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Research article

An application of the Baker method to a new conjecture on exponential Diophantine equations

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Abstract: Let n be a positive integer with $n > 1$ and let a, b be fixed coprime positive integers with $\min\{a, b\} > 2$. In this paper, using the Baker method, we proved that, for any n, if $a > \max\{15064b, b^{3/2}\}$, then the equation $(an)^x + (bn)^y = ((a + b)n)^z$ has no positive integer solutions (x, y, z) with $x > z > y$. Further, let A, B be coprime positive integers with $\min\{A, B\} > 1$ and $2|B$. Combining the above conclusion with some existing results, we deduced that, for any n, if $(a, b) = (A^2, B^2), A > \max\{123B, B^{3/2}\}$ and $B \equiv 2 \pmod 4$, then this equation has only the positive integer solution $(x, y, z) = (1, 1, 1)$. Thus, we proved that the conjecture proposed by Yuan and Han is true for this case.

Keywords: exponential Diophantine equation; application of Baker method; positive integer solution

1. Introduction

Let $\mathbb{N}$ be the sets of all positive integers. Let n be a positive integer and let a, b be fixed coprime positive integers with $\min\{a, b\} > 2$. Recently, Yuan and Han [1] proposed the following conjecture:

Conjecture 1.1. *For any positive integer n, if* $\min\{a, b\} \geq 4$*, then the equation*

$$(an)^x + (bn)^y = ((a + b)n)^z, x, y, z \in \mathbb{N} \tag{1.1}$$

has only the solution $(x, y, z) = (1, 1, 1)$.

The above conjecture has been proved in many cases for $n = 1$ (see [2]). However, for general n, it is still widely open.

Let A, B be coprime positive integers with $\min\{A, B\} > 1$ and $2|B$. In [1], Yuan and Han [1] deal with the solutions (x, y, z) of (1.1) for the case that $(a, b) = (A^2, B^2)$, then (1.1) can be rewritten as

$$(A^2n)^x + (B^2n)^y = ((A^2 + B^2)n)^z, x, y, z \in \mathbb{N}. \tag{1.2}$$

In this respect, they proved that, for any n, if $B \equiv 2 \pmod 4$, then (1.2) has no solutions (x, y, z) with $y > z > x$; in particular, if $(a, b) = (A^2, B^2)$ and $B = 2$, then Conjecture 1.1 is true. Very recently, Le and Soydan [3] proved that, for any n, if $A > B^3/8$, then (1.2) has no solutions (x, y, z) with $x > z > y$. Thus, they deduce that, for any n, if $(a, b) = (A^2, B^2)$, $A > B^3/8$ and $B \equiv 2 \pmod 4$, then Conjecture 1.1 is true. Their proof relies heavily on an upper bound for solutions of exponential Diophantine equations due to Scott and Styer [4].

In this paper, using the Baker method, we prove a general result as follows:

Theorem 1.2. *For any $n > 1$, if $a > \max\{15064b, b^{3/2}\}$, then* (1.1) *has no solutions* (x, y, z) *with* $x > z > y$.

Combining Theorem 1.2 with the above mentioned results of [1], we can obtain the following corollary:

Corollary 1.3. *For any $n > 1$, if $(a, b) = (A^2, B^2)$, $A > \max\{123B, B^{3/2}\}$ and $B \equiv 2 \pmod 4$, then Conjecture 1.1 is true.*

Obviously, Theorem 1.2 and Corollary 1.3 improve the corresponding results in [3].

2. Lemmas

Let $\mathbb{Z}, \mathbb{Q}, \mathbb{C}$ be the sets of all integers, rational numbers, and complex numbers, respectively. For any algebraic number α of degree d over $\mathbb{Q}$, let $h(\alpha)$ denote the absolute logarithmic height of α, then we have

$$h(\alpha) = \frac{1}{d}(\log |a_0| + \sum_{i=1}^{d} \log \max \left\{1, |\alpha^{(i)}|\right\}), \tag{2.1}$$

where a_0 is the leading coefficient of the minimal polynomial of α over $\mathbb{Z}$ and $\alpha^{(i)} (i = 1, \ldots, d)$ are the conjugates of α in $\mathbb{C}$. For $\alpha \neq 0$, let $\log \alpha$ be any determination of its logarithms.

