Electronic
Research Archive

http://www.aimspress.com/journal/era

ERA, 32(3): 1663–1691.
DOI: 10.3934/era.2024076
Received: 26 November 2023
Revised: 23 January 2024
Accepted: 06 February 2024
Published: 20 February 2024

Research article

Convergence of finite element solution of stochastic Burgers equation

Jingyun Lv[1,*] **and Xiaoyan Lu**[2]

[1] School of Fundamental Education, Beijing Polytechnic College, Beijing 100042, China
[2] Basic Courses Department of Beijing Polytechnic, Beijing 100176, China

* **Correspondence:** Email: lvjingyun2007@163.com.

Abstract: We explore the numerical approximation of the stochastic Burgers equation driven by fractional Brownian motion with Hurst index $H \in (1/4, 1/2)$ and $H \in (1/2, 1)$, respectively. The spatial and temporal regularity properties for the solution are obtained. The given problem is discretized in time with the implicit Euler scheme and in space with the standard finite element method. We obtain the strong convergence of semidiscrete and fully discrete schemes, performing the error estimates on a subset $\Omega_{k,h}$ of the sample space Ω with the Gronwall argument being used to overcome the difficulties, caused by the subtle interplay of the nonlinear convection term. Numerical examples confirm our theoretical findings.

Keywords: stochastic Burgers equation; fractional Brownian motion; finite element method; error estimates; strong convergence

1. Introduction

We consider the stochastic Burgers equation

$$d\boldsymbol{u} = (\nu\Delta\boldsymbol{u} - [\boldsymbol{u}\cdot\nabla]\boldsymbol{u})dt + \boldsymbol{\Psi}(t)dB^H(t), \quad \text{in } D\times(0,T), \tag{1.1}$$

$$\boldsymbol{u}(\boldsymbol{x},0) = \boldsymbol{u}_0(\boldsymbol{x}), \quad \text{in } D, \tag{1.2}$$

$$\boldsymbol{u}(\boldsymbol{x},t) = 0, \quad \text{on } \partial D\times(0,T), \tag{1.3}$$

where $D = [0,L]^2 \subset \mathbb{R}^2$, $\boldsymbol{u} = (u_1,u_2)$, $\nu > 0$, $L > 0, T > 0$. $\boldsymbol{\Psi}$ is a deterministic function. $B^H(t)$ is a fractional Brownian motion on a filtered probability space $(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t\}_{t\geq 0})$, $H \in (1/4,1/2)$ or $H \in (1/2,1)$. $\boldsymbol{u}_0$ is a given random variable.

The stochastic Burgers equation (SBE) has applications in many areas [1–5]. For $H = 1/2$ in (1.1), Bertini et al. [6], Brzeźniak et al. [7], and Kim [8] established the well-posedness results of the SBE. Catuogno and Olivera [9] proved the existence of a strong solution of the SBE. Twardowska and

Zabczyk [10] and Goldys and Maslowski [11] obtained the ergodicity results of the SBE. In [12], E et al. established a stationary solution of the SBE. Zou and Wang [13] considered the existence result of a fractional-order SBE. Zhou et al. [14] obtained strong solutions for the SBE. The SBE is a typical evolution equation, and recently a series of numerical methods, like the two-grid method [15–17], ADI method [18–20], finite difference method [21,22], spectral method [23], finite volume method [24,25], and extrapolation method [26], have been developed to solve it. From a computational point of view, Hairer and Voss [27] devoted their research to the finite difference approximations of the SBE. Hairer and Matetski [28] obtained the optimal convergence rate of the SBE. Blömker and Jentzen [29] studied Galerkin approximations of the SBE. Jentzen et al. [30] proposed and analyzed explicit space-time discrete numerical approximations for the SBE. Uma et al. [31] developed an approximate solution method for solving the SBE.

Fractional Brownian motion (fBm for short) is a type of Gaussian process that exhibits long-range dependence. It is characterized by its self-similarity and its correlation structure which is determined by a parameter called the Hurst index H. It plays a crucial role in various fields. Its applications can be broadly categorized into two main areas: finance and image/signal processing. In finance, fBm is utilized for modeling and simulating price movements in financial markets. It is particularly useful in capturing the long-term dependency and self-similarity observed in asset prices. Furthermore, fBm has notable applications in image and signal processing. Due to its ability to represent fractal-like structures, fBm is employed for texture synthesis and texture modeling, enabling the generation of realistic textures. Moreover, fBm is utilized for noise reduction, interpolation, and denoising in image processing, providing improved image quality and enhanced feature extraction capabilities. These applications have motivated theoretical and numerical investigations of stochastic differential equations (SDE) driven by fBm. For examples of theoretical results of the SDE driven by fBm, see, Wang et al. [32], Jiang et al. [33], Hinz [34], Pei et al. [35], Yang et al. [36], Zou et al. [37], and the references therein. However, there is less literature on studying numerical approximations for the SDE driven by fBm. For example, Cao et al. [38] considered finite element approximations for the SDE driven by fBm. Tudor [39] investigated Wong-Zakai type approximations. Qi and Lin [40] obtained the optimal error bound. Hong et al. [41] investigated the super-convergence result for the stochastic heat equation driven by fBm. The challenge of analyzing the convergence of SDE driven by fBm is that the Burkholder-Davis-Gundy inequality is unavailable. Therefore, it is necessary to take a different strategy to solve this problem. In the existing literature, we are not aware of any numerical investigation of (1.1)–(1.3). This paper aims to fill the gap.

The main difficulty of equations (1.1)–(1.3) is that the noise term causes the solutions $\boldsymbol{u}$ to be very "rough". Thus, new techniques and skills need to be explored. First of all, we establish the following energy estimates

$$\sup_{0\le t\le T} \mathbb{E}[\|\boldsymbol{u}(t)\|^2] + 4\nu\mathbb{E}\left[\int_0^t \|\nabla\boldsymbol{u}(s)\|^2 ds\right] \le C_{H,T},$$

and

$$\sup_{0\le t\le T} \mathbb{E}[\|\nabla\boldsymbol{u}(t)\|^2] \le C_{H,T},$$

where $C_{H,T} > 0$. Next, we prove the Hölder regularity results, that is, if $H \in (1/4, 1/2)$, it holds that

$$\mathbb{E}\left[\|\boldsymbol{u}(t_2) - \boldsymbol{u}(t_1)\|_\beta^2\right] \le C(t_2 - t_1)^{\min\{1-\beta/2,\ 3/2-\beta,\ 4H-1,\ 2H\}},$$

and if $H \in (1/2, 1)$, we obtain

$$\mathbb{E}[\|\boldsymbol{u}(t_2) - \boldsymbol{u}(t_1)\|_\beta^2] \leq C(t_2 - t_1)^{\min\{1-\beta/2,\ 3/2-\beta,\ 2H-\beta\}},$$

with the induced norms $\|\cdot\|_\beta = \|A^{\beta/2}\cdot\|$ for any $0 \leq \beta \leq 1$. For the detailed definitions of operator A and operator $A^{\beta/2}$, we refer the readers to Section 2. A key tool used here is a very careful treatment of multiple integrals.

To prove the convergence, the main problem here is caused by the subtle interplay of the nonlinear convection term and the stochastic forcing, which prevents a Gronwall argument in the context of expectations. We investigate the error estimates on a subset of Ω and use Markov's inequality to overcome the difficulties. Based on the regularity properties, and for a subset $\Omega_k \subset \Omega$, with $\mathbb{P}[\Omega_k] \to 1$ as $k \to 0$, if $H \in (1/4, 1/2)$, then

$$\mathbb{E}_{\Omega_k}\left[\max_{1\leq n\leq N}\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|^2 + \frac{\nu k}{2}\sum_{n=1}^{N}\|\nabla(\boldsymbol{u}(t_n) - \boldsymbol{u}^n)\|^2\right] \leq C(k^{\min\{1/2,4H-1\}+1-\varepsilon} + k^{4H-1+2\delta-\varepsilon} + k^{2H+2\delta-\varepsilon}),$$

and if $H \in (1/2, 1)$, we have

$$\mathbb{E}_{\Omega_k}\left[\max_{1\leq n\leq N}\|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|^2 + \frac{\nu k}{2}\sum_{n=1}^{N}\|\nabla(\boldsymbol{u}(t_n) - \boldsymbol{u}^n)\|^2\right] \leq C(k^{\min\{1/2,2H-1\}+1-\varepsilon} + k^{2H+2\delta-\varepsilon}),$$

where $\mathbb{E}_{\Omega_k}[\cdot] = \mathbb{E}[\mathbb{I}_{\Omega_k}\cdot]$, $\varepsilon > 0$ is arbitrarily small, and k is the time step. For a subset $\Omega_h \subset \Omega$, with $\mathbb{P}[\Omega_h] \to 1$ as $h \to 0$, where $h > 0$ is the space step, if $H \in (1/4, 1/2)$, we have

$$\mathbb{E}_{\Omega_{k,h}}[\max_{1\leq n\leq N}\|\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n\|^2] \leq C(k^{\min\{1/2,4H-1\}+1-\varepsilon} + k^{4H-1+2\delta-\varepsilon} + k^{2H+2\delta-\varepsilon} + h^{2-\epsilon} + k^{4H-1}h^{2-\epsilon}),$$

and for $H \in (1/2, 1)$, we get

$$\mathbb{E}_{\Omega_{k,h}}[\max_{1\leq n\leq N}\|\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n\|^2] \leq C(k^{\min\{1/2,2H-1\}+1-\varepsilon} + k^{2H+2\delta-\varepsilon} + h^{2-\epsilon} + k^{2H}h^{2-\epsilon}),$$

where $\Omega_{k,h} = \Omega_k \cap \Omega_h$, and both $\epsilon > 0$ and $\varepsilon > 0$ are arbitrarily small. It should be emphasized that the convergence analysis presented here requires various delicate error estimates, which is different from the case of the Wiener process.

Section 2 presents some preliminaries. The spatial and temporal regularity results of (1.1)–(1.3) are established in Section 3. Section 4 shows the convergence results for the time discretization scheme. Section 5 is devoted to the space-time discretization. We present the numerical experiments in Section 6.

2. Notations and preliminaries

We assume that $L^q(D)$ and $W^{k,p}(D)$ are Lebesgue and Sobolev spaces. Let $\mathbb{L}^q(D) := [L^q(D)]^2$, $\mathbb{W}^{k,p}(D) := [W^{k,p}(D)]^2$. We define the space

$$V = \{\boldsymbol{u} \in [H^1(D)]^2 : \boldsymbol{u} = 0 \text{ on } \partial D\}.$$

We define the operator A via $A\boldsymbol{u} = -\nu\Delta\boldsymbol{u}$, in the domain $D(A) = V \cap \mathbb{W}^{2,2}(D)$. We define the operator $A^{\frac{s}{2}}$, $s \in \mathbb{R}$, by

$$A^{\frac{s}{2}}x = \sum_{j=1}^{\infty} \gamma_j^{\frac{s}{2}} < x, e_j > e_j,\ x \in D(A^{\frac{s}{2}}),$$

$$D(A^{\frac{s}{2}}) = \{x \in V : \sum_{j=1}^{\infty} \gamma_j^{s} < x, e_j >^2 < \infty\},$$

where $\{(\gamma_j, e_j)\}_{j=1}^{\infty}$ are the pairs of eigenvalues and eigenfunctions of A.

Let $\dot{V}^s = D(A^{\frac{s}{2}})$ be the Hilbert space endowed with the norm

$$||x||_s := ||A^{\frac{s}{2}}x|| = \left(\sum_{j=1}^{\infty} \gamma_j^{s} < x, e_j >^2\right)^{\frac{1}{2}}, x \in \dot{V}^s.$$

We define the space $\mathcal{L}(V)$ by

$$\mathcal{L}(V) = \{f | f : V \to V \text{ is a bounded linear operator}\}$$

with norm $\|\cdot\|_{\mathcal{L}(V)}$.

It is also well-known that (see [42])

$$||A^{\gamma}e^{-tA}||_{\mathcal{L}(V)} \le Ct^{-\gamma},\ t > 0,\ \gamma \ge 0, \tag{2.1}$$

and

$$||A^{-\rho}(I - e^{-tA})||_{\mathcal{L}(V)} \le Ct^{\rho},\ t > 0,\ \rho \in [0, 1]. \tag{2.2}$$

Definition 2.1. [1, 43] A continuous Gaussian process $\{B^H(t), t \ge 0\}$ that satisfies

$$\mathbb{E}[B^H(t)B^H(s)] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}),\ t, s \in [0, \infty),$$

is called an fBm, where $H \in (0, 1)$ is the Hurst index. In particular, if $H = \frac{1}{2}$, then the process is in fact Brownian motion or a Wiener process.

Define $K_H(t, s)$ by

$$K_H(t,s) = \begin{cases} C_H(t-s)^{H-\frac{1}{2}} + C_H(\frac{1}{2} - H)\int_s^t (u-s)^{H-\frac{3}{2}}\left(1 - (\frac{s}{u})^{\frac{1}{2}-H}\right)du,\ H \in (0, \frac{1}{2}), & (2.3) \\ C_H(H - \frac{1}{2})s^{\frac{1}{2}-H}\int_s^t (u-s)^{H-\frac{3}{2}}u^{H-\frac{1}{2}}du,\ H \in (\frac{1}{2}, 1), & (2.4) \end{cases}$$

where $C_H = (\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)})^{\frac{1}{2}}$ is a constant.

By (2.3) and (2.4), we have

$$\frac{\partial K_H}{\partial t}(t,s) = C_H\left(H-\frac{1}{2}\right)\left(\frac{s}{t}\right)^{\frac{1}{2}-H}(t-s)^{H-\frac{3}{2}}. \tag{2.5}$$

Let $\mathcal{H}$ denote the reproducing kernel Hilbert space of fBm. For every $0<H<1,\ s<\tau$, we define the linear mapping $K_\tau^*: \mathcal{H} \to L^2([0,T])$ [43,44] by

$$(K_\tau^*\varphi)(s) = \varphi(s)K_H(\tau,s) + \int_s^\tau (\varphi(r)-\varphi(s))\frac{\partial K_H}{\partial r}(r,s)dr. \tag{2.6}$$

For $1/2<H<1$, we have a simpler expression

$$(K_\tau^*\varphi)(s) = \int_s^\tau \varphi(r)\frac{\partial K_H}{\partial r}(r,s)dr. \tag{2.7}$$

Define a linear bounded covariance operator as Q that satisfies $\mathrm{Tr}(Q)<\infty$. The corresponding eigenvalues and eigenfunctions are denoted by $\{(\lambda_k,e_k)\}_{k=1}^\infty$. We denote by $\mathcal{L}_2^0 := \mathcal{L}_2^0(Y,X)$ the space of all Q-Hilbert-Schmidt operators $\phi: Y\to X$ endowed with the norm $\|\phi\|^2_{\mathcal{L}_2^0} = \sum_{k=1}^{\infty}\|\lambda_k^{1/2}\phi e_k\|^2<\infty$, where Y and X are real separable Hilbert spaces.

