Electronic Research Archive

https://www.aimspress.com/journal/era

ERA, 32(3): 1574–1617.
DOI: 10.3934/era.2024072
Received: 20 November 2023
Revised: 29 January 2024
Accepted: 02 February 2024
Published: 19 February 2024

Research article

Dynamics and density function for a stochastic anthrax epidemic model

Bing Zhao[1], **Shuting Lyu**[1,*] **and Qimin Zhang**[1,2,*]

[1] School of Mathematics and Information Science, North Minzu University, Yinchuan 750021, China
[2] School of Mathematics and Statistics, Ningxia University, Yinchuan 750021, China

* **Correspondence:** Email: xjlsht@163.com, zhangqimin64@sina.com.

Abstract: In response to the pressing need to understand anthrax biology, this paper focused on the dynamical behavior of the anthrax model under environmental influence. We defined the threshold parameter R^s, when $R^s > 1$; the disease was almost certainly present and the model exists a unique ergodic stationary distribution. Subsequently, statistical features were employed to analyze the dynamic behavior of the disease. The exact representation of the probability density function in the vicinity of the quasi-equilibrium point was determined by the Fokker-Planck equation. Finally, some numerical simulations validated our theoretical results.

Keywords: stochastic anthrax model; ergodic stationary distribution; Fokker-Planck equation; probability density function

1. Introduction

Anthrax is a contagious disease caused by bacteria produced by gram-positive endospores and can be spread between animals and humans [1]. The disease can infect all warm-blooded animals, especially herbivores such as cattle, sheep, goats, and horses [2–5]. When herbivores ingest a sufficient number of spores from soil and plants, they are usually infected with anthrax. Once infected, the animal will die within 24–48 hours [6, 7]. Humans are infected with anthrax primarily through contact with diseased animals. Once the bacteria enters the human body, it can cause symptoms such as fever, skin lesions and breathing difficulties [7]. Anthrax spores have a strong ability to survive, which can result in contaminated soil and water becoming long-lasting sources of infection [8, 9]. Anthrax occurs worldwide in 2004, a massive outbreak of anthrax killed up to 500 animals in Zimbabwe, and 109 animal anthrax deaths were reported in North Dakota in 2005. Between August 2009 and October 2010, there was an outbreak of 273 human cutaneous anthrax cases and 140 animal anthrax deaths in Bangladesh [10–14]. In a limited resource environment, anthrax has caused great harm to the development of animal husbandry and human health. As a result, anthrax has attracted widespread attention, and some progress has been made in disease control and vaccine research. In order to efficiently control the spread of anthrax, it is imperative to investigate the dynamics of the disease.

In the past few decades, various mathematical models have been intensively studied on anthrax dynamics. In previous studies, Hahn and Furniss analyzed how anthrax spreads and determined the threshold of the disease [15]. In relation to animal migration, Friedman and Yakubu addressed the basic reproduction number of the model. They examined the sufficient conditions for the presence of an endemic equilibrium point in the diffusion model [16]. Mushayabasa et al. investigated the global asymptotic stability of the endemic equilibrium point by analyzing a model that considers the vectorial impacts of anthrax transmission [17]. In fact, the spread of the virus is linked to climate, season, temperature and humidity. Due to the complexity of the environment system, the parameters of the system inevitably change randomly due to perturbations, thus making it random. Incorporating randomness into mathematical models of biological and biochemical processes is essential for gaining a deeper comprehension of the mechanisms that regulate biological systems [18]. From [19, 20], Cai and Wang examined how changes in the environment affect the spread of diseases by analyzing the random fluctuations in a susceptible-infected-susceptible (SIS) model that includes media coverage. However, few studies have constructed stochastic processes into anthrax models, which facilitates the study in this paper.

Soil renovation, heavy rains, and flash floods can lead to a rise in spore levels in the surroundings, heightening the likelihood of infection [21]. It is crucial to take into account environmental disturbances when studying anthrax. Based on the existing research [7], we present a stochastic model of anthrax to explore the impact of stochastic disturbances on threshold dynamics. The stability of the new model and the probability density function in the case of disease persistence are worth studying. It is well known that the endemic equilibrium point and the basic reproduction number can reflect the persistence of the disease for identified systems. However, environmental perturbations are unpredictable, so there is no positive equilibrium in the stochastic system [22]. In this way, a stochastic anthrax model is investigated for the stationary distribution of disease persistence. In practice, the statistical nature of the model can lead to prevention and control the disease easier [22, 23]. However, the resolution of the high-dimensional Fokker-Planck equation presents a challenging task, and only a limited number of research endeavors have been dedicated to addressing the computation of the probability density function. To close this gap, in this paper we employ the four-dimensional Fokker-Planck equation to investigate the probability density function for the stochastic anthrax model. Our article has the following contributions:

- On the basis of existing research [7], we propose a stochastic anthrax model. Sufficient conditions for the existence of an ergodic stationary distribution of globally positive solutions are established.
- The probability density function expression has been explored by few studies due to the difficulty of solving the high-dimensional Fokker-Planck equation. Therefore, this study investigates the precise representation of the probability density function in the vicinity of the quasi-equilibrium point of the model when a persistent disease is present.

The rest of our paper is arranged as follows. Section 2 introduces the mathematical models and gives the relevant notation and lemma. Section 3 proves the unique ergodic stationary distribution existence when $R^s > 1$. In Section 4, when $R^s > 1$, the probability density function is obtained by solving the Fokker-Planck equation in four dimensions. Section 5 provides the numerical simulations for our results of the analysis. The main results are discussed in Section 6.

2. Model formulation and preliminaries

2.1. Anthrax model formulation

Understanding the transmission mechanisms of anthrax epidemics requires the use of mathematical models as an important tool. As it is known that there are different forms of anthrax models, here we consider the following anthrax model [7]:

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = rN(t)(1-\dfrac{N(t)}{K}) - \eta_i \dfrac{S(t)I(t)}{N(t)} - \eta_a A(t)S(t) - \eta_c C(t)S(t) - \mu S(t) + \tau I(t), \\ \dfrac{\mathrm{d}I(t)}{\mathrm{d}t} = \eta_i \dfrac{S(t)I(t)}{N(t)} + \eta_a A(t)S(t) + \eta_c C(t)S(t) - (\gamma+\mu+\tau)I(t), \\ \dfrac{\mathrm{d}A(t)}{\mathrm{d}t} = -\alpha A(t) + \beta C(t), \\ \dfrac{\mathrm{d}C(t)}{\mathrm{d}t} = (\gamma+\mu)I(t) - \delta(S(t)+I(t))C(t) - \kappa C(t), \end{cases} \tag{2.1}$$

where $S(t)$ and $I(t)$ are the number of susceptible and infected animals at time t, respectively. $N(t) = S(t) + I(t)$ denotes the total live animal population at time t, A denotes grams of spores in the environment, and C denotes the number of infected carcasses. The birth rate of animals is $r > 0$, K is the animal carrying capacity, and $rN(1 - \frac{N}{K})$ is born into the susceptible class per unit time. η_a is the transmission rate of environment to susceptible animals, η_c is the transmission rate of infected carcasses to susceptible animals, and η_i is the transmission rate of infected animals to susceptible animals, so $\eta_a > 0, \eta_c > 0, \eta_i > 0$. μ is the death rate, γ is the disease death rate, α is the spore decay rate, β is the spore growth rate, δ the is carcass consumption rate, κ is the carcass decay rate, and τ denotes the recovery rate of infected animals.

Assume that $\frac{\eta_i}{\gamma+\mu+\tau} < 1$, otherwise the anthrax increase is unbounded. On account of the total live animal population as $N(t) = S(t) + I(t)$, adding the equation $S(t)$ and $I(t)$ results in

$$\frac{\mathrm{d}N(t)}{\mathrm{d}t} = rN(t)(1-\frac{N(t)}{K}) - \mu N(t) - \gamma I(t) \leqslant rN(t)(1-\frac{N(t)}{K}) - \mu N(t),$$

and if $r > \mu$, this implies that $\limsup_{t\to\infty} N(t) \leqslant K(1-\frac{\mu}{r}) = S_0$.

Assuming $0 < N(0) \leqslant S_0$, the positive invariant is

$$\Gamma = \left\{ (S, I, A, C) \in \mathbb{R}^4_+ \middle| 0 \leqslant S + I \leqslant S_0,\ 0 \leqslant A \leqslant \frac{\beta S_0(\gamma+\mu)}{\alpha(\delta S_0+\kappa)},\ 0 \leqslant C \leqslant \frac{(\gamma+\mu)S_0}{\delta S_0+\kappa} \right\}.$$

The disease-free equilibrium point E_0 and the endemic equilibrium point E^* are obtained by solving the equation. $E_0 = (S_0, 0, 0, 0), E^* = (S^*, I^*, A^*, C^*)$, where

$$S^* = \frac{N^*}{D},\ I^* = N^* - S^*,\ A^* = \frac{\beta(\gamma+\mu)I^*}{\alpha(\delta N^*+\kappa)},\ C^* = \frac{(\gamma+\mu)I^*}{\delta N^*+\kappa},$$

$N^* = S_0 - \frac{K\gamma}{r}(1-\frac{1}{D})$ with $D = \frac{\eta_i}{\gamma+\mu+\tau} + \frac{\gamma+\mu}{\gamma+\mu+\tau}\frac{\eta_c N^*}{\delta N^*+\kappa} + \frac{\gamma+\mu}{\gamma+\mu+\tau}\frac{\beta\eta_a N^*}{\alpha(\delta N^*+\kappa)}$, and the basic reproduction number of Model (2.1) is

$$R_0 := \frac{\eta_i}{\gamma+\mu+\tau} + \frac{\gamma+\mu}{\gamma+\mu+\tau}\frac{\eta_c S_0}{\delta S_0+\kappa} + \frac{\gamma+\mu}{\gamma+\mu+\tau}\frac{\beta\eta_a S_0}{\alpha(\delta S_0+\kappa)}.$$

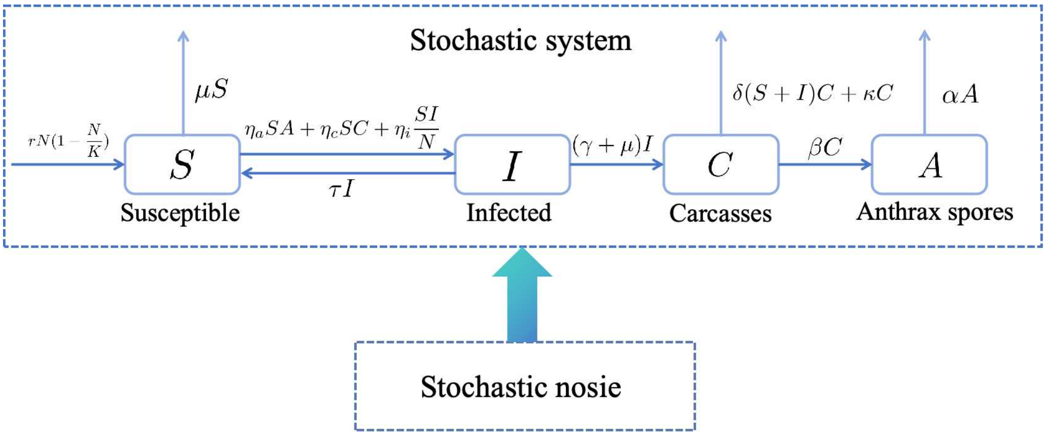

Figure 1. The flow chart of anthrax transmission.

According to Saad-Roy et al. [7], the main results about the stability of the equilibrium point are as follows:

(i) If $R_0 < 1$, the disease-free eqilibrium $E_0 = (S_0, 0, 0, 0)$ is globally asymptotically stable; thus a small outbreak of anthrax is eradicated. If $R_0 > 1$, the disease-free equilibrium is unstable; thus, anthrax persists.

(ii) If $R_0 > 1$, there has a unique endemic equilibrium with $E^* = (S^*, I^*, A^*, C^*)$, which is globally asymptotically stable in Γ.

In real life, the dynamic behaviors of anthrax are disturbed by random factors in nature. In this paper, we consider random noise on the basis of the Model (2.1) and adopt the method used by Imhof et al. [24]. Our hypothesis posits a direct correlation between stochastic fluctuations and $S(t), I(t), A(t), C(t)$. Thus, the stochastic model for anthrax transmission is to be described in the following form:

$$\begin{cases} dS(t) =[rN(t)(1-\frac{N(t)}{K}) - \eta_i \frac{S(t)I(t)}{N(t)} - \eta_a A(t)S(t) - \eta_c C(t)S(t) - \mu S(t) + \tau I(t)]dt + \sigma_1 S(t)dB_1(t), \\ dI(t) =[\eta_i \frac{S(t)I(t)}{N(t)} + \eta_a A(t)S(t) + \eta_c C(t)S(t) - (\gamma+\mu+\tau)I(t)]dt + \sigma_2 I(t)dB_2(t), \\ dA(t) =[-\alpha A(t) + \beta C(t)]dt + \sigma_3 A(t)dB_3(t), \\ dC(t) =[(\gamma+\mu)I(t) - \delta(S(t)+I(t))C(t) - \kappa C(t)]dt + \sigma_4 C(t)dB_4(t), \end{cases} \tag{2.2}$$

where σ_i $(i = 1, 2, 3, 4)$ represents the intensities of the white noises and the definition of $B_i(t)$ $(i = 1, 2, 3, 4)$ is based on complete probability space and is independent of standard Brownian motions. The transmission diagram illustrating the dynamics of anthrax transmission is shown in Figure 1. After giving the basic information of the anthrax model, the relevant lemma is given in the following section.

2.2. *Notations and lemmas*

Throughout this paper, we use $X(t) = (S(t), I(t), A(t), C(t))$ as the solution of Model (2.2) with the initial value $X(0) = (S(0), I(0), A(0), C(0))$. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geqslant 0}, \mathbb{P})$ be the probability space with the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions that are right continuous, and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets. Let $\mathbb{R}^4_+$ be an-dimensional Euclidean space and $U_n = [\frac{1}{n}\cdot n]\times[\frac{1}{n}\cdot n]\times[\frac{1}{n}\cdot n]\times[\frac{1}{n}\cdot n]$.

Our focus in this section is on the existence and uniqueness of a globally positive solution. In order to achieve this goal, Lemma 2.1 is introduced to demonstrate the existence of a positive solution to the Model (2.2). The proof of the existence theorem for positive solution can be obtained in terms of Mao and Bahar [25, 26], so the proof is omitted here.

Lemma 2.1. *For any initial value* $X(0) = (S(0), I(0), A(0), C(0)) \subseteq \mathbb{R}^4_+$*, Model (2.2) has a unique global positive solution* $X(t) = (S(t), I(t), A(t), C(t)), t \geqslant 0$*, and the solution will remain in* $\mathbb{R}^4_+$ *with probability one.*

Based on the existence of the positive solution, the next step we will concentrate on is the asymptotic estimate of the Model (2.2).

Lemma 2.2. *If for all* $\epsilon > 1$*, the following inequalities hold*

$$\begin{aligned}&\epsilon(\eta_i + \mu + \tau) > \tau + \frac{\epsilon(\epsilon-1)\sigma_1^2}{2} + 1,\ \epsilon^2(\gamma + \mu + \tau) > (\gamma + \mu + \tau) + \frac{\epsilon^2(\epsilon-1)\sigma_2^2}{2} + 1,\\ &\epsilon\alpha > (\epsilon-1)\beta + \frac{\epsilon(\epsilon-1)\sigma_3^2}{2} + 1,\ \epsilon\kappa > (\epsilon-1)(\gamma+\mu) + \beta + \frac{\epsilon(\epsilon-1)\sigma_4^2}{2} + 1,\end{aligned} \tag{2.3}$$

there exists a positive constant $\hat{T}(\epsilon)$ *such that the solution* $X(t) = (S(t), I(t), A(t), C(t))$ *with initial value* $X(0) \subseteq \mathbb{R}^4_+$ *has the following property* $\limsup_{t\to\infty} \mathbb{E}|X(t)| \leqslant \hat{T}(\epsilon)$*. For this expression, the solution* $X(t)$ *is stochastically ultimately bounded.*

The proof of Lemma 2.2 is shown in the Appendix.

Let $X(t)$ be a regular time-homogeneous Markov process in $\mathbb{R}^l$ described by the following stochastic differential eqaution: $\mathrm{d}X(t) = b(X)\mathrm{d}t + \sum_{r=1}^{k} g_r(X)\mathrm{d}B_r(t)$. The diffusion matrix is defined as follows: $B(X) = ((b_{i,j}(X))), b_{i,j}(X) = \sum_{r=1}^{k} g_r^i g_r^j(X)$, where $X(t)$ is nonsingular [27], then we have the following lemma.

Lemma 2.3 (Khasminskii [27])**.** *The Markov process* $X(t)$ *has a unique ergodic stationary distribution* $v(\cdot)$ *if there exists a bounded open domain* $U \subset \mathbb{R}^l$ *with regular boundary* Γ *and if it has the following properties:*

(i) There exists a positive $\bar{E}$ *such that* $\sum_{i,j=1}^{n} b_{ij}(X)\xi_i\xi_j \geqslant \bar{E}|\xi|^2$*,* $X \in U$*,* $\bar{E} \in \mathbb{R}^l$*;*

(ii) There exists a nonnegative C^2*-function* V *such that* LV *is negative for any* $X \in \mathbb{R}^l \backslash U$*.*

Thus the Markov process $X(t)$ *has a stationary distribution* $v(\cdot)$*. Let* $f(x)$ *be a function integrable with respect to the measure* π*, for all* $x \in \mathbb{R}^l$*. We can get*

$$\mathbb{P}\{\lim_{T\to\infty} \frac{1}{T}\int_0^T f(X(t))dt = \int_{\mathbb{R}^l} f(x)\pi(dx)\} = 1.$$

According to the Routh-Hurwitz stability principle and Zhou et al. [28], the following algebraic equation solving theory is given.

Lemma 2.4 (Zhou et al. [28]). *For the four-dimensional real matrix $D = diag(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, $Z = (a_{ij})_{4\times4}$, the characteristic polynomial of Z is $\omega_Z(\zeta) = \zeta^4 + s_1\zeta^3 + s_2\zeta^2 + s_3\zeta + s_4$. Assuming that $\Lambda = (v_{ij})_{4\times4}$ is a symmetric matrix, we have the following algebraic equation*

$$D^2 + Z\Lambda + \Lambda Z^T = 0.$$

If Z has all negative real part eigenvaues, that is,

$$s_1 > 0,\ s_3 > 0,\ s_1 s_2 - s_3 > 0,$$

then Λ is a positive definite matrix.

Using Lemmas 2.5–2.7, we will develop solutions for the four dimensions as described by Tan et al. [23] and Zhou et al. [29].

Lemma 2.5 (Zhou et al. [29]). *For the algebraic equation $G_0^2 + B_1X_1 + X_1B_1^T = 0$ and $G_0 = diag(1, 0, 0, 0)$,*

$$B_1 = \begin{bmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad X_1 = \begin{bmatrix} r_{11} & 0 & r_{13} & 0 \\ 0 & r_{22} & 0 & r_{24} \\ r_{13} & 0 & r_{33} & 0 \\ 0 & r_{24} & 0 & r_{44} \end{bmatrix}, \tag{2.4}$$

where $r_{22} = \frac{a_3}{2(a_1(a_2a_3 - a_1a_4) - a_3^2)}, r_{13} = -r_{22}, r_{33} = \frac{a_1}{a_3}r_{22}, r_{11} = \frac{a_2a_3 - a_1a_4}{a_3}r_{22}, r_{24} = -r_{33}, r_{44} = \frac{a_1a_2 - a_3}{a_3a_4}r_{22}$. If $a_1 > 0$, $a_3 > 0, a_4 > 0$ and $a_1a_2 - a_3 > 0, a_1(a_2a_3 - a_1a_4) - a_3^2 > 0$, then the matrix X_1 is postive definite.

Lemma 2.6 (Tan et al. [23]). *For the algebraic equation $G_0^2 + B_2X_2 + X_2B_2^T = 0$ and $G_0 = diag(1, 0, 0, 0)$,*

$$B_2 = \begin{bmatrix} -b_1 & -b_2 & -b_3 & -b_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b_{44} \end{bmatrix}, \quad X_2 = \begin{bmatrix} \widetilde{r}_{11} & 0 & \widetilde{r}_{13} & 0 \\ 0 & \widetilde{r}_{22} & 0 & 0 \\ \widetilde{r}_{13} & 0 & \widetilde{r}_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\widetilde{r}_{22} = \frac{1}{2(b_1b_2 - b_3)}, \widetilde{r}_{13} = -\widetilde{r}_{22}, \widetilde{r}_{11} = b_2\widetilde{r}_{22}, \widetilde{r}_{33} = \frac{b_1}{b_3}\widetilde{r}_{22}$. If $b_1 > 0, b_3 > 0$ and $b_1b_2 - b_3 > 0$, then the matrix X_2 is semi-postive definite.

Lemma 2.7 (Tan et al. [23]). *For the algebraic equation $G_0^2 + B_2X_2 + X_2B_2^T = 0$ and $G_0 = diag(1, 0, 0, 0)$,*

$$B_3 = \begin{bmatrix} -c_1 & -c_2 & -c_3 & -c_4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & c_{33} & c_{34} \\ 0 & 0 & c_{43} & c_{44} \end{bmatrix}, \quad X_3 = \begin{bmatrix} \widetilde{r}_{11} & 0 & 0 & 0 \\ 0 & \widetilde{r}_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $\widetilde{r}_{11} = \frac{1}{2c_1}, \widetilde{r}_{22} = \frac{1}{2c_1c_2}$. If $c_1 > 0$ and $c_2 > 0$, then the matrix X_3 is semi-postive definite.

For a stochastic model, stability is the basis for studying other properties, so the stationary distribution is an important research content. Thus, the ergodic stationary distribution of the Model (2.2) is studied in the following section.

3. Stationary distribution

Due to the complex structure of nonlinear systems, it is generally difficult to solve their state equations. Therefore, the Lyapunov method has been widely applied [30–35]. In this section, within the framework of the stochastic Lyapunov method, we will provide sufficient conditions for the ergodicity of the global positive solution and the existence of a stationary distribution.

Define

$$R^s := \frac{\eta_i}{\gamma+\mu+\tau+\frac{\sigma_2^2}{2}} + \frac{\gamma+(\mu+\frac{\sigma_1^2}{2})}{\gamma+\mu+\tau+\frac{\sigma_2^2}{2}}\frac{\eta_c S_0}{\delta S_0+(\kappa+\frac{\sigma_4^2}{2})} + \frac{\gamma+(\mu+\frac{\sigma_1^2}{2})}{\gamma+\mu+\tau+\frac{\sigma_2^2}{2}}\frac{\beta\eta_a S_0}{(\alpha+\frac{\sigma_3^2}{2})[\delta S_0+(\kappa+\frac{\sigma_4^2}{2})]},$$

where R^s is a stochastic threshold for Model (2.2). Next, its ergodicity is discussed using threshold.

