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Research article

A double time-delay Holling II predation model with weak Allee effect and age-structure

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Abstract: A double-time-delay Holling II predator model with weak Allee effect and age structure was studied in this paper. First, the model was converted into an abstract Cauchy problem. We also discussed the well-posedness of the model and the existence of the equilibrium solution. We analyzed the global stability of boundary equilibrium points, the local stability of positive equilibrium points, and the conditions of the Hopf bifurcation for the system. The conclusion was verified by numerical simulation.

Keywords: predation model; Allee effect; age-structure; time-delay effect; stability; Hopf bifurcation

1. Introduction

This paper considers the following double-time-delay Holling II predation model with weak Allee effect and age-structure:

$$\begin{cases} \frac{dV(t)}{dt} = \gamma V(t)\left(1-\frac{V(t)}{K}\right)\left(\frac{V(t)}{m}+1\right)-\frac{\alpha V(t)}{\beta+V(t)}\int_0^{+\infty} p(t,a)da, \\ \frac{\partial p(t,a)}{\partial t}+\frac{\partial p(t,a)}{\partial a} = -\sigma p(t,a), \\ p(t,0) = \eta\frac{\alpha V(t-\tau_1)}{\beta+V(t-\tau_1)}\int_0^{+\infty}\delta(a)\,p(t,a)da. \end{cases} \tag{1.1}$$

The boundary conditions are

$$V_0 = \chi \in C([-\tau_1,0],\mathbb{R}),\, p(0,a) = p_0(a) \in L^1_+(0,+\infty).$$

Here, $V(t)$ and $p(t,a)$ represent the number of predator densities at time t and age a, γ is the rate of prey' intrinsic growth, and $\Lambda = \gamma+\mu$, where Λ is the birth and μ is the mortality rate of the prey. K indicates the maximum environmental load of prey, while σ and η indicate the predator mortality rate and the conversion coefficient of predator intake to each prey. In addition, τ_1 is the time delay effect, and $0 < m < K$ indicates the survival threshold of the food bait in the weak Allee effect. Meanwhile $F(V(t)) = \frac{\alpha V(t)}{\beta + V(t)}$ represents the Holling II functional reaction function, where α represents the capture rate and β is a semi-saturated constant. In addition, the reproductive generation of species in the ecosystem generally takes some time to mature to have fertility, i.e., "age-dependent fertility". Fertility $\delta(a) \in L_+^\infty((0, +\infty), A)$ depends on age, for which it is often assumed that:

Assumption 1.1. Suppose that

$$\delta(a) = \begin{cases} \delta_*, & a \geq \tau_2, \\ 0, & a < \tau_2, \end{cases}$$

where $\tau_2 > 0, \delta_* > 0$, and $\int_0^{+\infty} \delta(a) e^{-\sigma a} da = 1$.

It is well known that there is a long history of mathematical modeling of the interactions between predators and bait, and that different biological properties are considered in classical models, thus developing various types of predation models. Besides, there is growing evidence that functional responses play a crucial role in the interaction between predators and prey. The Gauss predation math model is given as

$$\begin{cases} \frac{dV}{dt} = \gamma\left(1 - \frac{V}{K}\right) - F(V) P, \\ \frac{dP}{dt} = -dP + \eta F(V) P, \end{cases}$$

where, $V(t)$ indicates the species density of the prey at time t, and $P(t)$ indicates the species number of predators at time t. The normal constants K and γ denote the environmental capacity and inherent growth rate, respectively. d and η denote the predator mortality and the conversion coefficient of predator intake to each prey, respectively. The functional reaction function $F(x)$ indicates the feeding rate of predators feeding on their prey.

For functional reaction functions, the earliest function form is the Lotka-Volterra type functional response $F(x) = \alpha x$. This was followed by Holling II: $F(x) = \frac{\alpha x}{\beta + x}$, Holling III: $F(x) = \frac{\alpha x^2}{\beta + x^2}$, Holling IV: $F(x) = \frac{\alpha x^2}{\beta + x}$, and Ivlev-type: $F(x) = 1 - e^{-\alpha x}$. For example, a class of Leslie-Gower and Holling II predator models are proposed in [1], which gives the global stability of the bounds of understanding, the existence of attracting sets, and the equilibrium points of coexistence:

$$\begin{cases} \frac{dx}{dt} = x\left(r_1 - b_1 x - \frac{a_1 y}{x + k_1}\right), \\ \frac{dy}{dt} = y\left(r_2 - \frac{a_2 y}{x + k_2}\right). \end{cases}$$

However, in the real biological world, due to food digestion, reproduction, and other reasons, organisms need reaction time, so the time delay effect is a very important factor in many biological systems. In recent years, many researchers have studied the stability of predation models with time delay, and time delay systems, such as [2–7].

At the same time, recent literature has done extensive analysis of predator-prey models of age structure, see [8–10]. To study age structure models, a common method is to convert the original model to a time delay differential equation, such as [11]. Another critical method is to turn it into an abstract Cauchy problem, thus applying semi-group theory ([12,13]). In [8], Yang proposed a class of age-structure predation models containing the functional response of the Holling II with a prey refuge:

$$\begin{cases} \frac{dv_1(t)}{dt} = rv_1(t)\left(1 - \frac{v_1(t)}{K}\right) - \sigma_1 v_1(t) + \sigma_2 v_2(t) - \frac{mv_1(t)\int_0^{+\infty} u(t,a)da}{1+cv_1(t)}, \\ \frac{dv_2(t)}{dt} = \Lambda + \sigma_1 v_1(t) - \sigma_2 v_2(t) - \nu v_2(t), \\ \frac{\partial u(t,a)}{\partial t} + \frac{\partial u(t,a)}{\partial a} = -\mu u(t,a), \\ u(t,0) = \eta \frac{mv_1(t)\int_0^{+\infty} \beta(a)u(t,a)da}{1+cv_1(t)}, \end{cases}$$

and obtained the Hopf bifurcation of the model at the internal equilibrium point, indicating that the model has a special periodic orbit that bifurcations from the internal equilibrium point when the parameter τ exceeds the bifurcation threshold τ_0. The validity of the theoretical analysis is verified by numerical simulation.

In recent years, the Allee effect in predator-prey models has also been widely studied [14–17]. The Allee effect is defined as the relationship between population size and fitness. In a predator-prey model, the impact of the Allee effect on logistic growth is expressed by including an $V - m$ form multiplier, where m is the Allee threshold. The Allee effect is broadly divided into two categories: the strong Allee effect and the weak Allee effect [18]. The strong Allee effect indicates negative population growth when the population size is below a certain threshold. In contrast, the weak Allee effect indicates a positive population growth trend below a certain threshold. In [16], the author considered a kind of generalized Holling III-type functional reaction, predation model with weak Allee effect, and explored the existence conditions of model equilibrium and singularity, as well as some properties of equilibrium stability:

$$\begin{cases} \frac{dx}{dt} = \left(r\left(1 - \frac{x}{K}\right)(x - m) - \frac{qxy}{x^2+bx+a}\right)x, \\ \frac{dy}{dt} = s\left(1 - \frac{y}{nx}\right)y. \end{cases}$$

Inspired by the above work, in this paper we will study the dynamical behavior of the predation model (1.1) of the double time-delay Holling II functional response function with weak Allee effect and age structure.

The organization plan for this article is as follows: In Section 2, the transformation of Cauchy's problem is given, and the system well-posedness is obtained. In Section 3, the equilibrium solution of the system is studied, and the linearized system is obtained. In Section 4, the dynamical behavior of the system is studied. In Section 5, some numerical simulations and discussions are conducted.

2. Well-posedness

First we normalize τ_2 in the system (1.1), which gives

$$\tilde{t} = \frac{t}{\tau_2}, \tilde{a} = \frac{a}{\tau_2},$$

and we consider the distribution

$$\tilde{V}(\tilde{t}) = V(\tau_2 \tilde{t}), \tilde{p}(\tilde{t}, \tilde{a}) = \tau_2 p(\tau_2 \tilde{t}, \tau_2 \tilde{a}).$$

After the wave is removed, system (1.1) becomes

$$\begin{cases} \frac{dV(t)}{dt} = \tau_2 \left[\frac{\gamma}{mK} V(t)(K - V(t))(V(t) + m) - \frac{\alpha V(t)}{\beta + V(t)} \int_0^{+\infty} p(t,a) da \right], \\ \frac{\partial p(t,a)}{\partial t} + \frac{\partial p(t,a)}{\partial a} = -\tau_2 \sigma p(t,a), \\ p(t,0) = \tau_2 \eta \frac{\alpha V\left(t - \frac{\tau_1}{\tau_2}\right)}{\beta + V\left(t - \frac{\tau_1}{\tau_2}\right)} \int_0^{+\infty} \delta(a) p(t,a) da, \end{cases} \tag{2.1}$$

where

$$V_0 = \bar{\chi} \in C\left(\left[-\frac{\tau_1}{\tau_2}, 0\right], \mathbb{R}\right), p(0,a) = p_0(a) \in L^1((0,+\infty), \mathbb{R}).$$

The new function $\delta(a)$ is

$$\delta(a) := \delta_* 1_{[1,+\infty]}(a) = \begin{cases} \delta_*, & a \geq 1, \\ 0, & otherwise, \end{cases}$$

and

$$\int_{\tau_2}^{+\infty} \delta_* e^{-\sigma a} da = 1.$$

i.e., $\delta_* = \sigma e^{-\sigma \tau_2}, (\tau_2 > 0)$.

