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Research article

Asymptotic behavior of a Balakrishnan-Taylor suspension bridge

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Abstract: In this manuscript, we examine a nonlinear Cauchy problem aimed at describing the deformation of the deck of either a footbridge or a suspension bridge in a rectangular domain $\Omega = (0,\pi)\times(-d,d)$, with $d << \pi$, incorporating hinged boundary conditions along its short edges, as well as free boundary conditions along its remaining free edges. We establish the existence of solutions and the exponential decay of energy.

Keywords: Balakrishnan-Taylor suspension bridge; existence of solutions; exponential decay

1. Introduction

This paper is concerned with well-posedness and exponential stability, in $\Omega\times(0,+\infty)$, for the following Balakrishnan-Taylor suspension bridge

$$|z_t|^\kappa z_{tt} + \alpha\Delta^2 z_{tt} + \Delta^2 z - \left(-\xi_1 + \xi_2\|z_x\|^2 + \sigma(z_x, z_{xt})\right) z_{xx} + \Delta^2 z_t + \gamma(x) f(z_t) + h(z) = 0, \tag{1.1}$$

with the boundary conditions

$$\begin{cases} z(0,y,t) = z_{xx}(0,y,t) = z(\pi,y,t) = z_{xx}(\pi,y,t) = 0, \ (y,t)\in(-d,d)\times(0,+\infty), \\ z_{yy}(x,\pm d,t) + \mu z_{xx}(x,\pm d,t) = 0, \ (x,t)\in(0,\pi)\times(0,\infty), \\ z_{yyy}(x,\pm d,t) + (2-\mu) z_{xxy}(x,\pm d,t) = 0, \ (x,t)\in(0,\pi)\times(0,+\infty), \end{cases} \tag{1.2}$$

and initial conditions

$$z(x,y,0) = z_0(x,y), \ z_t(x,y,0) = z_1(x,y), \quad \text{in } \ \Omega\times(0,+\infty), \tag{1.3}$$

where κ and α are positive constants. The constant μ is the Poisson ratio and ranges in value from 0.1 to 0.2 for concrete and for metals it is about 0.3. Accordingly, we will make the assumption that $0 < \mu < \frac{1}{2}$.

The constant $\sigma > 0$ is the Balakrishnan-Taylor damping coefficient, with the understanding that ξ_1 is positive for compressed plates and negative for stretched plates ([1, Section 5]), $\xi_2 > 0$ relies on the material elasticity of the deck, and the term $\xi_2\|z_x\|^2$ measures the geometric nonlinearity of the plate due to its stretching.

Here, the notation $(\cdot,\cdot)$ stands for the inner product in $L^2(\Omega)$ and its corresponding norm will be denoted by $\|\cdot\|$.

The function $\gamma \in L^\infty(\Omega)$ satisfies

$$\gamma(x) \geq \gamma_0 > 0, \ \text{ a.e., in } \omega \ \text{ and } \ \gamma = 0 \text{ in } \Omega \setminus \omega,$$

where ω is an open subset of Ω.

In this paper, we take into account the following conditions:

(H1): $f : \mathbb{R} \to \mathbb{R}$ is a nondecreasing C^1 function such that there exist positive constants ε, c_1, c_2 and a strictly increasing function $F \in C^1([0, +\infty))$, with $F(0) = 0$, and F is a linear or strictly convex C^2 function on $(0, \varepsilon]$, such that

$$\begin{aligned} & s^2 + f^2(s) \leq F^{-1}(sf(s)) && \text{for all } |s| \leq \varepsilon, \\ & c_1|s| \leq |f(s)| \leq c_2|s| && \text{for all } |s| \geq \varepsilon. \end{aligned} \tag{1.4}$$

(H2): $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function with $h(0) = 0$.

(H3): $H(t) = \displaystyle\int_0^t h(s)\, ds$ is positive such that

$$sh(s) - H(s) \geq 0, \ \forall \ s \in \mathbb{R}.$$

Remark 1.1. *Hypothesis* **(H1)** *implies that $sf(s) > 0$ for all $s \neq 0$.*

Model (1.1) describes the vibrations of a thin and narrow rectangular plate in the presence of Balakrishnan-Taylor damping (the term $\sigma(z_x, z_{xt})$), strong damping (the term $\Delta^2 z_t$), and nonlinear localized damping (the term $\gamma(x)f(z_t)$). Bridges are important, and they have long been a part of daily life for people. Bridges allow for uninterrupted travel over rivers and dangerous terrain, saving time and fuel while also minimizing traffic congestion, shortening travel distances, and reducing the number of accidents that may otherwise occur on the road. Nonetheless, difficulties with stability and collapse brought on by natural disasters like earthquakes and strong winds have been encountered during bridge building. The study of suspension bridges has been an interest for many researchers who have made efforts to discover the best designs and models feasible to tackle such challenges. Early results concerning suspension bridges go back to the works of McKenna and Walter [2] and McKenna et al. [3], where the authors gave a model describing the dynamics of a suspension bridge and proved the existence of nonlinear oscillations. The asymptotic dynamics and global attractors for coupled suspension bridge equations were investigated by Bochicchio et al. [4] and Ma and Zhong [5], respectively. Recently, a new model for a suspension bridge through a plate was given in Ferrero and Gazzola [6]. Further details on suspension bridge models can also be found in [7]. The bending and stretching energies of the model given in [6] were analyzed in [1]. Later, Berchio et al. [8] discussed the structural instability of nonlinear plates modelling suspension bridges. The finite time blow-up and uniform stability of a suspension bridge has recently been the subject of various works, and special cases of (1.1)–(1.3) have been investigated. We mention that all works that will be discussed hereafter treated similar problems to (1.1)–(1.3), but without strong damping. In [9], Wang considered problem (1.1) when $\kappa = \alpha = \xi_1 = \xi_2 = \sigma = 0$, $\gamma = 1$, $h(z) = az$ ($a = a(x, y, t)$ is a sign-changing and bounded measurable function), and $f(z) = z$ and with an external force. The author provided necessary and sufficient conditions for the uniqueness and existence of global solutions and finite time blow-up of these solutions. Next, Liu et al. [10] extended the work of Wang [9] by taking $f(z) = |z|^{m-2}z(m \geq 2)$. In [11], the authors considered system (1.1) in the case where $\kappa = \alpha = \xi_1 = \xi_2 = \sigma = 0$ and $\gamma(x) = \gamma$, and with linear damping, i.e., $f(z_t) = z_t$. Using the multiplier techniques, the authors showed a uniform decay of energy. Afterwards, Cavalcanti et al. [12] (resp., [13]) studied problem (1.1) when $\kappa = \alpha = \sigma = h = 0$ with a localized linear (resp., nonlinear) damping distributed around a neighborhood of the boundary, and showed the exponential decay of energy in both cases. Let us also mention works [14–19], which dealt with suspension bridges and where other types of damping (structural and viscoelastic) are presented.