Theorem 3.1. *If $R^s > 1$, for any initial value $(S(0), I(0), A(0), C(0)) \in \mathbb{R}_+^4$, then Model (2.2) has a unique stationary distribution $v(\cdot)$ and it follows the ergodic property.*

Proof. The diffusion matrix of Model (2.2) is given by

$$B(X) = (b_{i,j}(X))_{4\times4} = \begin{bmatrix} \sigma_1^2S^2 & 0 & 0 & 0 \\ 0 & \sigma_2^2I^2 & 0 & 0 \\ 0 & 0 & \sigma_3^2A^2 & 0 \\ 0 & 0 & 0 & \sigma_4^2C^2 \end{bmatrix}, \tag{3.1}$$

where $X = (S, I, A, C)$. Choose $\bar{E} = \min_{(S I,A,C)\in\bar{U}\subset\mathbb{R}_+^4}\{\sigma_1^2S^2, \sigma_2^2I^2, \sigma_3^2A^2, \sigma_4^2C^2\}$, then we have $\bar{E} > 0$, where $\bar{U} = [\frac{1}{n}\cdot n]\times[\frac{1}{n}\cdot n]\times[\frac{1}{n}\cdot n]\times[\frac{1}{n}\cdot n]$. For any $(S, I, A, C) \in \bar{U}$ and $(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}_+^4$, from (3.1), we have

$$\sum_{i,j=1}^{4} b_{ij}(X)\xi_i\xi_j = \sigma_1^2S^2\xi_1^2 + \sigma_2^2I^2\xi_2^2 + \sigma_3^2A^2\xi_3^2, +\sigma_4^2C^2\xi_4^2 \geqslant \bar{E}|\xi|^2,$$

where $|\xi| = (\xi_1^2+\xi_2^2+\xi_3^2+\xi_4^2)^{\frac{1}{2}}$, then the condition (i) in Lemma 2.3 holds.

Construct the following function:

$$W_1 = S + I + A + C - m_1\ln S - m_2\ln I - m_2m_3\ln A - m_2m_4\ln C,$$

and the positive constants m_i $(i = 1, 2, 3, 4)$ will be determined at a later stage. By It$\hat{o}$'s fomular, it can be obtained that

$$\begin{aligned} \mathrm{L}W_1 = &-\frac{m_1}{S}\left[rN(1-\frac{N}{K}) - \eta_aAS - \eta_cCS - \eta_i\frac{SI}{N} - \mu S + \tau I\right] \\ &- \frac{m_2}{I}\left[\eta_aAS + \eta_cCS + \eta_i\frac{SI}{N} - (\gamma+\mu+\tau)I\right] - \frac{m_2m_3}{A}(\alpha A + \beta C) \\ &- \frac{m_2m_4}{C}[(\gamma+\mu)I - \delta(S+I)C - \kappa C] \\ &+ \frac{m_1\sigma_1^2S^2}{2S^2} + \frac{m_2\sigma_2^2I^2}{2I^2} + \frac{m_2m_3\sigma_3^2A^2}{2A^2} + \frac{m_2m_4\sigma_4^2C^2}{2C^2} \\ &+ rN(1-\frac{N}{K}) - \mu S - \alpha A + \beta C - \kappa C - \delta(S+I)C \end{aligned}$$

$$
\begin{aligned}
&\leqslant rN(1-\frac{N}{K})-\left[\frac{m_1}{S}rN(1-\frac{N}{K})+\frac{m_2\eta_i S}{N}+\delta NC\right]-\left[\frac{m_2m_4(\gamma+\mu)I}{C}+\frac{m_2\eta_c CS}{I}\right]\\
&-\left[\frac{m_1\tau I}{S}+\frac{m_2\eta_a AS}{I}+\frac{m_2m_3\beta C}{A}\right]+m_1\eta_c C+m_1\mu+m_2(\gamma+\mu+\tau)\\
&+m_2m_3\alpha+m_2m_4\delta(S+I)+m_2m_4\kappa+\frac{m_1\sigma_1^2}{2}+\frac{m_2\sigma_2^2}{2}+\frac{m_2m_3\sigma_3^2}{2}+\frac{m_2m_4\sigma_4^2}{2}\\
&+\frac{\eta_i m_1 I}{N}-\mu S-\alpha A+\beta A-\kappa C\\
&\leqslant rN(1-\frac{N}{K})-3\sqrt[3]{m_1m_2rN(1-\frac{N}{K})\eta_i\delta C}-3\sqrt[3]{m_1m_2^2m_3\tau\eta_a\beta C}\\
&-2\sqrt{m_2^2m_4(\gamma+\mu)\eta_c S}+m_1(\eta_a A+\eta_c C+\mu+\frac{\sigma_1^2}{2})+m_2(\gamma+\mu+\tau+\frac{\sigma_2^2}{2})\\
&+m_2m_3(\alpha+\frac{\sigma_3^2}{2})+m_2m_4[\delta(S+I)+\kappa+\frac{\sigma_4^2}{2}]-\mu S-\alpha A+\beta C-\kappa C+\frac{\eta_i m_1 I}{N}.
\end{aligned}
$$

Let

$$
\begin{aligned}
&m_4[\delta(S+I)+\kappa+\sigma_4^2]^2=(\gamma+\mu)\eta_c S,\ m_3(\alpha+\frac{\sigma_3^3}{2})^2=m_1\tau\eta_a\beta C,\\
&m_2\left[(\gamma+\mu+\tau+\frac{\sigma_2^2}{2})+\frac{rN(1-\frac{N}{K})\tau\eta_a\beta C}{(\eta_a A+\eta_c C+\mu+\frac{\sigma_1^2}{2})(\alpha+\frac{\sigma_3^2}{2})}+\frac{(\gamma+\mu)\eta_c S}{\delta(S+I)+\kappa+\frac{\sigma_4^2}{2}}\right]\\
&=m_1(\eta_a A+\eta_c C+\mu+\frac{\sigma_1^2}{2})=rN(1-\frac{N}{K}),
\end{aligned}
$$

then we can get

$$
\begin{aligned}
&m_1=\frac{rN(1-\frac{N}{K})}{\eta_a A+\eta_c C+\mu+\frac{\sigma_1^2}{2}},\ m_2=\frac{rN(1-\frac{N}{K})}{[(\gamma+\mu+\tau+\frac{\sigma_2^2}{2})+\frac{rN(1-\frac{N}{K})\tau\eta_a\beta C}{(\eta_a A+\eta_c C+\mu+\frac{\sigma_1^2}{2})(\alpha+\frac{\sigma_3^2}{2})}+\frac{(\gamma+\mu)\eta_c S}{\delta(S+I)+\kappa+\frac{\sigma_4^2}{2}}]},\\
&m_3=\frac{rN(1-\frac{N}{K})\tau\eta_a\beta C}{(\eta_a A+\eta_c C+\mu+\frac{\sigma_1^2}{2})(\alpha+\frac{\sigma_3^2}{2})^2},\ m_4=\frac{(\gamma+\mu)\eta_c S}{[\delta(S+I)+\kappa+\frac{\sigma_4^2}{2}]^2}.
\end{aligned}
$$

Thus,

$$
\mathrm{L}W_1\leqslant -3rN(1-\frac{N}{K})(\sqrt[3]{R^t}-1)+\frac{\eta_i m_1 I}{N}+\beta C.
$$

Define the function

$$
W(S,I,A,C)=M(W_1+W_2)+W_3+W_4+W_5+W_6,
$$

where $W_2=S+I, W_3=-\ln S, W_4=-\ln I, W_5=-\ln A$, and $W_6=\frac{1}{p+1}(S+I)^{p+1}$. Select a suitable constant $M>0$ satisfying the following condiction

$$
-3MrN(1-\frac{N}{K})(\sqrt[3]{R^s}-1)+F_1\leqslant -2,
$$

where

$$F_1 = \sup_{(S,I,A,C)\in\mathbb{R}_+^4} \Big\{ -\frac{\mu}{4}S^{p+1} - \frac{\mu+\gamma}{4}I^{p+1} + MrN(1-\frac{N}{K}) + (Mm_1+1)\eta_i + \frac{p}{2}(\sigma_1^2 S^{p+1} + \sigma_1^2 S^{p+1}) \\ -\frac{\mu}{2}S^{p+1} - \frac{\mu+\gamma}{2}I^{p+1} + 2\mu + \tau + \gamma + \frac{M\sigma_1^2 S^2}{2} + \frac{M\sigma_2^2 I^2}{2} + \frac{\sigma_1^2}{2} + \frac{\sigma_2^2}{2} + \frac{\sigma_3^2}{2} \Big\} < \infty.$$

It is easy to check that

$$\liminf_{n\to\infty (S,I,A,C)\in\mathbb{R}_+^4\setminus U_n} W(S,I,A,C) = +\infty,$$

where $U_n = [\frac{1}{n}\cdot n]\times[\frac{1}{n}\cdot n]\times[\frac{1}{n}\cdot n]\times[\frac{1}{n}\cdot n]$. Furthermore, $W(S,I,A,C)$ is a continuous function. Hence, (S,I,A,C) must have a minimum point $(\bar{S}_0,\bar{I}_0,\bar{A}_0,\bar{C}_0)$ in the interior of $\mathbb{R}_+^4$, then we define a nonnegative function $\bar{W}:\mathbb{R}_+^4\to\mathbb{R}$ as follows

$$\bar{W}(S,I,A,C) = W(S,I,A,C) - W(\bar{S}_0,\bar{I}_0,\bar{A}_0,\bar{C}_0).$$

By utilizing It$\hat{o}$'s formula, we can derive the following result

$$\begin{aligned}
\mathrm{L}W_2 =& rN(1-\frac{N}{K}) - \mu S - (\gamma+\mu)I + \frac{\sigma_1^2 S^2}{2} + \frac{\sigma_2^2 I^2}{2} \leqslant rN(1-\frac{N}{K}) + \frac{\sigma_1^2 S^2}{2} + \frac{\sigma_2^2 I^2}{2},\\
\mathrm{L}W_3 =& -\frac{1}{S}\left[rN(1-\frac{N}{K}) - \eta_a AS - \eta_c CS - \frac{\eta_i SI}{N} - \mu S + \tau I\right] + \frac{\sigma_1^2}{2}\\
\leqslant& -\frac{rN(1-\frac{N}{K})}{S} + \frac{\eta_i I}{N} + \eta_a A + \eta_c C + \mu + \frac{\sigma_1^2}{2},\\
\mathrm{L}W_4 =& -\frac{1}{I}\left[\frac{\eta_i I}{N} + \eta_a A + \eta_c C - (\gamma+\mu+\tau)I\right] + \frac{\sigma_2^2}{2} \leqslant -\frac{\eta_a A}{I} - \frac{\eta_c C}{I} + (\gamma+\mu+\tau) + \frac{\sigma_2^2}{2},\\
\mathrm{L}W_5 =& -\frac{1}{A}(-\alpha A + \beta C) + \frac{\sigma_3^2}{2} \leqslant A - \frac{\beta C}{A} + \frac{\sigma_3^2}{2},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{L}W_6 =& (S+I)^p\left[rN(1-\frac{N}{K}) - \mu S - (\gamma+\mu)I\right] + \frac{p}{2}(S+I)^{p-1}(\sigma_1^2 S^2 + \sigma_2^2 I^2)\\
\leqslant& rN(1-\frac{N}{K})(S+I)^p - \mu(S+I)^{p+1} - \gamma I^{p+1} + \frac{p}{2}(\sigma_1^2 S^{p+1} + \sigma_2^2 I^{p+1}).
\end{aligned}$$

Choose $0 < p < \min\{1, \frac{\mu}{\sigma_1^2}\wedge\frac{\mu+\gamma}{\sigma_2^2}\}$, then we can see that

$$\mathrm{L}W_6 \leqslant -\frac{\mu}{2}S^{p+1} - \frac{\mu+\gamma}{2}I^{p+1} + rN(1-\frac{N}{K})(S+I)^P + \frac{p}{2}(\sigma_1^2 S^{p+1} + \sigma_2^2 I^{p+1}) - \frac{\mu}{2}S^{p+1} - \frac{\mu+\gamma}{2}I^{p+1},$$

hence,

$$
\begin{aligned}
\mathrm{L}\bar{W} \leqslant & -3MrN(1-\frac{N}{K})(\sqrt[3]{R^t}-1)+\frac{m_1\eta_i IM}{N}+MrN(1-\frac{N}{K})+M\beta C \\
& +\eta_a A+\eta_c C+A+\frac{\eta_i I}{N}-\frac{rN(1-\frac{N}{K})}{S}-\frac{\eta_a A}{I}-\frac{\eta_c C}{I} \\
& -\frac{\mu}{2}S^{p+1}-\frac{\mu+\gamma}{2}I^{p+1}+rN(1-\frac{N}{K})(S+I)^P+\frac{p}{2}(\sigma_1^2 S^{p+1}+\sigma_2^2 I^{p+1}) \\
& -\frac{\mu}{2}S^{p+1}-\frac{\mu+\gamma}{2}I^{p+1}+2\mu+\gamma+\tau+\frac{M\sigma_1^2 S^2}{2}+\frac{M\sigma_2^2 I^2}{2}+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2}, \\
\leqslant & -3MrN(1-\frac{N}{K})(\sqrt[3]{R^t}-1)+MrN(1-\frac{N}{K})+M\beta C+(Mm_1+1)\eta_i \\
& -\frac{rN(1-\frac{N}{K})}{S}-\frac{\eta_a A}{I}-\frac{\eta_c C}{I}+\eta_a A+\eta_c C+A-\frac{\mu}{2}S^{p+1}-\frac{\mu+\gamma}{2}I^{p+1} \\
& +rN(1-\frac{N}{K})(S+I)^P+\frac{p}{2}(\sigma_1^2 S^{p+1}+\sigma_2^2 I^{p+1})-\frac{\mu}{2}S^{p+1}-\frac{\mu+\gamma}{2}I^{p+1} \\
& +2\mu+\gamma+\tau+\frac{M\sigma_1^2 S^2}{2}+\frac{M\sigma_2^2 I^2}{2}+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2}.
\end{aligned}
$$

Next, we form a condensed subset D in order to satisfy the condition (ii) as outlined in Lemma 2.3. Define the bounded closed set

$$
D := \{\xi \leqslant S \leqslant \frac{1}{\xi}, \xi^2 \leqslant I \leqslant \frac{1}{\xi^2}, \xi \leqslant A \leqslant \frac{1}{\xi}, \xi \leqslant C \leqslant \frac{1}{\xi}\},
$$

where ξ is a suffciently small constant. Choose ξ satisfying the following conditions

$$
\begin{aligned}
& -3MrN(1-\frac{N}{K})(\sqrt[3]{R^t}-1)+\beta M\xi+F_1 \leqslant -1,\ -\frac{rN(1-\frac{N}{K})}{\xi}+F_2 \leqslant -1,\ -\frac{\eta_a}{\xi}+F_2 \leqslant -1, \\
& -\frac{\eta_c}{\xi}+F_2 \leqslant -1,\ -\frac{\mu}{4\xi^{p+1}}+F_2 \leqslant -1,\ -\frac{\mu+\gamma}{4\xi^{2(p+1)}}+F_2 \leqslant -1,
\end{aligned}
$$

where

$$
\begin{aligned}
F_2 = \sup_{(S,I,A,C)\in\mathbb{R}_+^4} & \Big\{-\frac{\mu}{4}S^{p+1}-\frac{\mu+\gamma}{4}I^{p+1}+MrN(1-\frac{N}{K})+M\beta C+(Mm_1+1)\eta_i+\frac{p}{2}(\sigma_1^2 S^{p+1}+\sigma_1^2 S^{p+1}) \\
& -\frac{\mu}{2}S^{p+1}-\frac{\mu+\gamma}{2}I^{p+1}+2\mu+\tau+\gamma+\frac{M\sigma_1^2 S^2}{2}+\frac{M\sigma_2^2 I^2}{2}+\frac{\sigma_1^2}{2}+\frac{\sigma_2^2}{2}+\frac{\sigma_3^2}{2}\Big\} < \infty.
\end{aligned}
$$

Divide $\mathbb{R}_+^4 \setminus D$ into six domains:

$$
\begin{aligned}
& D_1 = \{(S,I,A,C) \in \mathbb{R}_+^4, 0 < C < \xi\},\ D_2 = \{(S,I,A,C) \in \mathbb{R}_+^4, 0 < S < \xi\}, \\
& D_3 = \{(S,I,A,C) \in \mathbb{R}_+^4, 0 < I < \xi^2, A \geqslant \xi\},\ D_4 = \{(S,I,A,C) \in \mathbb{R}_+^4, 0 < I < \xi^2, C \geqslant \xi\}, \\
& D_5 = \{(S,I,A,C) \in \mathbb{R}_+^4, 0 < S < \frac{1}{\xi}\},\ D_6 = \{(S,I,A,C) \in \mathbb{R}_+^4, 0 < I < \frac{1}{\xi^2}\}.
\end{aligned}
$$

We aim to demonstrate that the inequality $L\bar{W}(S, I, A, C) \leqslant -1$ holds true on the domain $\mathbb{R}^4_+ \setminus D$. This is equivalent to establishing its validity on the six specified domains.
Case 1: If $(S, I, A, C) \in D_1$, one can see that

$$\begin{aligned} L\bar{W} \leqslant & -3MrN(1-\frac{N}{K})(\sqrt[3]{R^t}-1)+\beta MC+F_1 \\ \leqslant & -3MrN(1-\frac{N}{K})(\sqrt[3]{R^t}-1)+\beta M\xi+F_1 \leqslant -1. \end{aligned}$$

Case 2: If $(S, I, A, C) \in D_2$, one can see that

$$L\bar{W} \leqslant -\frac{rN(1-\frac{N}{K})}{S}+F_2 \leqslant -\frac{rN(1-\frac{N}{K})}{\xi}+F_2 \leqslant -1.$$

Case 3: If $(S, I, A, C) \in D_3$, one can see that

$$L\bar{W} \leqslant -\frac{\eta_a A}{I}+F_2 \leqslant -\frac{\eta_a}{\xi}+F_2 \leqslant -1.$$

Case 4: If $(S, I, A, C) \in D_4$, one can see that

$$L\bar{W} \leqslant -\frac{\eta_c C}{I}+F_2 \leqslant -\frac{\eta_c}{\xi}+F_2 \leqslant -1.$$

Case 5: If $(S, I, A, C) \in D_5$, one can see that

$$L\bar{W} \leqslant -\frac{\mu}{4}S^{p+1}+F_2 \leqslant -\frac{\mu}{4\xi^{p+1}}+F_2 \leqslant -1.$$

Case 6: If $(S, I, A, C) \in D_6$, one can see that

$$L\bar{W} \leqslant -\frac{\mu+\gamma}{4}I^{p+1}+F_2 \leqslant -\frac{\mu+\gamma}{4\xi^{2(p+1)}}+F_2 \leqslant -1.$$

Hence, we obtain $L\bar{W} \leqslant -1$ for all $(S, I, A, C) \in \mathbb{R}^4_+ \setminus D$. According to Lemma 2.3, it can be deduced that the Model (2.2) possesses a unique stationary distribution denoted as $\mathrm{v}(\cdot)$. This serves to conclude the proof.

Remark 1. *As we know, a deterministic system disease persistence is measured according to the basic reproduction number R_0 and the endemic equilibrium point E^*. However, in a stochastic model, there is no positive equilibrium due to unpredictable noise perturbations. Therefore, the persistence of the disease in a stochastic system is reflected by the presence of an ergodicity stationary distribution. According to Theorem 3.1, Model* (2.2) *has a unique stationary distribution when $R^s > 1$. Comparing the expressions for R_0 and R^s, we find that $R^s \leqslant R_0$ and the presence of disease in both the model and the stochastic model can be assessed using a consistent measure of $R^s > 1$.*

Remark 2. *Model (2.2) characterizes the impact of random noise on the spread of anthrax, which is an extension of Model (2.1). When random noise is absent, i.e., $\sigma_i = 0$ $(i = 1, 2, 3, 4)$, Model (2.2) degenerates into Model (2.1).*

4. Probability density function

This section aims to derive the precise formulation of the density function for Model (2.2) at a quasi-equilibrium point. It is important to emphasize that the probability density function serves as a representation of the dynamic attributes of a stochastic system with statistical relevance. Our main goal is to ascertain the probability density function for the purpose of examining the dynamics of the model from a statistical perspective.