Based on the integral semigroup theory, the suitability of the solution of system (2.1) is discussed below. For this purpose, (2.1) will be rewritten as the abstract cauchy problem(ACP). First, two lemmas about operator semigroups are introduced.

Lemma 2.1. [19,20] Let $(G, \mathcal{T}(G))$ be the *Hille-Yosida* operator on the Banach space Y, $A \in \mathcal{L}(Y)$ is the set of all bounded linear operators on Y, and $C = G + A$ are the *Hille-Yosida* operators.

Lemma 2.2. [19,20] Let G_0 be part of the operator G on $Y_0 := \overline{\mathcal{T}(G)}$, defined as: $G_0 x = Gx$, where $x \in \mathcal{T}(G_0) = \{x \in \mathcal{T}(G) : Gx \in Y_0\}$. If $(G, \mathcal{T}(G))$ is the *Hille-Yosida* operator on the Banach space, then $(G_0, \mathcal{T}(G_0))$ generates a C_0-semigroup on Y_0.

First, let

$$V(t) = \int_0^{+\infty} v(t,a) da,$$

and model (2.1) is converted to:

$$\begin{cases} \frac{\partial v(t,a)}{\partial t} + \frac{\partial v(t,a)}{\partial a} = -\tau_2 \mu v(t,a), \\ v(0,a) = v_0 \in L^1((0,+\infty), \mathbb{R}), \end{cases}$$

and

$$v(t,0) = \tau_2 R(v(t,a), p(t,a)),$$

where

$$\begin{aligned} R(v(t,a), p(t,a)) = \Lambda \int_0^{+\infty} v(t,a) da + \frac{\gamma(K-m)}{mK} \left(\int_0^{+\infty} v(t,a) da \right)^2 \\ - \frac{\gamma}{mK} \left(\int_0^{+\infty} v(t,a) da \right)^3 - \frac{\alpha \int_0^{+\infty} v(t,a) da \int_0^{+\infty} p(t,a) da}{\beta + \int_0^{+\infty} v(t,a) da}. \end{aligned}$$

Further, let

$$w(t,a) = \begin{pmatrix} v(t,a) \\ p(t,a) \end{pmatrix},$$

then system (2.1) becomes

$$\begin{cases} \frac{\partial w(t,a)}{\partial t} + \frac{\partial w(t,a)}{\partial t} = -Mw\left(t,a\right), \\ w_0\left(\theta,a\right) = \begin{pmatrix} v_0\left(t,a\right) \\ p_0\left(t,a\right) \end{pmatrix} \in C\left(\left[-\frac{\tau_1}{\tau_2},0\right], L^1\left((0,+\infty), G^2\right)\right), \\ w\left(t,0\right) = A\left(w_t\left(\theta,a\right)\right), \end{cases} \tag{2.2}$$

where

$$M = \begin{pmatrix} \tau_2\mu & 0 \\ 0 & \tau_2 d \end{pmatrix}, A\left(w_t\left(\theta,a\right)\right) = \begin{pmatrix} \tau_2 R\left(v\left(t,a\right), p\left(t,a\right)\right) \\ \tau_2\eta\alpha\frac{V\left(t-\frac{\tau_1}{\tau_2}\right)\cdot\int_0^{+\infty}\delta(a)p(t,a)da}{\beta+V\left(t-\frac{\tau_1}{\tau_2}\right)} \end{pmatrix}.$$

Here, we introduce the Banach space

$$Y := \mathbb{R}^2 \times \mathcal{L}^1\left((0,+\infty), \mathbb{R}^2\right).$$

We have the following usual product norm

$$\left\|\begin{pmatrix} \varphi \\ g \end{pmatrix}\right\| = \|\varphi\|_{\mathbb{R}^2} + \|g\|_{\mathcal{L}^1}, \begin{pmatrix} \varphi \\ g \end{pmatrix} \in Y.$$

Further, $G : \mathcal{T}(G) \subset Y \to Y$ is defined as

$$G\begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} -g\left(0\right) \\ -g' - Mg \end{pmatrix},$$

with domain

$$\mathcal{T}(G) = \{0_{\mathbb{R}^2}\} \times W^{1,1}\left((0,+\infty), \mathbb{R}^2\right),$$

and then

$$Y_0 := \overline{\mathcal{T}(G)} = \{0_{\mathbb{R}^2}\} \times \mathcal{L}^1\left((0,+\infty), \mathbb{R}^2\right).$$

Next, we introduce the space

$$I_G := \left\{\begin{pmatrix} \zeta\left(\cdot\right) \\ \rho\left(\cdot\right) \end{pmatrix} \in I\left(\left[-\frac{\tau_1}{\tau_2}, 0\right], X\right) : \zeta\left(0\right) = 0\right\},$$

and define the map $H : I_G \to Y$ as

$$H\left(\begin{pmatrix} \zeta\left(\cdot\right) \\ \rho\left(\cdot\right) \end{pmatrix}\right) = \begin{pmatrix} A\left(\rho\left(\cdot\right)\right) \\ 0_{\mathcal{L}^1} \end{pmatrix},$$

where

$$A\left(\rho\left(\cdot\right)\right) = \begin{pmatrix} \tau_2 R\left(\rho_1\left(0\right)\left(a\right), \rho_2\left(0\right)\left(a\right)\right) \\ \tau_2\frac{\eta\alpha\int_0^{+\infty}\rho_1\left(-\frac{\tau_1}{\tau_2}\right)(a)da\cdot\int_0^{+\infty}\delta(a)\rho_2(0)(a)da}{\beta+\int_0^{+\infty}\rho_1\left(-\frac{\tau_1}{\tau_2}\right)(a)da} \end{pmatrix}, \rho\left(\cdot\right) = \begin{pmatrix} \rho_1\left(\cdot\right) \\ \rho_2\left(\cdot\right) \end{pmatrix}.$$

Let $h\left(t\right) = \begin{pmatrix} 0_{\mathbb{R}^2} \\ w\left(t\right) \end{pmatrix}$, where $w\left(t\right) = w\left(t\right)\left(a\right) = w\left(t,a\right)$, then system (2.2) becomes

$$\begin{cases} \frac{dh(t)}{dt} = Gh\left(t\right) + H\left(h_t\right), \\ h_0 = \chi \in I_G, \end{cases} \tag{2.3}$$

where $h_t \in I_G, h_t(\theta) = h(t+\theta)$, and $h_0(\theta) = \begin{pmatrix} 0 \\ w_0(\theta,\cdot) \end{pmatrix}$. Obviously this is an abstract time delay differential equation, which is further rewritten (2.3) as the ACP for applying the theory of the integral semigroup. Define $y \in I\left([0,+\infty] \times \left[-\frac{\tau_1}{\tau_2}, 0\right] : Y\right)$, where $y(t,\theta) = h(t+\theta), t \geq 0$, and $\theta \in \left[-\frac{\tau_1}{\tau_2}, 0\right]$. Thus, we can get the following equation:

$$\begin{cases} \frac{\partial y(t,\theta)}{\partial t} - \frac{\partial y(t,\theta)}{\partial \theta} = 0, \theta \in \left[-\frac{\tau_1}{\tau_2}, 0\right], \\ \frac{\partial y(t,0)}{\partial \theta} = Gy(t,0) + H(y(t,\cdot)), t \geq 0, \\ y(0,\cdot) = h_0 \in I_G. \end{cases} \tag{2.4}$$

Below we take the product of space Y and space I as the state space Z, so

$$Z = Y \times \mathrm{I}, \mathrm{I} := I\left(\left[-\frac{\tau_1}{\tau_2}, 0\right], Y\right)$$

and the usual product norm

$$\left\| \begin{pmatrix} y \\ \chi \end{pmatrix} \right\| = \|y\|_Y + \|\chi\|_C, \begin{pmatrix} y \\ \chi \end{pmatrix} \in Z.$$

Therefore, system (2.4) can be rewritten as the Cauchy problem of an abstract non-dense definition, and the linear operator $\mathcal{T} : \mathcal{T}(\mathcal{L}) \subset Z \to Z$ is defined as follows:

$$\mathcal{L}\begin{pmatrix} 0_Y \\ \chi \end{pmatrix} = \begin{pmatrix} -\chi'(0) \\ \chi' \end{pmatrix}, \begin{pmatrix} 0_Y \\ \chi \end{pmatrix} \in \mathcal{T}(\mathcal{L}),$$
$$\mathcal{T}(\mathcal{L}) = \{0_Y\} \times \left\{\chi \in I^1\left(\left[-\tfrac{\tau_1}{\tau_2}, 0\right], Y\right), \chi(0) \in \mathcal{T}(G)\right\}.$$

Since $Z_0 := \overline{\mathcal{T}(\mathcal{L})} = \{0_Y\} \times I_G \neq Z$, we obtain that $\mathcal{L}$ is apparently a non-dense linear operator defined in Z. Furthermore, we take the following operator $\mathcal{H} : Z_0 \to Z$:

$$\mathcal{H}\begin{pmatrix} 0_Y \\ \chi \end{pmatrix} = \begin{pmatrix} H(\chi) \\ 0_{I_G} \end{pmatrix}.$$

Finally, let $z(t) = \begin{pmatrix} 0 \\ y(t) \end{pmatrix}, y(t) = y(t)(\theta) = y(t,\theta)$. Then, (2.4) is the non-dense definition of the Cauchy problem

$$\begin{cases} \frac{dz(t)}{dt} = \mathcal{L}z(t) + \mathcal{H}(z(t)), \\ z(0) = \begin{pmatrix} 0_Y \\ h_0 \end{pmatrix} \in Z_0. \end{cases} \tag{2.5}$$

[19,21] studied the global existence and uniqueness of solutions containing (2.5).

