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Research article

Pointwise error estimate of conservative difference scheme for supergeneralized viscous Burgers' equation

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Abstract: This work focuses on exploring pointwise error estimate of three-level conservative difference scheme for supergeneralized viscous Burgers' equation. The cut-off function method plays an important role in constructing difference scheme and presenting numerical analysis. We study the conservative invariant of proposed method, which is energy-preserving for all positive integers p and q. Meanwhile, one could apply the discrete energy argument to the rigorous proof that the three-level scheme has unique solution combining the mathematical induction. In addition, we prove the L_2-norm and L_∞-norm convergence of proposed scheme in pointwise sense with separate and different ways, which is different from previous work in [1]. Numerical results verify the theoretical conclusions.

Keywords: supergeneralized viscous Burgers' equation; finite difference method; conservativity; pointwise error estimate; convergence

1. Introduction

In this paper, we shall present a incisive analysis of a finite difference method for solving the following supergeneralized viscous Burgers' equation in the domain $[0, L] \times [0, T]$:

$$u_t + u^p(1-u)^q u_x = \nu u_{xx}, \quad x \in (0, L), t \in (0, T], \tag{1.1}$$

$$u(x, 0) = \Psi(x), \quad x \in (0, L), \tag{1.2}$$

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t \in [0, T], \tag{1.3}$$

here L and T are positive constants, $\Psi(x)$ that satisfies $\Psi(0) = \Psi(L) = 0$ is smooth on $[0, L]$, $p \geq 1$ and $q \geq 0$ are two positive integers, and positive constant ν denotes the dynamic viscosity coefficient.

In the last few decades, Burgers' equation for the case of supergeneralized viscous Burgers' equation with $p = 1$ and $q = 0$ has attracted much attention from researchers. It is caused by numerous effective applications of Burgers' equation to many fields of science and engineering like shock wave theory, cosmology, gas dynamics, quantum field and traffic flow, see e.g., [2–6]. The supergeneralized viscous Burgers' equation is a typical evolution equation, and recently a series of numerical methods have been developed to solve it, e.g., finite difference method [7–11], finite volume method [12–14], ADI method [15–18], collocation method [19,20], two-grid method [21,22] and extrapolation method [23]. Meanwhile, as the other simplified form of supergeneralized viscous Burgers' equation with $p \geq 1$ and $q = 0$, the generalized Burgers' equation also plays an important role in applied mathematics and engineering, see e.g., [24–27]. Recently, Wang et al. [28] established two conservative fourth-order compact schemes for Burgers' equation. Zhang et al. [29,30] derived various efficient difference schemes for Burgers' type equations. Gao et al. [31] proposed a bounded high-order upwind scheme in the normalized-variable formulation for the modified Burgers' equations. Guo et al. [32] proposed a BDF3 finite difference scheme for the generalized viscous Burgers' equation. Hu et al. [33] considered an implicit difference scheme to study the local conservation properties for Burgers' equation. Pany et al. [34] investigated an H^1-Galerkin mixed finite element method to approximate the solution of the Burgers' equation. In addition, Jiwari et al. [35] studied a numerical scheme which is a composition of forward finite difference, quasilinearization process and uniform Haar wavelets for solving Burgers' equation. Wang et al. [36] used the weak Galerkin finite element method to study a class of time fractional generalized Burgers' equation. Wang et al. [37–39] presented an implicit robust difference method to solve the modified Burgers equation on graded meshes. Zhang et al. [40] provided a fourth-order compact difference scheme for time-fractional Burgers' equation. Zhang et al. [41] considered a conservative decoupled difference scheme for the rotation-two-component Camassa-Holm system. Sun et al. [42] obtained nonlinear discrete scheme for generalized Burgers' equation with the help of meshless method. Zhang et al. [1] constructed various difference schemes for generalized Burgers' equation only with one positive parameter $p \geq 1$.