Before proving the probability density function, two equivalent differential equation forms for Model (2.2) are given. Let $x_1 = \ln S, x_2 = \ln I, x_3 = \ln A, x_4 = \ln C$. By Itô's formula, an equivalent model can be obtained:

$$\begin{cases} dx_1 = [rN(1-\frac{N}{K})e^{-x_1} - \eta_i \frac{e^{x_2}}{e^{x_1}+e^{x_2}} - \eta_a e^{x_3} - \eta_c e^{x_4} + \tau\frac{e^{x_2}}{e^{x_1}} - \mu - \frac{\sigma_1^2}{2}]dt + \sigma_1 dB_1(t), \\ dx_2 = [\eta_i \frac{e^{x_1}}{e^{x_1}+e^{x_2}} + \eta_a \frac{e^{x_1}e^{x_3}}{e^{x_2}} + \eta_c \frac{e^{x_1}e^{x_4}}{e^{x_2}} - (\gamma+\mu+\tau) - \frac{\sigma_2^2}{2}]dt + \sigma_2 dB_2(t), \\ dx_3 = [\beta\frac{e^{x_4}}{e^{x_3}} - \alpha - \frac{\sigma_3^2}{2}]dt + \sigma_3 dB_3(t), \\ dx_4 = [(\gamma+\mu)\frac{e^{x_2}}{e^{x_4}} - \delta(e^{x_1}+e^{x_2}) - \kappa - \frac{\sigma_4^2}{2}]dt + \sigma_4 dB_4(t). \end{cases} \tag{4.1}$$

Let $b_1 = \mu + \frac{\sigma_1^2}{2}, b_2 = \gamma+\mu+\tau+\frac{\sigma_2^2}{2}, b_3 = \alpha + \frac{\sigma_3^2}{2}, b_4 = \kappa + \frac{\sigma_4^2}{2}$. If

$$\bar{R}^s := \frac{\eta_i \delta C[\delta(S+I)+b_4]b_3}{b_3(\eta_a A + \eta_c C + b_1)[(\gamma+\mu)\eta_c S + b_2\delta(S+I) + b_2 b_4] + \tau\eta_a\beta rN(1-\frac{N}{K})C[\delta(S+I)+b_4]} > 1$$

holds, Model (2.2) admits a quasi-steady $\bar{E}_1 = (\bar{S}_1, \bar{I}_1, \bar{A}_1, \bar{C}_1) = (e^{x_1^*}, e^{x_2^*}, e^{x_3^*}, e^{x_4^*})$, where $\bar{I}_1$ is the positive root of the equtaion

$$\begin{aligned} F(I) = &\Big[b_1^2 b_2 b_3 + b_1 b_2 b_3 \delta(\tau - b_2) + \eta_a\beta(\gamma+\mu)(b_2-\tau)(\tau-b_2) + \eta_c b_3(\gamma+\mu)(b_2-\tau)(\tau-b_2)\Big]\bar{I}_1^2 \\ &+ [b_1^2 b_2 b_3 b_4 + b_1 b_2 b_3 \delta rN(1-\frac{N}{K}) + b_1 b_3(b_2-\eta_i) + b_3\delta(b_2-\eta_i)(\tau-b_2) - b_1\eta_a\beta(\gamma+\mu) \\ &+ 2\eta_a\beta(\gamma+\mu)rN(1-\frac{N}{K})(b_2-\tau) + 2b_3\eta_c(\gamma+\mu)rN(1-\frac{N}{K})(b_2-\tau) + b_1 b_3\eta_c(\gamma+\mu)(b_2-\tau)]\bar{I}_1 \\ &- \Big\{b_3(\eta_i - b_2)\delta rN(1-\frac{N}{K}) + \eta_a\beta(\gamma+\mu)[rN(1-\frac{N}{K})]^2 + b_1\eta_a(\gamma+\mu)rN(1-\frac{N}{K}) \\ &+ b_3\eta_c(\gamma+\mu)[rN(1-\frac{N}{K})]^2 + b_1 b_3\eta_c(\gamma+\mu)rN(1-\frac{N}{K})\Big\}, \end{aligned}$$

and

$$\bar{S}_1 = \frac{rN(1-\frac{N}{K}) + \bar{I}_1(b_2-\tau)}{b_1}, \ \bar{C}_1 = \frac{b_1(\gamma+\mu)\bar{I}_1}{b_1 b_4 + b_1\bar{I}_1 + \delta[rN(1-\frac{N}{K}) + \bar{I}_1(b_2-\tau)]},$$
$$\bar{A}_1 = \frac{b_1(\gamma+\mu)\beta\bar{I}_1}{b_1 b_3 b_4 + b_1 b_3\bar{I}_1 + b_3\delta[rN(1-\frac{N}{K}) + \bar{I}_1(b_2-\tau)]}.$$

In addition, assume if $R^s > 1$, then $\bar{R}^s > 1$, which means that Model (2.2) admits a quasi-steady $\bar{E}_1$ under the condition $R^s > 1$. Let

$$y_i = x_i - x_i^* \ (i = 1,2,3,4), \tag{4.2}$$

where $x_1^* = \ln \bar{S}_1, x_2^* = \ln \bar{I}_1, x_3^* = \ln \bar{A}_1, x_4^* = \ln \bar{C}_1$, then the linearized system of (4.1) is as follows:

$$\begin{cases} \mathrm{d}y_1 = (-a_{11}y_1 - a_{12}y_2 - a_{13}y_3 - a_{14}y_4)\mathrm{d}t + \sigma_1 \mathrm{d}B_1(t), \\ \mathrm{d}y_2 = (a_{21}y_1 - a_{21}y_2 + a_{23}y_3 + a_{24}y_4)\mathrm{d}t + \sigma_2 \mathrm{d}B_2(t), \\ \mathrm{d}y_3 = (-a_{33}y_3 + a_{33}y_4)\mathrm{d}t + \sigma_3 \mathrm{d}B_3(t), \\ \mathrm{d}y_4 = (-a_{41}y_1 - a_{42}y_2 - a_{44}y_4)\mathrm{d}t + \sigma_4 \mathrm{d}B_4(t), \end{cases} \tag{4.3}$$

where

$$\begin{aligned} a_{11} &= \mu + \frac{\sigma_1^2}{2} + \frac{\eta_i \bar{I}_1^2}{\bar{N}_1^2} + \eta_a \bar{A}_1 + \eta_c \bar{C}_1, \ a_{12} = \frac{\eta_i \bar{S}_1 \bar{I}_1}{\bar{N}_1^2} - \frac{\tau \bar{I}_1}{\bar{S}_1}, \ a_{13} = \eta_a \bar{A}_1, \\ a_{14} &= \eta_c \bar{C}_1, \ a_{21} = \frac{\eta_i \bar{S}_1 \bar{I}_1}{\bar{N}_1^2} + \frac{\eta_a \bar{S}_1 \bar{A}_1 + \eta_c \bar{S}_1 \bar{C}_1}{\bar{I}_1}, \ a_{23} = \frac{\eta_a \bar{S}_1 \bar{A}_1}{\bar{I}_1}, \ a_{24} = \frac{\eta_c \bar{S}_1 \bar{C}_1}{\bar{I}_1}, \\ a_{33} &= \frac{\beta \bar{C}_1}{\bar{A}_1}, \ a_{41} = \delta \bar{S}_1, \ a_{42} = \delta \bar{I}_1 - \frac{(\gamma + \mu)\bar{I}_1}{\bar{C}_1}, \ a_{44} = \frac{(\gamma + \mu)\bar{I}_1}{\bar{C}_1}. \end{aligned}$$

By employing the above lemmas, we will demonstrate the probability density function of the Model (2.2) near the quasi-steady equilibrium $\bar{E}_1$.

Let $Y = (y_1, y_2, y_3, y_4)^T, D = \mathrm{diag}(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, and $B(t) = (B_1(t), B_2(t), B_3(t), B_4(t))^T$. We can rewrite Model (4.3) as the following form:

$$\mathrm{d}Y = ZY\mathrm{d}t + D\mathrm{d}B(t),$$

where

$$Z = \begin{bmatrix} -a_{11} & -a_{12} & -a_{13} & -a_{14} \\ a_{21} & -a_{21} & a_{23} & a_{24} \\ 0 & 0 & -a_{33} & a_{33} \\ -a_{41} & -a_{42} & 0 & -a_{44} \end{bmatrix},$$

then the linearized Model (4.3) can be simiplified to $\mathrm{d}Y = ZY\mathrm{d}t + D\mathrm{d}B(t)$. Based on the pertinent theory proposed by Zhou et al., it is postulated that there is a probability density function denoted as $\Phi(y_1, y_2, y_3, y_4)$ in the vicinity of the quasi-endemic equilibrium $\bar{E}_1$.

Theorem 4.1 (Zhou et al. [22])**.** *Assuming that $R^s > 1$ for any initial value $(S(0), I(0), A(0), C(0) \in \mathbb{R}_+^4$, the solution $(S(t), I(t), A(t), C(t))$ of the Model (2.2) follows the unique log-normal probability density function $\Phi(S, I, A, C)$ around the quasi-endemic equilibrium $\bar{E}_1$, which is described by*

$$\Phi(S, I, A, C) = (2\pi)^{-2} |\Lambda|^{-\frac{1}{2}} e^{-\frac{1}{2}(\ln \frac{S}{\bar{S}_1} + \ln \frac{I}{\bar{I}_1} + \ln \frac{A}{\bar{A}_1} + \ln \frac{C}{\bar{C}_1})\Lambda^{-1}(\ln \frac{S}{\bar{S}_1} + \ln \frac{I}{\bar{I}_1} + \ln \frac{A}{\bar{A}_1} + \ln \frac{C}{\bar{C}_1})^T},$$

where Λ is a positive definite matrix.

From Gardiner [36], the density function $U(Y) = U(y_1, y_2, y_3, y_4)$ of the quasi-stationary distribution of Model (4.3) at the origin $Y^* = (0, 0, 0, 0)$ can be described by the four-dimensional Fokker-Planck equation as follows:

$$\begin{aligned} -\sum_{i=1}^{4} \frac{\sigma_i^2}{2} \frac{\partial^2 U}{\partial y_i^2} &+ \frac{\partial}{\partial y_1}[(-a_{11}y_1 + a_{12}y_2 - a_{13}y_3 - a_{14}y_4)U] + \frac{\partial}{\partial y_2}[(a_{21}y_1 - a_{21}y_2 + a_{23}y_3 + a_{24}y_4)U] \\ &+ \frac{\partial}{\partial y_3}[(-a_{33}y_3 + a_{33}y_4)U] + \frac{\partial}{\partial y_4}[(-a_{41}y_1 + a_{42}y_2 - a_{44}y_4)U] = 0, \end{aligned}$$

which can be approximated by a Gaussian distribution

$$U(Y) = \bar{c}\exp\left\{-\frac{1}{2}(Y - Y^*)\bar{M}(Y - Y^*)^T\right\},$$

$\bar{M}$ is a real symmetric matrix, which is described by $\bar{M}D^2\bar{M} + Z^T\bar{M} + \bar{M}Z = 0$. If $\bar{M}$ is positive definite, let $\bar{M}^{-1} = \Lambda$, then

$$D^2 + Z\Lambda + \Lambda Z^T = 0. \tag{4.4}$$

According to Tian et al. [37], (4.4) can be rewriten as the sum of the four equations as follows:

$$D_i^2 + Z\Lambda_i + \Lambda_i Z^T = 0 \ (i = 1, 2, 3, 4),$$

where $D_1 = (\sigma_1, 0, 0, 0), D_2 = (0, \sigma_2, 0, 0), D_3 = (0, 0, \sigma_3, 0), D_4 = (0, 0, 0, \sigma_4), \Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4$, $D^2 = D_1^2 + D_2^2 + D_3^2 + D_4^2$. Consider the corresponding characteristic equation of Z as follows:

$$\omega_Z(\zeta) = \zeta^4 + s_1\zeta^3 + s_2\zeta^2 + s_3\zeta + s_4, \tag{4.5}$$

where

$$\begin{aligned}
s_1 =& a_{11} + a_{21} + a_{33} + a_{44},\\
s_2 =& a_{11}(a_{21} + a_{33} + a_{44}) + a_{21}(a_{12} + a_{33} + a_{44}) + a_{33}a_{44} + a_{24}a_{42} - a_{14}a_{41},\\
s_3 =& a_{11}a_{21}(a_{33} + a_{44}) + a_{33}a_{44}(a_{11} + a_{21}) + a_{12}a_{21}(a_{33} + a_{44}) - a_{14}a_{21}(a_{41} + a_{42})\\
&+ a_{24}a_{41}(a_{11} - a_{12}) - a_{33}a_{41}(a_{13} + a_{14}) + a_{23}a_{33}a_{42} + a_{24}a_{33}a_{41},\\
s_4 =& a_{21}a_{33}a_{44}(a_{11} + a_{12}) + a_{33}a_{42}(a_{11}a_{23} + a_{11}a_{24}a_{33}a_{41} - a_{14}a_{21} - a_{13}a_{21})\\
&- a_{12}a_{33}a_{41}(a_{23} + a_{24}) - a_{21}a_{33}a_{41}(a_{13} + a_{14}).
\end{aligned}$$

In s_2, the following formula can be calculated

$$\begin{aligned}
a_{21}a_{44} + a_{24}a_{42} - a_{14}a_{41} &= \frac{\eta_i(\gamma + \mu)\bar{S}_1\bar{I}_1^2}{\bar{C}_1} + \frac{\eta_a(\gamma + \mu)\bar{S}_1\bar{A}_1}{\bar{C}_1} > 0,\\
a_{12}a_{21} &= \frac{\eta_i\bar{I}_1^2}{\bar{N}_1^2}(\tau - \frac{\eta_i\bar{S}_1^2}{\bar{N}_1^2}) + (\eta_a\bar{A}_1 + \eta_c\bar{C}_1)(\tau - \frac{\eta_i\bar{S}_1^2}{\bar{N}_1^2}) > 0.
\end{aligned}$$

In s_3, the following formula can be calculated

$$\begin{aligned}
a_{21}a_{33}a_{44} + a_{23}a_{33}a_{42} - a_{14}a_{33}a_{41} - a_{13}a_{33}a_{41} &= \frac{\eta_i(\gamma + \mu)\beta\bar{S}_1\bar{I}_1^2}{\bar{N}_1^2\bar{A}_1} + \frac{\eta_c\beta\bar{S}_1\bar{C}_1}{\bar{A}_1}(\gamma + \mu - \delta\bar{C}_1) > 0,\\
a_{12}a_{21}a_{44} &= \frac{\eta_i(\gamma + \mu)\bar{I}_1^3}{\bar{N}_1^2\bar{C}_1}(\tau - \frac{\eta_i\bar{S}_1^2}{\bar{N}_1^2}) + \frac{\eta_a(\gamma + \mu)\bar{I}_1\bar{A}_1}{\bar{C}_1}(\tau - \frac{\eta_i\bar{S}_1^2}{\bar{N}_1^2}) + \eta_c(\gamma + \mu)\bar{I}_1(\tau - \frac{\eta_i\bar{S}_1^2}{\bar{N}_1^2}) > 0,\\
a_{12}a_{21}a_{33} &= \frac{\eta_i\beta\bar{I}_1^2\bar{C}_1}{\bar{N}_1^2\bar{A}_1}(\tau - \frac{\eta_i\bar{S}_1^2}{\bar{N}_1^2}) + \eta_a\beta\bar{C}_1(\tau - \frac{\eta_i\bar{S}_1^2}{\bar{N}_1^2}) + \frac{\eta_c\beta\bar{C}_1^2}{\bar{A}_1}(\tau - \frac{\eta_i\bar{S}_1^2}{\bar{N}_1^2}) > 0,\\
-a_{14}a_{21}a_{42} &= \frac{\eta_i\eta_c\bar{S}_1\bar{I}_1^2}{\bar{N}_1^2}(\gamma + \mu - \delta\bar{C}_1) + \eta_a\eta_c\bar{S}_1\bar{A}_1(\gamma + \mu - \delta\bar{C}_1) + \eta_c^2\bar{S}_1\bar{C}_1(\gamma + \mu - \delta\bar{C}_1) > 0.
\end{aligned}$$

In s_4, the following formula can be calculated

$$
\begin{aligned}
-a_{14}a_{21}a_{33}a_{42} =& \frac{\eta_i\eta_c\beta\bar{S}_1\bar{I}_1^2\bar{C}_1}{\bar{N}_1^2\bar{A}_1}(\gamma+\mu-\delta\bar{C}_1) + \eta_a\eta_c\delta\bar{S}_1\bar{C}_1(\gamma+\mu-\delta\bar{C}_1) + \frac{\eta_c^2\beta\bar{S}_1\bar{C}_1^2}{\bar{A}_1}(\gamma+\mu-\delta\bar{C}_1) > 0,\\
a_2a_{21}a_{33}a_{44} =& \frac{\eta_i\beta(\gamma+\mu)\bar{I}_1^3}{\bar{N}_1^2\bar{A}_1}(\tau - \frac{\eta_i\bar{S}_1^2}{\bar{N}_1^2}) + \eta_a\beta(\gamma+\mu)\bar{I}_1(\tau - \frac{\eta_i\bar{S}_1^2}{\bar{N}_1^2}) + \frac{\eta_c\beta(\gamma+\mu)\bar{I}_1\bar{C}_1}{\bar{A}_1}(\tau - \frac{\eta_i\bar{S}_1^2}{\bar{N}_1^2}) > 0,
\end{aligned}
$$

where $\tau - \frac{\eta_i\bar{S}_1^2}{\bar{N}_1^2} > 0$ and

$$
(\gamma+\mu-\delta\bar{C}_1) = \frac{b_1b_4(\gamma+\mu) + \left[b_1\bar{I}_1(\gamma+\mu) - \delta b_1\bar{I}_1(\gamma+\mu)\right] + \delta(\gamma+\mu)[rN(1-\frac{N}{K}) + \bar{I}_1(\tau-b_2)]}{b_1b_4 + b_1\bar{I}_1 + \delta[rN(1-\frac{N}{K}) + \bar{I}_1(\tau-b_2)]} > 0,
$$

while $s_1 > 0$, $s_2 > 0$, and $s_3 > 0$. We can calculate $s_1s_2 - s_3 > 0$ and $s_1(s_2s_3 - s_1s_4) - s_3^2 > 0$. As per the Routh-Hurwitz stability criterion [38], it can be deduced that the matrix Z possesses eigenvalues with negative real parts. According to the theory of matrix similar transformations, it is evident that $\omega_Z(\zeta)$ is a similarity invariant, indicating that s_1, s_2, s_3, and s_4 are also similarity invariants.

Next, the corresponding proof for the positive definiteness of Λ is divided into four steps.

Proof. **Step 1.** Consider the algebraic equation

$$
D_1^2 + Z\Lambda_1 + \Lambda_1 Z^T = 0, \tag{4.6}
$$

Let $Z_1 = U_1ZU_1^{-1}$, where

$$
U_1 = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & \frac{a_{41}}{a_{21}} & 0 & 1 \end{bmatrix},
$$

and

$$
Z_1 = \begin{bmatrix} -a_{11} & \frac{a_{14}a_{41}-a_{12}a_{21}}{a_{21}} & -a_{13} & -a_{14}\\ a_{21} & -\frac{a_{21}^2+a_{24}a_{41}}{a_{21}} & a_{23} & a_{24}\\ 0 & w_1 & -a_{33} & a_{33}\\ 0 & w_2 & \frac{a_{23}a_{41}}{a_{21}} & \frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}} \end{bmatrix},
$$

where $w_1 = -\frac{a_{33}a_{41}}{a_{21}}$, $w_2 = -a_{41} - a_{42} + \frac{a_{41}a_{44}}{a_{21}} - \frac{a_{24}a_{41}^2}{a_{21}^2}$.

Case 1: When $w_1 \neq 0$ and $w_2 \neq 0$, there exists the elimination matrix U_{11} such that $Z_{11} = U_{11}Z_1U_{11}^{-1}$, where

$$
U_{11} = \begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & -\frac{w_2}{w_1} & 1 \end{bmatrix},
$$

then there is

$$Z_{11}=\begin{bmatrix}-a_{11} & \frac{a_{14}a_{41}-a_{12}a_{21}}{a_{21}} & -a_{13} & -a_{14}\\ a_{21} & -\frac{a_{21}^2+a_{24}a_{41}}{a_{21}} & a_{23}+\frac{a_{24}w_2}{w_1} & a_{24}\\ 0 & w_1 & -a_{33}+\frac{a_{33}w_2}{w_1} & a_{33}\\ 0 & 0 & w_3 & \frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}}-\frac{a_{33}w_2}{w_1}\end{bmatrix},$$

where $w_3=\frac{w_2}{w_1}\frac{a_{33}w_1-a_{33}w_2}{w_1}+\frac{w_2}{w_1}\frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}}+\frac{a_{23}a_{41}}{a_{21}}$.

Case 1-1: If $w_3\neq 0$, in order to find the standard transformation matrix Q_{111} such that $T_{111}=Q_{111}Z_{11}Q_{111}^{-1}$, where

$$Q_{111}=\begin{bmatrix}m_1 & m_2 & m_3 & m_4\\ 0 & w_1w_3 & \frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}}-a_{33} & a_{33}w_3(\frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}}-\frac{a_{33}w_2}{w_1})^2\\ 0 & 0 & w_3 & \frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}}-\frac{a_{33}w_2}{w_1}\\ 0 & 0 & 0 & 1\end{bmatrix},$$

with

$$\begin{aligned}m_1=&a_{21}w_1w_3,\ m_2=(\frac{a_{24}a_{41}+a_{24}a_{41}-a_{21}a_{44-a_{21}^2}-a_{33}}{a_21})w_1w_3,\\ m_3=&w_3[a_{23}w_1+a_{24}w_2+a_{33}w_3+(\frac{a_{33}(w_2-w_1)}{w_2})(\frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}}-\frac{a_{33}w_2}{w_1})\\ &+(\frac{a_{33}(w_2-w_1)}{w_2})^2+(\frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}}-\frac{a_{33}w_2}{w_1})^2],\\ m_4=&[a_{33}w_3+(\frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}}-\frac{a_{33}w_2}{w_1})^2](\frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}}-\frac{a_{33}w_2}{w_1})\\ &+a_{24}w_1w_3+a_{33}w_3(\frac{a_{24}a_{41}-a_{21}a_{44}-a_{21}a_{33}}{a_{21}}).\end{aligned}$$

Thus, we can get

$$T_{111}=\begin{bmatrix}-s_1 & -s_2 & -s_3 & -s_4\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix},$$

Meanwhile, (4.6) has the following form:

$$\begin{aligned}&(Q_{111}U_{11}U_1)D_1^2(Q_{111}U_{11}U_1)^T\\ &+[(Q_{111}U_{11}U_1)Z(Q_{111}U_{11}U_1)^{-1}](Q_{111}U_{11}U_1)\Lambda_1(Q_{111}U_{11}U_1)^T\\ &+(Q_{111}U_{11}U_1)\Lambda_1(Q_{111}U_{11}U_1)^T[(Q_{111}U_{11}U_1)Z(Q_{111}U_{11}U_1)^{-1}]^T=0.\end{aligned}$$

That is, $D_0^2+T_{111}\Theta_1+\Theta_1T_{111}^T=0$, where $\Theta_1=\frac{1}{\rho_{111}^2}(Q_{111}U_{11}U_1)\Lambda_1(Q_{111}U_{11}U_1)^T$, $\rho_{111}=a_{21}w_1w_3\sigma_1$. According to Lemma 2.5, we can obtain the semipositive definite matrix

$$\Theta_1=\begin{bmatrix}\frac{s_2s_3-s_1s_4}{2[s_1(s_2s_3-s_1s_4)-s_3^2]} & 0 & -\frac{s_3}{2[s_1(s_2s_3-s_1s_4)-s_3^2]} & 0\\ 0 & \frac{s_3}{2[s_1(s_2s_3-s_1s_4)-s_3^2]} & 0 & \frac{-s_1}{2[s_1(s_2s_3-s_1s_4)-s_3^2]}\\ -\frac{s_3}{2[s_1(s_2s_3-s_1s_4)-s_3^2]} & 0 & \frac{s_1}{2[s_1(s_2s_3-s_1s_4)-s_3^2]} & 0\\ 0 & -\frac{s_1}{2[s_1(s_2s_3-s_1s_4)-s_3^2]} & 0 & \frac{s_1s_2-s_3}{2[s_1(s_2s_3-s_1s_4)-s_3^2]}\end{bmatrix},$$

Therefore, it can be calculated that

$$\Lambda_1 = \rho_{111}^2 (Q_{111}U_{11}U_1)^{-1}\Theta_1[(Q_{111}U_{11}U_1)^{-1}]^T \tag{4.7}$$

is positive definite.