For the definition of the Wiener integral $\int_0^t \sigma(s)dB^H(s)$, we refer to references [43,45]. Then, the stochastic integral satisfies

$$\mathbb{E}\left[\left\|\int_0^t \sigma(s)dB^H(s)\right\|^2\right] = \mathbb{E}\left[\left\|\int_0^t (K_t^*\sigma)(s)dB(s)\right\|^2\right] = \mathbb{E}\left[\int_0^t \left\|(K_t^*\sigma)(s)\right\|^2_{\mathcal{L}_2^0}ds\right], \tag{2.8}$$

and the inequality holds

$$\mathbb{E}\left[\sup_{0\le t\le T}\left\|\int_0^t \sigma(s)dB^H(s)\right\|^p\right] \le C(p)\mathbb{E}\left[\left(\int_0^t \left\|(K_t^*\sigma)(s)\right\|^2_{\mathcal{L}_2^0}ds\right)^{\frac{p}{2}}\right],\ p\in(1,\infty), \tag{2.9}$$

where $\{B(t), t\in[0,T]\}$ is a Wiener process, the function $\sigma:[0,T]\to\mathcal{L}_2^0$, and the constant $C(p)>0$.

We introduce the following assumptions:

(S_1) $\boldsymbol{u}_0 \in L^p(\Omega,\mathcal{F},\mathbb{P},V),\ p\ge 2$.

(S_2) For $\gamma\in[0,3],\ \delta\in[0,\gamma/2]$, the mapping $\boldsymbol{\Psi}:[0,T]\to\mathcal{L}_2^0$ satisfies

$$\|A^{(\gamma-1)/2}\boldsymbol{\Psi}(t)\|_{\mathcal{L}_2^0} \le C_T,\ t\in[0,T], \tag{2.10}$$

and

$$\|A^{(\gamma-1)/2}(\boldsymbol{\Psi}(t)-\boldsymbol{\Psi}(s))\|_{\mathcal{L}_2^0} \le C_T(t-s)^\delta, \tag{2.11}$$

where C_T is a constant.

Definition 2.2. (Strong solution) Let (S_1)–(S_2) be valid. An adapted V-valued process $\{\boldsymbol{u}(t)\}_{t\in[0,T]}$ is called a strong solution to (1.1)–(1.3) if $\boldsymbol{u}(\cdot,\omega) \in C([0,T];V) \cap L^2(0,T;\mathbb{W}^{2,2} \cap V)$ $\mathbb{P}$-a.s., and for $\forall t \in [0,T]$, $\forall \boldsymbol{v} \in V$,

$$\begin{aligned}(\boldsymbol{u}(t),\boldsymbol{v}) + \nu \int_0^t (\nabla \boldsymbol{u}(s), \nabla \boldsymbol{v}) ds + \int_0^t ([\boldsymbol{u}(s)\cdot\nabla]\boldsymbol{u}(s), \boldsymbol{v}) ds \\ = (\boldsymbol{u}_0, \boldsymbol{v}) + \left(\int_0^t \boldsymbol{\Psi}(s) dB^H(s), \boldsymbol{v}\right),\ \mathbb{P}-a.s.\end{aligned} \tag{2.12}$$

The following properties are valid:

$$([\boldsymbol{u}\cdot\nabla]\boldsymbol{v},\boldsymbol{w}) = -([\boldsymbol{u}\cdot\nabla]\boldsymbol{w},\boldsymbol{v}),\ ([\boldsymbol{u}\cdot\nabla]\boldsymbol{v},\boldsymbol{v}) = 0,\ \forall\, \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in \mathbb{W}^{1,2}. \tag{2.13}$$

Using integration by parts, we have

$$|([\boldsymbol{u}\cdot\nabla]\boldsymbol{v},\boldsymbol{w})| \le \|\boldsymbol{u}\|_{\mathbb{L}^4}\|\nabla\boldsymbol{v}\|_{\mathbb{L}^2}\|\boldsymbol{w}\|_{\mathbb{L}^4}. \tag{2.14}$$

From the above estimate, using the Gagliardo-Nirenberg-Sobolev inequality $\mathbb{W}^{1,2}(D) \subset \mathbb{L}^4(D)$ leads to

$$|([\boldsymbol{u}\cdot\nabla]\boldsymbol{v},\boldsymbol{w})| \le C\|\nabla\boldsymbol{u}\|_{\mathbb{L}^2}\|\nabla\boldsymbol{v}\|_{\mathbb{L}^2}\|\nabla\boldsymbol{w}\|_{\mathbb{L}^2}, \tag{2.15}$$

and applying Ladyzhenskaya's inequality $\|\boldsymbol{u}\|_{\mathbb{L}^4} \le C\|\boldsymbol{u}\|_{\mathbb{L}^2}^{1/2}\|\nabla\boldsymbol{u}\|_{\mathbb{L}^2}^{1/2}$ shows

$$|([\boldsymbol{u}\cdot\nabla]\boldsymbol{v},\boldsymbol{w})| \le C\|\boldsymbol{u}\|_{\mathbb{L}^2}^{1/2}\|\nabla\boldsymbol{u}\|_{\mathbb{L}^2}^{1/2}\|\nabla\boldsymbol{v}\|_{\mathbb{L}^2}\|\boldsymbol{w}\|_{\mathbb{L}^2}^{1/2}\|\nabla\boldsymbol{w}\|_{\mathbb{L}^2}^{1/2}. \tag{2.16}$$

More about the above properties of trilinear operator can be found in [46].

3. The regularity results

We establish the regularity properties of (2.12) in the spatial and temporal directions.

Theorem 3.1. We assume that (S_1)–(S_2) hold. If $H \in (1/4, 1/2)$ or $H \in (1/2, 1)$, then the solution $\boldsymbol{u}(t)$ to (1.1)–(1.3) satisfies

$$\sup_{0\le t\le T} \mathbb{E}\left[\|\boldsymbol{u}(t)\|^2\right] + 4\nu\mathbb{E}\left[\int_0^t \|\nabla\boldsymbol{u}(s)\|^2 ds\right] \le C(H,T)(1+\mathbb{E}[\|\boldsymbol{u}_0\|^2]), \tag{3.1}$$

and

$$\sup_{0\le t\le T} \mathbb{E}[\|\nabla\boldsymbol{u}(t)\|^2] \le C(H,T)(1+\mathbb{E}[\|\nabla\boldsymbol{u}_0\|^2]), \tag{3.2}$$

where $C(H,T) > 0$ is a constant.

Proof. (i). By (1.1)–(1.3), we have

$$\left(\frac{d\boldsymbol{u}(t)}{dt}, \boldsymbol{u}(t)\right) = \nu\,(\Delta\boldsymbol{u}(t),\boldsymbol{u}(t)) - ([\boldsymbol{u}(t)\cdot\nabla]\boldsymbol{u}(t),\boldsymbol{u}(t)) + \left(\boldsymbol{\Psi}(t)\frac{dB^H(t)}{dt}, \boldsymbol{u}(t)\right).$$

Applying (2.13), we deduce

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}(t)\|^2 + \nu\|\nabla\boldsymbol{u}(t)\|^2 = \left(\boldsymbol{\Psi}(t)\frac{dB^H(t)}{dt}, \boldsymbol{u}(t)\right). \tag{3.3}$$

Integrating (3.3) from 0 to t yields

$$\frac{1}{2}[\|\boldsymbol{u}(t)\|^2 - \|\boldsymbol{u}_0\|^2] + \nu\int_0^t \|\nabla\boldsymbol{u}(s)\|^2 ds = \int_0^t (\boldsymbol{\Psi}(s), \boldsymbol{u}(s))dB^H(s). \tag{3.4}$$

Using Young's inequality and (2.9), we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le t\le T}\int_0^t (\boldsymbol{\Psi}(s), \boldsymbol{u}(s))dB^H(s)\right] &\le C\mathbb{E}\left[\left(\left(\int_0^t \|\boldsymbol{u}(s)\|^2\|(K_t^*\boldsymbol{\Psi})(s)\|^2_{\mathcal{L}_2^0}ds\right)^{1/2}\right]\\
&\le C\mathbb{E}\left[\sup_{0\le t\le T}\|\boldsymbol{u}(t)\|(\int_0^t \|(K_t^*\boldsymbol{\Psi})(s)\|^2_{\mathcal{L}_2^0}ds)^{1/2}\right]\\
&\le \frac{1}{4}\mathbb{E}[\sup_{0\le t\le T}\|\boldsymbol{u}(t)\|^2] + C\mathbb{E}\left[\int_0^t \|(K_t^*\boldsymbol{\Psi})(s)\|^2_{\mathcal{L}_2^0}ds\right].
\end{aligned} \tag{3.5}$$

For $1/4 < H < 1/2$, by (2.6), the inequality ([32])

$$K_H(t,s) \le C_H(t-s)^{H-\frac{1}{2}}s^{H-\frac{1}{2}},$$

and the application of Hölder's inequality and (2.5), (2.10), and (2.11) gives

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \|(K_t^*\boldsymbol{\Psi})(s)\|^2_{\mathcal{L}_2^0}ds\right]\\
&= \mathbb{E}\left[\int_0^t \|\boldsymbol{\Psi}(s)K_H(t,s) + \int_s^t (\boldsymbol{\Psi}(r) - \boldsymbol{\Psi}(s))\frac{\partial K_H}{\partial r}(r,s)dr\|^2_{\mathcal{L}_2^0}ds\right]\\
&\le 2\mathbb{E}\left[\int_0^t \|\boldsymbol{\Psi}(s)K_H(t,s)\|^2_{\mathcal{L}_2^0}ds\right] + 2\mathbb{E}\left[\int_0^t\left(\int_s^t \|\boldsymbol{\Psi}(r) - \boldsymbol{\Psi}(s)\|^2_{\mathcal{L}_2^0}dr\right)\left(\int_s^t \|\frac{\partial K_H}{\partial r}(r,s)\|^2 dr\right)ds\right]\\
&\le C_H^2\int_0^t (t-s)^{2H-1}s^{2H-1}ds + C(H,\delta)\int_0^t (t-s)^{2\delta+1}\left(\int_s^t (\frac{s}{r})^{1-2H}(r-s)^{2H-3}dr\right)ds\\
&\le C_H^2\left(\int_0^t (t-s)^{4H-2}ds\right)^{\frac{1}{2}}\left(\int_0^t s^{4H-2}ds\right)^{\frac{1}{2}} + C(H,\delta)\int_0^t (t-s)^{2(H+\delta)-1}ds\\
&= C(H)t^{4H-1} + C(H,\delta)t^{2(H+\delta)},
\end{aligned} \tag{3.6}$$

where we use the fact $(s/r)^{1-2H} \le 1$ for $s \le r$ and $H \in (1/4, 1/2)$.

Similarly, for $H \in (1/2, 1)$, one can derive that

$$\begin{aligned}
\mathbb{E}\left[\int_0^t \|(K_t^*\boldsymbol{\Psi})(s)\|^2_{\mathcal{L}_2^0}ds\right] &= \mathbb{E}\left[\int_0^t \|\int_s^t \boldsymbol{\Psi}(r)\frac{\partial K_H}{\partial r}(r,s)dr\|^2_{\mathcal{L}_2^0}ds\right]\\
&\le \mathbb{E}\left[\int_0^t (\int_s^t \|\boldsymbol{\Psi}(r)\|^2_{\mathcal{L}_2^0}dr)(\int_s^t \|\frac{\partial K_H}{\partial r}(r,s)\|^2 dr)ds\right]
\end{aligned}$$

$$\leq C_H^2(H-\frac{1}{2})^2\int_0^t (t-s)(\int_s^t (\frac{s}{r})^{1-2H}(r-s)^{2H-3}dr)ds$$
$$\leq C(H)t^{2H-1}\int_0^t (t-s)^{2H-1}s^{1-2H}ds$$
$$\leq C(H)T^{2H}, \tag{3.7}$$

where we use the results $r^{2H-1} \leq t^{2H-1}$ for $H > 1/2$ and $r \leq t$, $(t-s)^{2H-1} \leq T^{2H-1}$ for $(s,t) \subset [0,T]$.

Therefore, (3.3)–(3.7) lead us to the proof of (3.1).

(ii). From (1.1), we have

$$\left(\frac{d\boldsymbol{u}(t)}{dt}, \Delta\boldsymbol{u}(t)\right) = \nu\left(\Delta\boldsymbol{u}(t), \Delta\boldsymbol{u}(t)\right) - ([\boldsymbol{u}(t)\cdot\nabla]\boldsymbol{u}(t), \Delta\boldsymbol{u}(t)) + \left(\boldsymbol{\Psi}(t)\frac{dB^H(t)}{dt}, \Delta\boldsymbol{u}(t)\right),$$

and by the Dirichlet condition, it can be concluded that

$$\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{u}(t)\|^2 + \nu\|\Delta\boldsymbol{u}(t)\|^2 = ([\boldsymbol{u}(t)\cdot\nabla]\boldsymbol{u}(t), \Delta\boldsymbol{u}(t)) + \left(\nabla\boldsymbol{\Psi}(t)\frac{dB^H(t)}{dt}, \nabla\boldsymbol{u}(t)\right). \tag{3.8}$$

Integrating (3.8) from 0 to t, we have

$$\frac{1}{2}[\|\nabla\boldsymbol{u}(t)\|^2 - \|\nabla\boldsymbol{u}_0\|^2] + \nu\int_0^t \|\Delta\boldsymbol{u}(s)\|^2ds$$
$$= \int_0^t ([\boldsymbol{u}(s)\cdot\nabla]\boldsymbol{u}(s), \Delta\boldsymbol{u}(s))ds + \int_0^t (\nabla\boldsymbol{\Psi}(s), \nabla\boldsymbol{u}(s))dB^H(s). \tag{3.9}$$

Using the inequality $|([\boldsymbol{u}(s)\cdot\nabla]\boldsymbol{u}(s), \Delta\boldsymbol{u}(s))| \leq C\|\nabla\boldsymbol{u}(s)\|^2\|\Delta\boldsymbol{u}(s)\|$ (see [47]) and Young's inequality, we have

$$\int_0^t ([\boldsymbol{u}(s)\cdot\nabla]\boldsymbol{u}(s), \Delta\boldsymbol{u}(s))\,ds$$
$$\leq \int_0^t C\|\nabla\boldsymbol{u}(s)\|^2\|\Delta\boldsymbol{u}(s)\|ds$$
$$\leq \sup_{0\leq s\leq T}\|\nabla\boldsymbol{u}(s)\|\left(\int_0^t C\|\nabla\boldsymbol{u}(s)\|\|\Delta\boldsymbol{u}(s)\|ds\right)$$
$$\leq \frac{1}{8}\sup_{0\leq s\leq T}\|\nabla\boldsymbol{u}(s)\|^2 + C\left(\int_0^t \|\nabla\boldsymbol{u}(s)\|\|\Delta\boldsymbol{u}(s)\|ds\right)^2$$
$$\leq \frac{1}{8}\sup_{0\leq s\leq T}\|\nabla\boldsymbol{u}(s)\|^2 + C\left(\int_0^t \|\nabla\boldsymbol{u}(s)\|^2ds\right)\left(\int_0^t \|\Delta\boldsymbol{u}(s)\|^2ds\right)$$
$$\leq \frac{1}{8}\sup_{0\leq s\leq T}\|\nabla\boldsymbol{u}(s)\|^2 + C(H,T,\boldsymbol{u}_0)\int_0^t \|\Delta\boldsymbol{u}(s)\|^2ds. \tag{3.10}$$

For the stochastic integral term in (3.9), by (2.10), using the same methods as in the derivation of (3.6) and (3.7), we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\int_0^t (\nabla\boldsymbol{\Psi}(s), \nabla\boldsymbol{u}(s))dB^H(s)\right] \leq C\mathbb{E}\left[\left(\int_0^t \|\nabla\boldsymbol{u}(s)\|^2\|(K_t^*\nabla\boldsymbol{\Psi})(s)\|^2_{\mathcal{L}_2^0}ds\right)^{1/2}\right]$$

$$\leq C\mathbb{E}\left[\sup_{0\leq s\leq T}\|\nabla \boldsymbol{u}(s)\|\left(\int_0^t \|(K_t^*\nabla\boldsymbol{\Psi})(s)\|_{\mathcal{L}_2^0}^2 ds\right)^{1/2}\right]$$
$$\leq \frac{1}{8}\mathbb{E}[\sup_{0\leq s\leq T}\|\nabla \boldsymbol{u}(s)\|^2] + C\mathbb{E}\left[\int_0^t \|(K_t^*\nabla\boldsymbol{\Psi})(s)\|_{\mathcal{L}_2^0}^2 ds\right]$$
$$\leq \frac{1}{8}\mathbb{E}[\sup_{0\leq s\leq T}\|\nabla \boldsymbol{u}(s)\|^2] + C(H,T). \tag{3.11}$$

Let $\nu > 0$ be sufficiently large such that $\nu - C(H,T,\boldsymbol{u}_0) > 0$. Taking the expectation, from (3.8)–(3.11), we finish the proof of (3.2).