The previous works are mainly concerned with the simple case of the parameter $p = 1$ for problem (1.1)–(1.3). Our scheme can extended the results in the previous work [1] with a positive integer $p \geq 1$. In this paper, the main contributions are as follows:

- We construct the discretization of the nonlinear term by a second-order operator in supergeneralized viscous Burgers' equation and provide complete theoretical analysis on the proposed scheme, including conservation, existence, uniqueness and convergence.
- We prove L_2-norm and L_∞-norm convergence in pointwise sense by the cut-off function method, which doesn't have any step ratio restrictions. The L_2-norm and L_∞-norm convergence are proved with separate and different ways, which is different from previous work in [1].

The rest of the paper is arranged as follows. We introduce some useful notations for discretization and construct our proposed scheme in Section 2. In Section 3, we present certain conclusions about conservative invariants and boundedness of the suggested numerical scheme, and we provide the proof of unique solvability and convergence. The numerical test in Section 4 is given to demonstrate the reliability of our analysis. A brief conclusion is followed in Section 5.

2. Derivation of the three-level difference scheme

Firstly, for any integer s, we denote set $N_s = \{i|1 \leq i \leq s, i \in Z\}$ and $N_s^0 = \{i|0 \leq i \leq s, i \in Z\}$. For two positive integers $\tilde{m}$ and $\tilde{n}$, define the spatial step $h = \frac{L}{\tilde{m}}$, and the temporal step $\tau = \frac{T}{\tilde{n}}$. Denote $x_i =$

ih, $i \in N_{\tilde{m}}^0$; $t_k = k\tau$, $k \in N_{\tilde{n}}^0$. We introduce the mesh $\tilde{\omega}_{LT} = \tilde{\omega}_L \times \tilde{\omega}_T$, where $\tilde{\omega}_L = \{x_i \,|\, i \in N_{\tilde{m}}^0\}$, and $\tilde{\omega}_T = \{t_k \,|\, k \in N_{\tilde{n}}^0\}$. Denote $x_{i+\frac{1}{2}} = \frac{1}{2}(x_i + x_{i+1}), i \in N_{\tilde{m}-1}^0$ and $t_{k+\frac{1}{2}} = \frac{1}{2}(t_k + t_{k+1}), k \in N_{\tilde{n}-1}^0$.

Let $\mathcal{J}_h = \{j \,|\, j = (j_0, j_1, \cdots, j_{\tilde{m}})\}$ and $\mathring{\mathcal{J}}_h = \{j \,|\, j \in \mathcal{J}_h, j_0 = j_{\tilde{m}} = 0\}$ be the spaces of grid functions on $\tilde{\omega}_L$. For $d, j \in \mathcal{J}_h$, introducing the following notations:

$$
\begin{aligned}
&\delta_x d_{i+\frac{1}{2}}^k = \frac{1}{h}(d_{i+1}^k - d_i^k), \quad \delta_x^2 d_i^k = \frac{1}{h^2}(d_{i-1}^k - 2d_i^k + d_{i+1}^k),\\
&\Delta_x d_i^k = \frac{1}{2h}(d_{i+1}^k - d_{i-1}^k), \quad d_i^{k+\frac{1}{2}} = \frac{1}{2}(d_i^k + d_i^{k+1}),\\
&\delta_t d_i^{k+\frac{1}{2}} = \frac{1}{\tau}(d_i^{k+1} - d_i^k), \quad (d, j) = h(\frac{1}{2}d_0 j_0 + \sum_{i=1}^{\tilde{m}-1} d_i j_i + \frac{1}{2}d_{\tilde{m}} j_{\tilde{m}}),\\
&d_i^{\bar{k}} = \frac{1}{2}(d_i^{k+1} + d_i^{k-1}), \quad \Delta_t d_i^k = \frac{1}{2\tau}(d_i^{k+1} - d_i^{k-1}),\\
&\|d\| = \sqrt{(d,d)}, \quad \|d\|_\infty = \max_{0\le i\le \tilde{m}} |d_i|,\\
&\psi(d, j)_i = d_i \Delta_x j_i + \Delta_x (dj)_i, \quad d_i^{\bar{k}} = \frac{d^{k+1} + d^{k-1}}{2},\\
&\langle d, j\rangle = h\sum_{i=0}^{\tilde{m}-1} (\delta_x d_{i+\frac{1}{2}})(\delta_x j_{i+\frac{1}{2}}), \quad |d|_1 = \sqrt{\langle d, d\rangle}.
\end{aligned}
$$