Finally, we recall some recent works on the plate equation that are related to our problem. In [20], the authors considered the equation

$$|z_t|^\kappa z_{tt} + \Delta^2 z + \Delta^2 z_{tt} - \int_0^t k(t-s)\Delta^2 z(s)\, ds = pz \ln |z|$$

in a bounded domain of $\mathbb{R}^2$, established the existence of the solutions, and proved explicit and general decay rate results. Next, Al-Mahdi [21] considered the same problem as in [20] but with infinite memory. He proved existence and general decay results with a wider class of relaxation functions. Later on, in [22], the authors proved similar outcomes to [21] by adding a nonlinear damping.

We also cite the recent works about the evolutive plate equation with partially hinged boundary conditions [23,24]. Motivated by all these works, our goal here is to prove the existence of global solutions as well as the exponential decay of energy of these solutions under the influence of a localized nonlinear damping distributed in a subset of Ω, combined with Balakrishnan-Taylor and strong damping.

The results presented here are new and different from previous works due to the presence of $\Delta^2 z_t$ and an external force source $h(z)$. Note that the external force generally promotes the blow-up of the solution.

The organization of the paper proceeds as follows: Section 2 is devoted to fixing notations, recalling some previous lemmas, and establishing a technical inequality. In Section 3, the well-posedness of system (1.1)–(1.3) is proved. The exponential stability is shown in the last part.

2. Preliminaries

We define the space

$$W = \{z \in H^2(\Omega) : z = 0 \text{ o}n \ \{0, \pi\} \times (-d, d)\}$$

along with the scalar product

$$\langle z, v \rangle = \int_\Omega \Big(z_{xx}v_{xx} - z_{xx}v_{yy} - z_{yy}v_{xx} + \mu(z_{xx}v_{yy} + z_{yy}v_{xx}) + 2(1-\mu)z_{xy}v_{xy} \Big)\, dx\, dy.$$

It is a known fact that $(W, \langle \cdot, \cdot \rangle)$ is a Hilbert space, with the norm $||.||_W^2$ being equivalent to the standard H^2 norm (see [6]).

Lemma 2.1. *([6]). Assume* $0 < \mu < \frac{1}{2}$ *and consider* $g \in L^2(\Omega)$*. Therefore, there is a unique function* $z \in W$ *such that*

$$\langle z, v\rangle = \int_\Omega gv \, dx \, dy, \tag{2.1}$$

for all $v \in W$.

The function z belonging to W and fulfilling (2.1) is referred to as the weak solution for the stationary problem

$$\begin{gathered} \Delta^2 z = g, \\ z(0, y) = z_{xx}(0, y) = z(\pi, y) = z_{xx}(\pi, y) = 0, \\ z_{yy}(x, \pm d) + \mu z_{xx}(x, \pm d) = z_{yyy}(x, \pm d) + (2 - \mu) z_{xxy}(x, \pm d) = 0. \end{gathered} \tag{2.2}$$

We also bring to mind the subsequent lemma:

Lemma 2.2. *([9]). Let* $z \in W$ *and consider that* $1 \leq q < +\infty$*. Hence, we have*

$$\|z\|_q \leq C_* \|z\|_W, \tag{2.3}$$

for some positive constant $C_* = C_*(\Omega, q) > 0$*, and, where* $\|\cdot\|_q$ *denotes the usual* $L^q(\Omega)$*-norm.*

The energy of the solutions of (1.1)–(1.3) is determined by

$$E(t) = \frac{1}{\kappa + 2} \int_\Omega |z_t|^{\kappa+2} \, dx \, dy + \frac{1}{2}\|z\|_W^2 + \frac{\alpha}{2}\|z_t\|_W^2 - \frac{\xi_1}{2}\|z_x\|^2 + \frac{\xi_2}{4}\|z_x\|^4 + \int_\Omega H(z) \, dx \, dy \tag{2.4}$$

and satisfies

$$E'(t) = -\|z_t\|_W^2 - \int_\Omega \gamma(x) f(z_t) z_t \, dx \, dy - \sigma \left(\frac{1}{2} \frac{d}{dt} \|z_x\|^2 \right)^2 \leq 0. \tag{2.5}$$

Remark 2.3. *An interesting observation is that, when* $\xi_1 < 0$*, the energy* $E(t)$ *remains nonnegative for all* $t \geq 0$*. This scenario, unlike real bridges, indicates a stretched plate instead of compressed one in terms of elasticity. However, in the more realistic case where* $\xi_1 > 0$*, typical of bridges, the energy is no longer guaranteed to be nonnegative. This aspect holds significant importance in the stabilization of distributed systems. To address this issue, we will draw upon concepts from [[1], Section 3]. We now introduce the following:*

$$\begin{aligned} H^1_*(\Omega) &:= \{z \in H^1(\Omega) : z = 0 \text{ on } \{0, \pi\} \times (-d, d)\}, \\ C^\infty_*(\Omega) &:= \{z \in C^\infty(\overline{\Omega}) : \exists\, \varepsilon > 0, z(x, y) = 0 \text{ if } x \in [0, \varepsilon] \cup [\pi - \varepsilon, \pi]\}. \end{aligned}$$