Let

$$\Omega := \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > -\varsigma\}, \varsigma := \min\{\tau_2\mu, \tau_2 d\} > 0,$$

and then the following conclusion holds:

Theorem 2.1. For G and $\mathcal{L}$, we can get:

(i) If $\lambda \in \Omega$, then $\lambda \in w(G)$ and

$$(\lambda - G)^{-1}\begin{pmatrix} \tilde{\varphi} \\ \tilde{g} \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix} \Leftrightarrow g(a) = e^{-\int_0^a (\lambda I + M)dl}\tilde{\varphi} + \int_0^a e^{-\int_0^a (\lambda I + M)dl}\tilde{g}(q)\,dq,$$

where $\begin{pmatrix} \tilde{\varphi} \\ \tilde{g} \end{pmatrix} \in Y, \begin{pmatrix} 0 \\ g \end{pmatrix} \in \mathcal{T}(G)$;

(ii) $w(\mathcal{L}) = w(G)$. Also, for the each $\lambda \in w(L)$, we can obtain the explicit formula of $\mathcal{L}$' resolvent

$$(\lambda - \mathcal{L})^{-1} \begin{pmatrix} \tilde{x} \\ \tilde{\chi} \end{pmatrix} = \begin{pmatrix} 0 \\ \chi \end{pmatrix} \Leftrightarrow \chi(\theta) = e^{\lambda\theta}(\lambda - G)^{-1}(\tilde{\chi}(0)) + \int_{\theta}^{0} e^{\lambda(\theta-q)}\tilde{\chi}(q)\,dq;$$

(iii) $\mathcal{L}$ and G are $Hille - Yosida$ operators on Z and Y, respectively.

Proof. The first two proof methods are shown in [22], and here we only need to prove (iii). Let $\lambda \in (-\varsigma, +\infty)$, then from (i) we can get

$$g(a) = e^{-(\lambda I+M)a}\tilde{\varphi} + \int_0^a e^{-(\lambda I+M)(a-q)}\tilde{g}(q)\,dq,$$

the integral for a, then

$$\begin{aligned}
\|g\|_{\mathcal{L}^1} &= \int_0^{+\infty} \left| e^{-(\lambda a+\tau_2\mu a)}\tilde{\alpha}_1 + \int_0^a e^{-(\lambda(a-q)+\tau_2\mu(a-q))}\tilde{g}_1(q)\,dq \right| da \\
&+ \int_0^{+\infty} \left| e^{-(\lambda a+\tau_2 d a)}\tilde{\alpha}_1 + \int_0^a e^{-(\lambda(a-q)+\tau_2 d(a-q))}\tilde{g}_2(q)\,dq \right| da \\
&\le 2\int_0^{+\infty} e^{-(\lambda+\varsigma)a}da\,|\tilde{\alpha}| + \sum_{i=1}^{2}\int_0^{+\infty}\int_0^a e^{-(\lambda+\varsigma)(a-q)}\,|\tilde{g}_i(q)|\,dqda \\
&= 2\int_0^{+\infty} e^{-(\lambda+\varsigma)a}da\,|\tilde{\alpha}| + \sum_{i=1}^{2}\int_0^{+\infty}\int_q^a e^{-(\lambda+\varsigma)(a-q)}da\,|\tilde{g}_i(q)|\,dq \\
&\le \frac{2}{\lambda+\varsigma}\left(|\tilde{\alpha}| + \|g\|_{\mathcal{L}^1}\right),
\end{aligned}$$

$$\left\|(\lambda I - G)^{-1}\right\| \le \frac{2}{\lambda+\varsigma}, (\lambda > -\varsigma).$$

This indicates that $(G, \mathcal{T}(G))$ is the *Hille-Yosida* operator. By the proof method in [21], we get that $(\mathcal{L}, \mathcal{T}(\mathcal{L}))$ is a *Hille-Yosida* operator.

Finally, let $Z_{0+} := Z_0 \cap Z_+$, and

$$\begin{aligned}
Z_+ &:= Y_+ \times I\left(\left[-\frac{\tau_1}{\tau_2}, 0\right], Y_+\right), \\
Y_+ &:= \mathbb{R}_+^2 \times \mathcal{L}^1\left((0, +\infty), \mathbb{R}_+^2\right).
\end{aligned}$$

Since both G and L are $Hille - Yosida$ operators, using the theory about the integral semigroup, we can get the well-posedness of model (2.5) as follows:

Theorem 2.2. There exists a unique continuous semigroup $\{U(t)\}_{t\ge 0}$ on Z_{0+} such that for any $z \in Z_{0+}$, we have that $t \to U(t)z$ is the unique integral solution for the following problem:

$$\begin{cases} \frac{dU(t)z}{dt} = \mathcal{L}U(t)z + \mathcal{H}(U(t)z), \\ U(0)z = z, \end{cases}$$

or

$$U(t) = z + \mathcal{L}\int_0^t U(q)z\,dq + \int_0^t \mathcal{H}(T(q)z)\,dq, t \ge 0.$$

3. Equilibrium points and linearized systems

Now we prove the existence and linearization of the equilibrium points of system (2.5). Suppose $\bar{z} = \begin{pmatrix} 0_Y \\ \bar{\chi} \end{pmatrix} \in \mathcal{T}(\mathcal{L})$ is the steady state solution of system (2.5), where

$$\bar{\chi} = \begin{pmatrix} \bar{\varsigma}(\cdot) \\ \bar{\rho}(\cdot) \end{pmatrix} \in I^1\left(\left[-\frac{\tau_1}{\tau_2}, 0\right], Y\right), \bar{\chi}(0) \in \mathcal{T}(\mathcal{L}), \bar{\rho}(\cdot) = \begin{pmatrix} \bar{\rho}_1(\cdot) \\ \bar{\rho}_2(\cdot) \end{pmatrix},$$

then we have $\mathcal{L}\bar{z} + \mathcal{H}(\bar{z}) = 0$, that is,

$$\begin{cases} H(\bar{\chi}) - \bar{\chi}'(0) + G\bar{\chi}(0) = 0, \\ \bar{\chi}' = 0. \end{cases} \tag{3.1}$$

The following conclusions can be obtained from (3.1):

Theorem 3.1. System (2.5) always has equilibrium:

$$\bar{z}_0 = \begin{pmatrix} 0_Y \\ \begin{pmatrix} \bar{\varsigma}_0(\cdot) \\ \begin{pmatrix} \bar{\rho}_{01}(\cdot) \\ \bar{\rho}_{02}(\cdot) \end{pmatrix} \end{pmatrix} \end{pmatrix}, \bar{z}_1 = \begin{pmatrix} 0_Y \\ \begin{pmatrix} \bar{\varsigma}_1(\cdot) \\ \begin{pmatrix} \bar{\rho}_{11}(\cdot) \\ \bar{\rho}_{12}(\cdot) \end{pmatrix} \end{pmatrix} \end{pmatrix},$$

where

$$\bar{\varsigma}_0(\theta) = \bar{\varsigma}_0(\theta) = 0_{\mathbb{R}^2}, \begin{pmatrix} \bar{\rho}_{01}(\theta)(a) \\ \bar{\rho}_{02}(\theta)(a) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \bar{\rho}_{11}(\theta)(a) \\ \bar{\rho}_{12}(\theta)(a) \end{pmatrix} = \begin{pmatrix} \tau_2\mu K e^{-\tau_2\mu a} \\ 0 \end{pmatrix}.$$

In addition, (2.5) has only the positive equilibrium solution

$$\bar{z}_* = \begin{pmatrix} 0_Y \\ \begin{pmatrix} \bar{\varsigma}(\cdot) \\ \begin{pmatrix} \bar{\rho}_1(\cdot) \\ \bar{\rho}_2(\cdot) \end{pmatrix} \end{pmatrix} \end{pmatrix},$$

if

$$\alpha\eta > 1, K(\alpha\eta - 1) > \beta,$$

where

$$\bar{\varsigma}(\theta) = 0_{\mathbb{R}^2}, \begin{pmatrix} \bar{\rho}_1(\theta)(a) \\ \bar{\rho}_2(\theta)(a) \end{pmatrix} = \begin{pmatrix} \frac{\tau_2\mu\beta}{\alpha\eta-1} e^{-\tau_2\mu a} \\ \frac{\tau_2\sigma\gamma\beta\eta\left[K(\alpha\eta-1)-\beta\right]\left[\beta+m(\alpha\eta-1)\right]}{mK(\alpha\eta-1)^3} e^{-\tau_2\sigma a} \end{pmatrix}.$$

Therefore, the following theorem holds for system (1.1):

Theorem 3.2. (i) System (1.1) always has equilibriums $E_0(0,0), E_1(m,0), E_2(K,0)$;