Let α_1, α_2 be two algebraic numbers with $\min\{|\alpha_1|, |\alpha_2|\} > 1$ and let β_1, β_2 be positive integers. Further, let

$$\Lambda = \beta_1 \log \alpha_1 - \beta_2 \log \alpha_2. \tag{2.2}$$

Lemma 2.1. *If α_1 and α_2 are multiplicatively independent and $\alpha_1, \alpha_2, \log \alpha_1, \log \alpha_2$ are real and positive, then*

$$\log |\Lambda| \geq -25.2D^4(\log A_1)(\log A_2)\left(\max\left\{1, \frac{10}{D}, 0.38 + \log K\right\}\right)^2,$$

where $D = [\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}]$,

$$\log A_j \geq \max\left\{\frac{1}{D}, \frac{|\log \alpha_1|}{D}, h(\alpha_j)\right\}, j = 1, 2,$$

$$K = \frac{\beta_1}{D \log A_2} + \frac{\beta_2}{D \log A_1}.$$

Proof. This is the special case of Corollary 2 of [5] for $m = 10$. □

Lemma 2.2. (Theorem 1.1 of [6], Proposition 3.1 of [1]) *Let* (x, y, z) *be a solution of* (1.1) *with* $(x, y, z) \neq (1, 1, 1)$, *then either* $x > z > y$ *or* $y > z > x$. *Moreover, if* $n > 1$, *then either*

$$x > z > y, \mathrm{rad}(n)|b, b = b_1b_2, b_1^y = n^{z-y}, b_1, b_2 \in \mathbb{N}, b_1 > 1, \gcd(b_1, b_2) = 1$$

or

$$y > z > x, \mathrm{rad}(n)|a, a = a_1a_2, a_1^x = n^{z-x}, a_1, a_2 \in \mathbb{N}, a_1 > 1, \gcd(a_1, a_2) = 1,$$

where $\mathrm{rad}(n)$ *is the product of all distinct prime divisors of* n.

Lemma 2.3. *Let* t *be a real number. If* $t > 7600$, *then* $t > 75.6(1.08 + \log t)^2$.

Proof. Let $f(t) = t - 75.6(1.08 + \log t)^2$ for $t > 1$, then we have $f'(t) = 1 - 151.2(1.08 + \log t)/t$, where $f'(t)$ is the derivative of $f(t)$. Since $f'(t) > 0$ for $t > 1500$, $f(t)$ is an increasing function for $t > 1500$. Therefore, since $f(7600) > 0$, we get $f(t) > 0$ for $t > 7600$. Thus,the lemma is proved. □

3. Proofs

Proof of Theorem 1.2. We now prove the first half of the theorem. Let $a > \max\{15064b, b^{3/2}\}$ and (x, y, z) be a solution of (1.1) with $x > z > y$. By Lemma 2.2, we have

$$b = b_1b_2, b_1^y = n^{z-y}, b_1, b_2 \in \mathbb{N}, \gcd(b_1, b_2) = 1 \tag{3.1}$$

and

$$a^x n^{x-z} + b_2^y = (a + b)^z. \tag{3.2}$$

By (3.2), we get

$$z \log(a + b) = x \log a + (x - z) \log n + \Lambda, \tag{3.3}$$

where

$$0 < \Lambda = \log\left(1 + \frac{b_2^y}{a^x n^{x-z}}\right). \tag{3.4}$$

Further, by (3.3), we have

$$0 < \Lambda = z \log\left(\frac{a + b}{a}\right) - (x - z) \log(an). \tag{3.5}$$

Notice that $x > z > y, a > b$, and $b \geq b_2$ by (3.1). We get $a^x n^{x-z} \geq a^x > a^y > b^y \geq b_2^y$. Hence, we see from (3.2) that

$$2a^x n^{x-z} > (a + b)^z. \tag{3.6}$$

Since $\log(1 + t) < t$ for any $t > 0$, by (3.4) and (3.6), we have

$$\Lambda < \frac{b_2^y}{a^x n^{x-z}} < \frac{2b_2^y}{(a + b)^z}. \tag{3.7}$$

Therefore, by (3.7), we get

$$\log(2b_2^y) > \log|\Lambda| + z \log(a + b). \tag{3.8}$$

Let

$$\alpha_1 = \frac{a+b}{a}, \alpha_2 = an, \beta_1 = z, \beta_2 = x - z. \tag{3.9}$$