When equipped with the Dirichlet norm below, $H^1_*(\Omega)$ *forms a normed space*

$$\|z\|_{H^1_*(\Omega)} := \left(\int_\Omega |\nabla z|^2 \, dx \, dy \right)^{1/2}. \tag{2.6}$$

Remark 2.4. *We introduce* $H^1_*(\Omega)$ *as the completion of* $C^\infty_*(\Omega)$ *with the norm* $\|\cdot\|_{H^1_*(\Omega)}$*. It is apparent that the inclusion* $W \hookrightarrow H^1_*(\Omega)$ *is compact and the optimal embedding constant can be expressed by*

$$\Lambda := \min_{z \in W} \frac{\|z\|_W^2}{\|z\|_{H^1_*(\Omega)}^2}.$$

Proposition 2.5. *([1]) Assume that* $0 < \xi_1 < \Lambda$. *Then,* $E(t) > 0$.

Proof. Using Remark 2.4, we obtain the Poincaré-type inequality

$$\|z\|^2_{H^1_*(\Omega)} \leq \Lambda^{-1}\|z\|^2_W, \quad \text{for all } z \in W. \tag{2.7}$$

Then, for all $z \in W$, and since

$$\|z_x\|^2 \leq \int_\Omega |\nabla z|^2 \, dx \, dy \leq \Lambda^{-1} \, \|z\|^2_W \tag{2.8}$$

we have

$$-\frac{\xi_1}{2}\|z_x\|^2 \geq -\frac{\xi_1}{2}\Lambda^{-1} \, \|z\|^2_W$$

and therefore

$$\frac{1}{2}\|z\|^2_W - \frac{\xi_1}{2}\|z_x\|^2 \geq \frac{1}{2}\|z\|^2_W \left(1 - \xi_1\Lambda^{-1}\right). \tag{2.9}$$

Thus, if $0 < \xi_1 < \Lambda$, it follows that $\frac{1}{2}\|z\|^2_W - \frac{\xi_1}{2}\|z_x\|^2 > 0$, and consequently $E(t) > 0$.

Within the scope of this paper, C represents a generic positive constant, and is not necessarily the same at different occurrences.

The proof of our main result relies heavily on the next proposition.

Proposition 2.6. *The solution of* (1.1) *verifies*

$$\int_\Omega \gamma(x) f(z_t) z \, dx \, dy \leq \frac{A}{2}\int_\Omega \gamma(x) f(z_t) z_t \, dx \, dy + \frac{B}{2}\|z\|^2_W \tag{2.10}$$

for some constants $A > 0$ *and* $0 < B < 2$.

Proof. Young's inequality and (2.3) lead to

$$\begin{aligned} \int_\Omega \gamma(x) f(z_t) z \, dx \, dy &\leq \frac{\|\gamma\|_\infty}{4\beta}\int_\Omega \gamma(x) f^2(z_t) \, dx \, dy + \beta \int_\Omega |z|^2 \, dx \, dy \\ &\leq \frac{\|\gamma\|_\infty}{4\beta}\int_\Omega \gamma(x) f^2(z_t) \, dx \, dy + \beta C_*^2 \|z\|^2_W \end{aligned} \tag{2.11}$$

for any $\beta > 0$.

The first term on the right hand side of (2.11) can be estimated as follows:

$$\begin{aligned} &\int_\Omega \gamma(x) f^2(z_t) \, dx \, dy = \int_{\{|z_t| \leq \varepsilon\}} \gamma(x) f^2(z_t) \, dx \, dy + \int_{\{|z_t| > \varepsilon\}} \gamma(x) f^2(z_t) \, dx \, dy \\ &\leq \int_{\{|z_t| \leq \varepsilon\}} \gamma(x) F^{-1}(f(z_t) z_t) \, dx \, dy + c_2 \int_{\{|z_t| > \varepsilon\}} \gamma(x) |f(z_t)||z_t| \, dx \, dy. \end{aligned}$$

Using hypothesis **(H1)** and the fact that $f(z_t)z_t > 0$, it holds that

$$\int_\Omega \gamma(x) f^2(z_t)\, dx\, dy \;\leq\; C \int_{\{|z_t|\leq\varepsilon\}} \gamma(x) f(z_t) z_t\, dx\, dy + c_2 \int_{\{|z_t|>\varepsilon\}} \gamma(x) f(z_t) z_t\, dx$$
$$\leq\; M \int_\Omega \gamma(x) f(z_t) z_t\, dx\, dy \tag{2.12}$$

where $M = \max\{C, c_2\}$.

By taking $\beta < \frac{1}{C_*^2}$, (2.10) is satisfied with $A = \frac{M\|\gamma\|_\infty}{2\beta}$ and $B = 2\beta C_*^2$.

3. Well-posedness

In this part, we shall use the Faedo-Galerkin approach to prove that system (1.1)–(1.3) is well-posed. We have the following result:

Theorem 3.1. *Assume* **(H1)–(H3)** *and* $0 < \xi_1 < \Lambda$. *Let* $(z_0, z_1) \in W \times W$. *Then, problems* (1.1)–(1.3) *is well-posed, i.e., for any* $T > 0$*, there exists*

$$z \in C^1([0, T], W),\; z_{tt} \in L^2([0, T], W)$$

satisfying

$$\int_\Omega |z_t|^\kappa z_{tt} w\, dx + \alpha\langle z_{tt}, w\rangle + \langle z, w\rangle + \int_\Omega \Big(-\xi_1 + \xi_2\|z_x\|^2 + \sigma(z_x, z_{xt})\Big) z_x w_x\, dx\, dy$$
$$+\langle z_t, w\rangle + \int_\Omega \gamma(x) f(z_t) w\, dx\, dy + \int_\Omega h(z) w\, dx\, dy = 0,\; \forall\, w \in W \tag{3.1}$$
$$z(x, y, 0) = z_0(x, y), \quad z_t(x, y, 0) = z_1(x, y)$$

for a.e., $t \in [0, T]$.