(ii) when $\alpha\eta > 1, K(\alpha\eta - 1) > \beta$, the positive equilibrium point $E_*(V_*, p_*(a))$ exists in the system, where

$$V_* = \frac{\beta}{\alpha\eta - 1}, p_*(a) = \frac{\sigma\gamma\beta\eta\left[K(\alpha\eta - 1) - \beta\right]\left[\beta + m(\alpha\eta - 1)\right]}{mK(\alpha\eta - 1)^3} e^{-\sigma a}.$$

Next, we linearize system (2.5) at the equilibrium point, set $\bar{z}$ as the steady state of the system (2.5), let $\varpi(t) = z(t) - \bar{z}$, and replace it with (2.5). Then,

$$\begin{cases} \frac{\varpi(t)}{dt} = L\varpi(t)(t) + \mathcal{H}(\varpi(t)(t) + \bar{z}) - \mathcal{H}(\varpi(t)(t)), t \geq 0, \\ \varpi(t)(0) = \begin{pmatrix} 0 \\ \omega_0 - \bar{\chi} \end{pmatrix} := \varpi(t)_0 \in \mathcal{T}(L). \end{cases}$$

Thus, the linearization system around $\bar{z}$ is as follows:

$$\begin{cases} \frac{d\varpi(t)(t)}{dt} = L\varpi(t)(t) + \mathcal{TH}(\varpi(t))(\varpi(t)(t)), t \geq 0, \\ \varpi(t)(0) = \varpi(t)_0 \in \mathcal{T}(L), \end{cases} \tag{3.2}$$

with

$$\mathcal{TH}(\bar{z})\begin{pmatrix} 0_Y \\ \chi \end{pmatrix} = \begin{pmatrix} \mathcal{T}H(\bar{\chi})(\chi) \\ 0_{C_G} \end{pmatrix}, \begin{pmatrix} 0_Y \\ \chi \end{pmatrix} \in \mathcal{T}(\mathcal{L}), \chi = \begin{pmatrix} \varsigma(\cdot) \\ \rho(\cdot) \end{pmatrix}, \tag{3.3}$$

and

$$\mathcal{T}H(\bar{\chi})(\chi) = \begin{pmatrix} \mathcal{T}H(\bar{\rho})(\rho) \\ 0_{\mathcal{L}^1} \end{pmatrix},$$

where

$$\begin{aligned} \mathcal{T}H(\bar{\rho})(\rho) = & \begin{pmatrix} \tau_2\Lambda + \frac{2\tau_2\gamma(K-m)}{mK}M_2 - \frac{3\tau_2\gamma}{mK}M_2^2 - \frac{\tau_2\alpha\beta N_2}{(\beta+M_2)^2} & -\frac{\tau_2\alpha M_2}{\beta+M_2} \\ 0 & 0 \end{pmatrix} \int_0^{+\infty} \rho(0)(a)\,da \\ & + \begin{pmatrix} 0 & 0 \\ \frac{\tau_2\alpha\eta\beta N_1}{(\beta+M_1)^2} & 0 \end{pmatrix} \int_0^{+\infty} \rho\left(-\frac{\tau_1}{\tau_2}\right)(a)\,da + \begin{pmatrix} 0 & 0 \\ 0 & \frac{\tau_2\alpha\eta M_1}{\beta+M_1} \end{pmatrix} \int_0^{+\infty} \delta(a)\rho(0)(a)\,da, \end{aligned}$$

$$\begin{aligned} M_1 &= \int_0^{+\infty} \bar{\rho}_1\left(-\tfrac{\tau_1}{\tau_2}\right)(a)\,da, M_2 = \int_0^{+\infty} \bar{\rho}_1(0)(a)\,da, \\ N_1 &= \int_0^{+\infty} \delta(a)\bar{\rho}_2(0)(a)\,da, N_2 = \int_0^{+\infty} \bar{\rho}_2(0)(a)\,da. \end{aligned}$$

According to Lemma 2.1 and Theorem 2.1, we get Theorems 3.3 and 3.4:

Theorem 3.3. $\mathcal{L} + \mathcal{TH}(\bar{z})$ is a Hille-Yosida operator.

Then, by Lemma 2.2, we can get the following:

Theorem 3.4. $(\mathcal{L}, \mathcal{T}(\mathcal{L})), (\mathcal{L} + \mathcal{TH}(\bar{z}), \mathcal{T}(\mathcal{L} + \mathcal{TH}(\bar{z})))$ generate C_0-semigroups $(\mathcal{I}(t))_{t\geq 0}$, $(\mathcal{J}(t))_{t\geq 0}$ on space Z_0, respectively.

Based on the proof of Theorem 2.1, the $Hille-Yosida$ estimate domain is $\|\mathcal{I}(t)\| \leq e^{-dt}$. Moreover, $\mathcal{TH}(\bar{z})\mathcal{I}(t) : Z_0 \to Z$ is clearly compact for any $t > 0$. Then, we have

$$\mathcal{J}(t) = e^{\mathcal{TH}(\bar{z})t}\mathcal{I}(t) = \mathcal{I}(t) + \sum_{k=1}^{+\infty} \frac{(\mathcal{TH}(\bar{z})t)^k}{k!}\mathcal{I}(t).$$

And then we can get $(\mathcal{J}(t))_{t\geq 0}$ is quasi-compact. According to [18], the quasi-compact related conclusion for strong continuous semigroups, when all eigenvalues of $\mathcal{L} + \mathcal{TH}(\bar{z})$ are negative, then for $\bar{d} > 0$, when $t \to +\infty$, $e^{\bar{d}t}\|\mathcal{J}(t)\| \to 0$.

Theorem 3.5. The solution semigroup $T(t)$ of the system (2.5) satisfies the following: the solution of the steady state $\bar{z}(t)$ is locally asymptotically stable (LAS), when all eigenvalues of $\mathcal{L} + \mathcal{TH}(\bar{z})$ have strictly negative real parts; the solution of the steady state $\bar{z}(t)$ is unstable, when the presence of $\mathcal{L} + \mathcal{TH}(\bar{z})$ has a strictly positive real eigenvalue.

4. Dynamics behavior

Obviously, E_0 and E_1 are unstable equilibrium points. Next we consider E_2's stability.

4.1. Stability at point E_2

4.1.1. Local stability at point E_2

Theorem 4.1. When $\frac{\alpha\eta K}{\beta+K} < 1$, the equilibrium state $\bar{z}_1$ of system (2.5), i.e., the equilibrium point $E_1(K,0)$ of system (1.1), is LAS; when $\frac{\alpha\eta K}{\beta+K} > 1$, $E_1(K,0)$ is unstable.

Proof. Let

$$\bar{z}(t) = z(t) - \bar{z}_1 = \begin{pmatrix} 0_Y & 0_{\mathbb{R}^2} & \tilde{w}(t)(\cdot) \end{pmatrix}^T,$$

where

$$\tilde{w}(t)(\cdot) = \begin{pmatrix} \tilde{v}(t)(\cdot) \\ \tilde{p}(t)(\cdot) \end{pmatrix} = \begin{pmatrix} v(t,\cdot) - \tau_2\mu K e^{-\tau_2\mu a} \\ p(t,\cdot) \end{pmatrix}.$$

From this, the linearized system $\bar{z}_1$ can be written as

$$\begin{cases} \frac{\partial\tilde{w}(t,a)}{\partial t} + \frac{\partial\tilde{w}(t,a)}{\partial a} = -M\tilde{w}(t,a), \\ \tilde{w}(t,0) = Q_1\int_0^{+\infty}\tilde{w}(t,a)\,da + Q_2\int_0^{+\infty}\delta(a)\,\tilde{w}(t,a)\,da, \end{cases}$$

where

$$Q_1 = \begin{pmatrix} \tau_2\Lambda + \frac{2\tau_2\gamma(K-m)}{m} - \frac{3\tau_2\gamma K}{m} & -\frac{\tau_2 mK}{\beta+K} \\ 0 & 0 \end{pmatrix}, Q_2 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{\tau_2\alpha\eta K}{\beta+K} \end{pmatrix},$$

which is equivalent to

$$\begin{cases} \tilde{V}'(t) = -\frac{\tau_2\gamma(m+K)}{m}\tilde{V}(t) - \frac{\tau_2 mK}{\beta+K}\int_0^{+\infty}\tilde{p}(t,a)\,da, \\ \frac{\partial\tilde{p}(t,a)}{\partial t} + \frac{\partial\tilde{p}(t,a)}{\partial a} = -\tau_2\sigma\tilde{p}(t,a), \\ \tilde{p}(t,0) = \frac{\tau_2\alpha\eta K}{\beta+K}\int_0^{+\infty}\delta(a)\,\tilde{p}(t,a)\,da, \end{cases} \tag{4.1}$$

where $\tilde{V}(t) = \int_0^{+\infty}\tilde{v}(t,a)\,da$.