Case 1-2: If $w_3 = 0$, we can find the standard transformation matrix Q_{112} such that $T_{112} = Q_{112}Z_{11}Q_{112}^{-1}$, where

$$Q_{112} = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ 0 & w_1 & \frac{a_{33}(w_2-w_1)}{w_1} & a_{33} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with

$$\begin{aligned} m_1 =& a_{21}w_1,\ m_2 = a_{33}w_2 - a_{33}w_1 - w_1\frac{a_{21}^2 + a_{24}a_{41}}{a_{21}},\ m_3 = a_{23}w_1 + a_{24}w_2 + (\frac{a_{33}w_2 - a_{33}w_1}{w_1})^2, \\ m_4 =& a_{24}w_1 + a_{33}(\frac{a_{33}w_2 - a_{33}w_1}{w_1}) + a_{33}(\frac{a_{24}a_{41} - a_{21}a_{44}}{a_{21}} - \frac{a_{33}w_2}{w_1}). \end{aligned}$$

We have

$$T_{112} = \begin{bmatrix} -b_1 & -b_2 & -b_3 & -b_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}} - \frac{a_{33}w_2}{w_1} \end{bmatrix},$$

where $b_i (i = 1, 2, 3, 4)$ are marks, and we only care about their traits. According to the uniqueness of $\omega_Z(\zeta)$, it can be obtained that

$$\begin{aligned} b_1 =& a_{11} + a_{33} + a_{44} - \frac{a_{21}a_{42}}{a_{41}} > 0, \\ b_2 =& a_{11}a_{33} + a_{11}a_{44} + a_{33}a_{44} + \frac{a_{21}^2 a_{42}}{a_{41}} + \frac{a_{21}^2 a_{42}^2}{a_{41}^2} - \frac{a_{11}a_{21}a_{42}}{a_{41}} - \frac{a_{21}a_{33}a_{42}}{a_{41}} - \frac{a_{21}a_{42}a_{44}}{a_{41}} > 0, \\ b_3 =& a_{11}a_{33}a_{44} + a_{12}a_{21}(a_{33} + a_{44}) + a_{24}a_{41}(a_{11} - a_{12}) - a_{33}a_{41}(a_{13} + a_{14}) + a_{23}a_{33}a_{42} + a_{24}a_{33}a_{41} \\ &+ \frac{a_{21}a_{33}a_{42}(a_{21} - a_{11})}{a_{41}} + \frac{a_{21}a_{42}a_{44}(a_{21} - a_{11})}{a_{41}} + \frac{a_{11}a_{21}^2 a_{42}^2}{a_{41}^2} + \frac{a_{21}^2 a_{33}a_{42}^2}{a_{41}^2} + \frac{a_{21}^2 a_{42}^2 a_{44}}{a_{41}^2} > 0. \end{aligned}$$

The following results can be obtained that $b_1b_2 - b_3 > 0$, then (4.6) can be transformed into the following form:

$$\begin{aligned} &(Q_{112}U_{11}U_1)D_1^2(Q_{112}U_{11}U_1)^T \\ &+ [(Q_{112}U_{11}U_1)Z(Q_{112}U_{11}U_1)^{-1}](Q_{112}U_{11}U_1)\Lambda_1(Q_{112}U_{11}U_1)^T \\ &+ (Q_{112}U_{11}U_1)\Lambda_1(Q_{112}U_{11}U_1)^T[(Q_{112}U_{11}U_1)Z(Q_{112}U_{11}U_1)^{-1}]^T = 0. \end{aligned}$$

That is, $D_0^2 + T_{112}\Theta_2 + \Theta_2 T_{112}^T = 0$, where $\Theta_2 = \frac{1}{\rho_{112}^2}(Q_{112}U_{11}U_1)\Lambda_1(Q_{112}U_{11}U_1)^T$, $\rho_{112} = a_{21}w_1\sigma_1$. According to the Lemma 2.6, we can obtain the semipositive definite matrix

$$\Theta_2 = \begin{bmatrix} \frac{b_2}{2(b_1b_2-b_3)} & 0 & -\frac{1}{2(b_1b_2-b_3)} & 0 \\ 0 & \frac{1}{2(b_1b_2-b_3)} & 0 & 0 \\ -\frac{1}{2(b_1b_2-b_3)} & 0 & \frac{b_1}{2b_3(b_1b_2-b_3)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

Therefore, by calculation,

$$\Lambda_1 = \rho_{111}^2 (Q_{111} U_{11} U_1)^{-1} \Theta_2 [(Q_{111} U_{11} U_1)^{-1}]^T \tag{4.8}$$

is semi-postive definite.

Case 2: When $w_1 \neq 0$ and $w_2 = 0$, the objective is to determine the standard transformation matrix Q_{12} that satisfies the equation $T_{12} = Q_{12} Z_1 Q_{12}^{-1}$, where

$$Q_{12} = \begin{bmatrix} -\frac{a_{23}a_{33}a_{41}^2}{a_{21}} & m_2 & m_3 & m_4 \\ 0 & -\frac{a_{23}a_{33}a_{41}^2}{a_{21}^2} & (-a_{33} + \frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}})\frac{a_{23}a_{41}}{a_{21}} & [a_{33} + (\frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}})^2]\frac{a_{23}a_{41}}{a_{21}} \\ 0 & 0 & \frac{a_{23}a_{41}}{a_{21}} & \frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with

$$\begin{aligned} m_2 =& (\frac{a_{21}^2 + a_{24}a_{41}}{a_{21}} + a_{33} - \frac{a_{24}a_{41} - a_{21}a_{44}}{a_{21}})\frac{a_{23}a_{33}a_{41}^2}{a_{21}^2}, \\ m_3 =& \frac{a_{23}a_{41}}{a_{21}}[a_{33}^2 - a_{33}\frac{a_{24}a_{41} - a_{21}a_{44}}{a_{21}} + (\frac{a_{24}a_{41} - a_{21}a_{44}}{a_{21}})^2], \\ m_4 =& -\frac{a_{23}a_{24}a_{33}a_{41}^2}{a_{21}^2} + (-a_{33} + \frac{a_{24}a_{41} - a_{21}a_{44}}{a_{21}})\frac{a_{23}a_{33}a_{41}}{a_{21}}. \end{aligned}$$

We can obtain that $T_{12} = T_{111}$, and (4.6) can be transformed into

$$D_0^2 + T_{12}\Theta_1 + \Theta_1 T_{12}^T = 0.$$

where $\Theta_1 = \frac{1}{\rho_{12}^2}(Q_{12}U_1)\Lambda_1(Q_{12}U_1)^T$, $\rho_{12} = -\frac{a_{23}a_{33}a_{41}^2}{a_{21}}\sigma_1$. Consequently,

$$\Lambda_1 = \rho_{12}^2 (Q_{12} U_1)^{-1} \Theta_1 [(Q_{12} U_1)^{-1}]^T \tag{4.9}$$

is a positive definite matrix.

Case 3: When $w_1 = 0$ and $w_2 \neq 0$, there exists the transformation matrix U_{13} such that $Z_{13} = U_{13} Z_1 U_{13}^{-1}$, where

$$U_{13} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

then we have

$$Z_{13} = \begin{bmatrix} -a_{11} & \frac{a_{14}a_{41}-a_{12}a_{21}}{a_{21}} & -a_{14} & -a_{13} \\ -a_{21} & -\frac{a_{21}^2+a_{24}a_{41}}{a_{21}} & a_{24} & a_{23} \\ 0 & w_2 & \frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}} & \frac{a_{23}a_{41}}{a_{21}} \\ 0 & 0 & a_{33} & -a_{33} \end{bmatrix},$$

we are able to find the standard transformation matrix Q_{13} such that $T_{13} = Q_{13} Z_{13} Q_{13}^{-1}$, where

$$Q_{13} = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ 0 & -a_{33}(\frac{a_{21}^2+a_{24}a_{41}}{a_{21}}) & a_{33}(-a_{33} + \frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}}) & a_{33}^2 + \frac{a_{23}a_{33}a_{41}}{a_{21}} \\ 0 & 0 & a_{33} & -a_{33} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with

$$m_1 = -a_{21}a_{33}w_2,\ m_2 = a_{33}(\frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}} - \frac{a_{21}^2+a_{24}a_{41}}{a_{21}} - a_{33})\frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}},$$
$$m_3 = a_{33}[a_{24}w_2 + \frac{a_{23}a_{33}a_{41}}{a_{21}} - a_{33}(\frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}}) + a_{33}^2 + (\frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}})^2],$$
$$m_4 = a_{23}a_{33}w_2 + \frac{a_{23}a_{33}a_{41}}{a_{21}}(\frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}} - a_{33}) - a_{33}(\frac{a_{23}a_{33}a_{41}}{a_{21}} + a_{33}^2).$$

We can obtain that $T_{13} = T_{111}$, and (4.6) can be transformed into

$$D_0^2 + T_{13}\Theta_1 + \Theta_1 T_{13}^T = 0,$$

where $\Theta_1 = \frac{1}{\rho_{13}^2}(Q_{13}U_{13}U_1)\Lambda_1(Q_{13}U_{13}U_1)^T$, $\rho_{13} = -a_{21}w_2\sigma_1$. Consequently,

$$\Lambda_1 = \rho_{13}^2(Q_{13}U_{13}U_1)^{-1}\Theta_1[(Q_{13}U_{13}U_1)^{-1}]^T \tag{4.10}$$

is a positive definite matrix.

Case 4: When $w_1 = 0$ and $w_2 = 0$, there exists the transformation matrix U_{14} such that $T_{14} = Q_{14}Z_1Q_{14}^{-1}$, where

$$Q_{14} = \begin{bmatrix} m_1 & m_2 & m_3 & m_4 \\ a_{21} & -\frac{a_{21}^2+a_{24}a_{41}}{a_{21}} & a_{23} & a_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with

$$m_1 = -a_{21}(\frac{a_{11}a_{21}+a_{21}^2+a_{24}a_{41}}{a_{21}}),\ m_2 = a_{14}a_{41} - a_{12}a_{21} + (\frac{a_{21}^2+a_{24}a_{41}}{a_{21}})^2,$$
$$m_3 = -a_{13}a_{21} - a_{23}a_{33} - \frac{a_{23}a_{21}^2}{a_{21}},\ m_4 = a_{23}a_{33} - a_{14}a_{21} - a_{24}a_{21} - a_{24}a_{44}.$$

We then have

$$T_{14} = \begin{bmatrix} -c_1 & -c_2 & -c_3 & -c_4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -a_{33} & a_{33} \\ 0 & 0 & \frac{a_{23}a_{41}}{a_{21}} & \frac{a_{24}a_{41}-a_{21}a_{44}}{a_{21}} \end{bmatrix},$$

$c_1 = a_{11} + a_{21} + \frac{a_{24}a_{41}}{a_{21}} > 0$, $c_2 = (a_{11}+a_{12})a_{21} + (a_{41}+a_{42})a_{24} - a_{14}a_{41} + \frac{a_{11}a_{24}a_{41}}{a_{21}} + \frac{a_{23}a_{33}a_{41}}{a_{21}} + \frac{a_{24}^2a_{41}^2}{a_{21}^2} > 0$. Thus, (4.6) can be transformed into

$$D_0^2 + T_{14}\Theta_3 + \Theta_3 T_{14}^T = 0,$$

where $\Theta_3 = \frac{1}{\rho_{14}^2}(Q_{14}U_1)\Lambda_1(Q_{14}U_1)^T$, $\rho_{14} = a_{21}\sigma_1$. With Lemma 2.7, we can obtain the semi-postive definite matrix

$$\Theta_3 = \begin{bmatrix} \frac{1}{2c_1} & 0 & 0 & 0 \\ 0 & \frac{1}{2c_1c_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, by calculation,

$$\Lambda_1 = \rho_{14}^2 (Q_{14}U_1)^{-1}\Theta_3[(Q_{14}U_1)^{-1}]^T \tag{4.11}$$

is semi-postive definite.

Step 2. For the following algebraic equation

$$D_2^2 + Z\Lambda_2 + \Lambda_2 Z^T = 0, \tag{4.12}$$

consider the corresponding order matrix J_2 and elimination matrix U_2:

$$J_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{a_{44}}{a_{33}} & 1 & 0 \\ 0 & \frac{a_{14}}{a_{33}} & 0 & 1 \end{bmatrix}.$$

Let $Z_2 = (U_2J_2)Z(U_2J_2)^{-1}$, then we can obtain that

$$Z_2 = \begin{bmatrix} a_{24} & \frac{a_{21}a_{33}+a_{21}a_{44}-a_{14}a_{23}}{a_{33}} & -a_{12} & a_{23} \\ a_{33} & a_{14} & 0 & -a_{33} \\ 0 & r_1 & -a_{42} & -a_{44} \\ 0 & r_2 & -a_{12} & -a_{13}-a_{14} \end{bmatrix},$$

where $r_1 = -a_{41} + \frac{a_{14}a_{44}+a_{42}a_{44}}{a_{33}}$, $r_2 = -a_{11} + \frac{a_{13}a_{14}+a_{12}a_{44}+a_{14}^2}{a_{33}}$.

Case 1: When $r_1 \neq 0, r_2 \neq 0$, there exists the elimination matrix U_{21} such that $Z_{21} = U_{21}Z_2U_{21}^{-1}$, where

$$U_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{r_2}{r_1} & 1 \end{bmatrix},$$

then we have

$$Z_{21} = \begin{bmatrix} a_{24} & a_{21} + \frac{a_{21}a_{44}-a_{14}a_{23}}{a_{33}} & -a_{12} + \frac{a_{23}r_2}{r_1} & a_{23} \\ a_{33} & a_{14} & -\frac{a_{33}r_2}{r_1} & -a_{33} \\ 0 & r_1 & -a_{42} - \frac{a_{44}r_2}{r_1} & -a_{44} \\ 0 & 0 & r_3 & -a_{13}-a_{14}+\frac{a_{44}r_2}{r_1} \end{bmatrix},$$

where $r_3 = (a_{42} + \frac{a_{44}r_2}{r_1})\frac{r_2}{r_1} - a_{12} - \frac{(a_{13}+a_{14})r_2}{r_1}$.

Case 1-1: If $r_3 \neq 0$, the standard transformation matrix Q_{211} can be determined in order to obtain the transformation T_{211} as $T_{211} = Q_{211}Z_{21}Q_{211}^{-1}$, where

$$Q_{211} = \begin{bmatrix} n_1 & n_2 & n_3 & n_4 \\ 0 & r_1r_3 & -(a_{13}+a_{14}+a_{42})r_3 & (\frac{a_{44}r_2}{r_1} - a_{13} - a_{14})^2 - a_{44}r_3 \\ 0 & 0 & r_3 & \frac{a_{44}r_2}{r_1} - a_{13} - a_{14} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with

$$\begin{aligned}
n_1 =& a_{33}r_1r_3,\ n_2 = (a_{13}-a_{42})r_1r_3,\\
n_3 =& [-a_{33}r_2 - a_{44}r_3 + (a_{42}+\frac{a_{44}r_2}{r_1})(a_{13}+a_{14}-\frac{a_{44}r_2}{r_1}) + (a_{42}+\frac{a_{44}r_2}{r_1})^2 + (\frac{a_{44}r_2}{r_1}-a_{13}-a_{14})^2]r_3,\\
n_4 =& -a_{33}r_1r_3 + a_{44}(a_{13}+a_{14}+a_{42})r_3 + [-a_{44}r_3 + (\frac{a_{44}r_2}{r_1}-a_{13}-a_{14})^2](\frac{a_{44}r_2}{r_1}-a_{13}-a_{14}).
\end{aligned}$$

As a result, we have $T_{211} = T_{111}$, and (4.12) has the following form:

$$\begin{aligned}
&(Q_{211}U_{21}U_2J_2)D_2^2(Q_{211}U_{21}U_2J_2)^T\\
&+[(Q_{211}U_{21}U_2J_2)Z(Q_{211}U_{21}U_2J_2)^{-1}](Q_{211}U_{21}U_2J_2)\Lambda_2(Q_{211}U_{21}U_2J_2)^T\\
&+(Q_{211}U_{21}U_2J_2)\Lambda_2(Q_{211}U_{21}U_2J_2)^T[(Q_{211}U_{21}U_2J_2)Z(Q_{211}U_{21}U_2J_2)^{-1}]^T = 0.
\end{aligned}$$

That is, $D_0^2 + T_{211}\tilde{\Theta}_1 + \tilde{\Theta}_1 T_{211} = 0$, where $\tilde{\Theta}_1 = \frac{1}{\rho_{211}^2}(Q_{211}U_{21}U_2J_2)\Lambda_2(Q_{211}U_{21}U_2J_2)^T$, $\rho_{211} = a_{33}r_1r_3\sigma_2$. According to Lemma 2.5, we can obtain the semipositive definite matrix

$$\tilde{\Theta}_1 = \begin{bmatrix}
\frac{s_2s_3-s_1s_4}{2[s_1(s_2s_3-s_1s_4)-s_3^2]} & 0 & -\frac{s_3}{2[s_1(s_2s_3-s_1s_4)-s_3^2]} & 0\\
0 & \frac{s_3}{2[s_1(s_2s_3-s_1s_4)-s_3^2]} & 0 & \frac{-s_1}{2[s_1(s_2s_3-s_1s_4)-s_3^2]}\\
-\frac{s_3}{2[s_1(s_2s_3-s_1s_4)-s_3^2]} & 0 & \frac{s_1}{2[s_1(s_2s_3-s_1s_4)-s_3^2]} & 0\\
0 & -\frac{s_1}{2[s_1(s_2s_3-s_1s_4)-s_3^2]} & 0 & \frac{s_1s_2-s_3}{2[s_1(s_2s_3-s_1s_4)-s_3^2]}
\end{bmatrix}.$$

Therefore, it can be obtained by calculation that

$$\Lambda_2 = \rho_{211}^2(Q_{211}U_{21}U_2J_2)^{-1}\tilde{\Theta}_1[(Q_{211}U_{21}U_2J_2)^{-1}]^T. \tag{4.13}$$

Case 1-2: If $r_3 = 0$, we can find the standard transformation matrix Q_{212} such that $T_{212} = Q_{212}Z_{21}Q_{212}^{-1}$, where

$$Q_{212} = \begin{bmatrix}
a_{33}r_1 & (a_{14}-a_{42}-\frac{a_{44}r_2}{r_1})r_1 & (a_{42}+\frac{a_{44}r_2}{r_1})^2 - a_{33}r_2 & (a_{13}+a_{14}+a_{42})a_{44} - a_{33}r_1\\
0 & r_1 & -(a_{42}+\frac{a_{44}r_2}{r_1}) & -a_{44}\\
0 & 0 & 1 & 0\\
0 & 0 & 0 & 1
\end{bmatrix},$$

then we have

$$T_{212} = \begin{bmatrix}
-\tilde{b}_1 & -\tilde{b}_2 & -\tilde{b}_3 & -\tilde{b}_4\\
1 & 0 & 0 & 0\\
0 & 1 & 0 & 0\\
0 & 0 & 0 & \frac{a_{44}r_2}{r_1} - a_{13} - a_{14}
\end{bmatrix},$$

where

$$
\begin{aligned}
\tilde{b}_1 =& \frac{1}{a_{14}a_{44}+a_{42}a_{44}-a_{33}a_{41}}(-a_{14}^2a_{44}-a_{12}a_{44}^2+a_{11}a_{33}a_{44}+a_{13}a_{42}^2+a_{13}a_{33}a_{41}+a_{14}^2a_{44})\\
&+\frac{1}{a_{14}a_{44}+a_{42}a_{44}-a_{33}a_{41}}(a_{11}a_{14}a_{44}+a_{11}a_{42}a_{44}-a_{11}a_{33}a_{41}+a_{14}a_{21}a_{44}-a_{14}a_{33}a_{41})\\
&+\frac{1}{a_{14}a_{44}+a_{42}a_{44}-a_{33}a_{41}}(a_{14}a_{33}a_{44}+a_{33}a_{42}a_{44}-a_{33}^2a_{41}+a_{14}a_{44}^2+a_{42}a_{44}^2)\\
&+\frac{1}{a_{14}a_{44}+a_{42}a_{44}-a_{33}a_{41}}(+a_{14}a_{42}a_{44}+a_{21}a_{42}a_{44}-a_{21}a_{33}a_{41}-a_{33}a_{41}a_{44})>0,\\
\tilde{b}_2 =& [a_{11}(a_{21}+a_{33}+a_{44})+a_{21}(a_{12}+a_{33}+a_{44})+a_{33}a_{44}+a_{24}a_{42}-a_{14}a_{41}]\\
&+\frac{1}{(a_{14}a_{44}+a_{42}a_{44}-a_{33}a_{41})^2}(-a_{14}^2a_{44}-a_{12}a_{44}^2+a_{11}a_{33}a_{44})(a_{13}a_{42}^2+a_{13}a_{33}a_{41}+a_{14}^2a_{44})\\
&+\frac{1}{(a_{14}a_{44}+a_{42}a_{44}-a_{33}a_{41})^2}(a_{13}a_{14}a_{44}+a_{14}^2a_{44}+a_{12}a_{44}^2-a_{11}a_{33}a_{44})(a_{14}a_{42}a_{44}-a_{14}a_{33}a_{41})\\
&+\frac{1}{(a_{14}a_{44}+a_{42}a_{44}-a_{33}a_{41})^2}(a_{13}a_{14}a_{44}+a_{14}^2a_{44}+a_{12}a_{44}^2-a_{11}a_{33}a_{44})\\
&+\frac{1}{(a_{14}a_{44}+a_{42}a_{44}-a_{33}a_{41})^2}(a_{13}a_{14}a_{44}+a_{14}^2a_{44}+a_{12}a_{44}^2-a_{11}a_{33}a_{44})>0,\\
\tilde{b}_3 =& \frac{a_{14}a_{44}+a_{42}a_{44}-a_{33}a_{41}}{a_{13}a_{14}a_{44}+a_{14}^2a_{44}+a_{12}a_{44}^2-a_{13}a_{33}a_{44}}[a_{33}a_{42}(a_{11}a_{23}+a_{11}a_{24}a_{33}a_{41}-a_{14}a_{21}-a_{13}a_{21})]\\
&+\frac{a_{14}a_{44}+a_{42}a_{44}-a_{33}a_{41}}{a_{13}a_{14}a_{44}+a_{14}^2a_{44}+a_{12}a_{44}^2-a_{13}a_{33}a_{44}}(+a_{21}a_{33}a_{44}(a_{11}+a_{12})+a_{21}a_{33}a_{41}(a_{13}+a_{14}))\\
&+\frac{a_{14}a_{44}+a_{42}a_{44}-a_{33}a_{41}}{a_{13}a_{14}a_{44}+a_{14}^2a_{44}+a_{12}a_{44}^2-a_{13}a_{33}a_{44}}[a_{12}a_{33}a_{41}(a_{23}+a_{24})]>0.
\end{aligned}
$$