Remark 3.2. *The function* z *satisfying* (3.1) *is called a weak solution of* (1.1)–(1.3).

Proof. Let $\{w_j\}_{j=1}^\infty$ be a basis of W and $E_p = span\{w_1, w_2, ..., w_p\},\; p \geq 1$. Let us write

$$z_0^p(x, y) = \sum_{j=1}^p a_j w_j(x, y), \qquad z_1^p(x, y) = \sum_{j=1}^p b_j w_j(x, y)$$

such that

$$z_0^p \to z_0 \text{ in } W, \text{ and } z_1^p \to z_1 \text{ in } W. \tag{3.2}$$

We will seek approximate solutions

$$z^p(x, y, t) = \sum_{j=1}^p c_j(t) w_j(x, y)$$

satisfying

$$\begin{gathered}\int_\Omega |z_t^p|^\kappa z_{tt}^p w_j\,dx + \alpha\langle z_{tt}^p, w_j\rangle + \langle z^p, w_j\rangle + \int_\Omega \left(-\xi_1 + \xi_2\|z_x^p\|^2 + \sigma(z_x^p, z_{xt}^p)\right) z_x^p (w_j)_x\,dx\,dy \\ +\langle z_t^p, w_j\rangle + \int_\Omega \gamma(x) f(z_t^p) w_j\,dx\,dy + \int_\Omega h(z^p) w_j\,dx\,dy = 0,\ \forall\ w_j \in E_p,\ j = 1,2,...,p, \qquad (3.3) \\ z^p(x,y,0) = z_0^p(x,y),\quad z_t^p(x,y,0) = z_1^p(x,y).\end{gathered}$$

Consequently, we obtain a system of ordinary differential equations for unknown functions $c_j(t)$ [20, 25]. By a classical ODEs result, system (3.3) possesses a solution z^p on $[0, t_p)$, $0 < t_p \le T$, for each $p \ge 1$.

Now, we multiply (3.3) by $c_j'(t)$ and sum over $j = 1, ..., p$ to get

$$\frac{d}{dt}E^p(t) = -\|z_t^p\|_W^2 - \int_\Omega \gamma(x) f(z_t^p) z_t^p\,dx\,dy - \sigma\left(\frac{1}{2}\frac{d}{dt}\|z_x^p\|^2\right)^2 \le 0 \qquad (3.4)$$

where

$$E^p(t) = \frac{1}{\kappa+2}\int_\Omega |z_t^p|^{\kappa+2}\,dx\,dy + \frac{1}{2}\|z^p\|_W^2 + \frac{\alpha}{2}\|z_t^p\|_W^2 - \frac{\xi_1}{2}\|z_x^p\|^2 + \frac{\xi_2}{4}\|z_x^p\|^4 + \int_\Omega H(z^p)\,dx\,dy. \qquad (3.5)$$

Our choice of initial conditions implies that $E^p(0)$ is uniformly bounded. Let us integrate (3.4) over $(0, t)$, $0 < t < t_p$, which leads to

$$E^p(t) + \int_0^t \|z_t^p\|_W^2\,ds + \int_0^t\int_\Omega \gamma(x) f(z_t^p) z_t^p\,dx\,dy\,ds + \sigma\int_0^t \left(\frac{1}{2}\frac{d}{dt}\|z_x^p\|^2\right)^2 ds \le E^p(0) \le M_1 \qquad (3.6)$$

where M_1 represents a positive constant that does not depend on either t or p.

Then, we can replace t_p by T and, in addition, we have

$$z^p,\ z_t^p \text{ are uniformly bounded in } L^\infty(0,T;W) \qquad (3.7)$$

Next, we multiply (3.3) by $c_j''(t)$ and we sum over $j = 1, ..., p$ to get

$$\begin{aligned}&\int_\Omega |z_t^p|^\kappa |z_{tt}^p|^2\,dx\,dy + \alpha\|z_{tt}^p\|_W^2 \\ &= -\int_\Omega \left(-\xi_1 + \xi_2\|z_x^p\|^2 + \sigma(z_x^p, z_{xt}^p)\right) z_x^p z_{xtt}^p\,dx\,dy - \langle z^p, z_{tt}^p\rangle \qquad (3.8) \\ &\quad -\langle z_t^p, z_{tt}^p\rangle - \int_\Omega \gamma(x) f(z_t^p) z_{tt}^p\,dx\,dy - \int_\Omega h(z^p) z_{tt}^p\,dx\,dy.\end{aligned}$$

With the help of The Cauchy-Schwarz inequality, Young's inequality, and (2.8), we find that

$$\begin{aligned}&\left|\int_\Omega \left(-\xi_1 + \xi_2\|z_x^p\|^2 + \sigma(z_x^p, z_{xt}^p)\right) z_x^p z_{xtt}^p\,dx\,dy\right| \\ &\le \beta\|z_{xtt}^p\|^2 + \frac{C}{4\beta}\left(\|z_x^p\|^2 + \|z_x^p\|^6 + \|z_x^p\|^4\|z_{xt}^p\|^2\right) \\ &\le \beta\Lambda^{-1}\|z_{tt}^p\|_W^2 + \frac{C\Lambda^{-1}}{4\beta}\left(\|z^p\|_W^2 + \|z^p\|_W^6 + \|z^p\|_W^8 + \|z_t^p\|_W^4\right) \qquad (3.9)\end{aligned}$$

for any $\beta > 0$.