Theorem 3.2. If (S_1)–(S_2) are valid and $\beta \in [0,1]$, the strong solution $\boldsymbol{u}(t)$ is Hölder continuous. Furthermore, if the Hurst index satisfies $H \in (1/4, 1/2)$, it holds that

$$\mathbb{E}[\|\boldsymbol{u}(t_2) - \boldsymbol{u}(t_1)\|_\beta^2] \leq C(t_2 - t_1)^{\min\{1-\beta/2,\ 3/2-\beta,\ 4H-1,\ 2H\}},$$

and for $H \in (1/2, 1)$, it holds that

$$\mathbb{E}[\|\boldsymbol{u}(t_2) - \boldsymbol{u}(t_1)\|_\beta^2] \leq C(t_2 - t_1)^{\min\{1-\beta/2,\ 3/2-\beta,\ 2H-\beta\}}.$$

Proof. Referring to some results available in [47], we can represent the strong solution to (1.1)–(1.3) as

$$\boldsymbol{u}(t) = e^{-tA}\boldsymbol{u}_0 + \int_0^t e^{-(t-s)A}[\boldsymbol{u}(s)\cdot\nabla]\boldsymbol{u}(s)ds + \int_0^t e^{-(t-s)A}\boldsymbol{\Psi}(s)dB^H(s). \tag{3.12}$$

For any $t_1 < t_2$, from (3.12), we have

$$\begin{aligned}\boldsymbol{u}(t_2) - \boldsymbol{u}(t_1) &= (e^{-t_2A} - e^{-t_1A})\boldsymbol{u}_0 \\ &\quad + \left(\int_0^{t_2} e^{-(t_2-s)A}[\boldsymbol{u}(s)\cdot\nabla]\boldsymbol{u}(s)ds - \int_0^{t_1} e^{-(t_1-s)A}[\boldsymbol{u}(s)\cdot\nabla]\boldsymbol{u}(s)ds\right) \\ &\quad + \left(\int_0^{t_2} e^{-(t_2-s)A}\boldsymbol{\Psi}(s)dB^H(s) - \int_0^{t_1} e^{-(t_1-s)A}\boldsymbol{\Psi}(s)dB^H(s)\right) \\ &:= I_a + I_b + I_c. \end{aligned} \tag{3.13}$$

Application of (2.1) and (2.2) leads to

$$\begin{aligned}\mathbb{E}[\|I_a\|_\beta] &= \mathbb{E}[\|A^{\beta/2}e^{-t_1A}(e^{-(t_2-t_1)A} - \mathbf{I})\boldsymbol{u}_0\|] \\ &= \mathbb{E}[\|e^{-t_1A}A^{-(1/2-\beta/4)}(e^{-(t_2-t_1)A} - \mathbf{I})A^{1/2+\beta/4}\boldsymbol{u}_0\|] \\ &\leq C(t_2-t_1)^{1/2-\beta/4}\mathbb{E}[\|\boldsymbol{u}_0\|_{1+\beta/2}]. \end{aligned} \tag{3.14}$$

Using the Sobolev embedding theorem, we can obtain

$$\|A^{-1/2}[\boldsymbol{u}\cdot\nabla]\boldsymbol{u}\| \leq C\|\boldsymbol{u}\|^2,\ \ \|A^{-1/4}[\boldsymbol{u}\cdot\nabla]\boldsymbol{u}\| \leq C\|A^{1/2}\boldsymbol{u}\|^2. \tag{3.15}$$

Furthermore,

$$I_b = \int_0^{t_1}(e^{-(t_2-s)A} - e^{-(t_1-s)A})[\boldsymbol{u}(s)\cdot\nabla]\boldsymbol{u}(s)ds + \int_{t_1}^{t_2} e^{-(t_2-s)A}[\boldsymbol{u}(s)\cdot\nabla]\boldsymbol{u}(s)ds$$

$$:= I_{b1} + I_{b2}. \tag{3.16}$$

For I_{b1} in (3.16), by (2.1), (2.2), (3.15), and (3.2), we have

$$\begin{aligned}
\mathbb{E}[\|I_{b1}\|_\beta] &\le \mathbb{E}\left[\int_0^{t_1} \|A^{\beta/2}(e^{-(t_2-s)A} - e^{-(t_1-s)A})[\boldsymbol{u}(s)\cdot\nabla]\boldsymbol{u}(s)\|ds\right] \\
&\le \mathbb{E}\left[\int_0^{t_1} \|A^{3/4+\beta/4}e^{-(t_1-s)A}\|\|A^{-(1/2-\beta/4)}(e^{-(t_2-t_1)A} - \mathbf{I})\|\|A^{-1/4}[\boldsymbol{u}(s)\cdot\nabla]\boldsymbol{u}(s)\|ds\right] \\
&\le C(t_2-t_1)^{1/2-\beta/4}\mathbb{E}\left[\int_0^{t_1}(t_1-s)^{-3/4-\beta/4}\|A^{1/2}\boldsymbol{u}(s)\|^2 ds\right] \\
&\le C(t_2-t_1)^{1/2-\beta/4}t_1^{1/4-\beta/4}\sup_{0\le s\le T}\mathbb{E}[\|\boldsymbol{u}(s)\|_1^2] \\
&\le C(H,T,\boldsymbol{u}_0)(t_2-t_1)^{1/2-\beta/4}.
\end{aligned} \tag{3.17}$$

For the term I_{b2}, we obtain

$$\begin{aligned}
\mathbb{E}[\|I_{b2}\|_\beta] &\le \mathbb{E}\left[\int_{t_1}^{t_2} \|A^{\beta/2}e^{-(t_2-s)A}[\boldsymbol{u}(s)\cdot\nabla]\boldsymbol{u}(s)\|ds\right] \\
&\le \mathbb{E}\left[\int_{t_1}^{t_2} \|A^{1/4+\beta/2}e^{-(t_2-s)A}\|\|A^{-1/4}[\boldsymbol{u}(s)\cdot\nabla]\boldsymbol{u}(s)\|ds\right] \\
&\le C\mathbb{E}\left[\int_{t_1}^{t_2}(t_2-s)^{-1/4-\beta/2}\|A^{1/2}\boldsymbol{u}(s)\|^2 ds\right] \\
&\le C(t_2-t_1)^{3/4-\beta/2}\sup_{0\le s\le T}\mathbb{E}[\|\boldsymbol{u}(s)\|_1^2] \\
&\le C(H,T,\boldsymbol{u}_0)(t_2-t_1)^{3/4-\beta/2}.
\end{aligned} \tag{3.18}$$

The term I_c also can be rearranged as

$$\begin{aligned}
I_c &= \int_0^{t_1}(e^{-(t_2-s)A} - e^{-(t_1-s)A})\boldsymbol{\Psi}(s)dB^H(s) + \int_{t_1}^{t_2} e^{-(t_2-s)A}\boldsymbol{\Psi}(s)dB^H(s) \\
&:= I_{c1} + I_{c2}.
\end{aligned} \tag{3.19}$$

For $1/4 < H < 1/2$, by (2.9), we get

$$\begin{aligned}
&\mathbb{E}[\sup_{0\le t_1\le T}\|I_{c1}\|_\beta^2] \\
&\le C\mathbb{E}\left[\int_0^{t_1}\|A^{\beta/2}K_{t_1}^*(e^{-(t_2-s)A} - e^{-(t_1-s)A})\boldsymbol{\Psi}(s)\|_{\mathcal{L}_2^0}^2 ds\right] \\
&= C\mathbb{E}[\int_0^{t_1}\|A^{\beta/2}\{(e^{-(t_2-s)A} - e^{-(t_1-s)A})\boldsymbol{\Psi}(s)K_H(t_1,s) \\
&\quad + \int_s^{t_1}[(e^{-(t_2-r)A} - e^{-(t_1-r)A})\boldsymbol{\Psi}(r) - (e^{-(t_2-s)A} - e^{-(t_1-s)A})\boldsymbol{\Psi}(s)]\frac{\partial K_H}{\partial r}(r,s)dr\}\|_{\mathcal{L}_2^0}^2 ds \\
&\le C\Big\{\mathbb{E}\left[\int_0^{t_1}\|A^{\beta/2}(e^{-(t_2-s)A} - e^{-(t_1-s)A})\boldsymbol{\Psi}(s)K_H(t_1,s)\|_{\mathcal{L}_2^0}^2 ds\right]
\end{aligned}$$

$$+\mathbb{E}\left[\int_0^{t_1}\left(\int_s^{t_1}\|A^{\beta/2}(e^{-(t_2-r)A}-e^{-(t_1-r)A})\mathbf{\Psi}(r)\|^2_{\mathcal{L}_2^0}dr\right)\left(\int_s^{t_1}\|\frac{\partial K_H}{\partial r}(r,s)\|^2dr\right)ds\right]$$
$$+\mathbb{E}\left[\int_0^{t_1}\left(\int_s^{t_1}\|A^{\beta/2}(e^{-(t_2-s)A}-e^{-(t_1-s)A})\mathbf{\Psi}(s)\|^2_{\mathcal{L}_2^0}dr\right)\left(\int_s^{t_1}\|\frac{\partial K_H}{\partial r}(r,s)\|^2dr\right)ds\right]\Bigg\}$$
$$:= C\{I_{c11}+I_{c12}+I_{c13}\}. \tag{3.20}$$

Using (2.1), (2.2), (2.10), and Hölder's inequality, one can obtain

$$\begin{aligned} I_{c11} &\le \int_0^{t_1}\|e^{-(t_1-s)A}A^{-(1/2-\beta/4)}(e^{-(t_2-t_1)A}-\mathbf{I})K_H(t_1,s)\|^2\|A^{(1/2+\beta/4)}\mathbf{\Psi}(s)\|^2_{\mathcal{L}_2^0}ds \\ &\le C_H^2(t_2-t_1)^{1-\beta/2}\int_0^{t_1}(t_1-s)^{2H-1}s^{2H-1}ds \\ &\le \frac{C_H^2t_1^{4H-1}}{4H-1}(t_2-t_1)^{1-\beta/2} \\ &\le C(H,T)(t_2-t_1)^{1-\beta/2}. \end{aligned} \tag{3.21}$$

Similarly, we get

$$\begin{aligned} I_{c12} &\le \int_0^{t_1}\left(\int_s^{t_1}\|A^{-(1/2-\beta/4)}(e^{-(t_2-r)A}-e^{-(t_1-r)A})\|^2\right. \\ &\qquad \left.\times\|A^{(1/2+\beta/4)}\mathbf{\Psi}(r)\|^2_{\mathcal{L}_2^0}dr\right)\left(\int_s^{t_1}\|\frac{\partial K_H}{\partial r}(r,s)\|^2dr\right)ds \\ &\le C_H^2\left(H-\frac{1}{2}\right)^2(t_2-t_1)^{1-\beta/2}\int_0^{t_1}(t_1-s)\left(\int_s^{t_1}(\frac{s}{r})^{1-2H}(r-s)^{2H-3}dr\right)ds \\ &\le C(H)(t_2-t_1)^{1-\beta/2}\int_0^{t_1}(t_1-s)^{2H-1}ds \\ &\le C(H,T)(t_2-t_1)^{1-\beta/2}, \end{aligned} \tag{3.22}$$

where we use the fact $(s/r)^{1-2H}\le 1$ for $r>s$ and $1/4<H<1/2$. In a similar manner, we have

$$\begin{aligned} I_{c13} &\le \int_0^{t_1}\left(\int_s^{t_1}\|A^{-(1/2-\beta/4)}(e^{-(t_2-s)A}-e^{-(t_1-s)A})\|^2\right. \\ &\qquad \left.\times\|A^{(1/2+\beta/4)}\mathbf{\Psi}(s)\|^2_{\mathcal{L}_2^0}dr\right)\left(\int_s^{t_1}\|\frac{\partial K_H}{\partial r}(r,s)\|^2dr\right)ds \\ &\le C_H^2(H-\frac{1}{2})^2(t_2-t_1)^{1-\beta/2}\int_0^{t_1}(t_1-s)\left(\int_s^{t_1}(\frac{s}{r})^{1-2H}(r-s)^{2H-3}dr\right)ds \\ &\le C(H,T)(t_2-t_1)^{1-\beta/2}. \end{aligned} \tag{3.23}$$

For the term I_{c2}, we obtain

$$\begin{aligned} &\mathbb{E}[\sup_{0\le t_2\le T}\|I_{c2}\|^2_\beta] \\ &\le C\mathbb{E}\left[\int_{t_1}^{t_2}\|A^{\beta/2}K^*_{t_2}(e^{-(t_2-s)A}\mathbf{\Psi}(s))\|^2_{\mathcal{L}_2^0}ds\right] \end{aligned}$$

$$
\begin{aligned}
&= C\mathbb{E}\Big[\int_{t_1}^{t_2}\|A^{\beta/2}\{e^{-(t_2-s)A}\mathbf{\Psi}(s)K_H(t_2,s)\\
&\quad+\int_s^{t_2}[e^{-(t_2-r)A}\mathbf{\Psi}(r)-e^{-(t_2-s)A}\mathbf{\Psi}(s)]\frac{\partial K_H}{\partial r}(r,s)dr\}\|^2_{\mathcal{L}_2^0}ds\Big]\\
&\le C\Big\{\mathbb{E}\left[\int_{t_1}^{t_2}\|A^{\beta/2}e^{-(t_2-s)A}\mathbf{\Psi}(s)K_H(t_2,s)\|^2_{\mathcal{L}_2^0}ds\right]\\
&\quad+\mathbb{E}\left[\int_{t_1}^{t_2}\left(\int_s^{t_2}\|A^{\beta/2}e^{-(t_2-r)A}\mathbf{\Psi}(r)\|^2_{\mathcal{L}_2^0}dr\right)\left(\int_s^{t_2}\|\frac{\partial K_H}{\partial r}(r,s)\|^2dr\right)ds\right]\\
&\quad+\mathbb{E}\left[\int_{t_1}^{t_2}\left(\int_s^{t_2}\|A^{\beta/2}e^{-(t_2-s)A}\mathbf{\Psi}(s)\|^2_{\mathcal{L}_2^0}dr\right)\left(\int_s^{t_2}\|\frac{\partial K_H}{\partial r}(r,s)\|^2dr\right)ds\right]\Big\}\\
&:= C\{I_{c21}+I_{c22}+I_{c23}\}.
\end{aligned}\tag{3.24}
$$

For the term I_{c21}, we have

$$
\begin{aligned}
I_{c21} &\le \int_{t_1}^{t_2}\|e^{-(t_2-s)A}K_H(t_2,s)\|^2\|A^{\beta/2}\mathbf{\Psi}(s)\|^2_{\mathcal{L}_2^0}ds\\
&\le C_H^2\int_{t_1}^{t_2}(t_2-s)^{2H-1}s^{2H-1}ds\\
&\le \frac{C_H^2}{4H-1}(t_2-t_1)^{2H-1/2}(t_2^{4H-1}-t_1^{4H-1})^{1/2}\\
&\le C(H,T)(t_2-t_1)^{4H-1},
\end{aligned}\tag{3.25}
$$

where we use the inequality $t_2^\alpha - t_1^\alpha \le (t_2-t_1)^\alpha$ for $0<\alpha<1$ and $t_2>t_1$.