Lemma 2.1. [28] Let $j \in \mathcal{J}_h$ and $r \in \mathring{\mathcal{J}}_h$, then

$$(\psi(j, r), r) = 0.$$

Lemma 2.2. [28] Set $j \in \mathring{\mathcal{J}}_h$, then

$$-(\delta_x^2 j, j) = |j|_1^2, \quad \|j\|_\infty \le \frac{\sqrt{L}}{2}|j|_1, \quad \|j\| \le \frac{L}{\sqrt{6}}|j|_1.$$

Lemma 2.3. Suppose that $U = (U_0, U_1, \ldots, U_{\tilde{m}})$, $u = (u_0, u_1, \ldots, u_{\tilde{m}}) \in \mathcal{J}_h$ and $g(u)$ is a second-order smooth function. Denote $e = (e_0, e_1, \ldots, e_{\tilde{m}})$ and $e_i = U_i - u_i$, $i \in N_{\tilde{m}}^0$. Then there are $\rho \in (0, 1)$ and $\zeta_i \in (y_i, r_i)$ such that

$$
\begin{aligned}
\delta_x(g(U) - g(u))_{i+\frac{1}{2}} \;=\;& g'(\rho u_{i+1} + (1-\rho)u_i)\delta_x e_{i+\frac{1}{2}}\\
&+g''(\zeta_i)[\rho(U_{i+1} - u_{i+1}) + (1-\rho)(U_i - u_i)]\delta_x U_{i+\frac{1}{2}}, \qquad (2.1)
\end{aligned}
$$

where

$$y_i = \min\{\rho u_{i+1} + (1-\rho)u_i, \rho U_{i+1} + (1-\rho)U_i\},$$

$$r_i = \max\{\rho u_{i+1} + (1-\rho)u_i, \rho U_{i+1} + (1-\rho)U_i\}.$$

Proof. Using the mean value theorem, one has

$$\delta_x(g(U) - g(u))_{i+\frac{1}{2}}$$

$$
\begin{aligned}
&= \frac{1}{h}[(g(U_{i+1}) - g(u_{i+1})) - (g(U_i) - g(u_i))] \\
&= \frac{1}{h}[(g(U_i + h\delta_x U_{i+\frac{1}{2}}) - g(u_i + h\delta_x u_{i+\frac{1}{2}})) - (g(U_i) - g(u_i))] \\
&= \frac{1}{h}[(g(U_i + h\delta_x U_{i+\frac{1}{2}}) - g(U_i)) - (g(u_i + h\delta_x u_{i+\frac{1}{2}}) - g(u_i))] \\
&= g'(U_i + \rho h\delta_x U_{i+\frac{1}{2}})\delta_x U_{i+\frac{1}{2}} - g'(u_i + \rho h\delta_x u_{i+\frac{1}{2}})\delta_x u_{i+\frac{1}{2}}.
\end{aligned}
$$