Additionally, Young's inequality leads to

$$\left|\langle z^p, z^p_{tt}\rangle\right| \le \|z^p\|_W\|z^p_{tt}\|_W \le \beta\|z^p_{tt}\|^2_W + \frac{1}{4\beta}\|z^p\|^2_W, \tag{3.10}$$

$$\left|\langle z^p_t, z^p_{tt}\rangle\right| \le \beta\|z^p_{tt}\|^2_W + \frac{1}{4\beta}\|z^p_t\|^2_W \tag{3.11}$$

and by (2.3) and the fact that h is lipschitz with $h(0) = 0$, we find

$$\begin{aligned}\Big|\int_\Omega h(z^p)z^p_{tt}\,dx\,dy\Big| &\le \beta\|z^p_{tt}\|^2 + \frac{1}{4\beta}\int_\Omega |h(z^p)|^2\,dx\,dy\\ &\le \beta C_*^2\|z^p_{tt}\|^2_W + \frac{C_h^2}{4\beta}\int_\Omega |z^p|^2\,dx\,dy\\ &\le \beta C_*^2\|z^p_{tt}\|^2_W + \frac{C_h^2C_*^2}{4\beta}\|z^p\|^2_W \end{aligned}\tag{3.12}$$

where C_h is the Lipschitz constant for the function h.

By proceeding as in the proof of Proposition 2.6, one gets that

$$\Big|\int_\Omega \gamma(x)f(z^p_t)z^p_{tt}\,dx\,dy\Big| \le \beta C_*^2\|z^p_{tt}\|^2_W + \frac{C}{4\beta}\int_\Omega \gamma(x)f(z^p_t)z^p_t\,dx\,dy. \tag{3.13}$$

Combining (3.8)–(3.13), we see that

$$\begin{aligned}&\int_\Omega |z^p_t|^\kappa|z^p_{tt}|^2\,dx\,dy + \left(\alpha - \beta(2 + \Lambda^{-1} + 2C_*^2)\right)\|z^p_{tt}\|^2_W\\ &\le \frac{1}{4\beta}\Big\{\left(1 + C\Lambda^{-1} + C_h^2C_*^2\right)\|z^p\|^2_W + \|z^p_t\|^2_W\Big\}\\ &\quad + \frac{C\Lambda^{-1}}{4\beta}\left(\|z^p\|^6_W + \|z^p\|^8_W + \|z^p_t\|^4_W\right).\end{aligned}\tag{3.14}$$

By using (2.9), (3.6), and the definition of $E^p(t)$, we easily see that

$$\|z^p\|^2_W \le \frac{2M_1}{1-\xi_1\Lambda^{-1}}, \quad \|z^p_t\|^2_W \le \frac{2M_1}{\alpha},$$

which yields

$$\int_0^T \|z^p\|^2_W dt \le \frac{2M_1T}{1-\xi_1\Lambda^{-1}}, \quad \int_0^T \|z^p_t\|^2_W dt \le \frac{2M_1T}{\alpha}, \quad \int_0^T \|z^p\|^6_W dt \le \left(\frac{2M_1}{1-\xi_1\Lambda^{-1}}\right)^3 T$$

and

$$\int_0^T \|z^p\|^8_W dt \le \left(\frac{2M_1}{1-\xi_1\Lambda^{-1}}\right)^4 T, \quad \int_0^T \|z^p_t\|^4_W dt \le \frac{4M_1^2T}{\alpha^2}.$$

Integrating (3.14) over $(0, T)$ and using the previous estimates, we derive that

$$\int_0^T\int_\Omega |z^p_t|^\kappa|z^p_{tt}|^2\,dx\,dy\,dt + \left(\alpha - \beta(2 + \Lambda^{-1} + 2C_*^2)\right)\int_0^T \|z^p_{tt}\|^2_W\,dt \le \frac{CT}{4\beta}. \tag{3.15}$$

Choosing β small enough in (3.15), we infer that

$$\int_0^T \|z_{tt}^p\|_W^2 \, dt \leq C \tag{3.16}$$

implying that

$$z_{tt}^p \text{ is uniformly bounded in } L^2(0,T;W). \tag{3.17}$$

From (3.7) and (3.17), we can find a subsequence of (z^p), which we continue to label as (z^p), satisfying

$$z^p \rightharpoonup z,\ z_t^p \rightharpoonup z_t, \quad \text{weakly star in } L^\infty(0,T;W) \text{ and weakly in } L^2(0,T;W) \tag{3.18}$$

and

$$z_{tt}^p \rightharpoonup z_{tt} \text{ weakly in } L^2(0,T;W). \tag{3.19}$$

Analysis of the nonlinear terms:

Due to the compact embedding $W \subset L^2(\Omega)$, it follows that, up to a subsequence,

$$z^p \to z \text{ strongly in } L^2(0,T;L^2(\Omega)),$$

$$z^p \to z \text{ a.e. in } \Omega \times (0,T).$$

Since h is Lipschitz continuous, it holds that

$$h(z^p) \to h(z) \quad \text{a.e., in} \quad \Omega \times (0,T). \tag{3.20}$$

On the other hand, the fact that h is Lipschitz and (z^p) is bounded in $L^\infty(0,T;L^2(\Omega))$ implies that $h(z^p)$ is bounded in $L^\infty(0,T;L^2(\Omega))$. This latter fact combined with (3.20) leads to

$$h(z^p) \rightharpoonup h(z) \text{ in } L^2(0,T;L^2(\Omega)).$$

By using (3.7), (3.17), and the Aubin-Lions theorem (see [19]), we have, up to a subsequence, that

$$z_t^p \to z_t \text{ strongly in } L^2(0,T;L^2(\Omega))$$

and hence

$$|z_t^p|^\kappa z_t^p \to |z_t|^\kappa z_t \text{ a.e. in } \Omega \times (0,T). \tag{3.21}$$

From (2.3) and (3.6), one gets that

$$\left\| \left| z_t^p \right|^\kappa z_t^p \right\|_{L^2(0,T;L^2(\Omega))}^2 \leq C_*^{2(\kappa+1)} \int_0^T \|z_t^p\|_W^{2(\kappa+1)} \, dt \leq C_*^{2(\kappa+1)} \left(\frac{2M_1}{\alpha} \right)^{\kappa+1} T. \tag{3.22}$$

Consequently, by (3.21), (3.22), and the Lion's lemma [19, 26], we see that

$$|z_t^p|^\kappa z_t^p \rightharpoonup |z_t|^\kappa z_t \text{ in } L^2(0,T;L^2(\Omega)). \tag{3.23}$$