We can obtain $\tilde{b}_1\tilde{b}_2-\tilde{b}_3>0$, then (4.12) can be transformed into the following form:

$$
\begin{aligned}
&(Q_{212}U_{21}U_2J_2)D_2^2(Q_{212}U_{21}U_2J_2)^T\\
&+[(Q_{212}U_{12}U_2J_2)Z(Q_{212}U_{21}U_2J_2)^{-1}](Q_{212}U_{21}U_2J_2)\Lambda_2(Q_{212}U_{21}U_2J_2)^T\\
&+(Q_{212}U_{21}U_2J_2)\Lambda_2(Q_{212}U_{21}U_2J_2)^T[(Q_{212}U_{21}U_2J_2)Z(Q_{212}U_{21}U_2J_2)^{-1}]^T=0.
\end{aligned}
$$

That is, $D_0^2+T_{212}\tilde{\Theta}_2+\tilde{\Theta}_2T_{212}^T=0$, where $\tilde{\Theta}_2=\frac{1}{\rho_{212}^2}(Q_{212}U_{21}U_2J_2)\Lambda_2(Q_{212}U_{21}U_2J_2)^T$, $\rho_{212}=a_{33}r_1\sigma_2$. According to Lemma 2.6, we can obtain the semipositive definite matrix

$$
\tilde{\Theta}_2=\begin{bmatrix}\frac{\tilde{b}_2}{2(\tilde{b}_1\tilde{b}_2-\tilde{b}_3)} & 0 & -\frac{1}{2(\tilde{b}_1\tilde{b}_2-\tilde{b}_3)} & 0\\ 0 & \frac{1}{2(\tilde{b}_1\tilde{b}_2-\tilde{b}_3)} & 0 & 0\\ -\frac{1}{2(\tilde{b}_1\tilde{b}_2-\tilde{b}_3)} & 0 & \frac{\tilde{b}_1}{2\tilde{b}_3(\tilde{b}_1\tilde{b}_2-\tilde{b}_3)} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.
$$

Therefore, it can be obtained by calculation that

$$
\Lambda_2=\rho_{212}^2(Q_{212}U_{21}U_2J_2)^{-1}\tilde{\Theta}_2[(Q_{212}U_{21}U_2J_2)^{-1}]^T \tag{4.14}
$$

is semi-postive definite.
Case 2: When $r_1 \neq 0$ and $r_2 = 0$, there exists the transformation matrix Q_{22} such that $T_{22} = Q_{22}Z_2Q_{22}^{-1}$, where

$$Q_{22} = \begin{bmatrix} -a_{12}a_{33}r_1 & a_{12}(a_{13}+a_{42})r_1 & n_3 & n_4 \\ 0 & -a_{12}r_1 & a_{12}(a_{13}+a_{14}+a_{42}) & (a_{13}+a_{14})^2 + a_{12}a_{44} \\ 0 & 0 & -a_{12} & -a_{13}-a_{14} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with

$$\begin{aligned} n_3 =& -a_{12}[a_{12}a_{44} + a_{42}(a_{13}+a_{14}) + a_{42}^2 + (a_{13}+a_{14})^2], \\ n_4 =& a_{12}a_{33}r_1 - a_{12}a_{44}(a_{13}+a_{14}+a_{42}) - [a_{12}a_{44} + (a_{13}+a_{14})^2](a_{13}+a_{14}). \end{aligned}$$

It can be obtained that $T_{22} = T_{111}$, and (4.12) can be transformed into

$$D_0^2 + T_{22}\tilde{\Theta}_1 + \tilde{\Theta}_1 T_{22}^T = 0,$$

where $\tilde{\Theta}_1 = \frac{1}{\rho_{22}^2}(Q_{22}U_2J_2)\Lambda_2(Q_{22}U_2J_2)^T$, $\rho_{22} = -a_{21}a_{33}r_1\sigma_2$. Consequently,

$$\Lambda_2 = \rho_{22}^2(Q_{22}U_2J_2)^{-1}\tilde{\Theta}_1[(Q_{22}U_2J_2)^{-1}]^T \tag{4.15}$$

is a positive definite matrix.
Case 3: When $r_1 = 0$ and $r_2 \neq 0$, there exists the transformation matrix U_{23} such that $Z_{23} = U_{23}Z_2U_{23}^{-1}$

$$U_{23} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

then we have

$$Z_{23} = \begin{bmatrix} a_{24} & a_{21} + \frac{a_{21}a_{44} - a_{14}a_{23}}{a_{33}} & a_{23} & -a_{21} \\ a_{33} & a_{14} & -a_{33} & 0 \\ 0 & r_2 & -a_{13}-a_{14} & -a_{12} \\ 0 & 0 & -a_{44} & -a_{42} \end{bmatrix},$$

We can find the standard transformation matrix Q_{23} such that $T_{23} = Q_{23}Z_{23}Q_{23}^{-1}$, where

$$Q_{23} = \begin{bmatrix} -a_{33}a_{44}r_2 & a_{44}(a_{13}+a_{42})r_2 & m_3 & m_4 \\ 0 & -a_{44}r_2 & a_{44}(a_{13}+a_{14}+a_{42}) & a_{42}^2 + a_{12}a_{44} \\ 0 & 0 & -a_{44} & -a_{42} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with

$$\begin{aligned} m_3 =& -a_{44}[-a_{33}r_2 + a_{12}a_{44} + a_{42}(a_{13}+a_{14}) + a_{24}^2 + (a_{13}+a_{14})^2] \\ m_4 =& a_{12}a_{33}r_1 - a_{12}a_{44}(a_{13}+a_{14}+a_{42}) - [a_{12}a_{44} + (a_{13}+a_{14})^2](a_{13}+a_{14}). \end{aligned}$$

It can be obtained by calculation that $T_{23} = T_{111}$, and (4.12) can be transformed into

$$D_0^2 + T_{23}\tilde{\Theta}_1 + \tilde{\Theta}_1 T_{23}^T = 0,$$

where $\tilde{\Theta}_1 = \frac{1}{\rho_{23}^2}(Q_{23}U_{23}U_2J_2)\Lambda_2(Q_{23}U_{23}U_2J_2)^T$, $\rho_{23} = -a_{33}a_{44}r_2\sigma_2$. Consequently,

$$\Lambda_2 = \rho_{23}^2(Q_{23}U_{23}U_2J_2)^{-1}\tilde{\Theta}_1[(Q_{23}U_{23}U_2J_2)^{-1}]^T \tag{4.16}$$

is a positive definite matrix.

Case 4: When $r_1 = 0$ and $r_2 = 0$, there exists the transformation matrix Q_{24} such that $T_{24} = Q_{24}Z_2Q_{24}^{-1}$, where

$$Q_{24} = \begin{bmatrix} a_{33}(a_{14}+a_{24}) & a_{21}a_{33}+a_{21}a_{44}-a_{14}a_{23}+a_{14}^2 & -a_{21}a_{33}+a_{12}a_{33} & (a_{13}+a_{23})a_{33} \\ a_{33} & a_{14} & 0 & -a_{33} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

then we have

$$T_{24} = \begin{bmatrix} -\tilde{c}_1 & -\tilde{c}_2 & -\tilde{c}_3 & -\tilde{c}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -a_{42} & -a_{44} \\ 0 & 0 & -a_{12} & -a_{13}-a_{14} \end{bmatrix},$$

where $c_1 = a_{11} + a_{21} + a_{33} + a_{44} - a_{13} - a_{14} - a_{42} > 0$, $c_2 = \frac{s_4}{a_{13}a_{42}+a_{14}a_{42}-a_{12}a_{44}} > 0$. Thus, (4.12) can be transformed into

$$D_0^2 + T_{24}\tilde{\Theta}_3 + \tilde{\Theta}_3T_{24}^T = 0,$$

where $\tilde{\Theta}_3 = \frac{1}{\rho_{24}^2}(Q_{24}U_2J_2)\Lambda_2(Q_{24}U_2J_2)^T$, $\rho_{24} = a_{33}\sigma_2$. With Lemma 2.7, we can obtain the positive semidefinite matrix

$$\tilde{\Theta}_3 = \begin{bmatrix} \frac{1}{2\tilde{c}_1} & 0 & 0 & 0 \\ 0 & \frac{1}{2\tilde{c}_1\tilde{c}_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, it can be obtained by calculation that

$$\Lambda_2 = \rho_{24}^2(Q_{24}U_2J_2)^{-1}\tilde{\Theta}_3[(Q_{24}U_2J_2)^{-1}]^T \tag{4.17}$$

is semi-postive definite.

Step 3. For the following algebraic equation

$$D_3^2 + Z\Lambda_3 + \Lambda_3Z^T = 0, \tag{4.18}$$

consider the corresponding order matrix J_3 and elimination matrix U_3 :

$$J_3 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad U_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{a_{23}}{a_{13}} & 1 \end{bmatrix}.$$

Let $Z_3 = (U_3J_3)Z(U_3J_3)^{-1}$. It can be obtained by calculation that

$$Z_3 = \begin{bmatrix} -a_{33} & 0 & a_{33} & 0 \\ -a_{13} & \frac{a_{12}a_{23}-a_{11}a_{13}}{a_{13}} & -a_{14} & -a_{12} \\ 0 & \frac{a_{23}a_{42}-a_{13}a_{41}}{a_{13}} & -a_{44} & -a_{42} \\ 0 & k_1 & k_2 & -\frac{a_{12}a_{23}+a_{13}a_{21}}{a_{13}} \end{bmatrix},$$

where $k_1 = a_{21} + \frac{a_{21}a_{23}-a_{11}a_{23}}{a_{13}} + \frac{a_{12}a_{23}^2}{a_{13}^2}$, $k_2 = a_{24} - \frac{a_{14}a_{23}}{a_{13}}$.
Case 1: When $k_1 = 0$ and $k_2 \neq 0$, there exists the standard transformation matrix Q_{31} such that $T_{31} = Q_{31}Z_3Q_{31}^{-1}$, where

$$Q_{31} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ 0 & (-a_{41} + \frac{a_{23}a_{42}}{a_{13}})k_2 & -(a_{44} + \frac{a_{12}a_{23}+a_{13}a_{21}}{a_{13}})k_2 & (\frac{a_{12}a_{23}+a_{13}a_{21}}{a_{13}})^2 - a_{42}k_2 \\ 0 & 0 & k_2 & \frac{a_{12}a_{23}+a_{13}a_{21}}{a_{13}} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with

$$\begin{aligned} q_1 &= -(-a_{41} + \frac{a_{23}a_{42}}{a_{13}})a_{13}k_2,\ q_2 = (-a_{11} - a_{44} + \frac{a_{13}a_{21}}{a_{13}})(-a_{41} + \frac{a_{23}a_{42}}{a_{13}})k_2, \\ q_3 &= [a_{14}a_{41} - \frac{a_{14}a_{23}a_{42}}{a_{13}} - a_{42}k_2 + \frac{a_{12}a_{23}a_{44} + a_{13}a_{21}a_{44}}{a_{13}} + a_{44}^2 + (\frac{a_{12}a_{23} + a_{13}a_{21}}{a_{13}})^2]k_2, \\ q_4 &= (a_{41} - \frac{a_{23}a_{42}}{a_{13}})a_{12}k_2 + (a_{44} + \frac{a_{12}a_{23} + a_{13}a_{21}}{a_{13}})a_{42}k_2 + [a_{42}k_2 - (\frac{a_{12}a_{23} + a_{13}a_{21}}{a_{13}})^2]\frac{a_{12}a_{23} + a_{13}a_{21}}{a_{13}}. \end{aligned}$$

We can obtain that $T_{31} = T_{111}$, and (4.18) can be transformed into

$$D_0^2 + T_{31}\bar{\Theta}_1 + \bar{\Theta}_1 T_{31}^T = 0,$$

where $\bar{\Theta}_1 = \frac{1}{\rho_{31}^2}(Q_{31}U_3J_3)\Lambda_3(Q_{31}U_3J_3)^T$, $\rho_{31} = (a_{13}a_{41} - a_{23}a_{42})k_2\sigma_3$, and $\bar{\Theta}_1 = \Theta_1$. Consequently,

$$\Lambda_3 = \rho_{31}^2(Q_{31}U_3J_3)^{-1}\bar{\Theta}_1[(Q_{31}U_3J_3)^{-1}]^T \tag{4.19}$$

is positive definite matrix.
Case 2: When $k_1 \neq 0$, $k_2 = 0$, there exists the transformation matrix U_{32} such that $Z_{32} = U_{32}Z_3U_{32}^{-1}$, where

$$U_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -\frac{a_{13}k_1}{a_{23}a_{42}-a_{13}a_{41}} & 1 \end{bmatrix},$$

then we have

$$Z_{32} = \begin{bmatrix} -a_{33} & 0 & a_{33} & 0 \\ -a_{13} & -a_{11} + \frac{a_{12}a_{23}}{a_{13}} & -a_{14} + \frac{a_{12}a_{13}k_1}{a_{23}a_{42}-a_{13}a_{41}} & a_{12} \\ 0 & -a_{41} + \frac{a_{23}a_{42}}{a_{13}} & -a_{44} + \frac{a_{13}a_{42}k_1}{a_{23}a_{42}-a_{13}a_{41}} & a_{42} \\ 0 & 0 & k_3 & \frac{a_{12}a_{23}+a_{13}a_{21}}{a_{13}} - \frac{a_{13}a_{42}k_1}{a_{23}a_{42}-a_{13}a_{41}} \end{bmatrix},$$

where $k_3 = \frac{(a_{13}a_{44}+a_{12}a_{23}+a_{13}a_{21})k_1}{a_{23}a_{42}-a_{13}a_{41}} - \frac{a_{13}^2 a_{42} k_1^2}{(a_{23}a_{42}-a_{13}a_{41})^2}$.
Case 2-1: If $k_3 \neq 0$, we can find the standard transformation matrix Q_{321} such that $T_{321} = Q_{321}Z_{32}T_{321}^{-1}$, where

$$Q_{321} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ 0 & (-a_{41} + \frac{a_{23}a_{24}}{a_{13}})k_3 & (-a_{44} + \frac{a_{12}a_{23}+a_{13}a_{21}}{a_{13}})k_3 & (\frac{a_{12}a_{23}+a_{13}a_{21}}{a_{13}} - \frac{a_{13}a_{42}k_1}{a_{23}a_{24}-a_{13}a_{41}})^2 + a_{42}k_3 \\ 0 & 0 & k_3 & \frac{a_{12}a_{23}+a_{13}a_{21}}{a_{13}} - \frac{a_{13}a_{42}k_1}{a_{23}a_{24}-a_{13}a_{41}} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with

$$\begin{aligned}
q_1 =&(a_{13}a_{41} - a_{23}a_{42})k_3,\ q_2 = (-a_{11} - a_{44} + \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{12}a_{23} + a_{13}a_{21}}{a_{13}})a_{42}k_3, \\
q_3 =&k_3[(-a_{41} + \frac{a_{23}a_{24}}{a_{13}})(-a_{14} + \frac{a_{12}a_{13}k_1}{a_{23}a_{42} - a_{13}a_{41}}) + a_{42}k_2 + (-a_{44} + \frac{a_{13}a_{42}k_1}{a_{23}a_{42} - a_{13}a_{41}})\frac{a_{12}a_{23} + a_{13}a_{21}}{a_{13}} \\
&- (-a_{44} + \frac{a_{13}a_{42}k_1}{a_{23}a_{42} - a_{13}a_{41}})\frac{a_{13}a_{42}k_1}{a_{23}a_{42} - a_{13}a_{41}} + (-a_{44} + \frac{a_{13}a_{42}k_1}{a_{23}a_{42} - a_{13}a_{41}})^2 \\
&+ (\frac{a_{12}a_{23} + a_{13}a_{21}}{a_{13}} - \frac{a_{13}a_{42}k_1}{a_{23}a_{42} - a_{13}a_{41}})^2], \\
q_4 =&[a_{42}k_3 + (\frac{a_{12}a_{23} + a_{13}a_{21}}{a_{13}} - \frac{a_{13}a_{42}k_1}{a_{23}a_{42} - a_{13}a_{41}})^2](\frac{a_{12}a_{23} + a_{13}a_{21}}{a_{13}} - \frac{a_{13}a_{42}k_1}{a_{23}a_{42} - a_{13}a_{41}}) \\
&+ (-a_{41} + \frac{a_{23}a_{24}}{a_{13}})a_{12}k_3 + (-a_{44} + \frac{a_{12}a_{23} + a_{13}a_{21}}{a_{13}})a_{42}k_3.
\end{aligned}$$

It can be obtained by calculation that $T_{321} = T_{111}$, and (4.18) can be transformed into

$$D_0^2 + T_{321}\bar{\Theta}_1 + \bar{\Theta}_1 T_{321}^T = 0,$$

where $\bar{\Theta}_1 = \frac{1}{\rho_{321}^2}(Q_{321}U_{32}U_3J_3)\Lambda_3(Q_{321}U_{32}U_3J_3)^T$, $\rho_{321} = (a_{13}a_{41} - a_{23}a_{42})k_3\sigma_3$. Consequently,

$$\Lambda_3 = \rho_{321}^2(Q_{321}U_{32}U_3J_3)^{-1}\bar{\Theta}_1[(Q_{321}U_{32}U_3J_3)^{-1}]^T \tag{4.20}$$

is positive definite matrix.
Case 2-2: If $k_3 = 0$, we can find the standard transformation matrix Q_{322} such that $T_{322} = Q_{322}Z_{32}Q_{322}^{-1}$, where

$$Q_{322} = \begin{bmatrix} q_1 & q_2 & q_3 & q_4 \\ 0 & -a_{41} + \frac{a_{23}a_{24}}{a_{13}} & -a_{44} + \frac{a_{13}a_{42}k_1}{a_{23}a_{42}-a_{13}a_{41}} & a_{42} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with

$$\begin{aligned}
q_1 =&a_{13}a_{41} - a_{23}a_{42},\ q_2 = (-a_{11} - a_{44} + \frac{a_{12}a_{23}}{a_{13}} + \frac{a_{13}a_{42}k_1}{a_{23}a_{42} - a_{13}a_{41}})(-a_{41} + \frac{a_{23}a_{24}}{a_{13}}), \\
q_3 =&(-a_{44} + \frac{a_{13}a_{42}k_1}{a_{23}a_{42} - a_{13}a_{41}})^2 + (-a_{14} + \frac{a_{12}a_{13}k_1}{a_{23}a_{42} - a_{13}a_{41}})(-a_{41} + \frac{a_{23}a_{24}}{a_{13}}), \\
q_4 =&(-a_{44} + \frac{a_{13}a_{42}k_1}{a_{23}a_{42} - a_{13}a_{41}})a_{42} + (-a_{41} + \frac{a_{23}a_{24}}{a_{13}})a_{12}.
\end{aligned}$$

We can find

$$T_{322}=\begin{bmatrix}-\bar{b}_1 & -\bar{b}_2 & -\bar{b}_3 & -\bar{b}_4\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & \frac{a_{12}a_{23}+a_{13}a_{21}}{a_{13}}-\frac{a_{13}a_{42}k_1}{a_{23}a_{42}-a_{13}a_{41}}\end{bmatrix},$$

which means that

$$\begin{aligned}\bar{b}_1=&a_{13}^2a_{21}a_{42}+a_{12}a_{23}^2a_{42}+a_{13}a_{21}a_{23}a_{42}+a_{13}a_{21}a_{23}a_{42}+a_{13}a_{23}a_{33}a_{42}\\ &+a_{13}a_{23}a_{41}a_{44}-a_{13}^2a_{41}(a_{11}+a_{21}+a_{33}+a_{44})>0,\\ \bar{b}_2=&(a_{13}^2a_{21}a_{42}+a_{12}a_{23}^2a_{42}+a_{13}a_{21}a_{23}a_{42}+a_{13}a_{23}a_{33}a_{42})(a_{13}^2a_{21}a_{42}+a_{12}a_{23}^2a_{42}+a_{13}a_{21}a_{23}a_{42})\\ &+(a_{13}a_{21}a_{23}a_{42}+a_{13}a_{23}a_{41}a_{44})(a_{13}^2a_{21}a_{42}+a_{12}a_{23}^2a_{42}+a_{13}a_{21}a_{23}a_{42}-a_{11}a_{13}a_{23}a_{42})\\ &-a_{13}^2a_{41}(a_{11}+a_{21}+a_{33}+a_{44})(a_{13}^2a_{21}a_{42}+a_{12}a_{23}^2a_{42}+a_{13}a_{21}a_{23}a_{42}-a_{11}a_{13}a_{23}a_{42})\\ &+a_{11}(a_{21}+a_{33}+a_{44})+a_{21}(a_{12}+a_{33}+a_{44})+a_{33}a_{44}+a_{24}a_{42}-a_{14}a_{41}>0,\\ \bar{b}_3=&\frac{a_{13}(a_{23}a_{42}-a_{13}a_{41})}{a_{13}^2a_{21}a_{42}+a_{12}a_{23}^2a_{42}+a_{13}a_{21}a_{23}a_{42}-a_{11}a_{13}a_{23}a_{42}}[a_{33}a_{42}(a_{11}a_{23}+a_{11}a_{24}a_{33}a_{41}-a_{14}a_{21}-a_{13}a_{21})]\\ &+\frac{a_{13}(a_{23}a_{42}-a_{13}a_{41})}{a_{13}^2a_{21}a_{42}+a_{12}a_{23}^2a_{42}+a_{13}a_{21}a_{23}a_{42}-a_{11}a_{13}a_{23}a_{42}}[a_{21}a_{33}a_{44}(a_{11}+a_{12})-a_{21}a_{33}a_{41}(a_{13}+a_{14})]\\ &+\frac{a_{13}(a_{23}a_{42}-a_{13}a_{41})}{a_{13}^2a_{21}a_{42}+a_{12}a_{23}^2a_{42}+a_{13}a_{21}a_{23}a_{42}-a_{11}a_{13}a_{23}a_{42}}[-a_{12}a_{33}a_{41}(a_{23}+a_{24})]>0,\end{aligned}$$

we can find that $\bar{b}_1\bar{b}_2-\bar{b}_3>0$, and (4.18) can be transformed into the following form :

$$\begin{aligned}&(Q_{322}U_{32}U_3J_3)D_3^2(Q_{322}U_{32}U_3J_3)^T\\ &+[(Q_{322}U_{32}U_3J_3)Z(Q_{322}U_{32}U_3J_3)^{-1}](Q_{322}U_{32}U_3J_3)\Lambda_3(Q_{322}U_{32}U_3J_3)^T\\ &+(Q_{322}U_{32}U_3J_3)\Lambda_3(Q_{322}U_{32}U_3J_3)^T[(Q_{322}U_{32}U_3J_3)Z(Q_{322}U_{32}U_3J_3)^{-1}]^T=0.\end{aligned}$$

That is, $D_0^2+T_{322}\bar{\Theta}_2+\bar{\Theta}_2T_{322}^T=0$, where $\bar{\Theta}_2=\frac{1}{\rho_{322}^2}(Q_{322}U_{32}U_3J_3)\Lambda_3(Q_{322}U_{32}U_3J_3)^T$, $\rho_{322}=(a_{13}a_{41}-a_{23}a_{42})\sigma_3$. According to Lemma 2.6, we are able to obtain the semipositive definite matrix

$$\bar{\Theta}_2=\begin{bmatrix}\frac{\bar{b}_2}{2(\bar{b}_1\bar{b}_2-\bar{b}_3)} & 0 & -\frac{1}{2(\bar{b}_1\bar{b}_2-\bar{b}_3)} & 0\\ 0 & \frac{1}{2(\bar{b}_1\bar{b}_2-\bar{b}_3)} & 0 & 0\\ -\frac{1}{2(\bar{b}_1\bar{b}_2-\bar{b}_3)} & 0 & \frac{\bar{b}_1}{2\bar{b}_3(\bar{b}_1\bar{b}_2-\bar{b}_3)} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Therefore, it can be obtained by calculation that

$$\Lambda_3=\rho_{322}^2(Q_{322}U_{32}U_3J_3)^{-1}\bar{\Theta}_2[(Q_{322}U_{32}U_3J_3)^{-1}]^T, \tag{4.21}$$

is semi-postive definite.