Again, applying the mean value theorem, we have

$$
\begin{aligned}
&\delta_x(g(U) - g(u))_{i+\frac{1}{2}} \\
= \;& g'(u_i + \rho h\delta_x u_{i+\frac{1}{2}})\delta_x e_{i+\frac{1}{2}} + [g'(U_i + \rho h\delta_x U_{i+\frac{1}{2}}) - g'(u_i + \rho h\delta_x u_{i+\frac{1}{2}})]\delta_x U_{i+\frac{1}{2}} \\
= \;& g'(\rho u_{i+1} + (1-\rho)u_i)\delta_x e_{i+\frac{1}{2}} \\
& + [g'(\rho U_{i+1} + (1-\rho)U_i) - g'(\rho u_{i+1} + (1-\rho)u_i)]\delta_x U_{i+\frac{1}{2}} \\
= \;& g'(\rho u_{i+1} + (1-\rho)u_i)\delta_x e_{i+\frac{1}{2}} + g''(\zeta_i)[\rho e_{i+1} + (1-\rho)e_i]\delta_x U_{i+\frac{1}{2}}.
\end{aligned}
$$

The proof is finished.

In order to construct a three-level conservative numerical scheme for supergeneralized viscous Burgers' equation (1.1)–(1.3), we first turn problem (1.1) into an equivalent form as follows:

$$
\begin{cases}
u_t + \sum\limits_{m=0}^{q} C_q^m \frac{(-1)^m}{p+m+2}(W_{(m)}u_x + (W_{(m)}u)_x) = \nu u_{xx}, \\
W_{(m)} = u^{p+m},
\end{cases} \tag{2.2}
$$

where C_q^m is the binomial coefficient, $0 \le m \le q$.

We denote $U_i^k = u(x_i, t_k)$, and let u_i^k denote the nodal approximation to the exact solution computed at the mesh point (x_i, t_k).

Considering (2.2) at the point (x_i, t_k), $i \in N_{\tilde{m}-1}$, $k \in N_{\tilde{n}-1}$, one gets

$$
\begin{cases}
\Delta_t U_i^k + \sum\limits_{m=0}^{q} C_q^m \frac{(-1)^m}{p+m+2}\psi(W_{(m)}^k, U^{\bar{k}})_i = \nu\delta_x^2 U_i^{\bar{k}} + P_i^k, \\
{W_{(m)}}_i^k = (U_i^k)^{p+m}.
\end{cases} \tag{2.3}
$$

By Taylor expansion, one gets

$$
|P_i^k| \le c_1(\tau^2 + h^2), \tag{2.4}
$$

where c_1 is a positive constant.

We consider (1.1) at the point (x_i, t_0), $i \in N_{\tilde{m}-1}$, noticing (1.2), and one gets

$$
u_t(x_i, t_0) = \upsilon\Psi(x_i)'' - (\Psi(x_i))^p(1 - \Psi(x_i))^q\Psi(x_i)', \quad i \in N_{\tilde{m}-1}.
$$

Denote

$$
r_i = \Psi(x_i) + \frac{\tau}{2}[\upsilon\Psi(x_i)'' - (\Psi(x_i))^p(1 - \Psi(x_i))^q\Psi(x_i)'], \tag{2.5}
$$

$$
{R_{(m)}}_i = (r_i)^{p+m}, \quad i \in N_{\tilde{m}-1}. \tag{2.6}
$$

Considering (2.2) at the point $(x_i, t_{\frac{1}{2}})$, $i \in N_{\tilde{m}-1}$, one gets

$$\delta_t U_i^{\frac{1}{2}} + \sum_{m=0}^{q} C_q^m \frac{(-1)^m}{p+m+2} \psi(R_{(m)}, U^{\frac{1}{2}})_i = \nu \delta_x^2 U_i^{\frac{1}{2}} + P_i^0, \tag{2.7}$$

and

$$|P_i^0| \le c_1(\tau^2 + h^2). \tag{2.8}$$

Noticing (1.2) and (1.3), we get

$$\begin{cases} U_i^0 = \Psi(x_i), & i \in N_{\tilde{m}-1}, \\ U_0^k = 0, & U_{\tilde{m}}^k = 0, \quad k \in N_{\tilde{n}}^0. \end{cases} \tag{2.9}$$