Concerning the term $\gamma(x) f(z_t^p)$, we can see from (3.6) that the sequence $\gamma(x) f(z_t^p) z_t^p$ is bounded in $L^2(0,T;L^2(\Omega))$. Then, by (2.12), $(f(z_t^p))_{p \geq 1}$ is bounded in $L_\gamma^2(\Omega \times (0,T))$, where L_γ^2 is the weighted Lebesgue space. Hence, we get, up to a subsequence, that

$$f(z_t^p) \rightharpoonup \chi \text{ in } L_\gamma^2(\Omega \times (0,T)). \tag{3.24}$$

The integration of (3.3) over the interval $(0, t)$ leads to

$$\frac{1}{\kappa+1}\int_{\Omega}|z_t^p|^{\kappa}z_t^p w_j\,dx\,dy+\alpha\langle z_t^p, w_j\rangle+\int_0^t\langle z^p, w_j\rangle ds+\int_0^t\langle z_t^p, w_j\rangle ds$$
$$+\int_0^t\int_{\Omega}\left(-\xi_1+\xi_2\|z_x^p\|^2+\sigma(z_x^p, z_{xt}^p)\right)z_x^p(w_j)_x\,dx\,dy\,ds+\int_0^t\int_{\Omega}\gamma(x)f(z_t^p)w_j\,dx\,dy\,ds$$
$$+\int_0^t\int_{\Omega}h(z^p)w_j\,dx\,dy\,ds=\frac{1}{\kappa+1}\int_{\Omega}|z_1^p|^{\kappa}z_1^p w_j\,dx\,dy+\alpha\langle z_1^p, w_j\rangle,\ \forall\ j=1,...,p.$$

By letting $p\to\infty$ and using [14], we get that

$$\frac{1}{\kappa+1}\int_{\Omega}|z_t|^{\kappa}z_t w_j\,dx\,dy+\alpha\langle z_t, w_j\rangle-\frac{1}{\kappa+1}\int_{\Omega}|z_1|^{\kappa}z_1 w_j\,dx\,dy-\alpha\langle z_1, w_j\rangle$$
$$=-\int_0^t\langle z, w_j\rangle ds-\int_0^t\langle z_t, w_j\rangle ds-\int_0^t\int_{\Omega}\left(-\xi_1+\xi_2\|z_x\|^2+\sigma(z_x, z_{xt})\right)z_x(w_j)_x\,dx\,dy\,ds$$
$$-\int_0^t\int_{\Omega}\gamma(x)\chi w_j\,dx\,dy\,ds-\int_0^t\int_{\Omega}h(z)w_j\,dx\,dy\,ds,\ \ \forall\ j=1,...,p. \tag{3.25}$$

Consequently, we conclude that (3.25) holds for all $w\in W$. Moreover, within (3.25), the expressions on the righthand side are absolutely continuous, and hence we can deduce that (3.25) is differentiable for almost every $t\geq 0$. Thus, we derive that

$$\int_{\Omega}|z_t|^{\kappa}z_{tt}w\,dx+\alpha\langle z_{tt}, w\rangle+\langle z, w\rangle+\int_{\Omega}\left(-\xi_1+\xi_2\|z_x\|^2+\sigma(z_x, z_{xt})\right)z_x w_x\,dx\,dy$$
$$+\langle z_t, w\rangle+\int_{\Omega}\gamma(x)\chi w\,dx\,dy+\int_{\Omega}h(z)w\,dx\,dy=0,\ \forall\ w\in W. \tag{3.26}$$

Next, by using the same ideas as in [18] (where it was showed that $\chi=|u_t|^m u_t$ in the proof of Theorem 3.2 of [18]), we prove that $\chi=f(z_t)$, and then (3.26) turns to

$$\int_{\Omega}|z_t|^{\kappa}z_{tt}w\,dx+\alpha\langle z_{tt}, w\rangle+\langle z, w\rangle+\int_{\Omega}\left(-\xi_1+\xi_2\|z_x\|^2+\sigma(z_x, z_{xt})\right)z_x w_x\,dx\,dy$$
$$+\langle z_t, w\rangle+\int_{\Omega}\gamma(x)f(z_t)w\,dx\,dy+\int_{\Omega}h(z)w\,dx\,dy=0,\ \forall\ w\in W.$$

In terms of the initial conditions, we may also apply (3.18) and (3.19) in a standard way to confirm that

$$z(x,y,0)=z_0(x,y)\ \text{ and }\ z_t(x,y,0)=z_1(x,y),\ (x,y)\in\Omega.$$

Moreover, we can see that

$$\|z_t\|_W^2\leq 2E(t)\leq 2E(0)$$

and

$$\|z\|_W^2\leq\frac{2}{1-\xi_1\Lambda^{-1}}\left(\frac{1}{2}\|z\|_W^2-\frac{\xi_1}{2}\|z_x\|^2\right)\leq\frac{2}{1-\xi_1\Lambda^{-1}}E(t)\leq\frac{2}{1-\xi_1\Lambda^{-1}}E(0)$$

which means that the solution $z(t)$ of (1.1)–(1.3) is bounded and global.

4. Exponential stability

We start this section by defining the functional

$$\theta(t) = \frac{1}{\kappa+1}\int_\Omega z|z_t|^\kappa z_t \, dx\, dy + \alpha\langle z, z_t\rangle + \frac{1}{2}\|z\|_W^2. \tag{4.1}$$

We hen have the following:

Proposition 4.1. *Suppose that* $0 < \xi_1 < \Lambda$. *Then, we have*

$$|\theta(t)| \le \lambda E(t), \ \forall\, t \ge 0 \tag{4.2}$$

for some constant $\lambda > 0$.