Let $\tilde{V}(t) = \tilde{V}_0 e^{\lambda t}$, $\tilde{p}(t,a) = \tilde{p}_0(a)\,e^{\lambda t}$, and substituting this into (4.1), the characteristic equations of (4.1)

$$\Delta_0(\lambda) = \left(\frac{\tau_2\alpha\eta K}{\beta+K}\int_0^{+\infty}\delta(a)\,e^{-(\lambda+\tau_2\sigma)a}da - 1\right)\left(\lambda + \frac{\tau_2\gamma(m+K)}{m}\right) = f_0(\lambda)\,g_0(\lambda) = 0.$$

Let $f_0(\lambda) = \frac{\tau_2\eta\alpha\beta K}{\beta+K}\int_0^{+\infty}\delta(a)\,e^{-(\lambda+\tau_2\sigma)a}da - 1$. Then

$$f_0(0) = \frac{\eta\alpha\beta K}{\beta+K} - 1, \lim_{\lambda\to\infty} f_0(\lambda) = -1.$$

Obviously, the root of $g_0(\lambda) = 0$ is negative, and for $f_0(\lambda) = \frac{\tau_2\alpha\eta K}{\beta+K}\int_0^{+\infty}\delta(a)\,e^{-(\lambda+\tau_2\sigma)a}da - 1$, we have

$$f_0(0) = \frac{\alpha\eta K}{\beta+K} - 1, \lim_{\lambda\to\infty} f_0(\lambda) = -1.$$

Because $f_0(\lambda)$ is strictly decreasing and satisfies continuous real functions, we have:
When $\frac{\eta\alpha K}{\beta+K}-1>0$, $f_0(\lambda)=0$ has at least one positive root, and E_2 is unstable.
When $\frac{\eta\alpha K}{\beta+K}-1<0$, $f_0(\lambda)=0$ has no complex solution with real root and no negative, suppose that $\lambda_0=\theta+\omega i$ is the solution, so

$$\begin{aligned}1=|f(\lambda_0)+1| &= \left|\tfrac{\tau_2\alpha\eta K}{\beta+K}\int_0^{+\infty}\delta(a)\,e^{-(\theta+\tau_2\sigma)a-\omega ai}da-1\right| \\ &\le \tfrac{\tau_2\alpha\eta K}{\beta+K}\int_0^{+\infty}\delta(a)\,e^{-(\theta+\tau_2\sigma)a}da \\ &= f_0(\theta)+1\le f_0(0)+1=\tfrac{\tau_2\alpha\eta K}{\beta+K}<1.\end{aligned}$$

Clearly, this is contradictory, so the solution of the characteristic equation must have negative real parts, that is, when $\frac{\eta\alpha K}{\beta+K}-1<0$, $f_0(\lambda)=0$, and E_2 is LAS.

4.1.2. Local stability at point E_2

This section demonstrates the global stability of E_2 using asymptotic autonomous semigroup theory.
Theorem 4.2. When $\frac{\eta\alpha K}{\beta+K}-1<0$, E_2 is globally asymptotically stable.
Proof. From $\frac{dV(t)}{dt}$ of system (2.1), we can obtain

$$\frac{dV}{dt}\le\tau_2\gamma V\left(1-\frac{V}{K}\right)\left(\frac{V}{m}+1\right),$$

and, by the comparison principle, we have:

$$\lim_{t\to\infty}(\sup V(t))\le K.$$

Therefore, for any $\kappa>0$, there exists t_1 such that $V\left(t-\frac{\tau_1}{\tau_2}\right)\le K+\kappa$, when $t\ge t_1+\frac{\tau_1}{\tau_2}$, and then

$$\begin{aligned}p(t,0)&\le\tau_2\eta\alpha\tfrac{K+\kappa}{\beta+(K+\kappa)}\int_0^{+\infty}\delta(a)\,p(t,a)\,da\\ &\le\tau_2\eta\alpha\int_0^{+\infty}\delta(a)\,p(t,a)\,da,\left(t\ge t_1+\tfrac{\tau_1}{\tau_2}\right).\end{aligned}$$

Now, we consider the following system:

$$\begin{cases}\frac{\partial\hat{p}}{\partial t}+\frac{\partial\hat{p}}{\partial a}=-\tau_2\sigma\hat{p},\\ \hat{p}(t,0)=\tau_2\eta\alpha\int_0^{+\infty}\delta(a)\,\hat{p}(t,a)\,da.\end{cases}\tag{4.2}$$

Using the same method as in Theorem 4.1, the solution of (4.2) exists in the form of $\hat{p}(t,a)=\hat{p}_0(a)\,e^{\lambda_0 t}$, where $\hat{p}_0(a)$ is non-negative and λ_0 is the root of the characteristic equation of (4.2), i.e.,

$$\Delta_0(\lambda_0)=\tau_2\eta\alpha\int_0^{+\infty}\delta(a)\,e^{-(\lambda_0+\tau_2\sigma)a}da-1=0.$$

From the second equation of (2.1), we can get

$$p(t,a)=\begin{cases}p(t-a,0)\,e^{-\tau_2\sigma a}, & a\le t,\\ p_0(a-t)\,e^{-\tau_2\sigma t}, & a<t,\end{cases}$$

namely $p(t,a)\le\hat{p}(t,a)$ for $t\ge t_1+\frac{\tau_1}{\tau_2}$. So, $p(t,a)\le\hat{p}_0(a)\,e^{\lambda_0 t}$.

From Theorem 4.1, when $\frac{\eta\alpha K}{\beta+K} < 1$, $\lim\limits_{t\to\infty} p(t,a) = 0$. Thus, when $t \to \infty$, $\frac{dV}{dt}$ of (2.1) converges to

$$\frac{d\hat{V}}{dt} = \tau_2\gamma\hat{V}\left(1 - \frac{\hat{V}}{K}\right)\left(\frac{\hat{V}}{m} + 1\right),$$

which illustrates that $\lim\limits_{t\to\infty} \hat{V}(t) = K$.

Applying the related theories from [23], we can get $\lim\limits_{t\to\infty} V(t) = K$. Hence, when $\frac{\eta\alpha K}{\beta+K} < 1$, E_2 is globally asymptotically stable.

4.2. Dynamics behavior at E_*

4.2.1. Stability at E_*

First, we need to obtain the characteristic equation of system (3.2). Let $\mathcal{K} = \mathcal{T}F(\bar{z})$, where $\bar{z}$ represents the equilibrium state of system (2.5).

Now, we note $\bar{A} = \lambda I - (\mathcal{L} + \mathcal{K})$, $\bar{B} = I - \mathcal{K}(\lambda I - \mathcal{L})^{-1}$, $\bar{C} = \lambda I - \mathcal{L}$. Suppose that $\lambda \in \Omega$. Because $\bar{C}$ is reversible, then $\bar{A}$ is equivalent to

$$\bar{A} = \bar{B}\bar{C}. \tag{4.3}$$

From this, we get that

$$\bar{A} \text{ is reversible and } \bar{B} \text{ is reversible.}$$

If $\bar{B}$ is reversible, then

$$\bar{A}^{-1} = \bar{B}^{-1}\bar{C}^{-1}$$

Applying Theorem 2.1 and (3.3), we can get that, for

$$\xi = \begin{pmatrix} \vartheta \\ g \end{pmatrix}, \tilde{\xi} = \begin{pmatrix} \tilde{\vartheta} \\ \tilde{g} \end{pmatrix} \in Y, \chi \in \begin{pmatrix} \varsigma(\cdot) \\ \rho(\cdot) \end{pmatrix}, \tilde{\chi} \in \begin{pmatrix} \tilde{\varsigma}(\cdot) \\ \tilde{\rho}(\cdot) \end{pmatrix} \in C^1\left(\left[-\frac{\tau_1}{\tau_2}, 0\right], Y\right),$$

we have

$$\bar{B}\begin{pmatrix} \xi \\ \chi \end{pmatrix} = \begin{pmatrix} \tilde{\xi} \\ \tilde{\chi} \end{pmatrix}, \tag{4.4}$$

which is equivalent to

$$\begin{cases} \xi - \mathcal{T}H(\tilde{\chi})\left(e^{\lambda\theta}(\lambda I - G)^{-1}(\chi(0) + \xi) + \int_\theta^0 e^{\lambda(\theta-s)}\chi(s)\,ds\right) = \tilde{\xi}, \\ \chi = \tilde{\chi}, \end{cases}$$

i.e.,

$$\begin{cases} \left(I - \mathcal{T}H(\tilde{\chi})\left(e^{\lambda\theta}(\lambda I - G)^{-1}\right)\right)\xi = \tilde{\xi} + \mathcal{T}H(\tilde{\chi})\left(e^{\lambda\theta}(\lambda I - G)^{-1}\chi(0) + \int_\theta^0 e^{\lambda(\theta-s)}\chi(s)\,ds\right), \\ \chi = \tilde{\chi}, \end{cases}$$

Let

$$\left(I - \mathcal{T}H(\tilde{\chi})\left(e^{\lambda\theta}(\lambda I - G)^{-1}\right)\right)\xi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}, \tag{4.5}$$

and

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = \begin{pmatrix} \tilde{\vartheta} + \mathcal{T}A(\bar{\rho})\left(e^{\lambda\theta}(\lambda I - G)^{-1}\tilde{\rho}(0) + \int_\theta^0 e^{\lambda(\theta-s)}\tilde{\rho}(s)\,ds\right) \\ \tilde{g} \end{pmatrix},$$

then we have

$$\begin{pmatrix} \vartheta \\ g \end{pmatrix} - \begin{pmatrix} \mathcal{T}A(\bar{\rho})\left[e^{\lambda\theta}\left(e^{-\int_0^a(\lambda I-G)^{-1}dl}\vartheta + \int_0^a e^{-\int_0^a(\lambda I-G)^{-1}dl}g(s)\,ds\right)\right] \\ 0 \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix}.$$