Omitting the small terms P_i^k in (2.3) and P_i^0 in (2.7), and replacing U_i^k by u_i^k, and ${W_{(m)}}_i^k$ by ${w_{(m)}}_i^k$, $i \in N_{\tilde{m}-1}$, $k \in N_{\tilde{n}-1}$, respectively. Thus, we can obtain the three-level difference approximation for (1.1)–(1.3) as follows

$$\Delta_t u_i^k + \sum_{m=0}^{q} C_q^m \frac{(-1)^m}{p+m+2} \psi(w_{(m)}^k, u^{\bar{k}})_i = \nu \delta_x^2 u_i^{\bar{k}}, \tag{2.10}$$

$$\delta_t u_i^{\frac{1}{2}} + \sum_{m=0}^{q} C_q^m \frac{(-1)^m}{p+m+2} \psi(R_{(m)}, u^{\frac{1}{2}})_i = \nu \delta_x^2 u_i^{\frac{1}{2}}, \tag{2.11}$$

$${w_{(m)}}_i^k = (u_i^k)^{p+m}, \quad i \in N_{\tilde{m}}^0, \quad k \in N_{\tilde{n}-1}, \tag{2.12}$$

$$u_i^0 = \Psi(x_i), \quad i \in N_{\tilde{m}-1}, \tag{2.13}$$

$$u_0^k = 0, \quad u_{\tilde{m}}^k = 0, \quad k \in N_{\tilde{n}}^0. \tag{2.14}$$

Noticing that substituting (2.12) into (2.10), the three-level difference scheme only contains one variable u_i^k.

3. The numerical analysis of three-level difference scheme

We now begin to consider the energy conservation and boundedness of solution of the three-level numerical scheme (2.10)–(2.14).

Theorem 3.1. *Suppose that $\{u_i^k, {w_{(m)}}_i^k \mid i \in N_{\tilde{m}}^0, k \in N_{\tilde{n}}^0\}$ is the solution of (2.10)–(2.14), we get*

$$\frac{1}{2}(\|u^1\|^2 + \|u^0\|^2) + \nu\tau |u^{\frac{1}{2}}|_1^2 = \|u^0\|^2, \tag{3.1}$$

$$\Upsilon^k = \Upsilon^0, \quad k \in N_{\tilde{n}-1}, \tag{3.2}$$

where

$$\Upsilon^k = \frac{1}{2}(\|u^{k+1}\|^2 + \|u^k\|^2) + 2\nu\tau \sum_{s=1}^{k} |u^{\bar{s}}|_1^2, \quad k \in N_{\tilde{n}-1}^0. \tag{3.3}$$

Proof. 1) Taking the inner product of (2.11) with $u^{\frac{1}{2}}$, one obtains

$$(\delta_t u^{\frac{1}{2}}, u^{\frac{1}{2}}) + \sum_{m=0}^{q} C_q^m \frac{(-1)^m}{p+m+2}(\psi(R_{(m)}, u^{\frac{1}{2}}), u^{\frac{1}{2}}) = \nu(\delta_x^2 u^{\frac{1}{2}}, u^{\frac{1}{2}}).$$

Since $u^{\frac{1}{2}} \in \mathring{\mathcal{J}}_h$, by Lemmas 2.1 and 2.2, one gets

$$\begin{aligned}
&(\delta_t u^{\frac{1}{2}}, u^{\frac{1}{2}}) = \frac{1}{2\tau}(\|u^1\|^2 - \|u^0\|^2),\\
&(\psi(R_{(m)}, u^{\frac{1}{2}}), u^{\frac{1}{2}}) = 0,\\
&-(\delta_x^2 u^{\frac{1}{2}}, u^{\frac{1}{2}}) = |u^{\frac{1}{2}}|_1^2.
\end{aligned}$$

Thus,

$$\frac{1}{2}(\|u^1\|^2 - \|u^0\|^2) + \nu\tau|u^{\frac{1}{2}}|_1^2 = 0. \tag{3.4}$$

Namely,

$$\frac{1}{2}(\|u^1\|^2 + \|u^0\|^2) + \nu\tau|u^{\frac{1}{2}}|_1^2 = \|u^0\|^2.$$

