\ A\left[(WA)^{\textcircled{\dagger}}\right]^4(WA)^2 &= A\left[(WA)^{\mathrm{D}}\right]^4(WA)^2\left[(WA)^2\right]^{\dagger}(WA)^2 \\ &= A(WA)^2\left[(WA)^6\right]^{\dagger}(WA)^2 = A^{\textcircled{w}_2,W}. \end{aligned}$$

(c) In this part of the example, our goal is to verify results of Theorem 2.2.

(c1) The statements involved in Theorem 2.2(iv) are verified as follows.

- In the case $m = 1$ verification is confirmed by

$$AWA^{\textcircled{w},W}WA^{\textcircled{w},W} = A^{\textcircled{w},W};$$

$$A^{\textcircled{w},W}(WA)^3W = \left(\begin{array}{c|c} 4\mathbf{I}_4 & 6\mathbf{I}_4 + 24\mathbf{D}_4^1 + 12\mathbf{D}_4^2 + 48\mathbf{D}_4^3 \\ \hline \mathbf{0}_4 & \mathbf{0}_4 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 4 & 0 & 0 & 0 & 6 & 24 & 12 & 48 \\ 0 & 4 & 0 & 0 & 0 & 6 & 24 & 12 \\ 0 & 0 & 4 & 0 & 0 & 0 & 6 & 24 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 6 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = (AW)^2;$$

$$[(WA)^2]^*(WA)^2WA^{\textcircled{w},W} = \left(\begin{array}{c|c} 8\mathbf{I}_4 & 12\mathbf{I}_4 + 6\mathbf{D}_4^1 \\ \hline \mathbf{0}_4 & \mathbf{0}_4 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 8 & 0 & 0 & 0 & 12 & 6 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 12 & 6 & 0 \\ 0 & 0 & 8 & 0 & 0 & 0 & 12 & 6 \\ 0 & 0 & 0 & 8 & 0 & 0 & 0 & 12 \\ \hline 12 & 0 & 0 & 0 & 18 & 9 & 0 & 0 \\ 6 & 12 & 0 & 0 & 9 & \frac{45}{2} & 9 & 0 \\ 24 & 6 & 12 & 0 & 36 & 27 & \frac{45}{2} & 9 \\ 12 & 24 & 6 & 12 & 18 & 45 & 27 & \frac{45}{2} \end{array}\right) = [(WA)^2]^*WA.$$

- In the case $m = 2$ results are confirmed by

$$AWA^{\textcircled{w}_2,W}WA^{\textcircled{w}_2,W} = A^{\textcircled{w}_2,W};$$
$$A^{\textcircled{w}_2,W}(WA)^3W = (AW)^2;$$
$$[(WA)^2]^*(WA)^3WA^{\textcircled{w}_2,W} = 2[(WA)^2]^*WA = [(WA)^2]^*(WA)^2.$$

- Representations in the case $m \geq 3$ are confirmed by

$$AWA^{\textcircled{w}_m,W}WA^{\textcircled{w}_m,W} = A^{\textcircled{w}_m,W};$$
$$A^{\textcircled{w}_m,W}(WA)^3W = (AW)^2;$$
$$[(WA)^2]^*(WA)^{m+1}WA^{\textcircled{w}_m,W} = m[(WA)^2]^*WA = [(WA)^2]^*(WA)^m.$$

(c2) The statements involved in Theorem 2.2(vi) are verified using verification of part (iv) and the following computation.

- In the case $m = 1$

$$AWA^{\textcircled{w},W} = \left(\begin{array}{c|c} \mathbf{I}_4 & 3/2\mathbf{I}_4 + 3/4\mathbf{D}_4^1 \\ \hline \mathbf{0}_4 & \mathbf{0}_4 \end{array}\right) = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{3}{2} & \frac{3}{4} & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \frac{3}{2} & \frac{3}{4} & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{3}{2} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = A^{\textcircled{\dagger},W}WA;$$

$$A^{\textcircled{w},W}WA = \left(\begin{array}{c|c} \mathbf{I}_4 & 3/2\mathbf{I}_4 + 3/4\mathbf{D}_4^1 + 3\mathbf{D}_4^2 + 3/2\mathbf{D}_4^3 \\ \hline \mathbf{0}_4 & \mathbf{0}_4 \end{array}\right)$$

$$= \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & \frac{3}{2} & \frac{3}{4} & 3 & \frac{3}{2} \\ 0 & 1 & 0 & 0 & 0 & \frac{3}{2} & \frac{3}{4} & 3 \\ 0 & 0 & 1 & 0 & 0 & 0 & \frac{3}{2} & \frac{3}{4} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{3}{2} \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right) = (A^{\textcircled{\dagger},W}W)^2AWA.$$