Similarly, we have

$$
\begin{aligned}
I_{c22} &= \int_{t_1}^{t_2}\left(\int_s^{t_2}\|e^{-(t_2-r)A}A^{\beta/2}\mathbf{\Psi}(r)\|^2_{\mathcal{L}_2^0}dr\right)\left(\int_s^{t_2}\|\frac{\partial K_H}{\partial r}(r,s)\|^2dr\right)ds\\
&\le C_H^2(H-\frac{1}{2})^2\int_{t_1}^{t_2}(t_2-s)(\int_s^{t_2}(\frac{s}{r})^{1-2H}(r-s)^{2H-3}dr)ds\\
&\le C(H)\int_{t_1}^{t_2}(t_2-s)^{2H-1}ds\\
&\le C(H)(t_2-t_1)^{2H},
\end{aligned}\tag{3.26}
$$

and

$$
\begin{aligned}
I_{c23} &= \int_{t_1}^{t_2}\left(\int_s^{t_2}\|A^{\beta/2}e^{-(t_2-s)A}\mathbf{\Psi}(s)\|^2_{\mathcal{L}_2^0}dr\right)\left(\int_s^{t_2}\|\frac{\partial K_H}{\partial r}(r,s)\|^2dr\right)ds\\
&\le C_H^2(H-\frac{1}{2})^2\int_{t_1}^{t_2}(t_2-s)\left(\int_s^{t_2}(\frac{s}{r})^{1-2H}(r-s)^{2H-3}dr\right)ds\\
&\le C(H)(t_2-t_1)^{2H}.
\end{aligned}\tag{3.27}
$$

Now, we consider $H\in(1/2,1)$ in (3.19). By Lagrange's mean value theorem, $\exists\ \zeta\in(s,t)\subset[0,T]$ such that $t^{4H-1}-s^{4H-1}=(4H-1)\zeta^{4H-2}(t-s)\le(4H-1)t^{4H-2}(t-s)$. Then, we have

$$
\mathbb{E}[\sup_{0\le t_1\le T}\|I_{c1}\|^2_\beta]
$$

$$
\begin{aligned}
&\leq C\mathbb{E}\left[\int_0^{t_1}\|A^{\beta/2}K_{t_1}^*(e^{-(t_2-s)A}-e^{-(t_1-s)A})\mathbf{\Psi}(s)\|_{\mathcal{L}_2^0}^2 ds\right]\\
&= C\mathbb{E}\left[\int_0^{t_1}\left\|\int_s^{t_1}[A^{\beta/2}(e^{-(t_2-r)A}-e^{-(t_1-r)A})\mathbf{\Psi}(r)]\frac{\partial K_H}{\partial r}(r,s)dr\right\|_{\mathcal{L}_2^0}^2 ds\right]\\
&\leq C\mathbb{E}[\int_0^{t_1}\left(\int_s^{t_1}\|A^{-(1/2-\beta/4)}(e^{-(t_2-r)A}-e^{-(t_1-r)A})A^{(1/2+\beta/4)}\mathbf{\Psi}(r)\|_{\mathcal{L}_2^0}^2 dr\right)\\
&\quad\times\left(\int_s^{t_1}\|\frac{\partial K_H}{\partial r}(r,s)\|^2 dr\right)ds]\\
&\leq C_H^2(H-\frac{1}{2})^2(t_2-t_1)^{1-\beta/2}\int_0^{t_1}(t_1-s)\left(\int_s^{t_1}(\frac{s}{r})^{1-2H}(r-s)^{2H-3}dr\right)ds\\
&\leq C(H)(t_2-t_1)^{1-\beta/2}\int_0^{t_1}(t_1-s)^{2H-3/2}(t_1^{4H-1}-s^{4H-1})^{1/2}s^{1-2H}ds\\
&\leq C(H)t_1^{2H-1}(t_2-t_1)^{1-\beta/2}\int_0^{t_1}(t_1-s)^{2H-1}s^{1-2H}ds\\
&\leq C(H,T)(t_2-t_1)^{1-\beta/2}.
\end{aligned}
\tag{3.28}
$$

For I_{c2}, owing to the fact $s^{1-2H}\leq t_1^{1-2H}$ with $s\geq t_1$ and $H>1/2$, we can then obtain

$$
\begin{aligned}
&\mathbb{E}[\sup_{0\leq t_2\leq T}\|I_{c2}\|_\beta^2]\\
&\leq C\mathbb{E}\left[\int_{t_1}^{t_2}\|A^{\beta/2}K_{t_2}^*(e^{-(t_2-s)A}\mathbf{\Psi}(s))\|_{\mathcal{L}_2^0}^2 ds\right]\\
&= C\mathbb{E}\left[\int_{t_1}^{t_2}\|\int_s^{t_2}A^{\beta/2}(e^{-(t_2-r)A}\mathbf{\Psi}(r))\frac{\partial K_H}{\partial r}(r,s)dr\|_{\mathcal{L}_2^0}^2 ds\right]\\
&\leq C\mathbb{E}\left[\int_{t_1}^{t_2}\left(\int_s^{t_2}\|A^{\beta/2}e^{-(t_2-r)A}\mathbf{\Psi}(r)\|_{\mathcal{L}_2^0}^2 dr\right)\left(\int_s^{t_2}\|\frac{\partial K_H}{\partial r}(r,s)\|^2 dr\right)ds\right]\\
&\leq C_H^2(H-\frac{1}{2})^2\int_{t_1}^{t_2}(t_2-s)^{1-\beta}\left(\int_s^{t_2}(\frac{s}{r})^{1-2H}(r-s)^{2H-3}dr\right)ds\\
&\leq C(H)t_1^{1-2H}\int_{t_1}^{t_2}(t_2-s)^{2H-3/2-\beta}(t_2^{4H-1}-s^{4H-1})^{1/2}ds\\
&\leq C(H)(\frac{t_2}{t_1})^{2H-1}\int_{t_1}^{t_2}(t_2-s)^{2H-1-\beta}ds\\
&\leq C(H,T)(t_2-t_1)^{2H-\beta}.
\end{aligned}
\tag{3.29}
$$

Combining with (3.13)–(3.29), the proofs of Theorem 3.2 are finished.

4. The implicit Euler scheme

We let $t_n = nk,\ k = T/N$, for $n = 0, 1, \cdots, N,\ N \in \mathbb{N}_+$. Furthermore, we let $\boldsymbol{u}^0 := \boldsymbol{u}_0$. We define an implicit Euler scheme by seeking a random variable $\boldsymbol{u}^n$ in V such that $\mathbb{P}$-a.s.

$$
(\boldsymbol{u}^n-\boldsymbol{u}^{n-1},\boldsymbol{v})+\nu k(\nabla\boldsymbol{u}^n,\nabla\boldsymbol{v})+k([\boldsymbol{u}^n\cdot\nabla]\boldsymbol{u}^n,\boldsymbol{v})=(\mathbf{\Psi}^n\Delta B_n^H,\boldsymbol{v}),\tag{4.1}
$$

for any $\boldsymbol{v} \in V$, where $\Delta B_n^H = B^H(t_n) - B^H(t_{n-1})$ denotes the fractional Brownian increments.

Lemma 4.1. We assume that (S_1)–(S_2) hold. For $\boldsymbol{u}_0 \in V$ and $0 \le s < t \le T$, we have

$$\max_{1\le n\le N} \mathbb{E}[\|\boldsymbol{u}^n\|^2] + \mathbb{E}\left[\sum_{n=1}^{N} \|\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\|^2\right] + \nu\mathbb{E}\left[k\sum_{n=1}^{N} \|\nabla \boldsymbol{u}^n\|^2\right] \le C(1 + \mathbb{E}\|\boldsymbol{u}_0\|^2). \tag{4.2}$$

Proof. Setting $\boldsymbol{v} = \boldsymbol{u}^n$ in (4.1), we have

$$(\boldsymbol{u}^n - \boldsymbol{u}^{n-1}, \boldsymbol{u}^n) + \nu k(\nabla \boldsymbol{u}^n, \nabla \boldsymbol{u}^n) + k([\boldsymbol{u}^n \cdot \nabla]\boldsymbol{u}^n, \boldsymbol{u}^n) = (\boldsymbol{\Psi}^n \Delta B_n^H, \boldsymbol{u}^n). \tag{4.3}$$

By $(a-b)a = \frac{1}{2}[a^2 - b^2 + (a-b)^2]$ and $([\boldsymbol{u}^n \cdot \nabla]\boldsymbol{u}^n, \boldsymbol{u}^n) = 0$, we can reformulate (4.3) as

$$\frac{1}{2}(\|\boldsymbol{u}^n\|^2 - \|\boldsymbol{u}^{n-1}\|^2 + \|\boldsymbol{u}^n - \boldsymbol{u}^{n-1}\|^2) + \nu k\|\nabla \boldsymbol{u}^n\|^2 = (\boldsymbol{\Psi}^n \Delta B_n^H, \boldsymbol{u}^n). \tag{4.4}$$

Using (2.9) and Young's inequality, we have

$$\begin{aligned} \mathbb{E}[\max_{1\le n\le N}(\boldsymbol{\Psi}^n \Delta B_n^H, \boldsymbol{u}^n)] &= \mathbb{E}\left[\max_{1\le n\le N}\left(\int_{t_{n-1}}^{t_n} \boldsymbol{\Psi}^n dB^H(s), \boldsymbol{u}^n\right)\right] \\ &\le \mu\mathbb{E}\left[\int_{t_{n-1}}^{t_n} \|K_{t_n}^* \boldsymbol{\Psi}^n\|_{\mathcal{L}_2^0}^2 ds\right] + C_\mu \mathbb{E}[\|\boldsymbol{u}^n\|^2]. \end{aligned} \tag{4.5}$$

For the case of $1/4 < H < 1/2$, we have

$$\begin{aligned} \mathbb{E}\left[\int_{t_{n-1}}^{t_n} \|K_{t_n}^* \boldsymbol{\Psi}^n\|_{\mathcal{L}_2^0}^2 ds\right] &\approx \mathbb{E}\left[\int_{t_{n-1}}^{t_n} \|\boldsymbol{\Psi}(t_n) K_H(t_n, s)\|_{\mathcal{L}_2^0}^2 ds\right] \\ &\le C_H^2 \left(\int_{t_{n-1}}^{t_n} (t_n - s)^{2H-1} s^{2H-1} ds\right) \\ &\le C_H^2 \left(\int_{t_{n-1}}^{t_n} (t_n - s)^{4H-2} ds\right)^{1/2} \left(\int_{t_{n-1}}^{t_n} s^{4H-2} ds\right)^{1/2} \\ &\le C(H)(t_n - t_{n-1})^{2H-1/2}(t_n^{4H-1} - t_{n-1}^{4H-1})^{1/2} \\ &\le C(H)k^{4H-1}. \end{aligned} \tag{4.6}$$

Similarly, for $H \in (1/2, 1)$, one can derive that

$$\begin{aligned} \mathbb{E}\left[\int_{t_{n-1}}^{t_n} \left\|K_{t_n}^* \boldsymbol{\Psi}^n\right\|_{\mathcal{L}_2^0}^2 ds\right] &\approx \mathbb{E}\left[\int_{t_{n-1}}^{t_n} \|\int_s^{t_n} \boldsymbol{\Psi}(t_n) \frac{\partial K_H}{\partial r}(r, s) dr\|_{\mathcal{L}_2^0}^2 ds\right] \\ &\le \mathbb{E}\left[\int_{t_{n-1}}^{t_n} \left(\int_s^{t_n} \|\boldsymbol{\Psi}(t_n)\|_{\mathcal{L}_2^0}^2 dr\right)\left(\int_s^{t_n} \|\frac{\partial K_H}{\partial r}(r, s)\|^2 dr\right) ds\right] \\ &\le C_H^2 (H - \frac{1}{2})^2 \int_{t_{n-1}}^{t_n} (t_n - s)\left(\int_s^{t_n} (\frac{s}{r})^{1-2H}(r-s)^{2H-3} dr\right) ds \\ &\le C(H) t_n^{2H-1} \int_{t_{n-1}}^{t_n} (t_n - s)^{2H-1} s^{1-2H} ds \\ &\le C(H)(\frac{n}{n-1})^{2H-1} \int_{t_{n-1}}^{t_n} (t_n - s)^{2H-1} ds \end{aligned}$$

$$\leq C(H)k^{2H}. \tag{4.7}$$

Summing up (4.4) from $l = 1$ to n, taking expectations and applying (4.5)–(4.7) gives

$$\mathbb{E}[\|\boldsymbol{u}^n\|^2] + \mathbb{E}[\sum_{l=1}^{n} \|\boldsymbol{u}^l - \boldsymbol{u}^{l-1}\|^2] + \nu\mathbb{E}[k\sum_{l=1}^{n} \|\nabla \boldsymbol{u}^l\|^2]$$
$$\leq C(H,T) + \mathbb{E}[\|\boldsymbol{u}^0\|^2] + C_\mu \mathbb{E}[\sum_{l=1}^{n} \|\boldsymbol{u}^l\|^2]. \tag{4.8}$$

Taking the maximum of (4.8) over $1 \leq n \leq N$, and using Gronwall's inequality, we finish the proof.