2) Taking the inner product of (2.10) with $u^{\bar{k}}$, one gets

$$(\Delta_t u^k, u^{\bar{k}}) + \sum_{m=0}^{q} C_q^m \frac{(-1)^m}{p+m+2}(\psi(w_{(m)}^k, u^{\bar{k}}), u^{\bar{k}}) = \nu(\delta_x^2 u^{\bar{k}}, u^{\bar{k}}).$$

Since $u^{\bar{k}} \in \mathring{\mathcal{J}}_h$, by Lemmas 2.1 and 2.2, we have

$$\begin{aligned}
&(\Delta_t u^k, u^{\bar{k}}) = \frac{1}{4\tau}(\|u^{k+1}\|^2 - \|u^{k-1}\|^2),\\
&(\psi(w_{(m)}^k, u^{\bar{k}}), u^{\bar{k}}) = 0,\\
&-(\delta_x^2 u^{\bar{k}}, u^{\bar{k}}) = |u^{\bar{k}}|_1^2.
\end{aligned}$$

Thus,

$$\frac{1}{4}(\|u^{k+1}\|^2 - \|u^{k-1}\|^2) + \nu\tau|u^{\bar{k}}|_1^2 = 0. \tag{3.5}$$

Above equality can be rewritten as

$$\frac{1}{2}(\Upsilon^k - \Upsilon^{k-1}) = 0, \quad k \in N_{\tilde{n}-1}.$$

Thus,

$$\Upsilon^k = \Upsilon^0, \quad k \in N_{\tilde{n}-1}.$$

Corollary 3.2. *Let $\{u_i^k, {w_{(m)}}_i^k \mid i \in N_{\tilde{m}}^0, k \in N_{\tilde{n}}^0\}$ represent the solution of (2.10)–(2.14). Then one has*

$$\frac{1}{2}(\|u^{k+1}\|^2 + \|u^k\|^2) + \nu\tau|u^{\frac{1}{2}}|_1^2 + 2\nu\tau\sum_{s=1}^{k}|u^{\bar{s}}|_1^2 = \|u^0\|^2, \quad k \in N_{\tilde{n}-1}^0.$$

Proof. According to Theorem 3.1,

$$\begin{aligned}\Upsilon^k = \Upsilon^0 &= \frac{1}{2}(\|u^1\|^2 + \|u^0\|^2)\\ &= \|u^0\|^2 - \nu\tau|u^{\frac{1}{2}}|_1^2.\end{aligned}$$

Thus,

$$\Upsilon^k + \nu\tau|u^{\frac{1}{2}}|_1^2 = \|u^0\|^2.$$

Corollary 3.3. *Let $\{u_i^k, {w_{(m)}}_i^k \mid i \in N_{\tilde{m}}^0, k \in N_{\tilde{n}}^0\}$ represent the solution of (2.10)–(2.14). Then the computed solution u_i^k can satisfy*

$$\|u^k\| \le \|u^0\|, \quad k \in N_{\tilde{n}}.$$

Proof. From (3.4) and (3.5) in Theorem 3.1, we can get Corollary 3.3 directly.

Furthermore, we will carry out the proof of existence and uniqueness of the solution of (2.10)–(2.14).

Theorem 3.4. *The solution of (2.10)–(2.14) exists and it is unique.*

Proof. According to (2.13) and (2.14), u^0 has been determined uniquely. From (2.11) and (2.14), establishing a linear system with respect to u^1, and considering the corresponding homogeneous system

$$\frac{1}{\tau}u_i^1 + \frac{1}{2}\sum_{m=0}^{q} C_q^m \frac{(-1)^m}{p+m+2}\psi(R_{(m)}, u^1)_i = \frac{1}{2}\nu\delta_x^2 u_i^1, \quad i \in N_{\tilde{m}-1}, \tag{3.6}$$

$$u_0^1 = 0, \quad u_{\tilde{m}}^1 = 0. \tag{3.7}$$

Taking the inner product of (3.6) with u^1, one has

$$\frac{1}{\tau}\|u^1\|^2 + \frac{1}{2}\sum_{m=0}^{q} C_q^m \frac{(-1)^m}{p+m+2}(\psi(R_{(m)}, u^1), u^1) = \frac{1}{2}\nu(\delta_x^2 u^1, u^1).$$