From this, we can derive that

$$\begin{cases} \left(I - \mathcal{T}A(\bar{\rho})\left(e^{\lambda\theta}e^{-\int_0^a(\lambda I+\mathcal{T})dl}\right)\right)\vartheta = \chi_1 + \mathcal{T}A(\bar{\rho})\left(e^{\lambda\theta}e^{-\int_s^a(\lambda I+\mathcal{T})dl}g(s)\,ds\right). \\ g = \chi_2. \end{cases} \tag{4.6}$$

Let

$$\Delta(\lambda) = I - \mathcal{T}A(\bar{\rho})\left(e^{\lambda\theta}e^{-\int_0^a(\lambda I+\mathcal{T})dl}\right), \bar{\Phi}(\lambda,\chi_2) = \mathcal{T}A(\bar{\rho})\left(e^{\lambda\theta}\int_0^a e^{-\int_s^a(\lambda I+\mathcal{T})dl}\chi_2(s)\,ds\right).$$

From the first equation of (4.6), we can get $\Delta(\lambda)\vartheta = \chi_1 + \bar{\Phi}(\lambda,\chi_2)$. That is, when $\Delta(\lambda)$ is reversible, we have

$$\vartheta = (\Delta(\lambda))^{-1}\left(\chi_1 + \bar{\Phi}(\lambda,\chi_2)\right).$$

Thus, $\bar{A}$ is reversible, i.e.,

$$\bar{B} \text{ is reversible} \Leftrightarrow \Delta(\lambda) \text{ is reversible}.$$

So, Theorem 4.3 can be obtained:

Theorem 4.3. The following conclusion holds: $\sigma(\mathcal{L}+\mathcal{K}) \cap \Omega = \sigma_P(\mathcal{L}+\mathcal{K}) \cap \Omega = \{\lambda \in \Omega : \det(\Delta(\lambda)) = 0\}$, and if $\lambda \in w(\mathcal{L}+\mathcal{K}) \cap \Omega$, then the resolvents formula is

$$(\lambda I - (\mathcal{L}+\mathcal{K}))^{-1}\begin{pmatrix} \tilde{x} \\ \tilde{\chi} \end{pmatrix} = \begin{pmatrix} 0 \\ e^{\lambda\theta}(\lambda I - G)^{-1}(\tilde{\chi}(0) + x) + \int_\theta^0 e^{\lambda(\theta-s)}\tilde{\chi}(s)\,ds \end{pmatrix}, \tag{4.7}$$

where

$$x = \begin{pmatrix} (\Delta(\lambda))^{-1}\left[\tilde{\vartheta} + \mathcal{T}A(\bar{\rho})\left(e^{\lambda\theta}(\lambda I - G)^{-1}\tilde{\rho}(0) + \int_\theta^0 e^{\lambda\theta-\lambda s}\tilde{\rho}(s)\,ds\right) + \bar{\Phi}(\lambda,\tilde{g})\right] \\ \tilde{g} \end{pmatrix}. \tag{4.8}$$

Proof. Because $\lambda \in \Omega$ with $\det(\Delta(\lambda)) \neq 0$, we can get that $\left(I - \mathcal{T}H(\bar{\chi})\left(e^{\lambda\theta}(\lambda I - G)^{-1}\right)\right)$ is reversible, and then from (4.5) we can get

$$\left(I - \mathcal{T}H(\bar{\chi})\left(e^{\lambda\theta}(\lambda I - G)^{-1}\right)\right)^{-1}\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = x,$$

and

$$x = \begin{pmatrix} (\Delta(\lambda))^{-1}\left(\chi_1 + \bar{\Theta}(\lambda,\chi_2)\right) \\ \chi_2 \end{pmatrix}.$$

Thus, $\bar{B}$ is reversible for any $\begin{pmatrix} \tilde{x} \\ \tilde{\chi} \end{pmatrix} \in Z$,

$$\bar{B}\begin{pmatrix} \tilde{x} \\ \tilde{\chi} \end{pmatrix} = \begin{pmatrix} x \\ \chi \end{pmatrix},$$

where x as shown in formula (4.8), $\chi = \tilde{\chi}$. Therefore, we obtain (4.7) and

$$\{\lambda \in \Omega : \det(\Delta(\lambda)) = 0\} \subset w(\mathcal{L} + \mathcal{K}) \cap \Omega, \sigma(\mathcal{L} + \mathcal{K}) \cap \Omega \subset \{\lambda \in \Omega : \det(\Delta(\lambda)) = 0\}.$$

Further, take $\lambda \in \Omega : \det(\Delta(\lambda)) = 0$, and according to (4.3), there exists $\begin{pmatrix} 0 \\ \chi_0 \end{pmatrix} \in \mathcal{T}(L), \begin{pmatrix} 0 \\ \chi_0 \end{pmatrix} \neq 0$, so

$$(\lambda - (\mathcal{L} + \mathcal{K}))\begin{pmatrix} 0 \\ \chi_0 \end{pmatrix} = 0, \tag{4.9}$$

is established if and only if there exists $\begin{pmatrix} \breve{x}_0 \\ \breve{\chi}_0 \end{pmatrix} \in Z$, and $\begin{pmatrix} \breve{x}_0 \\ \breve{\chi}_0 \end{pmatrix} \neq 0$ satisfies

$$\left[I - \mathcal{K}(\lambda I - \mathcal{L})^{-1}\right]\begin{pmatrix} \breve{x}_0 \\ \breve{\chi}_0 \end{pmatrix} = 0. \tag{4.10}$$

Let the $\begin{pmatrix} \tilde{\xi} \\ \tilde{\chi} \end{pmatrix}$ of the Eq (4.4) be equal to 0, and then we can obtain the existence of $\begin{pmatrix} \breve{x}_0 \\ \breve{\chi}_0 \end{pmatrix} \in Z\backslash\{0\}$ is equivalent to (4.10) , where $\breve{x}_0 = \begin{pmatrix} \breve{\vartheta}_0 \\ \breve{f}_0 \end{pmatrix}$ satisfies

$$\begin{cases} \Delta(\lambda)\breve{\vartheta}_0 = 0, \\ \breve{g}_0 = 0, \\ \breve{\chi}_0 = 0. \end{cases}$$

Thus, (4.9) has solutions if and only if there exists $\breve{\vartheta}_0 \neq 0$, so $\Delta(\lambda)\breve{\vartheta}_0 = 0$, and $\lambda \in \sigma_P(\mathcal{L} + \mathcal{K})$. Therefore, $\{\lambda \in \Omega : \det(\Delta(\lambda)) = 0\} \subset \sigma_P(\mathcal{L} + \mathcal{K})$.

The above analysis shows that $\det(\Delta(\lambda)) = 0$ is the characteristic equation of (3.2) about $\bar{z}_*$.

Below, we analyze the stability at E_* and the Hopf bifurcation's existence. Due to the complexity of $\tau_1 \neq \tau_2$, only $\tau_1 = \tau_2 = \tau$ is considered below, and then the characteristic equation is:

$$\det(\Delta(\lambda)) = \frac{\lambda^2 + \tau p_1 \lambda + \tau^2 p_0 + \left(\tau q_1 \lambda + \tau^2 q_0\right)e^{-\lambda}}{(\lambda + \sigma\tau)(\lambda + \mu\tau)} = \frac{f(\lambda)}{g(\lambda)} = 0,$$

where

$$\begin{aligned} p_1 &= \sigma - \left[\gamma + \tfrac{2\beta\gamma(K-m)}{mK(\alpha\eta-1)} - \tfrac{3\gamma\beta^2}{mK(\alpha\eta-1)^2} - \tfrac{\gamma\left[K(\alpha\eta-1)-\beta\right]\left[\beta+m(\alpha\eta-1)\right]}{Km\eta\alpha(\alpha\eta-1)}\right], \\ p_0 &= -\sigma\left[\gamma\tfrac{2\beta\gamma(K-m)}{mK(\alpha\eta-1)} - \tfrac{3\gamma\beta^2}{mK(\alpha\eta-1)^2} - \tfrac{\gamma\left[K(\alpha\eta-1)-\beta\right]\left[\beta+m(\alpha\eta-1)\right]}{Km\eta\alpha(\alpha\eta-1)}\right], \\ q_1 &= -\sigma, \\ q_0 &= \sigma\left[\gamma + \tfrac{2\beta\gamma(K-m)}{mK(\alpha\eta-1)} - \tfrac{3\gamma\beta^2}{mK(\alpha\eta-1)^2}\right]. \end{aligned}$$

Let $\lambda = \tau\vartheta$, then

$$f(\lambda) = f(\tau\vartheta) = \tau^2\left[\vartheta^2 + p_1\vartheta + p_0 + (q_1\vartheta + q_0)\,e^{-\tau\vartheta}\right] = \tau^2 h(\vartheta). \tag{4.11}$$

Because of $\mathrm{g}(\lambda) \neq 0$, there is

$$\{\lambda \in \Omega : \det(\Delta(\lambda)) = 0\} = \{\tau\vartheta \in \Omega : h(\vartheta) = 0\}.$$

First, when $\tau = 0$, we have $h(\vartheta) = \vartheta^2 + (p_1 + q_1)\vartheta + (p_0 + q_0) = 0$, where $p_0 + q_0 = \frac{\sigma\gamma[K(\alpha\eta-1)-\beta][\beta+m(\alpha\eta-1)]}{Km\eta\alpha(\alpha\eta-1)} > 0$, hence we get the following theorem:

Theorem 4.4. When $\tau = 0$, if $p_1 + q_1 > 0$, then E_* is locally asymptotically stable; otherwise, it is unstable.