Theorem 4.1. We assume that (S_1)–(S_2) hold. We let $\boldsymbol{u}(t_n)$ and $\boldsymbol{u}^n$ be the solutions to (2.12) and (4.1), respectively. For the Hurst index $H \in (1/4, 1/2)$, it holds that

$$\mathbb{E}_{\Omega_k}\left[\max_{1\leq n\leq N} \|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|^2 + \frac{\nu k}{2}\sum_{n=1}^{N} \|\nabla(\boldsymbol{u}(t_n) - \boldsymbol{u}^n)\|^2\right]$$
$$\leq C(k^{\min\{1/2,4H-1\}+1-\varepsilon} + k^{4H-1+2\delta-\varepsilon} + k^{2H+2\delta-\varepsilon}),$$

and for $H \in (1/2, 1)$, we have

$$\mathbb{E}_{\Omega_k}\left[\max_{1\leq n\leq N} \|\boldsymbol{u}(t_n) - \boldsymbol{u}^n\|^2 + \frac{\nu k}{2}\sum_{n=1}^{N} \|\nabla(\boldsymbol{u}(t_n) - \boldsymbol{u}^n)\|^2\right]$$
$$\leq C(k^{\min\{1/2,2H-1\}+1-\varepsilon} + k^{2H+2\delta-\varepsilon}),$$

where $\Omega_k \subset \Omega$ such that $\mathbb{P}[\Omega_k] \to 1$ as $k \to 0$, $\varepsilon > 0$ is arbitrarily small.

Proof. We define the error $\boldsymbol{e}^n = \boldsymbol{u}(t_n) - \boldsymbol{u}^n$. From Eqs (2.12) and (4.1), we obtain

$$(\boldsymbol{e}^n - \boldsymbol{e}^{n-1}, \boldsymbol{v}) + \nu\int_{t_{n-1}}^{t_n} (\nabla(\boldsymbol{u}(s) - \boldsymbol{u}^n), \nabla \boldsymbol{v})ds + \int_{t_{n-1}}^{t_n} ([\boldsymbol{u}(s)\cdot\nabla]\boldsymbol{u}(s), \boldsymbol{v})ds$$
$$- \int_{t_{n-1}}^{t_n} ([\boldsymbol{u}^n\cdot\nabla]\boldsymbol{u}^n, \boldsymbol{v})ds = \left(\int_{t_{n-1}}^{t_n} \boldsymbol{\Psi}(s)dB^H(s) - \boldsymbol{\Psi}^n\Delta B_n^H, \boldsymbol{v}\right),\ \forall\, \boldsymbol{v} \in V. \tag{4.9}$$

Setting $\boldsymbol{v} = \boldsymbol{e}^n$ in (4.9), and using $(a-b)a = \frac{1}{2}(a^2 - b^2) + \frac{1}{2}(a-b)^2$, it is easy to get that

$$\mathbb{E}[(\boldsymbol{e}^n - \boldsymbol{e}^{n-1}, \boldsymbol{e}^n)] = \frac{1}{2}(\mathbb{E}[\|\boldsymbol{e}^n\|^2] - \mathbb{E}[\|\boldsymbol{e}^{n-1}\|^2]) + \frac{1}{2}\mathbb{E}[\|\boldsymbol{e}^n - \boldsymbol{e}^{n-1}\|^2]. \tag{4.10}$$

By means of Young's inequality, we have

$$-\nu\mathbb{E}\left[\int_{t_{n-1}}^{t_n} (\nabla(\boldsymbol{u}(s) - \boldsymbol{u}^n), \nabla \boldsymbol{e}^n)ds\right] = -\nu\mathbb{E}\left[\int_{t_{n-1}}^{t_n} \{\|\nabla \boldsymbol{e}^n\|^2 + (\nabla(\boldsymbol{u}(s) - \boldsymbol{u}(t_n)), \nabla \boldsymbol{e}^n)\}ds\right]$$
$$\leq -\frac{\nu k}{2}\mathbb{E}[\|\nabla \boldsymbol{e}^n\|^2] + \frac{\nu}{2}\mathbb{E}[\int_{t_{n-1}}^{t_n} \|\nabla(\boldsymbol{u}(s) - \boldsymbol{u}(t_n))\|^2 ds. \tag{4.11}$$

For the case $1/4 < H < 1/2$, we have

$$\mathbb{E}\left[\int_{t_{n-1}}^{t_n} \|\nabla(\boldsymbol{u}(s) - \boldsymbol{u}(t_n))\|^2 ds\right] \leq \int_{t_{n-1}}^{t_n} C(t_n - s)^{\min\{1/2,4H-1,2H\}} ds$$

$$\leq Ck^{\min\{1/2,4H-1\}+1}, \tag{4.12}$$

and for $1/2 < H < 1$, we have

$$\mathbb{E}\left[\int_{t_{n-1}}^{t_n} \|\nabla(\boldsymbol{u}(s) - \boldsymbol{u}(t_n))\|^2 ds\right] \leq \int_{t_{n-1}}^{t_n} C(t_n - s)^{\min\{1/2,2H-1\}} ds$$
$$\leq Ck^{\min\{1/2,2H-1\}+1}. \tag{4.13}$$

For the convection term in (4.9), we get

$$\begin{aligned}
&\int_{t_{n-1}}^{t_n} ([\boldsymbol{u}^n \cdot \nabla]\boldsymbol{u}^n - [\boldsymbol{u}(s) \cdot \nabla]\boldsymbol{u}(s), \boldsymbol{e}^n) ds \\
&= \int_{t_{n-1}}^{t_n} ([\boldsymbol{e}^n \cdot \nabla]\boldsymbol{u}^n, \boldsymbol{e}^n) ds + \int_{t_{n-1}}^{t_n} ([(\boldsymbol{u}(t_n) - \boldsymbol{u}(s)) \cdot \nabla]\boldsymbol{u}(t_n), \boldsymbol{e}^n) ds \\
&+ \int_{t_{n-1}}^{t_n} ([\boldsymbol{u}(s) \cdot \nabla](\boldsymbol{u}(t_n) - \boldsymbol{u}(s)), \boldsymbol{e}^n) ds \\
&=: L_1 + L_2 + L_3.
\end{aligned} \tag{4.14}$$

Using (2.14) and Ladyzhenskaya's inequality, we have

$$\begin{aligned}
\mathbb{E}[L_1] &= \mathbb{E}\left[\int_{t_{n-1}}^{t_n} ([\boldsymbol{e}^n \cdot \nabla]\boldsymbol{u}^n, \boldsymbol{e}^n) ds\right] \\
&\leq \mathbb{E}\left[\int_{t_{n-1}}^{t_n} \|\boldsymbol{e}^n\|_{\mathbb{L}^4}^2 \|\nabla \boldsymbol{u}^n\|_{\mathbb{L}^2} ds\right] \\
&\leq C_\nu k \mathbb{E}[\|\nabla \boldsymbol{u}^n\|^2 \|\boldsymbol{e}^n\|^2] + \frac{\nu k}{8} \mathbb{E}[\|\nabla \boldsymbol{e}^n\|^2].
\end{aligned} \tag{4.15}$$

For $1/4 < H < 1/2$, by (2.14) and Young's inequality, we have

$$\begin{aligned}
\mathbb{E}[L_2] &= \mathbb{E}\left[\int_{t_{n-1}}^{t_n} ([(\boldsymbol{u}(t_n) - \boldsymbol{u}(s)) \cdot \nabla]\boldsymbol{u}(t_n), \boldsymbol{e}^n) ds\right] \\
&\leq \mathbb{E}\left[\int_{t_{n-1}}^{t_n} C\|\nabla(\boldsymbol{u}(t_n) - \boldsymbol{u}(s))\|_{\mathbb{L}^2} \|\nabla \boldsymbol{u}(t_n)\|_{\mathbb{L}^2} \|\nabla \boldsymbol{e}^n\|_{\mathbb{L}^2}^2 ds\right] \\
&\leq \mathbb{E}\left[\int_{t_{n-1}}^{t_n} (C_\nu \|\nabla(\boldsymbol{u}(t_n) - \boldsymbol{u}(s))\|^2 \|\nabla \boldsymbol{u}(t_n)\|^2 + \frac{\nu}{8} \|\nabla \boldsymbol{e}^n\|^2) ds\right] \\
&\leq C_\nu^\dagger k^{\min\{1/2,4H-1\}+1} + \frac{\nu k}{8} \mathbb{E}[\|\nabla \boldsymbol{e}^n\|^2],
\end{aligned} \tag{4.16}$$

and for $1/2 < H < 1$, it holds that

$$\begin{aligned}
\mathbb{E}[L_2] &= \mathbb{E}\left[\int_{t_{n-1}}^{t_n} ([(\boldsymbol{u}(t_n) - \boldsymbol{u}(s)) \cdot \nabla]\boldsymbol{u}(t_n), \boldsymbol{e}^n)\, ds\right] \\
&\leq C_\nu^\dagger k^{\min\{1/2,2H-1\}+1} + \frac{\nu k}{8} \mathbb{E}[\|\nabla \boldsymbol{e}^n\|^2].
\end{aligned} \tag{4.17}$$

For the term L_3 and if $1/4 < H < 1/2$, we have

$$\begin{aligned}\mathbb{E}[L_3] &= \mathbb{E}\left[\int_{t_{n-1}}^{t_n} ([\boldsymbol{u}(s)\cdot\nabla](\boldsymbol{u}(t_n)-\boldsymbol{u}(s)), \boldsymbol{e}^n)ds\right]\\
&\le \mathbb{E}\left[\int_{t_{n-1}}^{t_n} C\|\nabla\boldsymbol{u}(s)\|\|\|\nabla(\boldsymbol{u}(t_n)-\boldsymbol{u}(s))\|\|\|\nabla\boldsymbol{e}^n\|^2 ds\right]\\
&\le \mathbb{E}\left[\int_{t_{n-1}}^{t_n} (C_\nu\|\nabla\boldsymbol{u}(s)\|^2\|\nabla(\boldsymbol{u}(t_n)-\boldsymbol{u}(s))\|^2 + \frac{\nu}{8}\|\nabla\boldsymbol{e}^n\|^2)ds\right]\\
&\le C_\nu^\dagger k^{\min\{1/2,4H-1\}+1} + \frac{\nu k}{8}\mathbb{E}[\|\nabla\boldsymbol{e}^n\|^2],\end{aligned} \tag{4.18}$$

and for $1/2 < H < 1$, we get

$$\begin{aligned}\mathbb{E}[L_3] &= \mathbb{E}\left[\int_{t_{n-1}}^{t_n} ([\boldsymbol{u}(s)\cdot\nabla](\boldsymbol{u}(t_n)-\boldsymbol{u}(s)), \boldsymbol{e}^n)ds\right]\\
&\le C_\nu^\dagger k^{\min\{1/2,2H-1\}+1} + \frac{\nu k}{8}\mathbb{E}[\|\nabla\boldsymbol{e}^n\|^2].\end{aligned} \tag{4.19}$$

To bound the stochastic integral, using (2.9) and Young's inequality, we have

$$\begin{aligned}&\mathbb{E}\left[\max_{1\le n\le N}\left(\int_{t_{n-1}}^{t_n} (\boldsymbol{\Psi}(s)-\boldsymbol{\Psi}^n)dB^H(s), \boldsymbol{e}^n\right)\right]\\
&\approx \mathbb{E}\left[\max_{1\le n\le N}(\int_{t_{n-1}}^{t_n} (\boldsymbol{\Psi}(s)-\boldsymbol{\Psi}(t_n))dB^H(s), \boldsymbol{e}^n)\right]\\
&\le \mathbb{E}\left[C_\mu \max_{1\le n\le N}\left\|\int_{t_{n-1}}^{t_n} (\boldsymbol{\Psi}(s)-\boldsymbol{\Psi}(t_n))dB^H(s)\right\|^2 + \mu\|\boldsymbol{e}^n\|^2\right]\\
&\le C\mathbb{E}\left[\int_{t_{n-1}}^{t_n} \|K_{t_n}^*(\boldsymbol{\Psi}(s)-\boldsymbol{\Psi}(t_n))\|^2_{\mathcal{L}_2^0}ds\right] + \mu\mathbb{E}[\|\boldsymbol{e}^n\|^2].\end{aligned} \tag{4.20}$$

For $1/4 < H < 1/2$, by (S_2), we have

$$\begin{aligned}&\mathbb{E}\left[\int_{t_{n-1}}^{t_n} \|K_{t_n}^*(\boldsymbol{\Psi}(s)-\boldsymbol{\Psi}(t_n))\|^2_{\mathcal{L}_2^0}ds\right]\\
&= \mathbb{E}\left[\int_{t_{n-1}}^{t_n} \|(\boldsymbol{\Psi}(s)-\boldsymbol{\Psi}(t_n))K_H(t_n,s) + \int_s^{t_n}(\boldsymbol{\Psi}(r)-\boldsymbol{\Psi}(s))\frac{\partial K_H}{\partial r}(r,s)dr\|^2_{\mathcal{L}_2^0}ds\right]\\
&\le 2\mathbb{E}\left[\int_{t_{n-1}}^{t_n} \|(\boldsymbol{\Psi}(s)-\boldsymbol{\Psi}(t_n))K_H(t_n,s)\|^2_{\mathcal{L}_2^0}ds\right]\\
&\quad + 2\mathbb{E}\left[\int_{t_{n-1}}^{t_n} \|\int_s^{t_n}(\boldsymbol{\Psi}(r)-\boldsymbol{\Psi}(s))\frac{\partial K_H}{\partial r}(r,s)dr\|^2_{\mathcal{L}_2^0}ds\right]\\
&\le C_H^2\left(\int_{t_{n-1}}^{t_n}(t_n-s)^{2H+2\delta-1}s^{2H-1}ds\right)\\
&\quad + 2\mathbb{E}[\int_{t_{n-1}}^{t_n}\left(\int_s^{t_n}\|\boldsymbol{\Psi}(r)-\boldsymbol{\Psi}(s)\|^2_{\mathcal{L}_2^0}dr\right)\left(\int_s^{t_n}\|\frac{\partial K_H}{\partial r}(r,s)\|^2dr\right)ds\end{aligned}$$

$$
\begin{aligned}
&\leq C_H^2 \left(\int_{t_{n-1}}^{t_n} (t_n - s)^{4H+4\delta-2} ds \right)^{1/2} \left(\int_{t_{n-1}}^{t_n} s^{4H-2} ds \right)^{1/2} \\
&\quad + C_H^2 (H - \frac{1}{2})^2 \int_{t_{n-1}}^{t_n} (t_n - s)^{2\delta+1} \left(\int_s^{t_n} (\frac{s}{r})^{1-2H} (r-s)^{2H-3} dr \right) ds \\
&\leq C(H)(t_n - t_{n-1})^{2H+2\delta-1/2} \left(t_n^{4H-1} - t_{n-1}^{4H-1} \right)^{1/2} + C(H)(t_n - t_{n-1})^{2H+2\delta} \\
&\leq C(H)(k^{4H+2\delta-1} + k^{2H+2\delta}). \qquad (4.21)
\end{aligned}
$$