Proof. Thanks to the Cauchy-Schwarz inequality, Young's inequality, and (2.3), we obtain

$$\begin{aligned}
|\theta(t)| &\le \frac{1}{\kappa+1}\int_\Omega |z||z_t|^{\kappa+1}\, dx\, dy + \alpha\|z\|_W\|z_t\|_W + \frac{1}{2}\|z\|_W^2 \\
&\le \frac{1}{2(\kappa+1)}\int_\Omega |z_t|^{2(\kappa+1)}\, dx\, dy + \frac{1}{2(\kappa+1)}\int_\Omega |z|^2\, dx\, dy + \frac{\alpha}{2}\|z_t\|_W^2 + \frac{1}{2}(1+\alpha)\|z\|_W^2 \\
&\le \frac{C_*^{2(\kappa+1)}}{2(\kappa+1)}\|z_t\|_W^{2(\kappa+1)} + \frac{C_*^2}{2(\kappa+1)}\|z\|_W^2 + \frac{\alpha}{2}\|z_t\|_W^2 + \frac{1}{2}(1+\alpha)\|z\|_W^2 \\
&\le \frac{C_*^{2(\kappa+1)}}{2(\kappa+1)}\left(\frac{2E(0)}{\alpha}\right)^\kappa \|z_t\|_W^2 + \frac{1}{2}\left(1+\alpha+\frac{C_*^2}{\kappa+1}\right)\|z\|_W^2 + \frac{\alpha}{2}\|z_t\|_W^2 \\
&\le \lambda E(t),
\end{aligned}$$

with $\lambda = \max\left\{\left(1+\alpha+\frac{C_*^2}{\kappa+1}\right)\frac{1}{1-\xi_1\Lambda^{-1}}, \alpha + \frac{C_*^{2(\kappa+1)}}{\kappa+1}\left(\frac{2E(0)}{\alpha}\right)^\kappa\right\}$.

Let η be a positive constant satisfying, for the moment,

$$\eta < \frac{1}{\lambda}. \tag{4.3}$$

From (4.2) and (4.3), the functional $\mathcal{L}$ defined by

$$\mathcal{L}(t) = E(t) + \eta\theta(t)$$

satisfies $\mathcal{L}(t) \sim E(t)$, and more precisely we have

$$(1-\lambda\eta)E(t) \le \mathcal{L}(t) \le (1+\lambda\eta)E(t). \tag{4.4}$$

Lemma 4.2. *Assume (***H1***)–(***H3***) and* $0 < \xi_1 < \Lambda$. *Therefore, there exists a constant* $\delta > 0$ *such that*

$$\mathcal{L}'(t) \le -\delta E(t), \quad \forall\, t \ge 0. \tag{4.5}$$

Proof. By combining (1.1) and (2.5), we infer that

$$\begin{aligned}
\mathcal{L}^{'}(t) &= E^{'}(t) + \eta\theta^{'}(t) \\
&= -\|z_t\|_W^2 - \int_\Omega \gamma(x) f(z_t) z_t \, dx\, dy - \sigma\left(\frac{1}{2}\frac{d}{dt}\|z_x\|^2\right)^2 \\
&\quad +\eta\int_\Omega |z_t|^\kappa z_{tt} z \, dx\, dy + \frac{\eta}{\kappa+1}\int_\Omega |z_t|^{\kappa+2}\, dx\, dy + \eta\alpha\|z_t\|_W^2 + \eta\alpha\langle z, z_{tt}\rangle + \eta\langle z, z_t\rangle \\
&= -\|z_t\|_W^2 - \int_\Omega \gamma(x) f(z_t) z_t \, dx\, dy - \sigma\left(\frac{1}{2}\frac{d}{dt}\|z_x\|^2\right)^2 + \frac{\eta}{\kappa+1}\int_\Omega |z_t|^{\kappa+2}\, dx\, dy \\
&\quad -\eta\alpha\langle z, z_{tt}\rangle - \eta\|z\|_W^2 - \eta\left(-\xi_1 + \xi_2\|z_x\|^2 + \sigma(z_x, z_{xt})\right)\|z_x\|^2 \\
&\quad -\eta\langle z, z_t\rangle + \eta\alpha\|z_t\|_W^2 + \eta\alpha\langle z, z_{tt}\rangle + \eta\langle z, z_t\rangle \\
&\quad -\eta\int_\Omega \gamma(x) f(z_t) z \, dx\, dy - \eta\int_\Omega h(z) z \, dx\, dy
\end{aligned}$$

$$\begin{aligned}
&= -\frac{\eta}{\kappa+2}\int_\Omega |z_t|^{\kappa+2}\, dx\, dy + \frac{\eta}{\kappa+2}\int_\Omega |z_t|^{\kappa+2}\, dx\, dy + \frac{\eta}{\kappa+1}\int_\Omega |z_t|^{\kappa+2}\, dx\, dy \\
&\quad -\|z_t\|_W^2 + \eta\alpha\|z_t\|_W^2 + \eta\xi_1\|z_x\|^2 - \eta\xi_2\|z_x\|^4 - \eta\int_\Omega H(z)\, dx\, dy \\
&\quad -\sigma\left((z_x, z_{xt})\right)^2 - \eta\sigma\|z_x\|^2 (z_x, z_{xt}) - \eta\|z\|_W^2 \\
&\quad -\int_\Omega \gamma(x) f(z_t) z_t \, dx\, dy - \eta\int_\Omega \gamma(x) f(z_t) z \, dx\, dy - \eta\int_\Omega [h(z)z - H(z)]\, dx\, dy. \qquad (4.6)
\end{aligned}$$

By (2.3), we can easily check that

$$\int_\Omega |z_t|^{\kappa+2}\, dx\, dy \le C_*^{\kappa+2}\left(\frac{2E(0)}{\alpha}\right)^{\frac{\kappa}{2}} \|z_t\|_W^2.$$

Therefore, it holds that

$$\begin{aligned}
&\frac{\eta}{\kappa+2}\int_\Omega |z_t|^{\kappa+2}\, dx\, dy + \frac{\eta}{\kappa+1}\int_\Omega |z_t|^{\kappa+2}\, dx\, dy - \|z_t\|_W^2 + \eta\alpha\|z_t\|_W^2 \\
&\le -\left(1 - \eta(\alpha + M_2)\right)\|z_t\|_W^2 \qquad (4.7)
\end{aligned}$$

where $M_2 = C_*^{\kappa+2}\left(\frac{2E(0)}{\alpha}\right)^{\frac{\kappa}{2}}\left(\frac{1}{\kappa+2} + \frac{1}{\kappa+1}\right)$.