4.2.2. Hopf bifurcation at E_*

This section considers the Hopf bifurcation problem when $\tau > 0$. Since the age structure model studied is infinite and the central manifold theory needs to be applied to the abstract non-dense Cauchy problem, we can simplify the system considering the finite-dimensional equation on the central manifold. Thus, the Hopf bifurcation theorem of Hassard remains valid. Therefore, we will use Hassard's theorem directly below to explore the existence of the Hopf bifurcation.

Let $\tau > 0$, so the root of $h(\vartheta) = 0$ has continuous dependence on τ_0. As τ increases, the root of $h(\vartheta) = 0$ can pass through the imaginary axis to the right. Let $\vartheta = i\omega\,(\omega > 0)$ be the purely imaginary roots of $h(\vartheta) = 0$ and substitute it into $h(\vartheta) = 0$, which gives

$$-\omega^2 + ip_1\omega + p_0 + iq_1\omega e^{-i\omega\tau} + q_0 e^{-i\omega\tau} = 0,$$

disassociating the real part and imaginary part,

$$\begin{cases} -\omega^2 + p_0 = -q_1\omega\sin\omega\tau - q_0\cos\omega\tau, \\ p_1\omega = q_0\sin\omega\tau - q_1\omega\cos\omega\tau, \end{cases}$$

i.e.,

$$\begin{cases} \sin\omega\tau = \frac{q_1\omega^3+(p_1q_0-p_0q_1)\omega}{q_0^2+q_1^2\omega^2}, \\ \cos\omega\tau = \frac{(q_0-p_1q_1)\omega^2-p_0q_0}{q_0^2+q_1^2\omega^2}, \end{cases}$$

and

$$\left(\omega^2 - p_0\right)^2 + p_1^2\omega^2 = q_1^2\omega^2 + q_0^2,$$

which is

$$\omega^4 + \left(p_1^2 - 2p_0 - q_1^2\right)\omega^2 + \left(p_0^2 - q_0^2\right) = 0. \tag{4.12}$$

Let $\Theta = \omega^2$, then above equation becomes

$$\Theta^2 + \left(p_1^2 - 2p_0 - q_1^2\right)\Theta + \left(p_0^2 - q_0^2\right) = 0.$$

Due to $p_0 + q_0 > 0$,

$$p_1^2 - 2p_0 - q_1^2 = \left[\gamma + \frac{2\beta\gamma(K-m)}{mK(\alpha\eta-1)} - \frac{3\gamma\beta^2}{mK(\alpha\eta-1)^2} - \frac{\gamma[K(\alpha\eta-1)-\beta][\beta+m(\alpha\eta-1)]}{Km\eta\alpha(\alpha\eta-1)}\right]^2 > 0,$$

and, when $p_0 - q_0 < 0$, the above equation has the sole positive root, and denote it as Θ_*. That means (4.12) has the only positive root $\omega_* = \sqrt{\Theta_*}$, hence $h(\vartheta) = 0, (\tau = \tau_k)$ has a pair of purely imaginary roots, with

$$\tau_k = \begin{cases} \frac{1}{\omega_*}\left(\arccos \frac{(q_0 - p_1 q_1)\omega_*^2 - p_0 q_0}{q_0^2 + q_1^2\omega_*^2} + 2k\pi\right), & c \geq 0, \\ \frac{1}{\omega_*}\left(2\pi - \arccos \frac{(q_0 - p_1 q_1)\omega_*^2 - p_0 q_0}{q_0^2 + q_1^2\omega_*^2} + 2k\pi\right), & c < 0, \end{cases}$$

and

$$c = \frac{q_1\omega_*^3 + (p_1 q_0 - p_0 q_1)\,\omega_*}{q_0^2 + q_1^2\omega_*^2}.$$

Lemma 4.5. In the case of Assumption 1.1 holding, when $\alpha\eta > 1$, $p_1 + q_1 > 0$ and $p_0 - q_0 < 0$, then

$$\left.\frac{dh(\vartheta)}{d\vartheta}\right|_{\vartheta = i\omega_*} \neq 0,$$

and, at this time, $\vartheta = i\omega_*$ is the unique root of $h(\vartheta) = 0$.

Proof. From (4.11), we can get that

$$\frac{dh(\vartheta)}{d\vartheta} = 2i\omega_* + p_1 + q_1 e^{-i\omega_*\tau_k} - q_0\tau_k e^{-i\omega_*\tau_k} - iq_1\omega_*\tau_k e^{-i\omega_*\tau_k}.$$

Because of $h(\vartheta) = 0$, we derive that

$$\left[2\vartheta + p_1 + q_1 e^{-\vartheta\tau} - \tau\,(q_1\vartheta + q_0)\,e^{-\vartheta\tau}\right]\frac{dh(\vartheta)}{d\vartheta} = \vartheta\,(q_1\vartheta + q_0)\,e^{-\vartheta\tau}.$$

If $\left.\frac{dh(\vartheta)}{d\vartheta}\right|_{\vartheta = i\omega_*} = 0$ is correct, then $i\omega_*\,(iq_1\omega_* + q_0)\,e^{-i\omega_*\tau} = 0$, i.e., $iq_1\omega_* + q_0 = 0$, hence $q_1 = q_0 = 0$. Because $q_1 = -\sigma < 0$, which contradicts the conclusion, $\left.\frac{dh(\vartheta)}{d\vartheta}\right|_{\vartheta = i\omega_*} \neq 0$.

Let $\vartheta(\tau) = \bar{\alpha}(\tau) + i\bar{\omega}(\tau)$ become the root of $h(\vartheta) = 0$ where $\bar{\alpha}(\tau_k) = 0, \bar{\omega}(\tau_k) = \omega_*$. Evaluating τ on both sides of $h(\vartheta) = 0$, we get that

$$\begin{aligned} \left.\left(\frac{d\vartheta}{d\tau}\right)^{-1}\right|_{\vartheta = i\omega_*} &= \left.\frac{2\vartheta + p_1 + q_1 e^{-\vartheta\tau} - \tau\,(q_1\vartheta + q_0)\,e^{-\vartheta\tau}}{\vartheta\,(q_1\vartheta + q_0)\,e^{-\vartheta\tau}}\right|_{\vartheta = i\omega_*} \\ &= \left.\left(-\frac{\tau}{\vartheta} + \frac{q_1}{\vartheta\,(q_1\vartheta + q_0)} - \frac{2\vartheta + p_1}{\vartheta\,(\vartheta^2 + p_1\vartheta + p_0)}\right)\right|_{\vartheta = i\omega_*}, \end{aligned}$$

so

$$\begin{aligned} \mathrm{Re}\left(\left.\left(\frac{d\vartheta}{d\tau}\right)^{-1}\right|_{\vartheta = i\omega_*}\right) &= \frac{-q_1^2}{q_1^2\omega_*^2 + q_0^2} + \frac{2\omega_*^2 + p_1^2 - 2p_0}{p_1^2\omega_*^2 + (p_0 - \omega_*^2)^2} \\ &= \frac{2\omega_*^2 + p_1^2 - 2p_0 - q_1^2}{q_1^2\omega_*^2 + q_0^2}. \end{aligned}$$

Besides,

$$\omega_*^2 = \frac{-\left(p_1^2 - 2p_0 - q_1^2\right) + \sqrt{\left(p_1^2 - 2p_0 - q_1^2\right)^2 - 4\left(p_0^2 - q_0^2\right)}}{2}.$$

Replace ω_*^2 with $\mathrm{Re}\left(\left(\frac{d\vartheta}{d\tau}\right)^{-1}\Big|_{\vartheta=i\omega_*}\right)$, so

$$
\begin{aligned}
sign\left(\left(\frac{d\mathrm{Re}\,(\vartheta)}{d\tau}\right)^{-1}\Bigg|_{\tau=\tau_k}\right) &= sign\left(\mathrm{Re}\left(\left(\frac{d\vartheta}{d\tau}\right)^{-1}\Bigg|_{\vartheta=i\omega_*}\right)\right) \\
&= sign\left(\frac{2\omega_*^2 + p_1^2 - 2p_0 - q_1^2}{q_1^2\omega_*^2 + q_0^2}\right) > 0.
\end{aligned}
$$

According to the correlation theorem of the Hopf bifurcation in [24], we get Theorem 4.6:

Theorem 4.6. In the case of Assumption 1.1 holding, when $\alpha\eta > 1, K(\alpha\eta - 1) > \beta$, $p_1 + q_1 > 0$, and $p_0 - q_0 < 0$, then

(i) when $\tau \in [0, \tau_0)$, E_* is asymptotically stable, and when $\tau > \tau_0$, it is unstable;

(ii) when $\tau = \tau_k$, system (1.1) undergoes a Hopf bifurcation at the equilibrium E_*.