For the case $1/2 < H < 1$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_{t_{n-1}}^{t_n} \| K_{t_n}^* (\boldsymbol{\Psi}(s) - \boldsymbol{\Psi}(t_n)) \|_{\mathcal{L}_2^0}^2 ds \right] \\
&= \mathbb{E}\left[\int_{t_{n-1}}^{t_n} \left\| \int_s^{t_n} (\boldsymbol{\Psi}(s) - \boldsymbol{\Psi}(t_n)) \frac{\partial K_H}{\partial r}(r,s) dr \right\|_{\mathcal{L}_2^0}^2 ds \right] \\
&\leq \mathbb{E}\left[\int_{t_{n-1}}^{t_n} \left(\int_s^{t_n} \| \boldsymbol{\Psi}(t_n) - \boldsymbol{\Psi}(s) \|_{\mathcal{L}_2^0}^2 dr \right) \left(\int_s^{t_n} \| \frac{\partial K_H}{\partial r}(r,s) \|^2 dr \right) ds \right] \\
&\leq C_H^2 (H - \frac{1}{2})^2 \int_{t_{n-1}}^{t_n} (t_n - s)^{2\delta+1} \left(\int_s^{t_n} (\frac{s}{r})^{1-2H} (r-s)^{2H-3} dr \right) ds \\
&\leq C(H) t_n^{2H-1} \int_{t_{n-1}}^{t_n} (t_n - s)^{2H+2\delta-1} s^{1-2H} ds \\
&\leq C(H) (\frac{n}{n-1})^{2H-1} \int_{t_{n-1}}^{t_n} (t_n - s)^{2H+2\delta-1} ds \\
&\leq C(H) k^{2H+2\delta}. \qquad (4.22)
\end{aligned}
$$

By summing up (4.9) with $\boldsymbol{v} = \boldsymbol{e}^n$ from $n = 1$ to N, as well as considering the estimates (4.10)–(4.22), for $1/4 < H < 1/2$, we have

$$
\begin{aligned}
&\mathbb{E}\left[\max_{1\leq n\leq N} \{ \|\boldsymbol{e}^n\|^2 + 2\sum_{n=1}^N \|\boldsymbol{e}^n - \boldsymbol{e}^{n-1}\|^2 + \frac{\nu k}{2} \sum_{n=1}^N \|\nabla \boldsymbol{e}^n\|^2 \} \right] \\
&\quad \leq Ck\mathbb{E}\left[\sum_{n=1}^N (\|\nabla \boldsymbol{u}^n\|^2 + 1) \|\boldsymbol{e}^n\|^2 \right] + C(k^{\min\{1/2, 4H-1\}+1} + k^{4H-1+2\delta} + k^{2H+2\delta}),
\end{aligned}
$$

and for $1/2 < H < 1$, we get

$$
\begin{aligned}
&\mathbb{E}\left[\max_{1\leq n\leq N} \{ \|\boldsymbol{e}^n\|^2 + 2\sum_{n=1}^N \|\boldsymbol{e}^n - \boldsymbol{e}^{n-1}\|^2 + \frac{\nu k}{2} \sum_{n=1}^N \|\nabla \boldsymbol{e}^n\|^2 \} \right] \\
&\quad \leq Ck\mathbb{E}\left[\sum_{n=1}^N (\|\nabla \boldsymbol{u}^n\|^2 + 1) \|\boldsymbol{e}^n\|^2 \right] + C(k^{\min\{1/2, 2H-1\}+1} + k^{2H+2\delta}).
\end{aligned}
$$

Due to the expectations of $\{\|\nabla \boldsymbol{u}^n\|^2\}_{n=1}^N$, we may not apply the discrete Gronwall inequality. We consider a subset

$$
\Omega_k = \{\omega \in \Omega \mid \max_{1\leq n\leq N} \|\nabla \boldsymbol{u}^n\|^2 \leq k\} \subset \Omega,
$$

where $k > 0$, and application of Markov's inequality leads to

$$\mathbb{P}\left[\max_{1\le n\le N} \|\nabla \boldsymbol{u}^n\|^2\right] \le k\} \ge 1 - \frac{\mathbb{E}[\max_{1\le n\le N} \|\nabla \boldsymbol{u}^n\|^2]}{k},\ \forall\ k > 0.$$

We define the indicator function $\mathbb{I}_{\Omega_k}$ by

$$\mathbb{I}_{\Omega_k} = \begin{cases} 1,\ \omega \in \Omega_k, \\ 0,\ \omega \notin \Omega_k. \end{cases}$$

Using the discrete Gronwall inequality, we have

$$\begin{aligned}\mathbb{E}\left[\mathbb{I}_{\Omega_k} \max_{1\le n\le N}\{\|\boldsymbol{e}^n\|^2 + 2\sum_{n=1}^{N} \|\boldsymbol{e}^n - \boldsymbol{e}^{n-1}\|^2 + \frac{\nu k}{2}\sum_{n=1}^{N} \|\nabla \boldsymbol{e}^n\|^2\}\right] \\ \le Ce^{C_T k}(k^{\min\{1/2,4H-1\}+1} + k^{4H-1+2\delta} + k^{2H+2\delta}),\end{aligned}$$

and

$$\begin{aligned}\mathbb{E}\left[\mathbb{I}_{\Omega_k} \max_{1\le n\le N}\{\|\boldsymbol{e}^n\|^2 + 2\sum_{n=1}^{N} \|\boldsymbol{e}^n - \boldsymbol{e}^{n-1}\|^2 + \frac{\nu k}{2}\sum_{n=1}^{N} \|\nabla \boldsymbol{e}^n\|^2\}\right] \\ \le Ce^{C_T k}(k^{\min\{1/2,2H-1\}+1} + k^{2H+2\delta}).\end{aligned}$$

For any $\varepsilon > 0$, $C_T^{-1}\ln(k^{-\varepsilon}) > 0$, and we denote by $\mathbb{E}_{\Omega_k}[\cdot] = \mathbb{E}[\mathbb{I}_{\Omega_k}\cdot]$, such that

$$\mathbb{P}[\Omega_k] \to 1 \ \text{ as } \ k \to 0,$$

and then if $1/4 < H < 1/2$, we obtain

$$\mathbb{E}_{\Omega_k}\left[\max_{1\le n\le N} \|\boldsymbol{e}^n\|^2 + \frac{\nu k}{2}\sum_{n=1}^{N} \|\nabla \boldsymbol{e}^n\|^2\right] \le C(k^{\min\{1/2,4H-1\}+1-\varepsilon} + k^{4H-1+2\delta-\varepsilon} + k^{2H+2\delta-\varepsilon}),$$

and for $1/2 < H < 1$, it holds that

$$\mathbb{E}_{\Omega_k}\left[\max_{1\le n\le N} \|\boldsymbol{e}^n\|^2 + \frac{\nu k}{2}\sum_{n=1}^{N} \|\nabla \boldsymbol{e}^n\|^2\right] \le C(k^{\min\{1/2,2H-1\}+1-\varepsilon} + k^{2H+2\delta-\varepsilon}).$$

The proofs are finished.

5. Space-time discretization

We let $\{\mathcal{T}_h\}_{0<h<1}$ be a regular family of triangulations of D with the maximal mesh size of h. We define the finite element spaces V_h by

$$V_h = \{\boldsymbol{u}_h \in [H^1(D)]^2 : \boldsymbol{u}_h = 0 \text{ on } \partial D,\ \boldsymbol{u}_h|_{\mathcal{T}} \text{ is a linear function},\ \forall \mathcal{T} \in \mathcal{T}_h\}.$$

P_h is defined as the standard L^2-projection operator, i.e.,

$$(P_h\psi, \boldsymbol{v}_h) = (\psi, \boldsymbol{v}_h),\ \forall\ \boldsymbol{v}_h \in V_h,$$

we have (see [48])

$$\|P_h\psi - \psi\| + h\|\nabla(P_h\psi - \psi)\| \leq Ch^{r+1}\|\psi\|_{r+1},\ \forall\ \psi \in H_0^1(D) \cap H^{r+1}(D). \tag{5.1}$$

The fully discrete scheme of (1.1)–(1.3) is to seek $\boldsymbol{u}_h^n \in L^2(\Omega, V_h)$ such that

$$(\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) + \nu k(\nabla \boldsymbol{u}_h^n, \nabla \boldsymbol{v}_h) + k([\boldsymbol{u}_h^n \cdot \nabla]\boldsymbol{u}_h^n, \boldsymbol{v}_h) = (P_h \boldsymbol{\Psi}^n \Delta B_n^H, \boldsymbol{v}_h), \tag{5.2}$$

for any $\boldsymbol{v}_h \in V_h$, where ΔB_n^H denotes the fractional Brownian increments.

Theorem 5.1. We assume that (S_1)–(S_2) hold. Let $\boldsymbol{u}_h^n$ and $\boldsymbol{u}^n$ be the solutions to (5.2) and (4.1), respectively. For $H \in (1/4, 1/2)$, it holds that

$$\mathbb{E}_{\Omega_h}\left[\max_{1\leq n\leq N} \|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|^2 + \frac{\nu k}{2}\sum_{n=1}^N \|\nabla(\boldsymbol{u}^n - \boldsymbol{u}_h^n)\|^2\right] \leq C(h^{2-\epsilon} + k^{4H-1}h^{2-\epsilon}),$$

and if $H \in (1/2, 1)$, one can arrive at

$$\mathbb{E}_{\Omega_h}\left[\max_{1\leq n\leq N} \|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|^2 + \frac{\nu k}{2}\sum_{n=1}^N \|\nabla(\boldsymbol{u}^n - \boldsymbol{u}_h^n)\|^2\right] \leq C(h^{2-\epsilon} + k^{2H}h^{2-\epsilon}),$$

where $\Omega_h \subset \Omega$ such that $\mathbb{P}[\Omega_h] \to 1$ as $h \to 0$, $\epsilon > 0$ is arbitrarily small.

Proof. Setting $\mathbf{E}^n = \boldsymbol{u}^n - \boldsymbol{u}_h^n$, from (4.1) and (5.2), we get

$$\begin{aligned}&(\mathbf{E}^n - \mathbf{E}^{n-1}, \boldsymbol{v}_h) + \nu k(\nabla \mathbf{E}^n, \nabla \boldsymbol{v}_h) + k([\boldsymbol{u}^n \cdot \nabla]\boldsymbol{u}^n - [\boldsymbol{u}_h^n \cdot \nabla]\boldsymbol{u}_h^n, \boldsymbol{v}_h)\\ &\quad = (\boldsymbol{\Psi}^n \Delta B_n^H - P_h \boldsymbol{\Psi}^n \Delta B_n^H, \boldsymbol{v}_h).\end{aligned} \tag{5.3}$$

Taking $\boldsymbol{v}_h = \mathbf{E}^n$, we reformulate (5.3) as

$$\begin{aligned}&(\mathbf{E}^n - \mathbf{E}^{n-1}, \mathbf{E}^n) + \nu k\|\nabla \boldsymbol{E}^n\|^2 + k([\boldsymbol{E}^n \cdot \nabla]\boldsymbol{u}^n, \boldsymbol{E}^n) + k([\boldsymbol{u}_h^n \cdot \nabla]\boldsymbol{E}^n, \boldsymbol{E}^n)\\ &\quad = (\boldsymbol{\Psi}^n \Delta B_n^H - P_h \boldsymbol{\Psi}^n \Delta B_n^H, \boldsymbol{E}^n).\end{aligned} \tag{5.4}$$

For the convection term in (5.4), we have

$$k([\boldsymbol{E}^n \cdot \nabla]\boldsymbol{u}^n, \boldsymbol{E}^n) \leq k\|\boldsymbol{E}^n\|_{\mathbb{L}^4}^2 \|\nabla \boldsymbol{u}^n\|_{\mathbb{L}^2} \leq Ck\|\nabla \boldsymbol{u}^n\|^2\|\boldsymbol{E}^n\|^2 + \frac{\nu k}{8}\|\nabla \boldsymbol{E}^n\|^2, \tag{5.5}$$

and

$$k([\boldsymbol{u}_h^n \cdot \nabla]\boldsymbol{E}^n, \boldsymbol{E}^n) \leq k\|\boldsymbol{u}_h^n\|_{\mathbb{L}^4}\|\nabla \boldsymbol{E}^n\|_{\mathbb{L}^2}\|\boldsymbol{E}^n\|_{\mathbb{L}^4} \leq Ck\|\boldsymbol{u}_h^n\|_{\mathbb{L}^4}^4\|\boldsymbol{E}^n\|^2 + \frac{\nu k}{8}\|\nabla \boldsymbol{E}^n\|^2. \tag{5.6}$$

For the stochastic integral, if $1/4 < H < 1/2$, utilizing (5.1) and (2.9), similar to the derivation of (4.6), we then have

$$\begin{aligned}&\mathbb{E}[\max_{1\leq n\leq N}(\boldsymbol{\Psi}^n \Delta B_n^H - P_h \boldsymbol{\Psi}^n \Delta B_n^H, \boldsymbol{E}^n)]\\ &\leq C\mathbb{E}\left[\int_{t_{n-1}}^{t_n} \|K_{t_n}^*(P_h \boldsymbol{\Psi}^n - \boldsymbol{\Psi}^n)\|_{\mathcal{L}_2^0}^2 ds\right] + \mu\mathbb{E}[\|\boldsymbol{E}^n\|^2]\end{aligned}$$

$$\leq Ch^2\mathbb{E}\left[\int_{t_{n-1}}^{t_n}\|K_{t_n}^* A^{1/2}\boldsymbol{\Psi}^n\|_{\mathcal{L}_2^0}^2 ds\right]+\mu\mathbb{E}[\|\boldsymbol{E}^n\|^2]$$
$$\leq C(H)k^{4H-1}h^2+\mu\mathbb{E}[\|\boldsymbol{E}^n\|^2], \tag{5.7}$$

and if $1/2 < H < 1$, similar to the derivation of (4.7), it follows that

$$\mathbb{E}[\max_{1\leq n\leq N}(\boldsymbol{\Psi}^n\Delta B_n^H - P_h\boldsymbol{\Psi}^n\Delta B_n^H, \boldsymbol{E}^n)] \leq C(H)k^{2H}h^2+\mu\mathbb{E}[\|\boldsymbol{E}^n\|^2]. \tag{5.8}$$