By assumption **(H3)**, we see that

$$\int_\Omega [h(z)z - H(z)]\, dx\, dy \ge 0.$$

Moreover, we have

$$\begin{aligned}&(z_x, z_{xt})^2 + \eta\|z_x\|^2(z_x, z_{xt})\\ = \ &\left((z_x, z_{xt}) + \frac{\eta}{2}\|z_x\|^2\right)^2 - \frac{\eta^2}{4}\|z_x\|^4\end{aligned}$$

and then we obtain

$$-\sigma(z_x, z_{xt})^2 - \sigma\eta\|z_x\|^2(z_x, z_{xt}) \leq \frac{\sigma\eta^2}{4}\|z_x\|^4.$$

Finally, using Proposition 2.6, we have

$$\left|\int_\Omega \gamma(x)f(z_t)z\, dx\, dy\right| \ \leq \ \frac{A}{2}\int_\Omega \gamma(x)f(z_t)z_t\, dx\, dy + \frac{B}{2}\|z\|_W^2.$$

Using the estimates above, we find that

$$\begin{aligned}\mathcal{L}'(t) \ \leq \ & -(1-\eta(1+M_2))\|z_t\|_W^2 - \frac{\eta}{\kappa+2}\int_\Omega |z_t|^{\kappa+2}\, dx\, dy + \eta\xi_1\|z_x\|^2 - \eta\xi_2\|z_x\|^4\\ & -\eta\int_\Omega H(z)\, dx\, dy + \frac{\sigma\eta^2}{4}\|z_x\|^4 - \int_\Omega \gamma(x)f(z_t)z_t\, dx\, dy\\ & +\frac{\eta A}{2}\int_\Omega \gamma(x)f(z_t)z_t\, dx\, dy + \frac{\eta B}{2}\|z\|_W^2 - \eta\|z\|_W^2\\ \leq \ & -(1-\eta(1+M_2))\|z_t\|_W^2 - \frac{\eta}{\kappa+2}\int_\Omega |z_t|^{\kappa+2}\, dx\, dy - \eta\left(1-\frac{B}{2}\right)\|z\|_W^2 + \eta\xi_1\|z_x\|^2\\ & -\eta\left(\xi_2 - \frac{\sigma\eta}{4}\right)\|z_x\|^4 - \eta\int_\Omega H(z)\, dx\, dy - \left(1-\frac{\eta A}{2}\right)\int_\Omega \gamma(x)f(z_t)z_t\, dx\, dy.\end{aligned}$$

At this point, we pick up η satisfying (4.3) and

$$\eta < \min\left\{\frac{1}{1+M_2}, \frac{4\xi_2}{\sigma}, \frac{2}{A}\right\}$$

and since $B < 2$, then it holds that $1-\eta(1+M_2) > 0$, $1-\frac{B}{2} > 0$, $\xi_2 - \frac{\sigma\eta}{4} > 0$ and $1 - \frac{\eta A}{2} > 0$. By taking

$$\delta = \min\left\{(1-\eta(1+M_2))\frac{2}{\alpha}, \eta, 2\eta(1-\frac{B}{2}), \frac{4\eta}{\xi_2}\left(\xi_2 - \frac{\sigma\eta}{4}\right)\right\}$$

we get the desired inequality (4.5).

The following theorem establishes the uniform stability of system (1.1)–(1.3).

Theorem 4.3. *Assume (***H1***)–(***H3***) and* $0 < \xi_1 < \Lambda$. *Then, the energy of the solutions of* (1.1)–(1.3) *decays exponentially, i.e., there exist positive constants b and ν such that*

$$E(t) \leq bE(0)e^{-\nu t}, \quad t \geq 0. \tag{4.8}$$

Proof. By using (4.4) and (4.5), one finds that

$$\mathcal{L}'(t) \leq -\delta E(t) \leq -\frac{\delta}{1+\lambda\eta}\mathcal{L}(t)$$

which implies that

$$\frac{\mathcal{L}'(s)}{\mathcal{L}(s)} \leq -\frac{\delta}{1+\lambda\eta}, \quad \forall\ s \geq 0.$$

Integrating the last inequality over $(0, t)$, one has

$$\ln\left(\frac{\mathcal{L}(t)}{\mathcal{L}(0)}\right) \leq -\frac{\delta}{1+\lambda\eta}t.$$

Consequently,

$$\mathcal{L}(t) \leq \mathcal{L}(0)e^{-\frac{\delta t}{1+\lambda\eta}}.$$

Once more, relying on (4.4), we arrive at

$$E(t) \leq \frac{1}{1-\lambda\eta}\mathcal{L}(t) \leq \frac{1}{1-\lambda\eta}\mathcal{L}(0)e^{-\frac{\delta t}{1+\lambda\eta}} \leq \frac{1+\lambda\eta}{1-\lambda\eta}E(0)e^{-\frac{\delta t}{1+\lambda\eta}}.$$

Hence, (4.8) holds true with $b = \frac{1+\lambda\eta}{1-\lambda\eta}$ and $\nu = \frac{\delta}{1+\lambda\eta}$.

5. Conclusions

In this paper, we focus on the study of the asymptotic behavior of the energy associated with a nonlinear problem in a rectangular domain, subject to Balakrishnan-Taylor, strong, and localized nonlinear damping and with the presence of a source term. This equation describes the deformation of the deck of either a footbridge or a suspension bridge, which is hinged along its short edges and has free vibrations on the remaining portion of the boundary. As a future work, we can study this problem with the presence of fractional damping [27–29].

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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