Case 3: When $k_1=0$ and $k_2=0$, there exists the transformation matrix Q_{33} such that $T_{33}=Q_{33}Z_3Q_{33}^{-1}$, where

$$Q_{33}=\begin{bmatrix}a_{13}(a_{11}+a_{33}-\frac{a_{12}a_{23}}{a_{13}}) & (-a_{11}-\frac{a_{12}a_{23}}{a_{13}})^2 & n_3^* & n_4^*\\ -a_{13} & -a_{11}+\frac{a_{12}a_{23}}{a_{13}} & -a_{14} & -a_{12}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},$$

with $n_3^* = -a_{13}a_{33} + a_{14}(a_{11} + a_{44} - \frac{a_{12}a_{23}}{a_{13}})$ and $n_4^* = a_{12}(a_{11} + \frac{a_{13}a_{21}}{a_{13}}) + a_{14}a_{42}$, then we can find

$$T_{33} = \begin{bmatrix} -\bar{c}_1 & -\bar{c}_2 & -\bar{c}_3 & -\bar{c}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -a_{44} & -a_{42} \\ 0 & 0 & 0 & -\frac{a_{12}a_{23}+a_{13}a_{21}}{a_{13}} \end{bmatrix},$$

where

$$\begin{aligned}
\bar{c}_1 =& a_{11} + a_{33} - \frac{a_{12}a_{23}}{a_{13}} > 0, \\
\bar{c}_2 =& \frac{a_{13}}{a_{12}a_{23} + a_{13}a_{21} + a_{13}a_{44}}[a_{21}a_{33}a_{44}(a_{11} + a_{12}) + a_{33}a_{42}(a_{11}a_{23} + a_{11}a_{24}a_{33}a_{41} - a_{14}a_{21} - a_{13}a_{21})] \\
& - \frac{a_{13}}{a_{12}a_{23} + a_{13}a_{21} + a_{13}a_{44}}[a_{12}a_{33}a_{41}(a_{23} + a_{24}) + a_{21}a_{33}a_{41}(a_{13} + a_{14})] > 0.
\end{aligned}$$

Thus, (4.18) can be transformed into

$$D_0^2 + T_{33}\bar{\Theta}_3 + \bar{\Theta}_3 T_{33}^T = 0,$$

where $\bar{\Theta}_3 = \frac{1}{\rho_{33}^2}(Q_{33}U_3J_3)\Lambda_3(Q_{33}U_3J_3)^T$, $\rho_{33} = -a_{13}\sigma_3$. With Lemma 2.7, we can obtain the positive semi-definite matrix

$$\bar{\Theta}_3 = \begin{bmatrix} \frac{1}{2\bar{c}_1} & 0 & 0 & 0 \\ 0 & \frac{1}{2\bar{c}_1\bar{c}_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, it can be obtained by calculation that

$$\Lambda_3 = \rho_{33}^2(Q_{33}U_3J_3)^{-1}\bar{\Theta}_3[(Q_{33}U_3J_3)^{-1}]^T \tag{4.22}$$

is semi-postive definite.

Step 4. For the following algebraic equation

$$D_4^2 + Z\Lambda_4 + \Lambda_4 Z^T = 0, \tag{4.23}$$

consider the corresponding order matrix J_4 and elimination matrix U_4:

$$J_4 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad U_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -\frac{a_{21}}{a_{12}} & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Let $Z_4 = (U_4J_4)Z(U_4J_4)^{-1}$. It can be obtained that

$$Z_4 = \begin{bmatrix} 0 & -\frac{a_{21}a_{44}}{a_{12}} & -a_{44} & -a_{41} \\ -a_{12} & -a_{13} - \frac{a_{14}a_{21}}{a_{12}} & -a_{41} & -a_{11} \\ 0 & \frac{a_{13}a_{21}}{a_{12}} + \frac{a_{14}a_{21}^2}{a_{12}^2} & a_{24} + \frac{a_{21}a_{41}}{a_{12}} & a_{21} + \frac{a_{11}a_{21}}{a_{12}} \\ 0 & p_1 & p_2 & 0 \end{bmatrix},$$

where $p_1 = -a_{33} + \frac{a_{21}a_{33}}{a_{12}}$, $p_2 = a_{33}$.

Case 1: When $p_1 = 0$ and $p_2 \neq 0$, it is possible to determine the standard transformation matrix Q_{41} in order to satisfy the equation $T_{41} = Q_{41}Z_4Q_{41}^{-1}$, where

$$Q_{41} = \begin{bmatrix} l_1 & l_2 & l_3 & l_4 \\ 0 & (\frac{a_{13}a_{21}}{a_{12}} + \frac{a_{14}a_{21}^2}{a_{12}^2})a_{33} & a_{24}a_{33} + \frac{a_{21}a_{33}a_{41}}{a_{12}} & a_{21}a_{33} + \frac{a_{11}a_{21}a_{33}}{a_{12}} \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with

$$\begin{aligned} l_1 =& (-a_{13}a_{21} - \frac{a_{14}a_{21}^2}{a_{12}})a_{33}, \ l_2 = (-a_{13} + a_{24} + \frac{a_{21}a_{41} - a_{14}a_{21}}{a_{12}})(\frac{a_{13}a_{21}}{a_{12}} + \frac{a_{14}a_{21}^2}{a_{12}^2})a_{33}, \\ l_3 =& a_{33}[-a_{41}(\frac{a_{13}a_{21}}{a_{12}} + \frac{a_{14}a_{21}^2}{a_{12}^2}) + a_{21}a_{33} + \frac{a_{11}a_{21}a_{33}}{a_{12}} + (a_{24} + \frac{a_{21}a_{41}}{a_{12}})^2], \\ l_4 =& -a_{11}a_{33}(\frac{a_{13}a_{21}}{a_{12}} + \frac{a_{14}a_{21}^2}{a_{12}^2}) + (a_{24} + \frac{a_{21}a_{41}}{a_12})(a_{21} + \frac{a_{11}a_{21}}{a_{12}})a_{33}, \end{aligned}$$

then it can be obtained by calculation that $T_{41} = T_{111}$, and (4.23) can be transformed into

$$D_0^2 + T_{41}\hat{\Theta}_1 + \hat{\Theta}_1 T_{41}^T = 0,$$

where $\hat{\Theta}_1 = \frac{1}{\rho_{41}^2}(Q_{41}U_4J_4)\Lambda_4(Q_{41}U_4J_4)^T$, $\rho_{41} = (a_{13}a_{21} - \frac{a_{14}a_{21}^2}{a_{12}})a_{33}\sigma_4$, and $\hat{\Theta}_1 = \Theta_1$. Consequently,

$$\Lambda_4 = \rho_{41}^2(Q_{41}U_4J_4)^{-1}\hat{\Theta}_1[(Q_{41}U_4J_4)^{-1}]^T \tag{4.24}$$

is positive definite matrix.

Case 2: When $p_1 \neq 0$ and $p_2 = 0$, there exists the transformation matrix U_{42}, then $Z_{42} = U_{42}Z_4U_{42}^{-1}$, where

$$U_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & \frac{a_{12}(a_{12}a_{33} - a_{21}a_{33})}{a_{12}a_{13}a_{21} + a_{14}a_{21}^2} & 1 \end{bmatrix},$$

then we can find

$$Z_{42} = \begin{bmatrix} 0 & -\frac{a_{21}a_{44}}{a_{12}} & -a_{44} + \frac{a_{12}a_{33}a_{41}(a_{12}-a_{21})}{a_{12}a_{13}a_{21}+a_{14}a_{21}^2} & -a_{41} \\ -a_{12} & -a_{13} - \frac{a_{14}a_{21}}{a_{12}} & -a_{41} + \frac{a_{11}a_{12}a_{33}(a_{12}-a_{21})}{a_{12}a_{13}a_{21}+a_{14}a_{21}^2} & -a_{11} \\ 0 & \frac{a_{12}a_{13}a_{21}+a_{14}a_{21}^2}{a_{12}^2} & \frac{a_{12}a_{24}+a_{21}a_{41}}{a_{12}} - \frac{(a_{12}+a_{12})(a_{12}a_{33}-a_{21}a_{33})}{a_{12}a_{13}+a_{14}a_{21}} & \frac{a_{21}(a_{11}+a_{12})}{a_{12}} \\ 0 & 0 & p_3 & \frac{(a_{11}+a_{12})(a_{12}-a_{21})}{a_{12}a_{13}+a_{14}a_{21}} \end{bmatrix},$$

where $p_3 = a_{33} + \frac{a_{33}(a_{12}-a_{21})(a_{12}a_{24}+a_{21}a_{41})}{a_{12}a_{13}+a_{14}a_{21}} - \frac{a_{12}a_{33}^2(a_{11}+a_{12})(a_{12}-a_{21})^2}{a_{21}(a_{12}a_{13}+a_{14}a_{21})^2}$.

Case 2-1: If $p_3 \neq 0$, we can find the standard transformation matrix Q_{421} such that $T_{421} = Q_{421}Z_{42}Q_{421}^{-1}$, where

$$Q_{421} = \begin{bmatrix} l_1 & l_2 & l_3 & l_4 \\ 0 & \frac{p_3a_{21}(a_{12}a_{13}+a_{14}a_{21})}{a_{12}^2} & l_5 & \frac{(a_{11}+a_{12})^2(a_{12}-a_{21})^2}{(a_{12}a_{13}+a_{14}a_{21})^2} + \frac{p_3a_{21}(a_{11}+a_{12})}{a_{12}} \\ 0 & 0 & p_3 & \frac{(a_{11}+a_{12})(a_{12}-a_{21})}{(a_{12}a_{13}+a_{14}a_{21})} \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with

$$
\begin{aligned}
l_1 =& \frac{-a_{21}a_{33}(a_{12}-a_{21})(a_{12}a_{24}+a_{24}a_{41})}{a_{12}},\\
l_2 =& [\frac{a_{21}(-a_{12}a_{13}-a_{14}a_{21}+a_{21}a_{24}+a_{21}a_{41})(a_{12}a_{13}+a_{14}a_{21})}{a_{12}^3}+\frac{a_{21}(a_{11}+a_{12})(a_{12}-a_{21})}{a_{12}^2}]p_3\\
&-[\frac{a_{21}a_{33}(a_{11}+a_{12})(a_{12}-a_{21})}{a_{12}^2}]p_3,\\
l_3 =& [\frac{-a_{41}a_{21}(a_{12}a_{13}+a_{14}a_{21})}{a_{12}^2}+\frac{a_{11}a_{33}(a_{12}-a_{21})}{a_{12}}-\frac{a_{33}(a_{11}+a_{12})^2(a_{12}-a_{21})^2}{(a_{12}a_{13}+a_{14}a_{21})^2}]p_3\\
&+[\frac{(a_{11}+a_{12})(a_{12}-a_{21})(a_{12}a_{24}+a_{21}a_{41})}{a_{12}(a_{12}a_{13}+a_{14}a_{21})}+(\frac{a_{12}a_{24}+a_{21}a_{41}}{a_{12}}-\frac{(a_{11}+a_{12})(a_{12}a_{33}-a_{21}a_{33})}{a_{12}a_{13}+a_{14}a_{21}})^2\\
&+\frac{p_3a_{21}(a_{11}+a_{12})}{a_{12}}+\frac{(a_{11}+a_{12})^2(a_{12}-a_{21})^2}{(a_{12}a_{13}+a_{14}a_{21})^2}]p_3,\\
l_4 =& (\frac{a_{12}a_{24}+a_{21}a_{41}}{a_{12}}-\frac{(a_{11}+a_{12})(a_{12}a_{33}-a_{21}a_{33})}{a_{12}a_{13}+a_{14}a_{21}}+\frac{(a_{11}+a_{12})(a_{12}-a_{21})}{a_{12}a_{13}+a_{14}a_{21}})\frac{p_3a_{21}(a_{11}+a_{12})}{a_{12}}\\
&-\frac{a_{11}a_{21}a_{33}(a_{12}-a_{21})(a_{12}a_{24}+a_{21}a_{41})}{a_{12}^2}+\frac{p_3a_{21}(a_{11}+a_{12})}{a_{12}}\frac{(a_{11}+a_{12})(a_{12}-a_{21})}{a_{12}a_{13}+a_{14}a_{21}}\\
&+\frac{(a_{11}+a_{12})^2(a_{12}-a_{21})^2}{(a_{12}a_{13}+a_{14}a_{21})^2}\frac{(a_{11}+a_{12})(a_{12}-a_{21})}{a_{12}a_{13}+a_{14}a_{21}},\\
l_5 =& [\frac{a_{12}a_{24}+a_{21}a_{41}}{a_{12}}-\frac{(a_{11}+a_{12})(a_{12}a_{33}-a_{21}a_{33})}{a_{12}a_{13}+a_{14}a_{21}}+\frac{(a_{11}+a_{12})(a_{12}-a_{21})}{a_{12}a_{13}+a_{14}a_{21}}],
\end{aligned}
$$

then we can obtain that $T_{421}=T_{111}$, and (4.23) can be transformed into

$$D_0^2+T_{421}\hat{\Theta}_1+\hat{\Theta}_1T_{421}^T=0,$$

where $\hat{\Theta}_1=\frac{1}{\rho_{421}^2}(Q_{421}U_{42}U_4J_4)\Lambda_4(Q_{421}U_{42}U_4J_4)^T$, $\rho_{421}=\frac{-a_{21}(a_{12}a_{13}+a_{14}a_{21})}{a_{12}}\sigma_4$. Consequently,

$$\Lambda_4=\rho_{421}^2(Q_{421}U_{42}U_4J_4)^{-1}\hat{\Theta}_1[(Q_{421}U_{42}U_4J_4)^{-1}]^T \tag{4.25}$$

is positive definite matrix.

Case 2-2: If $p_3=0$, we are able to find the standard transformation matrix Q_{422} such that $T_{422}=Q_{422}Z_{42}Q_{422}^{-1}$, where

$$Q_{422}=\begin{bmatrix} l_1 & l_2 & l_3 & l_4\\ 0 & \frac{a_{12}a_{13}a_{21}+a_{14}a_{21}^2}{a_{12}^2} & \frac{a_{12}a_{24}+a_{21}a_{41}}{a_{12}}-\frac{a_{33}(a_{11}+a_{12})(a_{12}-a_{21})}{a_{12}a_{13}+a_{14}a_{21}} & \frac{a_{21}(a_{11}+a_{12})}{a_{12}}\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},$$

with

$$\begin{aligned}
l_1 =& -\frac{a_{21}(a_{12}a_{13}+a_{14}a_{21})}{a_{12}},\\
l_2 =& [-a_{13}-\frac{a_{12}a_{24}+a_{21}a_{41}-a_{14}a_{21}}{a_{12}}-\frac{(a_{11}+a_{12})(a_{12}a_{13}-a_{21}a_{33})}{a_{12}a_{13}+a_{14}a_{21}}]\frac{a_{21}(a_{12}a_{13}+a_{14}a_{21})}{a_{12}^2},\\
l_3 =& [\frac{a_{12}a_{24}+a_{21}a_{41}}{a_{12}}-\frac{a_{33}(a_{11}+a_{12})(a_{12}-a_{21})}{a_{12}a_{13}+a_{14}a_{21}}]^2-\frac{a_{41}(a_{12}a_{33}+a_{14}a_{21})}{a_{12}^2}+\frac{a_{11}a_{33}(a_{12}-a_{21})}{a_{12}},\\
l_4 =& \frac{a_{21}(a_{11}+a_{12})}{a_{12}}[\frac{a_{12}a_{24}+a_{21}a_{41}}{a_{12}}-\frac{(a_{11}+a_{12})(a_{12}a_{33}-a_{21}a_{33})}{a_{12}a_{13}+a_{14}a_{21}}+\frac{(a_{11}+a_{12})(a_{12}-a_{21})}{a_{12}a_{13}+a_{14}a_{21}}]\\
&-\frac{a_{11}a_{21}(a_{12}a_{13}+a_{14}a_{21})}{a_{12}^2}.
\end{aligned}$$

We then have

$$T_{422}=\begin{bmatrix}-\hat{b}_1 & -\hat{b}_2 & \hat{b}_3 & \hat{b}_4\\ 1 & 0 & 0 & \hat{b}_5\\ 0 & 1 & 0 & \hat{b}_6\\ 0 & 0 & 0 & \hat{b}_7\end{bmatrix},$$

which means that

$$\begin{aligned}
\hat{b}_1 =& \frac{1}{a_{12}a_{13}+a_{14}a_{21}}[(a_{12}a_{13}+a_{14}a_{21})(a_{11}+a_{21}+a_{33}+a_{44})+(a_{11}+a_{12})(a_{12}-a_{21})]>0,\\
\hat{b}_2 =& \frac{(a_{11}+a_{12})(a_{12}-a_{21})}{(a_{12}a_{13}+a_{14}a_{21})^2}[(a_{12}a_{13}+a_{14}a_{21})(a_{11}+a_{21}+a_{33}+a_{44})+(a_{11}+a_{12})(a_{12}-a_{21})]\\
&+a_{11}(a_{21}+a_{33}+a_{44})+a_{21}(a_{12}+a_{33}+a_{44})+a_{33}a_{44}+a_{24}a_{42}-a_{14}a_{41}>0,\\
\hat{b}_3 =& \frac{(a_{11}+a_{12})(a_{12}-a_{21})}{a_{12}a_{13}+a_{14}a_{21}}[a_{11}(a_{21}+a_{33}+a_{44})+a_{21}(a_{12}+a_{33}+a_{44})+a_{33}a_{44}+a_{24}a_{42}-a_{14}a_{41}]\\
&+\frac{(a_{11}+a_{12})^2(a_{12}-a_{21})^2}{(a_{12}a_{13}+a_{14}a_{21})^3}[(a_{12}a_{13}+a_{14}a_{21})(a_{11}+a_{21}+a_{33}+a_{44})+(a_{11}+a_{12})(a_{12}-a_{21})]\\
&+a_{11}a_{21}(a_{33}+a_{44})+a_{33}a_{44}(a_{11}+a_{21})+a_{12}a_{21}(a_{33}+a_{44})-a_{14}a_{21}(a_{41}+a_{42})\\
&+a_{24}a_{41}(a_{11}-a_{12})-a_{33}a_{41}(a_{13}+a_{14})+a_{23}a_{33}a_{42}+a_{24}a_{33}a_{41}>0,
\end{aligned}$$

$\hat{b}_1\hat{b}_2-\hat{b}_3>0$, and (4.23) can be transformed into the following form :

$$\begin{aligned}
&(Q_{422}U_{42}U_4J_4)D_4^2(Q_{422}U_{42}U_4J_4)^T\\
&+[(Q_{422}U_{42}U_4J_4)Z(Q_{422}U_{42}U_4J_4)^{-1}](Q_{422}U_{42}U_4J_4)\Lambda_4(Q_{422}U_{42}U_4J_4)^T\\
&+(Q_{422}U_{42}U_4J_4)\Lambda_4(Q_{422}U_{42}U_4J_4)^T[(Q_{422}U_{42}U_4J_4)Z(Q_{422}U_{42}U_4J_4)^{-1}]^T=0.
\end{aligned}$$

That is, $D_0^2+T_{422}\hat{\Theta}_2+\hat{\Theta}_2T_{422}^T=0$, where $\hat{\Theta}_2=\frac{1}{\rho_{422}^2}(Q_{422}U_{42}U_4J_4)\Lambda_4(Q_{422}U_{42}U_4J_4)^T$, $\rho_{422}=\frac{-a_{21}(a_{12}a_{13}+a_{14}a_{21})}{a_{21}}\sigma_4$. According to Lemma 2.6, we can find the semipositive definite matrix

$$\hat{\Theta}_2=\begin{bmatrix}\frac{\hat{b}_2}{2(\hat{b}_1\hat{b}_2-\hat{b}_3)} & 0 & -\frac{1}{2(\hat{b}_1\hat{b}_2-\hat{b}_3)} & 0\\ 0 & \frac{1}{2(\hat{b}_1\hat{b}_2-\hat{b}_3)} & 0 & 0\\ -\frac{1}{2(\hat{b}_1\hat{b}_2-\hat{b}_3)} & 0 & \frac{\hat{b}_1}{2\hat{b}_3(\hat{b}_1\hat{b}_2-\hat{b}_3)} & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Therefore, we can obtain that

$$\Lambda_4 = \rho_{422}^2 (Q_{422} U_{42} U_4 J_4)^{-1} \hat{\Theta}_2 [(Q_{422} U_{42} U_4 J_4)^{-1}]^T \tag{4.26}$$

is semi-postive definite.