Due to the result $\|\boldsymbol{E}^0\| = \|\boldsymbol{u}_0 - P_h\boldsymbol{u}_0\| \leq Ch\|\boldsymbol{u}_0\|_1$, by summing up (5.4) from $n = 1$ to N, as well as considering the estimates (5.5)–(5.8), and for $1/4 < H < 1/2$, we have

$$\mathbb{E}\left[\max_{1\leq n\leq N}\{\|\boldsymbol{E}^n\|^2+2\sum_{n=1}^{N}\|\boldsymbol{E}^n-\boldsymbol{E}^{n-1}\|^2+\frac{\nu k}{2}\sum_{n=1}^{N}\|\nabla\boldsymbol{E}^n\|^2\}\right]$$
$$\leq Ck\mathbb{E}\left[\sum_{n=1}^{N}(\|\nabla\boldsymbol{u}^n\|^2+\|\boldsymbol{u}_h^n\|_{\mathbb{L}^4}^4+1)\|\boldsymbol{E}^n\|^2\right]+C(h^2+k^{4H-1}h^2), \tag{5.9}$$

and if $1/2 < H < 1$, we have

$$\mathbb{E}\left[\max_{1\leq n\leq N}\{\|\boldsymbol{E}^n\|^2+2\sum_{n=1}^{N}\|\boldsymbol{E}^n-\boldsymbol{E}^{n-1}\|^2+\frac{\nu k}{2}\sum_{n=1}^{N}\|\nabla\boldsymbol{E}^n\|^2\}\right]$$
$$\leq Ck\mathbb{E}\left[\sum_{n=1}^{N}(\|\nabla\boldsymbol{u}^n\|^2+\|\boldsymbol{u}_h^n\|_{\mathbb{L}^4}^4+1)\|\boldsymbol{E}^n\|^2\right]+C(h^2+k^{2H}h^2). \tag{5.10}$$

We consider a subset

$$\Omega_h = \{\omega\in\Omega \mid \max_{1\leq n\leq N}\|\nabla\boldsymbol{u}^n\|^2+\max_{1\leq n\leq N}\|\boldsymbol{u}_h^n\|_{\mathbb{L}^4}^4 \leq \ln(h^{-\epsilon})\}\subset\Omega,\ \text{for any } \epsilon>0,$$

which satisfies Markov's inequality, together with

$$\mathbb{P}[\Omega_h]\to 1 \ \text{ as } \ h\to 0.$$

Denote by $\mathbb{E}_{\Omega_h}[\cdot] = \mathbb{E}[\mathbb{I}_{\Omega_h}\cdot]$, from (5.3)–(5.10), using the discrete Gronwall inequality, for $1/4 < H < 1/2$, we have

$$\mathbb{E}_{\Omega_h}\left[\max_{1\leq n\leq N}\|\boldsymbol{E}^n\|^2+\frac{\nu k}{2}\sum_{n=1}^{N}\|\nabla\boldsymbol{E}^n\|^2\right]\leq C(h^{2-\epsilon}+k^{4H-1}h^{2-\epsilon}),$$

and for $1/2 < H < 1$, then

$$\mathbb{E}_{\Omega_h}\left[\max_{1\leq n\leq N}\|\boldsymbol{E}^n\|^2+\frac{\nu k}{2}\sum_{n=1}^{N}\|\nabla\boldsymbol{E}^n\|^2\right]\leq C(h^{2-\epsilon}+k^{2H}h^{2-\epsilon}).$$

The proofs are finished.

We set $\Omega_{k,h} = \Omega_k\cap\Omega_h$. By using the triangle inequality, then the global error estimates for fully discrete method are given as follows:

Theorem 5.2. We assume that (S_1)–(S_2) hold. Let $\boldsymbol{u}(t_n)$ and $\boldsymbol{u}_h^n$ be the solutions to (2.12) and (5.2), respectively. For the Hurst index $H \in (1/4, 1/2)$, there holds

$$\begin{aligned}&\mathbb{E}_{\Omega_{k,h}}[\max_{1\le n\le N} \|\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n\|^2]\\&\le C(k^{\min\{1/2,4H-1\}+1-\varepsilon} + k^{4H-1+2\delta-\varepsilon} + k^{2H+2\delta-\varepsilon} + h^{2-\epsilon} + k^{4H-1}h^{2-\epsilon}),\end{aligned}$$

and for $H \in (1/2, 1)$, we have

$$\begin{aligned}&\mathbb{E}_{\Omega_{k,h}}[\max_{1\le n\le N} \|\boldsymbol{u}(t_n) - \boldsymbol{u}_h^n\|^2]\\&\le C(k^{\min\{1/2,2H-1\}+1-\varepsilon} + k^{2H+2\delta-\varepsilon} + h^{2-\epsilon} + k^{2H}h^{2-\epsilon}),\end{aligned}$$

where $\Omega_{k,h} \subset \Omega$ such that $\mathbb{P}[\Omega_{k,h}] \to 1$ as $k \to 0$ and $h \to 0$, both $\epsilon > 0$ and $\varepsilon > 0$ are arbitrarily small.

Remark. The absence of a nonlinear convective term $[\boldsymbol{u} \cdot \nabla]\boldsymbol{u}$ leads to $\Omega_{k,h} \equiv \Omega$, that is $\varepsilon = \epsilon = 0$ in Theorem 5.2, since the Gronwall inequality may now be utilized directly. We use Newton's iterative algorithm when calculating the nonlinear term in numerical calculations.

6. Numerical experiments

We present the approximation of fBm $B^H(t) = \int_0^t K_H(t,s)dB(s)$, where $B(t)$ is a Brownian motion (see [1]). Let $t_n = nk$ for $n = 0, 1, \ldots, N$ and $k = T/N$, then we approximate $B^H(t)$ by

$$B^H(t_n) = \sum_{i=0}^{n} \frac{B(t_{i+1}) - B(t_i)}{t_{i+1} - t_i} \int_{t_i}^{t_{i+1}} K_H(t_n, s)ds.$$

It should be pointed out that the integral $\int_{t_i}^{t_{i+1}} K_H(t_n,s)ds$ is approximated by $K_H(t_n, \frac{t_{i+1}+t_i}{2})$.

The first example provided is to show the convergence rates, and the second example is used to show the numerical simulations about the impacts of fBm on the Burgers equation.

6.1. Numerical test 1

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + \sin(t)\frac{dB^H(t)}{dt},\\ \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) + \sin(t)\frac{dB^H(t)}{dt},\\ u(x,y,0) = 1,\ v(x,y,0) &= -1,\ (x,y) \in \Omega,\\ u(x,y,t) = 0,\ v(x,y,t) &= 0,\ (x,y) \in \partial\Omega,\end{aligned}$$

where $\Omega = \{(x,y)|0 \le x, y \le 2\}$.

We take the temporal interval $[0, 1]$ and the viscosity coefficient $\nu = 0.2$. The errors $\mathbb{E}[\|\boldsymbol{e}^n\|] := (\mathbb{E}[\|\boldsymbol{u}_h^n - \boldsymbol{u}(t_n)\|^2])^{1/2}$ in the sense of the L^2-norm are computed by Monte Carlo method over 200 samples, where the "true" solution $\boldsymbol{u}(t_n)$ is approximated by a solution computed by small time step $k = \frac{1}{160}$ and space step $h = \frac{1}{200}$.

Table 1. Numerical results in the temporal direction with $h = \frac{1}{200}$.

k	$H = 0.4$		$H = 0.6$		$H = 0.9$	
	Error	Rate	Error	Rate	Error	Rate
1/10	6.3578e-01	-	6.1176e-01	-	5.9768e-01	-
1/20	5.2096e-01	0.2874	4.0651e-01	0.5897	3.5655e-01	0.7453
1/40	4.2716e-01	0.2869	2.7256e-01	0.5832	2.1224e-01	0.7468
1/80	3.4708e-01	0.2911	1.7933e-01	0.5901	1.2612e-01	0.7482

Table 2. Numerical results in the spatial direction with $k = \frac{1}{160}$.

h	$H = 0.4$		$H = 0.6$		$H = 0.9$	
	Error	Rate	Error	Rate	Error	Rate
1/5	5.6538e-01	-	4.5943e-01	-	4.2265e-01	-
1/10	3.0233e-01	0.9032	2.4211e-01	0.9243	2.1603e-01	0.9683
1/20	1.6207e-01	0.9013	1.2519e-01	0.9379	1.1202e-01	0.9579
1/40	8.4251e-02	0.9155	6.5280e-02	0.9384	5.5515e-02	0.9762

In Table 1, if the Hurst index H satisfies $1/4 < H < 1/2$, the convergence order is close to the theoretical convergence order $O(k^{\min\{2H-1/2,H\}-\varepsilon})$, and for Hurst index $1/2 < H < 1$, the orders are near to $O(k^{\min\{3/4,H\}-\varepsilon})$. Table 2 shows that the optimal order of spatial error estimation is consistent with the theoretical result of $O(h^{1-\epsilon})$.

6.2. Numerical test 2

Here, let us consider the two-dimensional stochastic Burger equations with $\mathbf{\Psi}(t) = t$ in (1.1) on the spatial domain $\Omega = [0, 2] \times [0, 2]$ and temporal interval $[0, 1]$:

$$
\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= \nu\Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) + t\frac{dB^H(t)}{dt},\\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} &= \nu\Big(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\Big) + t\frac{dB^H(t)}{dt},\\
u(x, y, 0) = \cos(\pi x) + \sin(\pi y),\ v(x, y, 0) &= x + y,\ (x, y) \in \Omega,\\
u(x, y, t) = 0,\ v(x, y, t) &= 0,\ (x, y) \in \partial\Omega.
\end{aligned}
$$

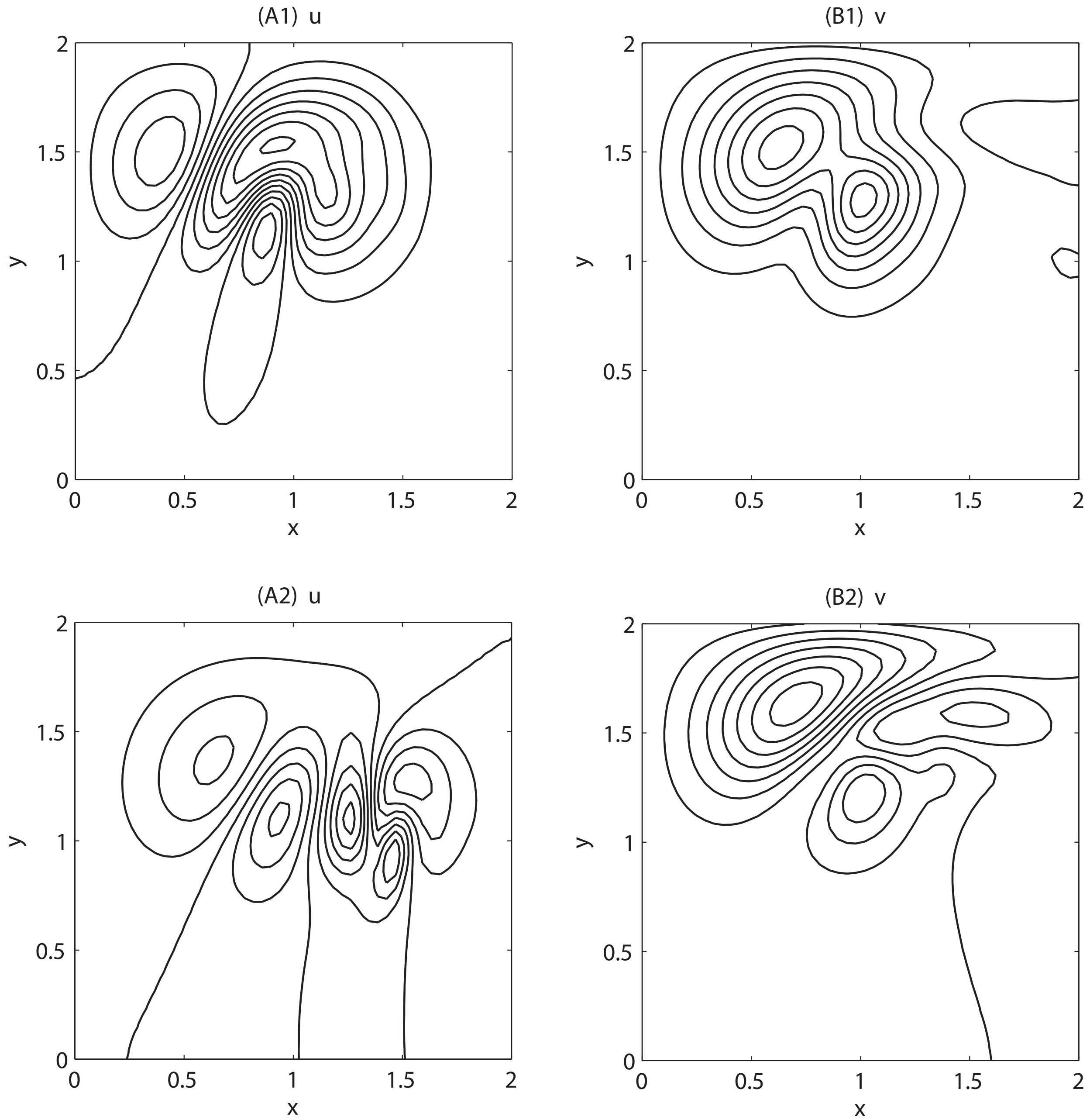

Figure 1. (A1) and (B1) show the numerical solutions u and v for the corresponding deterministic equation ($\mathbf{\Psi}(t) = 0$). Plots of (A2) and (B2) exhibit the mean values of u and v to the stochastic Burger equation with Hurst index $H = 0.3$.