By (3.5) and (3.9), Λ can be rewritten as (2.2). We see from (3.9) that α_1 and α_2 are multiplicatively independent rational numbers with $\min\{\alpha_1, \alpha_2\} > 1$. By (2.1) and (3.9), we have

$$[\mathbb{Q}(\alpha_1, \alpha_2) : \mathbb{Q}] = 1, \tag{3.10}$$

$$h(\alpha_1) = \log(a+b), h(\alpha_2) = \log(an). \tag{3.11}$$

Since $\Lambda > 0$ by (3.5), applying Lemma 2.1, we get from (3.5), (3.9), (3.10), and (3.11) that

$$\log|\Lambda| \geq -25.2(\log(a+b))(\log(an))(\max\{10, 0.38 + \log K\})^2, \tag{3.12}$$

where

$$K = \frac{z}{\log(an)} + \frac{x-z}{\log(a+b)}. \tag{3.13}$$

Therefore, by (3.8) and (3.12), we have

$$\log(2b_2^y) + 25.2(\log(a+b))(\log(an))(\max\{10, 0.38 + \log K\})^2 > z\log(a+b).$$

Hence, we obtain

$$\frac{\log(2b_2^y)}{(\log(a+b))(\log(an))} + 25.2(\max\{10, 0.38 + \log K\})^2 > \frac{z}{\log(an)}. \tag{3.14}$$

Since $z > y$ and $a > b^{3/2}$, if $2b_2^y > (a+b)^{2z/3}$, then from (3.1) we get $a^{2/3} > b$ and

$$2a^{2y/3} > 2b^y \geq 2b_2^y \geq (a+b)^{2z/3} > a^{2z/3} \geq a^{2(y+1)/3} > 2a^{2y/3},$$

which is a contradiction. So, we have $2b_2^y < (a+b)^{2z/3}$, which implies that

$$\frac{\log(2b_2^y)}{(\log(a+b))(\log(an))} < \frac{2z}{3\log(an)}. \tag{3.15}$$

Hence, by (3.14) and (3.15), we get

$$25.2(\max\{10, 0.38 + \log K\})^2 > \frac{z}{3\log(an)}. \tag{3.16}$$

When $10 \geq 0.38 + \log K$ by (3.13), we have

$$\frac{z}{\log(an)} < K \leq e^{9.62} < 15064. \tag{3.17}$$

When $10 < 0.38 + \log K$ by (3.16), we get

$$75.6(0.38 + \log K)^2 > \frac{z}{\log(an)}. \tag{3.18}$$

Since

$$\frac{z}{\log(an)} > \frac{x-z}{\log((a+b)/a)} > \frac{x-z}{\log(a+b)} \tag{3.19}$$

by (3.5), we see from (3.13) and (3.19) that

$$K < \frac{2z}{\log(an)}. \tag{3.20}$$

Further, by (3.18) and (3.20), we have

$$75.6\left(1.08 + \log\left(\frac{z}{\log(an)}\right)\right)^2$$

$$> 75.6\left(0.38 + \log\left(\frac{2z}{\log(an)}\right)\right)^2 > \frac{z}{\log(an)}. \tag{3.21}$$

Applying Lemma 2.3 to (3.21), we get

$$\frac{z}{\log(an)} < 7600. \tag{3.22}$$

The combination of (3.17) and (3.22) yields

$$\frac{z}{\log(an)} < 15064. \tag{3.23}$$

On the other hand, by (3.5), we have

$$\log(an) \le (x-z)\log(an) < z\log\left(\frac{a+b}{a}\right) < \frac{zb}{a}.$$

Therefore, we get

$$\frac{a}{b} < \frac{z}{\log(an)}. \tag{3.24}$$

Hence, by (3.23) and (3.24), we obtain

$$\frac{a}{b} < 15064. \tag{3.25}$$

However, since $a > 15064b$, (3.25) is false. Thus, the theorem is proved. □

Proof of Corollary 1.3. Let $(a, b) = (A^2, B^2)$. By Theorem 1.2, if $A > \max\{123B, B^{3/2}\}$, then (1.2) has no solutions (x, y, z) with $x > z > y$. On the other hand, by [1], if $B \equiv 2 \pmod 4$, then (1.2) has no solutions (x, y, z) with $y > z > x$. Therefore, by Lemma 2.2, if $A > \max\{123B, B^{3/2}\}$ and $B \equiv 2$ (mod 4), then (1.2) has no solutions (x, y, z) with $(x, y, z) \neq (1, 1, 1)$. Thus, the corollary is proved. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The author declares there is no conflict of interest.

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