5. Numerical simulation

This section uses the MATLAB software to simulate the model numerically. First, the parameters are:

$$\gamma = 1, K = 20, \alpha = 1.01, \beta = 4, \sigma = 0.02, \eta = 1.1, m = 5.$$

By calculating, we can get $\frac{\eta\alpha K}{b+K} - 1 = -0.074 < 0$, which satisfies the condition of Theorem 4.2, that $E_2\,(20, 0)$ is globally asymptotically stable at this time.

Let $\tau_1 = 1, \tau_2 = 2$. The available time series diagrams and phase diagrams are shown in Figure 1, and E_2 are globally stable at this time.

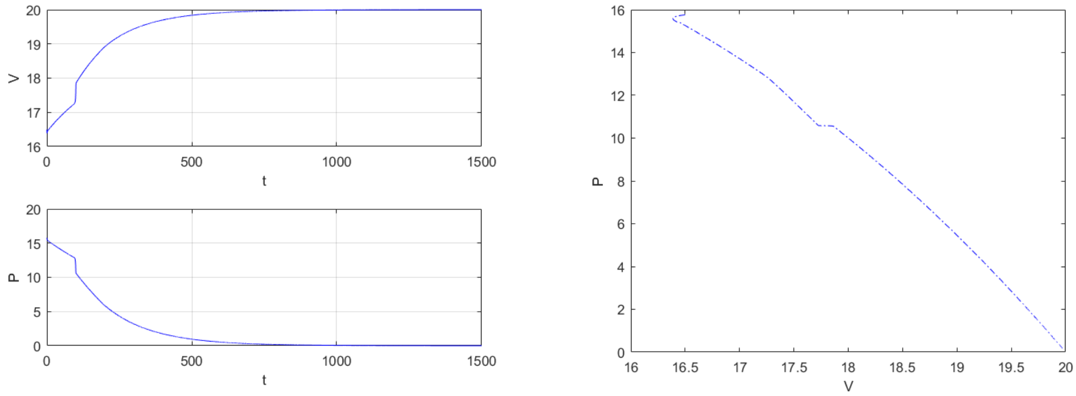

Figure 1. Sequence diagram of $V(t)$ and $p(t, a)$ over time, and phase diagram of $V(t)$ and $p(t, a)$ when E_2 is globally stable.

Next, let the parameters become

$$\gamma = 1, K = 20, \alpha = 1.082, \beta = 1.09, \sigma = 0.4, \eta = 1.0045, m = 5,$$

and set $V(0) = 14$, $p(0,a) = 16e^{-a}$. By calculating, we have

$$\begin{aligned}&V_* = 12.548,\, p_*(a) = 6.593e^{-0.4a},\, \int_0^{+\infty} p_*(a)\,da = 16.48,\\ &\eta\alpha - 1 = 0.086 > 0,\, K(\eta\alpha - 1) - b = 0.647 > 0,\\ &p_1 + q_1 = 0.027 > 0,\, p_0 - q_0 = -0.005 < 0,\end{aligned}$$

satisfying the condition of Theorem 4.6, and we can get $\tau_0 = 3.185$. First, let $\tau_1 = \tau_2 = 2$. $E_*\left(12.548, 6.593e^{-0.4a}\right)$ is asymptotically stable at this time, and the available time series diagrams and phase diagrams are shown in Figure 2. Second, let $\tau_1 = \tau_2 = 4$, E_* pass through a Hopf bifurcation. For aesthetics, we modify the initial value to $V(0) = 25$, $p(0,a) = 5.395e^{-a}$. The system has periodic solutions, and the available time series diagrams and phase diagrams are shown in Figure 3.

Finally, we observe the dynamics of the system under the current parameter conditions $\tau_1 \neq \tau_2$, and let $\tau_1 = 1, \tau_2 = 0$. E_* is asymptotically stable at this time, and the available time series diagrams and phase diagrams are shown in Figure 4.

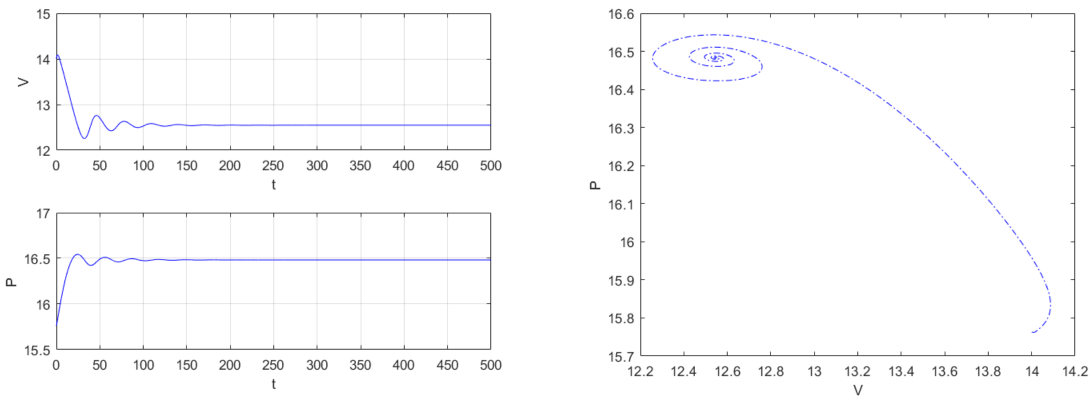

Figure 2. Sequence diagram of $V(t)$ and $p(t,a)$ over time, and phase diagram of $V(t)$ and $p(t,a)$ when E_* are asymptotically stable.

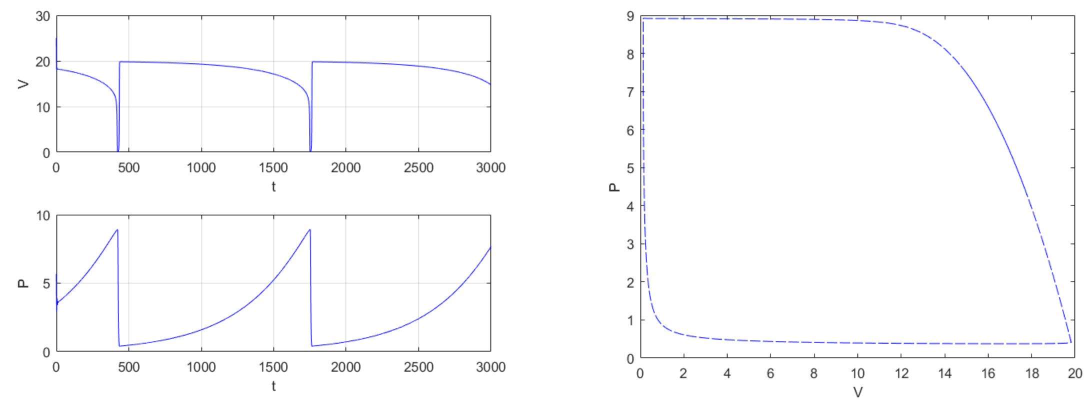

Figure 3. Sequence diagram of $V(t)$ and $p(t,a)$ over time, and phase diagram of $V(t)$ and $p(t,a)$ when E_* are unstable.

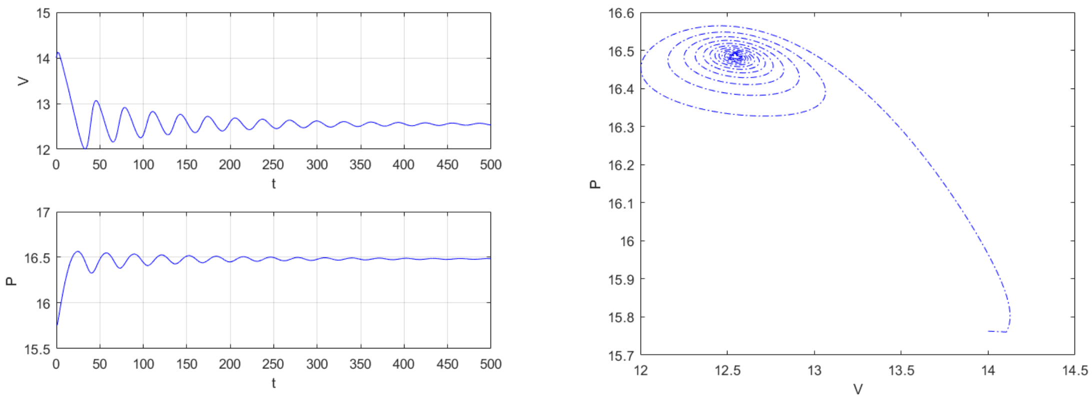

Figure 4. Sequence diagram of $V(t)$ and $p(t,a)$ over time, and phase diagram of $V(t)$ and $p(t,a)$ when E_* are asymptotically stable.

As can be seen from the above analysis, the time delay effects of predation processes, energy conversion, reproductive reproduction, etc., can cause changes in the dynamics behavior of the predation system over a later period of time. At a time when the delay is less than a certain threshold, the final size of the two species is in a state of coexistence and tends to stabilize. When this threshold is exceeded, the two species still coexist, but, because the system undergoes a Hopf bifurcation, the number of the two species is subject to periodic oscillations. We know that the weak Allee effect affects the population size of the predator system by influencing its time delay threshold, which is essential for the study of the predator system.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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