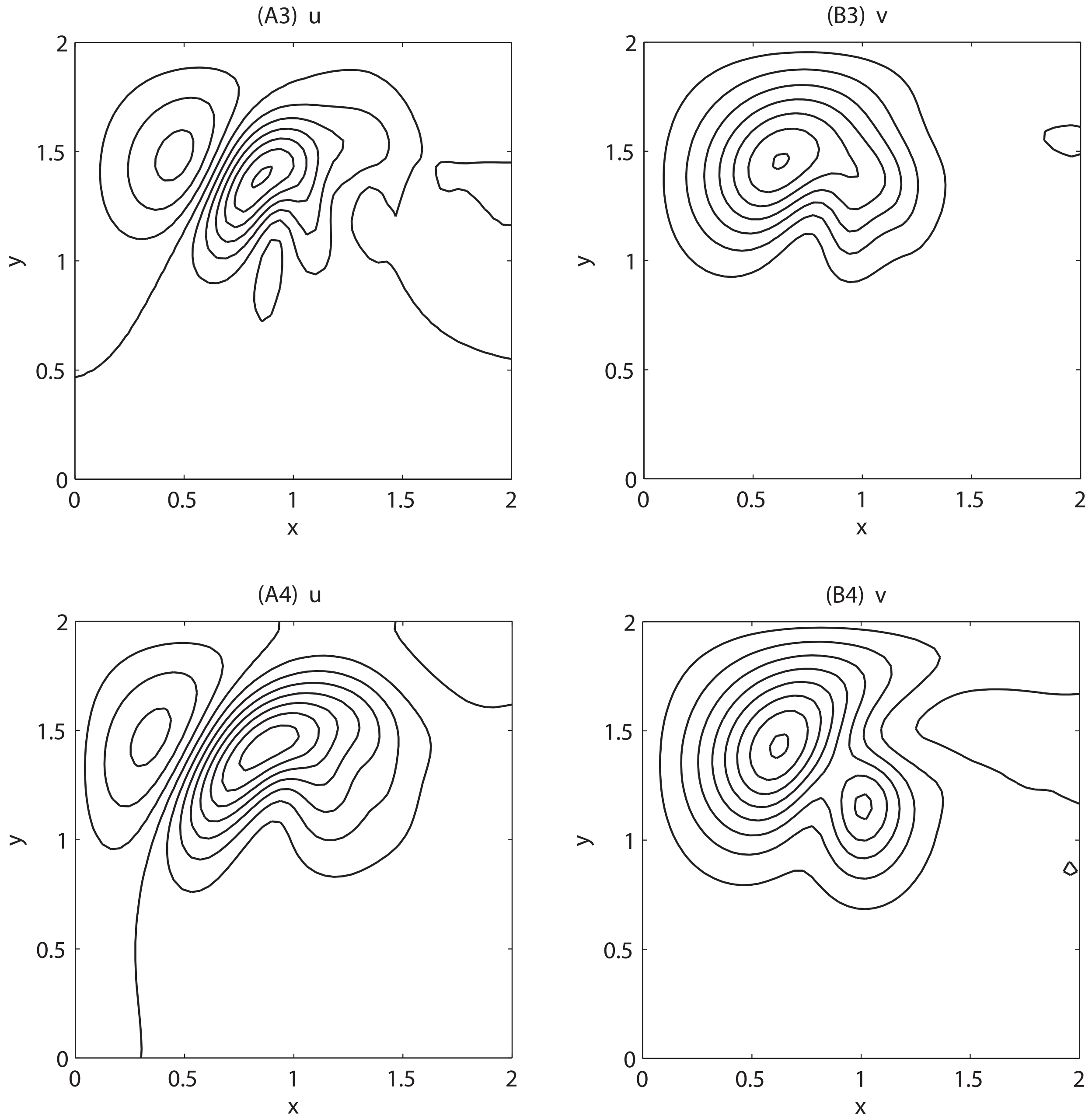

Figure 2. Plots of (A3), (B3), (A4) and (B4) show the mean values of u and v to the stochastic Burger equation with Hurst indexes $H = 0.6$ and $H = 0.8$, respectively.

We choose the viscosity coefficient $\nu = 0.1$, $h = 1/40$, and $k = 0.05$. Figure 1 (A1 and B1) shows the numerical results of the corresponding deterministic equation ($\mathbf{\Psi}(t) = 0$). Figure 1 (A2 and B2) and Figure 2 (A3 and B4) exhibit the mean values of u and v to the stochastic Burger equation with Hurst indexes $H = 0.3, 0.6, 0.8$, respectively.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgements

This work is supported by R&D Program of Beijing Municipal Education Commission (KM202310853001), Research Project on Youth Foundation of Beijing Polytechnic College (BGY2021KY-05QT), Key Research Project of Beijing Polytechnic College (BGY2023KY-47Z), Research Project of Beijing Polytechnic College (BGY2023KY-50) and Beijing Natural Science Foundation (No.1224036).

Conflicts of interest

The authors declare no conflict of interest.

References

1. Y. S. Mishura, *Stochastic Calculus for Fractional Brownian Motion and Related Processes*, Springer, 2008. https://doi.org/10.1007/978-3-540-75873-0
2. Y. Shi, X. H. Yang, Pointwise error estimate of conservative difference scheme for supergeneralized viscous Burgers' equation, *Electron. Res. Arch.*, **32** (2024), 1471–1497. https://doi.org/10.3934/era.2024068
3. J. W. Wang, X. X. Jiang, X. H. Yang, H. X. Zhang, A nonlinear compact method based on double reduction order scheme for the nonlocal fourth-order PDEs with Burgers' type nonlinearity, *J. Appl. Math. Comput.*, **70** (2024), 489–511. https://doi.org/10.1007/s12190-023-01975-4
4. J. W. Wang, H. X. Zhang, X. H. Yang, A predictor-corrector compact difference scheme for a class of nonlinear Burgers equations, *J. Hunan Univ. Technol.*, **38** (2024), 98–104. https://doi.org/10.3969/j.issn.1673-9833.2024.01.014
5. J. W. Wang, X. X. Jiang, H. X. Zhang, A BDF3 and new nonlinear fourth-order difference scheme for the generalized viscous Burgers' equation, *Appl. Math. Lett.*, (2024), 109002. https://doi.org/10.1016/j.aml.2024.109002
6. L. Bertini, N. Cancrini, G. Jona-Lasinio, The stochastic Burgers equation, *Commun. Math. Phys.*, **165** (1994), 211–232. https://link.springer.com/article/10.1007/BF02099769
7. Z. Brzeźniak, M. Capinski, F. Flandoli, Stochastic partial differential equations and turbulence, *Math. Models Methods Appl. Sci.*, **1** (1991), 41–59. https://doi.org/10.1142/S0218202591000046
8. J. U. Kim, On the stochastic Burgers equation with a polynomial nonlinearity in the real line, *Discrete Contin. Dyn. Syst., Ser. A.*, **6** (2006), 835–866. https://doi.org/10.3934/dcdsb.2006.6.835
9. P. Catuogno, C. Olivera, Strong solution of the stochastic Burgers equation, *Appl. Anal.*, **93** (2014), 646–652. https://doi.org/10.1080/00036811.2013.797074

10. K. Twardowska, J. Zabczyk, Qualitative properties of solutions to stochastic Burgers' system of equations, *Stoch. Partial Differ. Equations Appl. VII*, **245** (2006), 311–322. https://doi.org/10.1201/9781420028720.ch25

11. B. Goldys, B. Maslowski, Exponential ergodicity for stochastic Burgers and 2D Navier-Stokes equations, *J. Funct. Anal.*, **226** (2005), 230–255. https://doi.org/10.1016/j.jfa.2004.12.009

12. E. Weinan, K. M. Khanin, A. E. Mazel, Y. G. Sinai, Invariant measure for Burgers equation with stochastic forcing, *Ann. Math.*, **151** (2000), 877–960. https://doi.org/10.48550/arXiv.math/0005306

13. G. Zou, B. Wang, Stochastic Burgers' equation with fractional derivative driven by multiplicative noise, *Comput. Math. Appl.*, **74** (2017), 3195–3208. https://doi.org/10.1016/j.camwa.2017.08.023

14. G. Zhou, L. Wang, J. L. Wu, Global well-posedness of 2D stochastic Burgers equations with multiplicative noise, *Statist. Probab. Lett.*, **182** (2022), 109315. https://doi.org/10.1016/j.spl.2021.109315

15. C. J. Li, H. X. Zhang, X. H. Yang, A new α-robust nonlinear numerical algorithm for the time fractional nonlinear KdV equation, *Commun. Anal. Mech.*, **16** (2024), 147–168. https://doi.org/10.3934/cam.2024007

16. H. X. Zhang, X. X. Jiang, F. R. Wang, X. H. Yang, The time two-grid algorithm combined with difference scheme for 2D nonlocal nonlinear wave equation, *J. Appl. Math. Comput.*, (2024), 1–25. https://doi.org/10.1007/s12190-024-02000-y

17. F. Wang, X. H. Yang, H. X. Zhang, L. Wu, A time two-grid algorithm for the two dimensional nonlinear fractional PIDE with a weakly singular kernel, *Math. Comput. Simulat.*, **199** (2022), 38–59. https://doi.org/10.1016/j.matcom.2022.03.004

18. W. Xiao, X. H. Yang, Z. Z. Zhou, Pointwise-in-time α-robust error estimate of the ADI difference scheme for three-dimensional fractional subdiffusion equations with variable coefficients, *Commun. Anal. Mech.*, **16** (2024), 53–70. https://doi.org/10.3934/cam.2024003

19. X. H. Yang, W. Qiu, H. Chen, H. X. Zhang, Second-order BDF ADI Galerkin finite element method for the evolutionary equation with a nonlocal term in three-dimensional space, *Appl. Numer. Math.*, **172** (2022), 497–513. https://doi.org/10.1016/j.apnum.2021.11.004

20. Z. Y. Zhou, H. X. Zhang, X. H. Yang, J. Tang, An efficient ADI difference scheme for the nonlocal evolution equation with multi-term weakly singular kernels in three dimensions, *Int. J. Comput. Math.*, **100** (2023), 1719–1736. https://doi.org/10.1080/00207160.2023.2212307

21. Q. Q. Tian, H. X. Zhang, X. H. Yang, X. X. Jiang, An implicit difference scheme for the fourth-order nonlinear non-local PIDEs with a weakly singular kernel, *Comput. Appl. Math.*, **41** (2022), 328. https://doi.org/10.1007/s40314-022-02040-9

22. L. Wu, H. X. Zhang, X. H. Yang, F. Wang, A second-order finite difference method for the multi-term fourth-order integral-differential equations on graded meshes, *Comput. Appl. Math.*, **41** (2022), 313. https://doi.org/10.1007/s40314-022-02026-7

23. X. H. Yang, L. Wu, H. X. Zhang, A space-time spectral order sinc-collocation method for the fourth-order nonlocal heat model arising in viscoelasticity, *Appl. Math. Comput.*, **457** (2023), 128192. https://doi.org/10.1016/j.amc.2023.128192

24. X. H. Yang, Z. M. Zhang, On conservative, positivity preserving, nonlinear FV scheme on distorted meshes for the multi-term nonlocal Nagumo-type equations, *Appl. Math. Lett.*, **150** (2024), 108972. https://doi.org/10.1016/j.aml.2023.108972

25. X. H. Yang, H. X. Zhang, Q. Zhang, G. Y. Yuan, Simple positivity-preserving nonlinear finite volume scheme for subdiffusion equations on general non-conforming distorted meshes, *Nonlinear Dyn.*, **108** (2022), 3859–3886. https://doi.org/10.1007/s11071-022-07399-2

26. C. J. Li, H. X. Zhang, X. H. Yang, A high-precision Richardson extrapolation method for a class of elliptic Dirichlet boundary value calculation, *J. Hunan Univ. Technol.*, **38** (2024), 91–97. https://doi.org/10.3969/j.issn.1673-9833.2024.01.013

27. M. Hairer, J. Voss, Approximations to the stochastic Burgers equation, *J. Nonlinear Sci.*, **21** (2011), 897–920. https://doi.org/10.1007/s00332-011-9104-3

28. M. Hairer, K. Matetski, Optimal rate of convergence for stochastic Burgers-type equations, *Stoch. Partial Differ. Equations: Anal. Comput.*, **4** (2016), 402–437. https://doi.org/10.1007/s40072-015-0067-5

29. D. Blömker, A. Jentzen, Galerkin approximations for the stochastic Burgers equation, *SIAM J. Numer. Anal.*, **51** (2013), 694–715. https://doi.org/10.1137/110845756

30. A. Jentzen, D. Salimova, T. Welti, Strong convergence for explicit space-time discrete numerical approximation methods for stochastic Burgers equations, *J. Math. Anal. Appl.*, **469** (2019), 661–704. https://doi.org/10.1016/j.jmaa.2018.09.032

31. D. Uma, H. Jafari, S. R. Balachandar, S. G. Venkatesh, An approximate solution for stochastic Burgers equation driven by white noise, *Comput. Appl. Math.*, **41** (2022), 1–26. https://doi.org/10.1007/s40314-022-02018-7

32. G. Wang, M. Zeng, B. Guo, Stochastic Burgers' equation driven by fractional Brownian motion, *J. Math. Anal. Appl.*, **371** (2010), 210–222. https://doi.org/10.1016/j.jmaa.2010.05.015

33. Y. Jiang, T. Wei, X. Zhou, Stochastic generalized Burgers equations driven by fractional noises, *J. Differ. Equations*, **252** (2012), 1934–1961. https://doi.org/10.1016/j.jde.2011.07.032

34. M. Hinz, Burgers' system with a fractional Brownian random force, *Stochastics*, **83** (2011), 67–106. https://doi.org/10.1080/17442508.2010.514050

35. B. Pei, Y. Xu, J. L. Wu, Stochastic averaging for stochastic differential equations driven by fractional Brownian motion and standard Brownian motion, *Appl. Math. Lett.*, **100** (2020), 106006. https://doi.org/10.1016/j.aml.2019.106006

36. M. Yang, H. B. Gu, Riemann-Liouville fractional stochastic evolution equations driven by both Wiener process and fractional Brownian motion, *J. Inequal. Appl.*, **8** (2021), 1–19. https://doi.org/10.1186/s13660-020-02541-3

37. G. Zou, G. Lv, J. Wu, Stochastic Navier-Stokes equations with Caputo derivative driven by fractional noises. *J. Math. Anal. Appl.*, **461** (2018), 595–609. https://doi.org/10.1016/j.jmaa.2018.01.027

38. Y. Cao, J. Hong, Z. Liu, Finite element approximations for second-order stochastic differential equation driven by fractional Brownian motion, *IMA J. Numer. Anal.*, **38** (2018), 184–197. https://doi.org/10.1093/imanum/drx004

39. C. Tudor, Wong-Zakai type approximations for stochastic differential equations driven by a fractional Brownian motion, *Z. Anal. Anwend.*, **28** (2009), 165–182. https://doi.org/10.4171/ZAA/1378

40. R. Qi, Q. Lin, Time-stepping error bound for a stochastic parabolic volterra equation disturbed by fractional Brownian motions, *Numer. Math. Theor. Meth. Appl.*, **12** (2019), 778–796. https://doi.org/10.4208/nmtma.OA-2017-0153

41. J. Hong, C. Huang, Super-convergence analysis on exponential integrator for stochastic heat equation driven by additive fractional Brownian motion, preprint, arXiv:2007.02223v1. https://doi.org/10.48550/arXiv.2007.02223

42. R. Kruse, *Strong and Weak Approximation of Semilinear Stochastic Evolution Equations*, Springer, 2014. https://doi.org/10.1007/978-3-319-02231-4

43. B. B. Mandelbrot, J. W. Van Ness, Fractional Brownian motions, fractional noises and applications, *SIAM Rev.*, **10** (1968), 422–437. https://doi.org/10.1137/1010093

44. T. E. Duncan, B. Maslowski, B. Pasik-Duncan, Semilinear stochastic equations in a Hilbert space with a fractional Brownian motion, *SIAM J. Math. Anal.*, **40** (2009), 2286–2315. DOI:10.1137/08071764X

45. C. Prévôt, M. Röckner, *A Concise Course on Stochastic Partial Differential Equations*, Springer, 2007. https://doi.org/10.1007/978-3-540-70781-3

46. R. Témam, *Navier-Stokes Equations and Nonlinear Functional Analysis*, CBMS-NSF Regional Conference Series in Applied Mathematics, 1983. https://doi.org/10.1137/1.9781611970050.fm

47. Z. Dong, T. G. Xu, One-dimensional stochastic Burgers' equation driven by Lévy processes, *J. Funct. Anal.*, **243** (2007), 631–678. https://doi.org/10.1016/j.jfa.2006.09.010

48. V. Thomée, *Galerkin Finite Element Methods for Parabolic Problems*, Springer-Verlag, Berlin, 1984. https://doi.org/10.1007/3-540-33122-0