- In the case $m \geq 2$ results are confirmed by

$$AWA^{\textcircled{w}_m,W} = AWA^{\textcircled{w}_2,W} = A^{\textcircled{w},W}WA = (A^{\textcircled{\dagger},W}W)^2AWA = (A^{\textcircled{\dagger},W}W)^m(AW)^{m-1}A;$$
$$A^{\textcircled{w}_m,W}WA = A^{\textcircled{w}_2,W}WA = AWA^{\textcircled{w}_2,W} = (A^{\textcircled{\dagger},W}W)^3(AW)^2A = (A^{\textcircled{\dagger},W}W)^{m+1}(AW)^mA.$$

Example 5.2. Consider A and W from Example 5.1 and the vector $b = \begin{pmatrix} 2 & 2 & 0 & 1 & 1 & 2 & 0 & 1 \end{pmatrix}^{\mathrm{T}}$ with intention to verify Theorem 4.1.

In the case $m = 1$ of (4.1), the general solution to $\left[(WA)^2\right]^* (WA)^2Wx = \left[(WA)^2\right]^* WAb$ is equal to

$$x_1 = A^{\textcircled{w},W}b + \left(I - A^{\textcircled{w},W}WAW\right)u,$$

where $u = \begin{pmatrix} u_1 & u_2 & u_3 & u_4 & u_5 & u_6 & u_7 & u_8 \end{pmatrix}^{\mathrm{T}}$ is a vector of unknown variables. Symbolic calculation gives

$$x_1 = \begin{pmatrix} -\frac{3u_5}{2} - 6u_6 - 3u_7 - 12u_8 + \frac{5}{2} \\ -\frac{3u_6}{2} - 6u_7 - 3u_8 + \frac{5}{2} \\ -\frac{3u_7}{2} - 6u_8 + \frac{3}{8} \\ \frac{1}{4}(5 - 6u_8) \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix}.$$

Obtained vector x_1 is verified using $\left[(WA)^2\right]^* (WA)^2 W x_1 = \left[(WA)^2\right]^* WAb = \begin{pmatrix} 40 \\ 40 \\ 6 \\ 20 \\ 60 \\ 90 \\ 159 \\ \frac{429}{2} \end{pmatrix}$.

In the case $m \geq 2$ of (4.1), the general solution to $\left[(WA)^2\right]^* (WA)^{m+1} W x = \left[(WA)^2\right]^* (WA)^m b$ is equal to

$$x_m = A^{\circledR_m, W} b + \left(I - A^{\circledR_m, W} WAW\right) u.$$

Symbolic calculus produces

$$x_m = \begin{pmatrix} -\frac{3u_5}{2} - 6u_6 - 3u_7 - 12u_8 + \frac{13}{4} \\ -\frac{3u_6}{2} - 6u_7 - 3u_8 + 4 \\ -\frac{3u_7}{2} - 6u_8 + \frac{3}{8} \\ \frac{1}{4}(5 - 6u_8) \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{pmatrix}.$$

Correctness of the vector x_m is verified using

$$\left[(WA)^2\right]^* (WA)^{m+1} W x_2 = \left[(WA)^2\right]^* (WA)^m b = (m-1) \begin{pmatrix} 104 \\ 128 \\ 12 \\ 40 \\ 156 \\ 270 \\ 426 \\ 609 \end{pmatrix}.$$

6. Concluding remarks

In this research, we present an extension of the m-weak group inverse (or m-WGI) on the set of rectangular matrices, called the W-weighted m-WGI (or W-m-WGI). The W-m-WGI class presents a new, wider class of generalized inverses since this class involves the m-WGI, W-weighted weak group, and W-weighted Drazin inverse as special cases. Various characterizations and representations of W-m-WGI are developed. Usability of the W-m-WGI class in solving some constrained and unconstrained matrix equations and linear systems is considered. Some new properties of the weighted generalized group inverse and some known properties of the W-weighted Drazin inverse are obtained as corollaries. The given numerical examples confirm the obtained results.

There is increasing interest in the investigation of the WGI and its generalizations, and so for further research in the near future, it may be interesting to consider its generalizations to Hilbert space operators or tensors, iterative methods for approximation of W-m-WGI, or recurrent neural network (RNN) models for computing W-m-WGI.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

P. S. Stanimirović is an editorial board member for Electronic Research Archive and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

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