Case 3: When $p_1 = 0$ and $p_2 = 0$, there exists the transformation matrix Q_{42} such that $T_{43} = Q_{42} Z_4 Q_{42}$, where

$$Q_{42} = \begin{bmatrix} l_1 & l_2 & l_3 & l_4 \\ -a_{12} & -a_{13} - \frac{a_{14}a_{21}}{a_{12}} & -a_{41} & -a_{11} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with

$$l_1 = a_{12}a_{13} + a_{14}a_{21},\ l_2 = a_{21}a_{44} + (a_{13} + \frac{a_{14}a_{21}}{a_{12}})^2,\ l_3 = a_{12}a_{44} - a_{41}(-a_{13} - \frac{a_{14}a_{21}}{a_{12}} + a_{24} + \frac{a_{21}a_{41}}{a_{12}}),$$
$$l_4 = a_{12}a_{41} + a_{11}a_{13} + \frac{a_{11}a_{14}a_{21}}{a_{12}} - a_{21}a_{41} - \frac{a_{11}a_{21}a_{41}}{a_{12}}.$$

We then have

$$T_{43} = \begin{bmatrix} -\hat{c}_1 & -\hat{c}_2 & -\hat{c}_3 & -\hat{c}_4 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & a_{24} + \frac{a_{21}a_{41}}{a_{21}} & a_{21} + \frac{a_{11}a_{21}}{a_{12}} \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where

$$\begin{aligned} \hat{c}_1 =& a_{11} + a_{21} + a_{33} + a_{44} + a_{24} + \frac{a_{21}a_{41}}{a_{12}} > 0, \\ \hat{c}_2 =& a_{21}a_{33}a_{44}(a_{11} + a_{12}) + a_{33}a_{42}(a_{11}a_{23} + a_{11}a_{24}a_{33}a_{41} - a_{14}a_{21} - a_{13}a_{21}) - a_{12}a_{33}a_{41}(a_{23} + a_{24}) \\ &+ (a_{11} + a_{21} + a_{33} + a_{44} + a_{24} + \frac{a_{21}a_{41}}{a_{12}})(a_{24} + \frac{a_{21}a_{41}}{a_{12}}) - a_{21}a_{33}a_{41}(a_{13} + a_{14}) > 0. \end{aligned}$$

Thus, (4.23) can be transformed into

$$D_0^2 + T_{43}\hat{\Theta}_3 + \hat{\Theta}_3 T_{43}^T = 0,$$

where $\hat{\Theta}_3 = \frac{1}{\rho_{43}^2}(Q_{43}U_4J_4)\Lambda_4(Q_{43}U_4J_4)^T$, $\rho_{43} = -a_{12}\sigma_4$. With Lemma 2.7, we can obtain the positive semidefinite matrix $\hat{\Theta}_3$

$$\hat{\Theta}_3 = \begin{bmatrix} \frac{1}{2\hat{c}_1} & 0 & 0 & 0 \\ 0 & \frac{1}{2\hat{c}_1\hat{c}_2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore, it can be obtained by calculation that

$$\Lambda_4 = \rho_{43}^2 (Q_{43} U_4 J_4)^{-1} \hat{\Theta}_3 [(Q_{43} U_4 J_4)^{-1}]^T \tag{4.27}$$

is semi-postive definite.

In summary, Λ_3, Λ_4 are positive definite and Λ_1, Λ_1 are at least semi-positive definite. Hence, $\Lambda = \Lambda_1 + \Lambda_2 + \Lambda_3 + \Lambda_4$ is a positive definite matrix, and there is a unique and precise density function surrounding the quasi-endemic equilibrium $\bar{E}$ when $R^s > 1$. Considering the transformation of $(y_1, y_2, y_3, y_4) = (\ln \frac{S}{S_1}, \ln \frac{I}{I_1}, \ln \frac{A}{A_1}, \ln \frac{C}{C_1})$ and (S, I, A, C), the unique ergodic stationary distribution $\nu(\cdot)$ of Model (2.2) approximately follows a unique log-normal probability density function

$$\Phi(S, I, A, C) = (2\pi)^{-2} |\Lambda|^{-\frac{1}{2}} e^{-\frac{1}{2}(\ln \frac{S}{S_1} + \ln \frac{I}{I_1} + \ln \frac{A}{A_1} + \ln \frac{C}{C_1})\Lambda^{-1}(\ln \frac{S}{S_1} + \ln \frac{I}{I_1} + \ln \frac{A}{A_1} + \ln \frac{C}{C_1})^T}.$$

This completes the proof of Theorem 4.1.

Remark 3. *By combining Theorems 3.1 and 4.1, it is possible to find evidence that the stationary distribution of Model* (2.2) *around the equilibrium* $\bar{E}_1$ *is compatible with the probability density function* $\Phi(S, I, A, C)$. *Hence, a reasonable stochastic criterion for disease persistence can be achieved when* $R^s > 1$. *From this we can see that accurately representing the density function can be an efficient method for preventing and managing numerous epidemics.*

5. Numerical simulations

In this section, we apply theoretical findings to analyze the dynamics of anthrax and investigate the impact of environmental disturbance on the transmission of the disease. Using by Milstein's method [39], the discretization equation of Model (2.2) is given by

$$\begin{cases} S_{k+1} = S_k + \left[rN_k(1 - \dfrac{N_k}{K}) - \eta_a A_k S_k - \eta_c C_k S_k - \eta_i \dfrac{S_k I_k}{N_k} - \mu S_k + \tau I_k \right] \Delta t \\ \qquad + \sigma_1 S_k \sqrt{\Delta t} \chi_k + \dfrac{\sigma_1^2}{2} S_k^2 (\chi_k^2 - 1) \Delta t, \\ I_{k+1} = I_k + \left[\eta_a A_k S_k + \eta_c C_k S_k + \eta_i \dfrac{S_k I_k}{N_k} - (\gamma + \mu + \tau) I_k \right] \Delta t + \sigma_2 I_k \sqrt{\Delta t} \pi_k + \dfrac{\sigma_2^2}{2} I_k^2 (\pi_k^2 - 1) \Delta t, \\ A_{k+1} = A_k + (-\alpha A_k + \beta C_k) \Delta t + \sigma_3 A_k \sqrt{\Delta t} \omega_k + \dfrac{\sigma_3^2}{2} A_k^2 (\omega_k^2 - 1) \Delta t, \\ C_{k+1} = C_k + \left[(\gamma + \mu) I_k - \delta (S_k + I_k) C_k - \kappa C_k \right] \Delta t + \sigma_4 C_k \sqrt{\Delta t} \zeta_k + \dfrac{\sigma_4^2}{2} C_k^2 (\zeta_k^2 - 1) \Delta t, \end{cases} \tag{5.1}$$

where the time increment is $\Delta t > 0$, and $\chi_k, \pi_k, \omega_k, \zeta_k$ $(k = 1, ..., n)$ are the independent Gaussian random variables following the distribution $N(0, 1)$. As stated by Saad-Roy et al. [7] and Mushayabasa et al. [17], Table 1 displays the relevant biological parameters and the initial value of Model (2.2). Next, our objective is to perform empirical examples to concentrate on the following three aspects:

1. If $R^s > 1$, there is a unique ergodicity property stationary distribution.
2. Stochastic fluctuations have an impact on disease persistence in the Model (2.2).
3. Focus on the probability density function of Model (2.2).

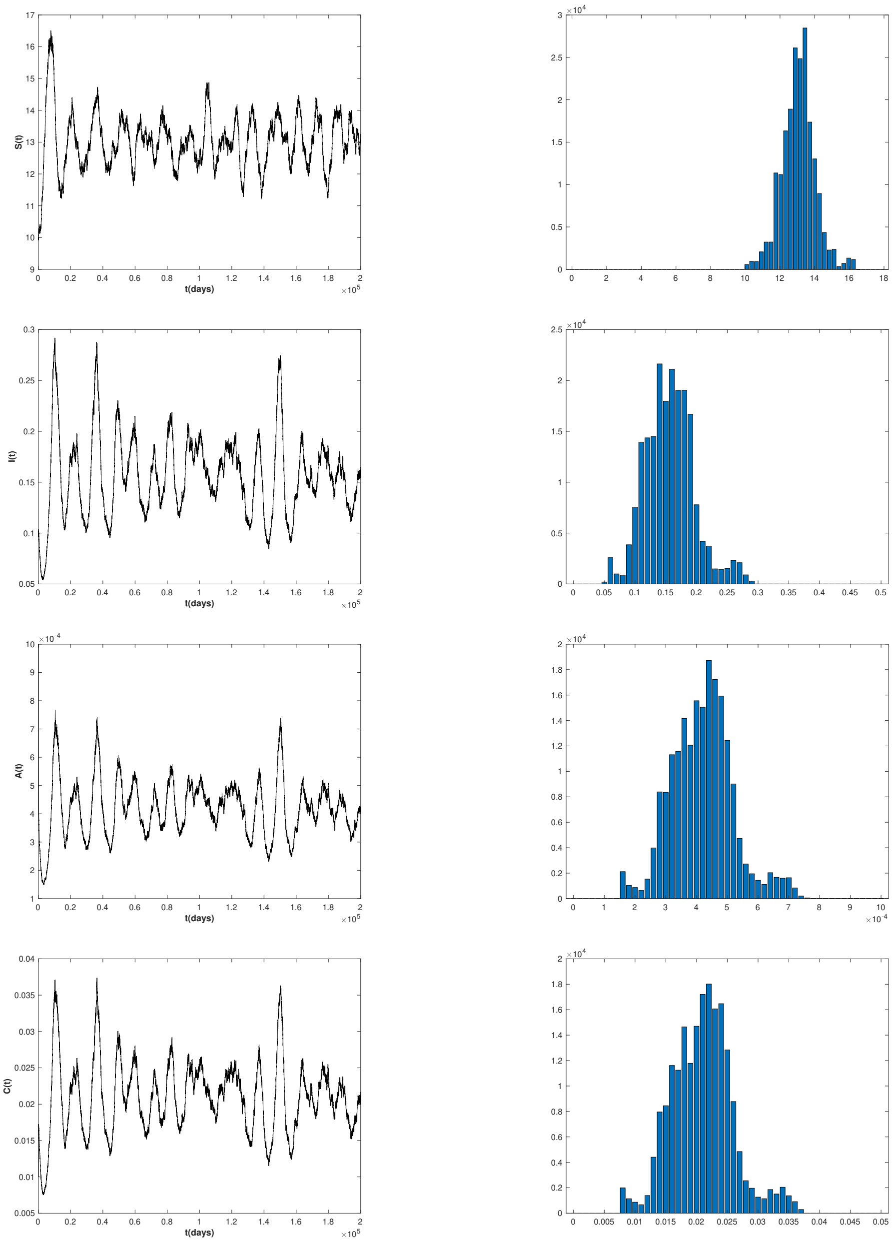

Figure 2. The left column shows the paths of $S(t)$, $I(t)$, $A(t)$ and $C(t)$ for Model (2.2) with initial values $(S(0), I(0), A(0), C(0)) = (10, 0.1, 0.001, 0.01)$ under noise intensities $\sigma_1 = 0.005, \sigma_2 = 0.005, \sigma_3 = 0.005, \sigma_4 = 0.005$, respectively. The right column shows the histograms of $S(t)$, $I(t)$, $A(t)$ and $C(t)$.

Table 1. List of biological parameters and initial of Model (2.2).

Parameters	Description	Value	Units	Ref.
K	Animal carrying capacity	100	animals	[7]
r	Birth rate	1/300	Day^{-1}	[7]
μ	Death rate	1/600	Day^{-1}	[7]
κ	Carcass decay rate	1/10	Day^{-1}	[7,40]
α	Spore decay rate	1/10	Day^{-1}	[7,17]
γ	Disease death rate	1/10	Day^{-1}	[7,14]
η_i	Transmission rate of infected animals	1/100	Day^{-1}.	[7]
η_a	Transmission rate of environment to susceptible animals	1/2	Day^{-1} gm spore^{-1}.	[7]
η_c	Transmission rate of infected carcasses to susceptible animals	1/10	Day^{-1} carcass^{-1}	[7]
τ	Recovery rate of infected animals	1/10	Day^{-1}	[7,41]
β	Spore growth rate	1/500	Spores $\text{carcass}^{-1}\text{day}^{-1}$	[7]
δ	Carcass consumption rate	1/20	Day^{-1} animal^{-1}	[7,41]

5.1. Dynamical behaviors of Model (2.2) under $R^s > 1$

The main parameters are given in Table 1. Let the stochastic perturbations be $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (0.005, 0.005, 0.005, 0.005)$ in Model (2.2) then we can calculate that

$$R_0 := \frac{\eta_i}{\gamma+\mu+\tau} + \frac{\gamma+\mu}{\gamma+\mu+\tau}\frac{\eta_c S_0}{\delta S_0+\kappa} + \frac{\gamma+\mu}{\gamma+\mu+\tau}\frac{\beta\eta_a S_0}{\alpha(\delta S_0+\kappa)} = 4.3560 > 1,$$

$$\begin{aligned} R^s :=& \frac{\eta_i}{\gamma+\mu+\tau+\frac{\sigma_2^2}{2}} + \frac{\gamma+(\mu+\frac{\sigma_1^2}{2})}{\gamma+\mu+\tau+\frac{\sigma_2^2}{2}}\frac{\eta_c S_0}{\delta S_0+(\kappa+\frac{\sigma_4^2}{2})} + \frac{\gamma+(\mu+\frac{\sigma_1^2}{2})}{\gamma+\mu+\tau+\frac{\sigma_2^2}{2}}\frac{\beta\eta_a S_0}{(\alpha+\frac{\sigma_3^2}{2})[\delta S_0+(\kappa+\frac{\sigma_4^2}{2})]} \\ =& 1.1545 > 1. \end{aligned}$$

The disease has a tendency to persist in both deterministic and stochastic models according to Theorem 3.1. It can be inferred that Model (2.2) has a stationary distribution $\nu(\cdot)$. In Figure 2, we give the time series diagram of the model solution and its corresponding histogram. In the time series diagram in Figure 2, the curve fluctuates up and down, but the overall trend is stable. The corresponding interval of frequency can be seen in the histogram on the right. It is shown that the Model (2.2) has a unique ergodic stationary distribution, which is consistent with the result of Theorem 3.1. This indicates that the disease will endure. In real life, anthrax is not easy to be completely eradicated, but in a relatively stable state. Next, the dynamical behavior is observed when the parameters of the stochastic and deterministic models are identical. It is shown that the stochastic path fluctuates around the deterministic path, which indicates that the stochastic Model (2.2) has an ergodic stationary distribution. Figure 3 confirms this.

5.2. Impact of random noises $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ on the disease persistence

Mathematical limitations make it tough to establish sufficient conditions for disease extinction in the stochastic anthrax model [42,43]. Therefore, for a comprehensive discussion, we present numerical simulation of disease elimination with high noise. For example, to prevent the spread of anthrax, people can increase the culling rate or vaccination rate, which will effectively eliminate the spread of anthrax.

We use numerical simulations to prove that excessive environmental noise can lead to the extinction of anthrax. Let $\sigma = \sigma_1 = \sigma_2 = \sigma_3 = \sigma_4$, σ be taken as 0.002, 0.005, and 0.01, respectively. The other parameters and initial value are identical to those in Table 1 and the simple calculations show that $R^s > 1$. It can be concluded that the disease persists; if we let $\sigma = 0.02, 0.05, 0.1$, studies have shown that large enough environment noise will eradicate the disease, and the numerical simulation clearly shows our conclusion in Figure 4.

5.3. The probability density function of Model (2.2)

In this part, the numerical representation of the probability density function of Model (2.2) will be our focus. From Theorem 3.1, we can calculate that if $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = (0.005, 0.005, 0.005, 0.005)$, the Model (2.2) has a unique stationary distribution. Moreover, through meticulous computation, it is determined that the variables' values in Table 2 are all positive. By the definition of $\bar{E}_1$, we obtain that $(\bar{S}_1, \bar{I}_1, \bar{A}_1, \bar{C}_1) = (24.8194, 0.4094, 0.00044, 0.02852)$. According to Theorem 4.1, we know that the distribution has a probability density function around the quasi-steady state $\bar{E}_1$, and the stationary distribution $v(\cdot)$ obeys a log-normal density function $\Phi(S, I, A, C)$. Its function expression is:

$$\Lambda = 10^{-3} \times \begin{bmatrix} 0.0511 & -0.0271 & 0.0134 & 0.0283 \\ -0.0271 & 1.4113 & 0.4421 & 0.1271 \\ 0.0134 & 0.4421 & 0.2216 & 0.0932 \\ 0.0283 & 0.1271 & 0.0932 & 0.1014 \end{bmatrix},$$

and the corresponding marginal density functions are :

$$F(S) = \frac{\partial \Phi}{\partial S} = 0.08 \exp[-9.79(\ln(S - 24.8194))^2], \quad F(I) = \frac{\partial \Phi}{\partial I} = 2.22 \exp[-0.35(\ln(I - 0.4094))^2],$$
$$F(A) = \frac{\partial \Phi}{\partial A} = 0.35 \exp[-2.26(\ln(A - 0.00044))^2], \quad F(C) = \frac{\partial \Phi}{\partial C} = 0.16 \exp[-4.93(\ln(C - 0.2852))^2].$$

By constructing frequency histograms and fitting curves for the variables S, I, A, and C, the marginal density functions $F(S)$, $F(I)$, $F(A)$, and $F(C)$ were plotted. The numerical findings depicted in Figure 5 validate the assertion that the fitting curves closely align with the corresponding theoretical marginal density functions.

Table 2. Value of variables.

w_1	w_2	w_3	r_1	r_2	r_3
-0.8739	1.7655	-0.8033	-17.2838	-0.0217	-0.0003
k_1	k_2	k_3	p_1	p_2	p_3
11.1646	-0.0001	-0.0352	-16.5267	0.1312	-10.7746

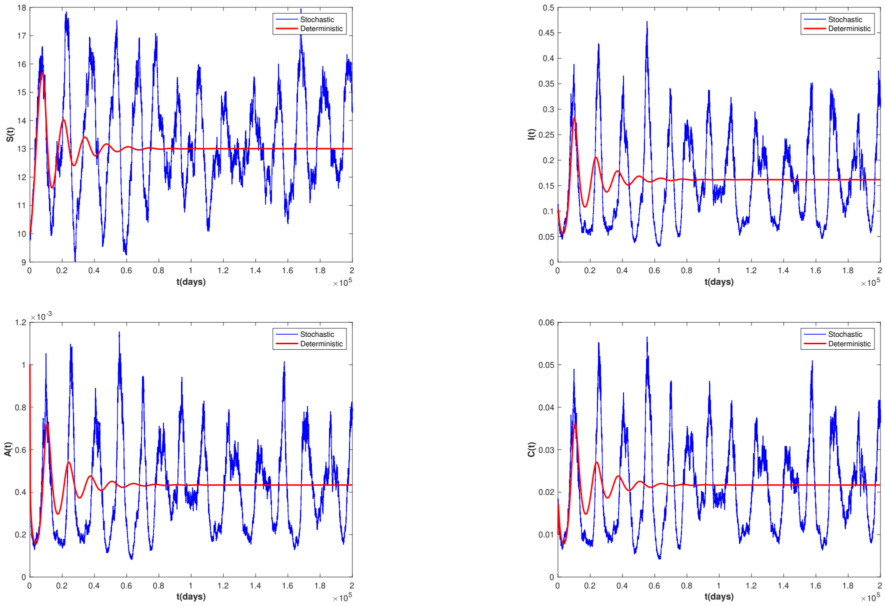

Figure 3. The column shows the paths of S, I, A, C, respectively, with $\sigma_i = 0.005$ $(i = 1, 2, 3, 4)$. The red line represents the result of deterministic Model (2.1), and the blue line represents the result of stochastic Model (2.2), repectively.

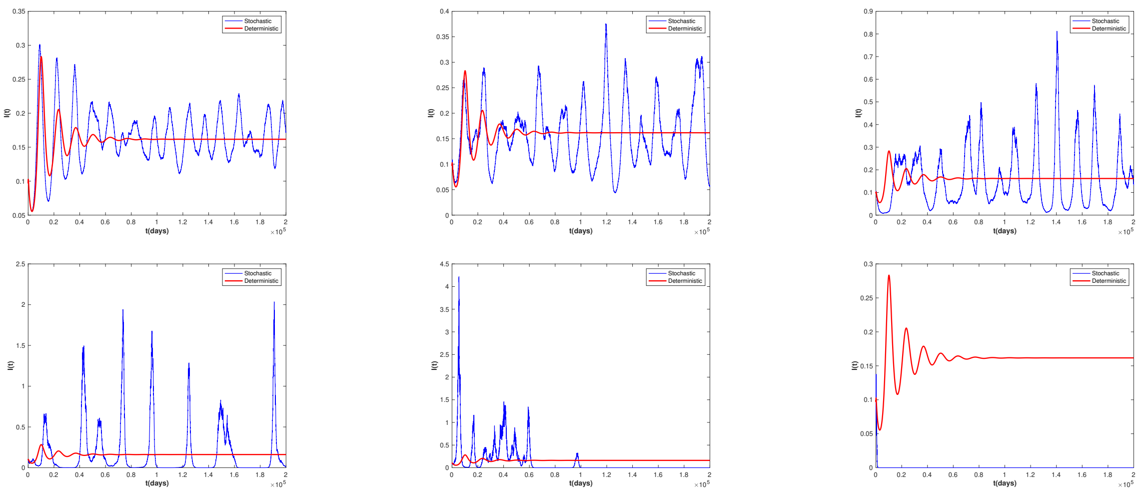

Figure 4. The paths of $I(t)$ for stochastic Model (2.2) with different noise intensities$(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$. The red line represents the result of deterministic Model (2.1), and the blue line represents the result of stochastic Model (2.2), repectively.

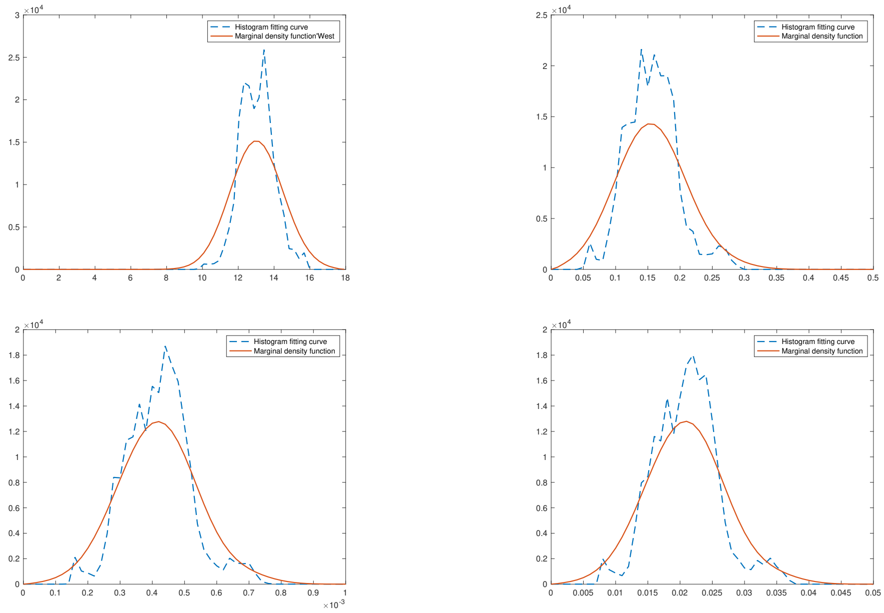

Figure 5. The marginal density functions and frequency fitting curves of S, I, A, and C in Model (2.2), respectively. All of the parameter values are the same as in the above Figure.

6. Conclusions and discussion

To explore the possibility of an ergodic stationary distribution and a representation of the probability density function, we propose the stochastic anthrax model in this study. We will discuss our primary contributions in the two areas.

By considering the effect of external environmental perturbations, we focus on the stochastic anthrax model. First, we calculate the basic reproduction number of the deterministic model R_0, if $R_0 < 1$, which means that the anthrax will be extinct in the animal population, while $R_0 > 1$ implies that anthrax will persistence. For stochastic Model (2.2), the uniform boundedness of the model is presented. Second, we demonstrate that the stochastic anthrax model has a unique ergodic stationary distribution at the threshold $R_s > 1$ in Lemma 2.3 and Theorem 3.1 by building appropriate Lyapunov functions and applying ergodic theory. From the similar threshold and basic reproduction number, $R^s > 1$ can be seen as a unified criterion to ensure the persistence of disease in deterministic and stochastic models. Comparing the expressions for R_0 and R^s, we find that $R^s \leqslant R_0$, so $R^s > 1$ can be viewed as a uniform measure to determine the persistence of the disease in both the model and the stochastic model. In addition, by setting up noise perturbations of different intensity, we found that the higher the noise level, the faster the disease extinction. That is, external disturbances play an important role in disease development, such as environmental elimination, culling of infected animals, and vaccination.

Understanding the statistical characteristics is crucial for managing infectious diseases, but it is challenging to ascertain the statistical properties of disease persistence within an ergodic stationary distribution. To study the dynamic behavior of disease, we construct the endemic equilibrium point, through the equivalence between the Model (4.1) and the corresponding linear Model (4.3) and deduced the log-normal accurate expression of $\Phi(S, I, A, C)$ four-dimensional density function. Furthermore, the covariance matrix Λ is determined through the use of an algebraic equation $D^2 + Z\Lambda + \Lambda Z^T = 0$. The precision of Λ can be verified by applying the stability theory to the zero solution of the linear equation, as discussed in the previous research [39]. Furthermore, our method and theory can demonstrate that Λ is positive definite, even in cases where D is a semi-definite matrix.

Finally, there are several important themes that deserve further study. To begin, the stochastic model established in this paper only considers the effect of white noise. However, Lévy noise can describe the model more accurately when there is a sudden change in the environment. On the other hand, time delay is an important factor affecting the stability of ecological dynamic systems, as time delay may alter the stability of the system and lead to system fluctuations [43,44]. Therefore, considering Lévy noise and delay effects has theoretical value and practical significance. We leave these for future work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The research was supported by the Ningxia Natural Science Foundation (No. 2022AAC03265), and Natural Science Foundation of China (No. 12161068).

Conflict of interest

The authors declare there is no conflicts of interest.

References

1. N. A. Logan, M. Rodríguez-Díaz, Bacillus spp. and related genera, *Princ. Pract. of Clini. Bacteriol.*, (2006), 139–158. https://doi.org/10.1002/9780470017968.ch9
2. P. R. Murray, K. S. Rosenthal, M. A. Pfaller, *Medical Microbiology*, Elsevier Health Sciences, 2021.
3. P. C. Hanna, Ireland J A W, Understanding Bacillus anthracis pathogenesis, *Trends. Microbiol.*, (1999), 180–182. https://doi.org/10.1016/S0966-842X(99)01507-3
4. M. E. Bales, A. L. Dannenberg, P. S. Brachman, A. F. Kaufmann, P. C. Klatsky, D. A. Ashford, Epidemiologic responses to anthrax outbreaks: a review of field investigations, 1950–2001, *Emerging Infect. Dis.*, **8** (2002), 1163. https://doi.org/10.3201/eid0810.020223
5. J. R. Ezzel, W. C. L. JW, Bacillus anthracis, *Pathog. Bact. Infect. Anim.*, (1993), 36–43. https://doi.org/10.1137/0149110
6. N. A. Suverly, B. Kvasnicka, R. Torrell, *Anthrax: a guide for livestock producers*, University of Nevada-Reno, 2001. https://doi.org/10.1111/j.1863-2378.2008.01135.x
7. C. M. Saad-Roy, P. Van den Driessche, A. A. Yakubu, A mathematical model of anthrax transmission in animal populations, *Bull. Math. Biol.*, **79** (2017), 303–324. https://doi.org/10.1007/s11538-016-0238-1

8. S. S. Lewerin, M. Elvander, T. Westermark, L. N. Hartzell, A. K. Norström, S. Ehrs, et al., Anthrax outbreak in a Swedish beef cattle herd-1st case in 27 years: Case report, *Acta Vet. Scand.*, **52** (2010), 1–8. https://doi.org/10.1186/1751-0147-52-7
9. P. R. Furniss, B. D. Hahn, A mathematical model of an anthrax epizoötic in the Kruger National Park, *Appl. Math. Modell.*, **5** (1981), 130–136. https://doi.org/10.1016/0307-904X(81)90034-2
10. S. V. Shadomy, A. E. Idrissi, E. Raizman, Anthrax outbreaks: a warning for improved prevention, control and heightened awareness, *Rome*, 2016.
11. S. B. Clegg, P. C. B. Turnbull, C. M. Foggin, P. M. Lindeque, Massive outbreak of anthrax in wildlife in the Malilangwe Wildlife Reserve, Zimbabwe, *Vet. Rec.*, **160** (2007), 113–118. https://doi.org/10.1136/vr.160.4.113
12. M. N. Mongoh, N. W. Dyer, C. L. Stoltenow, M. L. Khaitsa, Risk factors associated with anthrax outbreak in animals in North Dakota, 2005: A retrospective case-control study, *Public Health Rep.*, **123** (2008), 352–359. https://doi.org/10.1177/003335490812300315
13. A. Chakraborty, S. U. Khan, M. A. Hasnat, S. Parveen, M. Saiful Islam, A. Mikolon, et al., Anthrax outbreaks in Bangladesh, 2009–2010, *Am. J. Trop. Med. Hyg.*, **86** (2012), 703. https://doi.org/10.4269/ajtmh.2012.11-0234
14. W. Beyer, P. C. B. Turnbull, Anthrax in animals, *Mol. Aspects Med.*, **30** (2009), 481–489. https://doi.org/10.1016/j.mam.2009.08.004
15. B. D. Hahn, P. R. Furniss, A deterministic model of an anthrax epizootic: threshold results, *Ecol. Modell.*, **20** (1983), 233–241. https://doi.org/10.1016/0304-3800(83)90009-1
16. A. Friedman, A. A. Yakubu, Anthrax epizootic and migration: Persistence or extinction, *Math. Biosci.*, **241** (2013), 137–144. https://doi.org/10.1016/j.mbs.2012.10.004
17. S. Mushayabasa, T. Marijani, M. Masocha, Dynamical analysis and control strategies in modeling anthrax, *Comput. Appl. Math.*, **36** (2017), 1333–1348. https://doi.org/10.1007/s40314-015-0297-1
18. X. Li, G. Song, Y. Xia , C.Yuan, Dynamical behaviors of the tumor-immune system in a stochastic environment, *SIAM J. Appl. Math.*, **79** (2019), 2193–2217. https://doi.org/10.1137/19M1243580
19. Y. Cai, Y. Kang, M. Banerjee, W. Wang, A stochastic epidemic model incorporating media coverage, *Commun. Math. Sci.*, **14** (2016), 893–910. https://doi.org/10.1186/s13662-018-1925-z
20. Y. Zhao, D. Jiang, The threshold of a stochastic SIS epidemic model with vaccination, *Appl. Math. Comput.*, **243** (2014), 718–727. https://doi.org/10.1016/j.amc.2014.05.124
21. Q. Liu, D. Jiang, T. Hayat, B. Ahmad, Analysis of a delayed vaccinated SIR epidemic model with temporary immunity and Lévy jumps, *Nonlinear Anal. Hybrid Syst.*, **27** (2018), 29–43. https://doi.org/10.1016/j.nahs.2017.08.002
22. B. Zhou, D. Jiang, Y. Dai, T. Hayat, Stationary distribution and density function expression for a stochastic SIQRS epidemic model with temporary immunity, *Nonlinerar. Dyn.*, **105** (2021), 931–955. https://doi.org/10.1007/s11071-020-06151-y
23. Y. Tan, Y. Cai, X. Sun, K. Wang, R. Yao, W. Wang, A stochastic SICA model for HIV/AIDS transmission, *Chaos, Solitons Fractals*, **165** (2022), 112768. https://doi.org/10.1016/j.chaos.2022.112768
24. L. Imhof, S. Walcher, Exclusion and persistence in deterministic and stochastic chemostat models, *J. Differ. Equations*, **217** (2005), 26–53. https://doi.org/10.1016/j.jde.2005.06.017

25.X. Mao, Stationary distribution of stochastic population systems, *Syst. Control Lett.*, **60** (2011), 398–405. https://doi.org/10.1016/j.sysconle.2011.02.013

26.A. Bahar, X. Mao, Stochastic delay lotka–volterra model, *J. Math. Anal. Appl.*, **292** (2004), 364–380. https://doi.org/10.1016/j.jmaa.2003.12.004

27.R. Khasminskii, *Stochastic stability of differential equations*, Springer Science and Business Media, 2011.

28.B. Zhou, D. Jiang, Y. Dai, T. Hayat, Ergodic property, extinction, and density function of an SIRI epidemic model with nonlinear incidence rate and high-order stochastic perturbations, *Math. Methods. Appl. Sci.*, **45** (2022), 1513–1537. https://doi.org/10.1002/mma.7870

29.B. Zhou, X. Zhang, D. Jiang, Dynamics and density function analysis of a stochastic SVI epidemic model with half saturated incidence rate, *Chaos, Solitons Fractals*, **137** (2020), 109865. https://doi.org/10.1016/j.chaos.2020.109865

30.C. Xu, W. Ou, Y. Pang, Q. Cui, M. U. Rahman, M. Farman, et al., Hopf bifurcation control of a fractional-order delayed turbidostat model via a novel extended hybrid controller, *Match-Commun. Math. Comput. Chem.*, **91** (2024), 367–413. https://doi.org/10.46793/match.91-2.367X

31.C. Xu, Z. Liu, Y. Pang, A. Akgül, Stochastic analysis of a COVID-19 model with effects of vaccination and different transition rates: Real data approach, *Chaos, Solitons Fractals*, **170** (2023), 113395. https://doi.org/10.1016/j.chaos.2023.113395

32.C. Xu, Y. Pang, Z. Liu, J. Shen, M. Liao, P. Li, Insights into COVID-19 stochastic modelling with effects of various transmission rates: simulations with real statistical data from UK, Australia, Spain, and India, *Phys. Scr.*, **99** (2024), 025218. https://doi.org/10.1088/1402-4896/ad186c

33.C. Xu, Y. Zhao, J. Lin, Y. Pang, Z. Liu, J. Shen, et al., Mathematical exploration on control of bifurcation for a plankton–oxygen dynamical model owning delay, *J. Math. Chem.*, (2023), 1–31. https://doi.org/10.1007/s10910-023-01543-y

34.W. Ou, C. Xu, Q. Cui, Y. Pang, Z. Liu, J. Shen, et al., Hopf bifurcation exploration and control technique in a predator-prey system incorporating delay, *AIMS Math.*, **9** (2024), 1622–1651. https://doi.org/10.3934/math.2024080

35.Q. Cui, C. Xu, W. Ou, Y. Pang, Z. Liu, P. Li, et al., Bifurcation behavior and hybrid controller design of a 2D Lotka–Volterra commensal symbiosis system accompanying delay, *Mathematics*, **11** (2023), 4808. https://doi.org/10.3390/math11234808

36.C. W. Gardiner, *Handbook of stochastic methods*, Berlin: springer, 1985.

37.X. Tian, C, Ren, Linear equations, superposition principle and complex exponential notation, *Coll. Phys.*, **23** (2004), 23–25

38.H. Roozen, An asymptotic solution to a two-dimensional exit problem arising in population dynamics, *SIAM J. Appl. Math.*, **49** (1989), 1793–1810. https://doi.org/10.1137/0149110

39.D. J. Higham, An algorithmic introduction to numerical simulation of stochastic differential equations, *SIAM Rev.*, **43** (2001), 525–546. https://doi.org/10.1137/S0036144500378302

40.D. C. Dragon, B. T. Elkin, J. S. Nishi, T. R. Ellsworth, A review of anthrax in Canada and implications for research on the disease in northern bison, *J. Appl. Microbiol.*, **87** (1999), 208–213. https://doi.org/10.1046/j.1365-2672.1999.00872.x

41. World Health Organization, International Office of Epizootics, *Anthrax in humans and animals*, World Health Organization, 2008.

42. Z. Shi, D. Jiang, X. Zhang, A. Alsaedi, A stochastic SEIRS rabies model with population dispersal: Stationary distribution and probability density function, *Appl. Math. Comput.*, **427** (2022), 127189. https://doi.org/10.1016/j.amc. 2022.127189

43. B. Han, B. Zhou, D. Jiang, T. Hayat, A. Alsaedi, Stationary solution, extinction and density function for a high-dimensional stochastic SEI epidemic model with general distributed delay, *Appl. Math. Comput.*, **405** (2021), 126236. https://doi.org/10.1016/j.amc.2021.126236

44. H. Yang, F. Wu, P. E. Kloeden, Stationary distribution of stochastic population dynamics with infinite delay, *J. Differ. Equations*, **340** (2022), 205–226. https://doi.org/10.1016/j.jde.2022.08.035

Appendix 1 : The proof of Lemma 2.1

Proof. Denote the Lyapunov function : $V(S,I,A,C) = S^{\epsilon} + \epsilon I^{\epsilon} + A^{\epsilon} + C^{\epsilon}$, for $\epsilon > 1$, applying Itô's formula

$$\mathrm{D}V(S,I,A,C) = \mathrm{L}V(S,I,A,C)\mathrm{d}t + \epsilon\sigma_1 S^{\epsilon}\mathrm{d}B_1(t) + \epsilon\sigma_2 I^{\epsilon}\mathrm{d}B_2(t) + \epsilon\sigma_3 A^{\epsilon}\mathrm{d}B_3(t) + \epsilon\sigma_4 C^{\epsilon}\mathrm{d}B_4(t),$$

where

$$\begin{aligned}
&\mathrm{L}V(S,I,A,C)\\
&= \epsilon S^{\epsilon-1}[rN(1-\frac{N}{K}) - \eta_a AS - \eta_c CS - \eta_i\frac{SI}{N} - \mu S + \tau I]\\
&+ \epsilon^2 I^{\epsilon-1}[\eta_a AS + \eta_c CS + \eta_i\frac{SI}{N} - (\gamma+\mu)I - \tau I] + \epsilon A^{\epsilon-1}[-\alpha A + \beta C]\\
&+ \epsilon C^{\epsilon-1}[(\gamma+\mu)I - \delta(S+I)C - \kappa C] + \frac{\epsilon(\epsilon-1)}{2}\sigma_1^2 S^{\epsilon} + \frac{\epsilon^2(\epsilon-1)}{2}\sigma_2^2 I^{\epsilon}\\
&+ \frac{\epsilon(\epsilon-1)}{2}\sigma_3^2 A^{\epsilon} + \frac{\epsilon(\epsilon-1)}{2}\sigma_4^2 C^{\epsilon}\\
&\leqslant -S^{\epsilon}[\epsilon\eta_i + \mu\epsilon + \tau(\epsilon-1) - \frac{\epsilon(\epsilon-1)}{2}\sigma_1^2 S^{\epsilon}]\\
&- I^{\epsilon}[-(\gamma+\mu+\tau)(1-\epsilon^2) - \frac{\epsilon^2(\epsilon-1)}{2}\sigma_2^2 I^{\epsilon}] - A^{\epsilon}[\alpha\epsilon - \beta(\epsilon-1) - \frac{\epsilon(\epsilon-1)}{2}\sigma_3^2 A^{\epsilon}]\\
&- C^{\epsilon}[-(\epsilon-1)(\gamma+\mu) - \beta + \kappa\epsilon - \frac{\epsilon(\epsilon-1)}{2}\sigma_4^2 C^{\epsilon}] + \epsilon rN(1-\frac{N}{K})S^{\epsilon-1}.
\end{aligned}$$

Let

$$\begin{aligned}
&F(S,I,A,C)\\
&= -S^{\epsilon}[\epsilon\eta_i + \mu\epsilon + \tau(\epsilon-1) - 1 - \frac{\epsilon(\epsilon-1)}{2}\sigma_1^2 S^{\epsilon}] - I^{\epsilon}[-(\gamma+\mu+\tau)(1-\epsilon^2) - 1 - \frac{\epsilon^2(\epsilon-1)}{2}\sigma_2^2 I^{\epsilon}]\\
&- A^{\epsilon}[\alpha - \beta(\epsilon-1) - 1 - \frac{\epsilon(\epsilon-1)}{2}\sigma_3^2 A^{\epsilon}] - C^{\epsilon}[-(\epsilon-1)(\gamma+\mu) - \beta + \kappa\epsilon - 1 - \frac{\epsilon(\epsilon-1)}{2}\sigma_4^2 C^{\epsilon}]\\
&+ \epsilon rN(1-\frac{N}{K})S^{\epsilon-1}.
\end{aligned}$$

It follows from (2.3) that there exists the positive constant $\hat{T}_2$ such that $F(S,I,A,C) \leqslant \hat{T}_2$. Hence, we can obtain that

$$LV(S,I,A,C) \leqslant F(S,I,A,C) - V(S,I,A,C) \leqslant \hat{T}_2 - V(S,I,A,C),$$

which yields that $dV(S,I,A,C) \leqslant [\hat{T}_2 - V(S,I,A,C)]dt + \epsilon\sigma_1 S^{\epsilon}dB_1(t) + \epsilon\sigma_2 I^{\epsilon}dB_2(t) + \epsilon\sigma_3 A^{\epsilon}dB_3(t) + \epsilon\sigma_4 C^{\epsilon}dB_4(t)$. Integrating both sides of the equation $d[e^t V(S,I,A,C)]$ and taking expectations leads to the following

$$\mathbb{E}[e^t V(S,I,A,C)] \leqslant V(S(0),I(0),A(0),C(0)) + \hat{T}_2 e^t.$$

Thus, $\limsup_{t\to\infty} \mathbb{E}V(S,I,A,C) \leqslant \hat{T}_2$, as a result, $\limsup_{t\to\infty} \mathbb{E}|X(t)|^{\epsilon} \leqslant \frac{\hat{T}^2}{4^{(1-\frac{\epsilon}{2})\wedge 0}} := \hat{T}(\epsilon)$. In addition, when $\epsilon = 2$, we have $\limsup_{t\to\infty} \mathbb{E}|X(t)|^2 \leqslant \hat{T}_2$, which is the second moment of the solution of Model (2.2). Now, for any $\varepsilon > 0$, let $\Psi = \sqrt{\frac{\hat{T}_2}{\varepsilon}}$. By Chebyshev's inequality, $\mathbb{P}\{|X(t)| > \Psi\} \leqslant \frac{\mathbb{E}|X(t)|^2}{\Psi^2}$, which yields that $\limsup_{t\to\infty} \mathbb{P}\{|X(t)| > \Psi\} \leqslant \frac{\hat{T}_2}{\Psi^2} = \varepsilon$. Therefore, we have $\limsup_{t\to\infty} \mathbb{P}\{|X(t)| \leqslant \Psi\} \geqslant 1 - \varepsilon$. In other words, the solution of Model (2.2) is stochastically ultimately bounded.

Appendix 2 : Theory in obtaining standardized transformation matrix

Through the use of invertible linear transformations, we will derive the corresponding standardized transformation matrices of standard B_1. The matrix of B_1: For the algebraic equation $D_1^2 + Z\Lambda_1 + \Lambda_1 Z^T = 0$, where $D = \mathrm{diag}(\sigma_1, 0, 0, 0)$ and

$$Z = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & 0 & a_{43} & a_{44} \end{pmatrix}.$$

We assume that $a_{21} \neq 0$, $a_{32} \neq 0$ and $a_{43} \neq 0$. Define $X = (x_1, x_2, x_3, x_4)$, which follows $dX = ZXdt$. Considering the following vector $Y = (y_1, y_2, y_3, y_4)^T$,

$$\begin{aligned}
&y_4 = x_4, y_3 = y_4' = a_{43}x_3 + a_{44}x_4,\\
&y_2 = y_3' = a_{32}a_{43}x_2 + (a_{33} + a_{44})a_{43}x_3 + (a_{44}^2 + a_{34}a_{43})x_4,\\
&y_1 = y_2' = a_{21}a_{32}a_{43}x_1 + [(a_{22} + a_{33} + a_{44})a_{32}a_{43}]x_2 + [a_{43}(a_{23}a_{32} + a_{34}a_{43} + a_{33}a_{44}) + a_{33}^2 + a_{44}^2]x_3\\
&\quad + [a_{24}a_{32}a_{43} + (a_{33} + a_{44})a_{34}a_{43} + (a_{34}a_{43} + a_{44}^2)a_{44}]x_4\\
&\quad := n_1x_1 + n_2x_2 + n_3x_3 + n_4x_4,
\end{aligned}$$

then the corresponding strandardized transformation matrix is given by

$$N = \begin{pmatrix} n_1 & n_2 & n_3 & n_4 \\ 0 & a_{32}a_{43} & (a_{33} + a_{44})a_{43} & a_{44}^2 + a_{34}a_{43} \\ 0 & 0 & a_{43} & a_{44} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

We derive that $Y = NX$, which means $dY = NdX = NZXdt = (NAN^{-1})Ydt$, then we have

$$dY = d\begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} = \begin{pmatrix} -a_1 & -a_2 & -a_3 & -a_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} dt.$$

Therefore, we can obtain that the corresponding standard B_1 matrix is $NZN^{-1} := Z_1$, which refers to Lemma 2.5. Let $\rho = a_{21}a_{32}a_{43}\sigma$ and $\Theta = \rho^{-2}N\Lambda N^T$.