By Lemmas 2.1 and 2.2, one gets

$$\begin{aligned}&(\psi(R_{(m)}, u^1), u^1) = 0,\\ &(\delta_x^2 u^1, u^1) = -|u^1|_1^2.\end{aligned}$$

Therefore,

$$\frac{1}{\tau}\|u^1\|^2 + \frac{1}{2}\nu|u^1|_1^2 = 0.$$

It is easy to obtain

$$\|u^1\| = 0.$$

It implies that (2.11) and (2.14) determine u^1 uniquely.

Assume that u^k and u^{k-1} have been known. By (2.10), (2.12) and (2.14), we get the following linear homogeneous system of equations with respect to u^{k+1}:

$$\frac{1}{2\tau}u_i^{k+1} + \frac{1}{2}\sum_{m=0}^{q} C_q^m \frac{(-1)^m}{p+m+2}\psi(w_{(m)}^k, u^{k+1})_i = \frac{1}{2}\nu\delta_x^2 u_i^{k+1}, \quad i \in N_{\tilde{m}-1}, \tag{3.8}$$

$$u_0^{k+1} = 0, \quad u_{\tilde{m}}^{k+1} = 0. \tag{3.9}$$

Taking the inner product of (3.8) with u^{k+1}, one has

$$\frac{1}{2\tau}\|u^{k+1}\|^2 + \frac{1}{2}\sum_{m=0}^{q} C_q^m \frac{(-1)^m}{p+m+2}(u^{k+1}, \psi(w_{(m)}^k, u^{k+1})) = \frac{1}{2}\nu(u^{k+1}, \delta_x^2 u^{k+1}).$$

By Lemmas 2.1 and 2.2, one gets

$$(u^{k+1}, \psi(w_{(m)}^k, u^{k+1})) = 0,$$
$$(u^{k+1}, \delta_x^2 u^{k+1}) = -|u^{k+1}|_1^2.$$

Therefore,

$$\frac{1}{2\tau}\|u^{k+1}\|^2 + \frac{1}{2}\nu|u^{k+1}|_1^2 = 0.$$

It is easy to obtain

$$\|u^{k+1}\| = 0.$$

Consequently, it implies that u^{k+1} solved by (2.10), (2.12) and (2.14) is unique.

Based on mathematical induction, (2.10)–(2.14) is uniquely solvable, and this completes the proof.

In order to establish the convergence of (2.10)–(2.14), we will introduce the cut-off function method next.

Denote

$$M = \max_{(x,t)\in[0,L]\times[0,T]} |u(x,t)|, \quad \tilde{c_1} = \max_{(x,t)\in[0,L]\times[0,T]} \{|u_x(x,t)|\}. \tag{3.10}$$

Define a group of second-order smooth functions

$$g_m(u) = \begin{cases} u^{p+m}, & |u| \le M+1, \\ 0, & |u| \ge M+2, \end{cases}$$

where $0 \le m \le q$.

Denote

$$\max_{u\in R, 0\le m\le q} |g_m(u)| = \hat{c}_0, \ \max_{u\in R, 0\le m\le q} |g_m^{'}(u)| = \hat{c}_1, \ and \ \max_{u\in R, 0\le m\le q} |g_m^{''}(u)| = \hat{c}_2.$$

Based on the cut-off function method, we construct a new difference scheme as follows:

$$\Delta_t u_i^k + \sum_{m=0}^{q} C_q^m \frac{(-1)^m}{p+m+2} \psi(w_{(m)}^k, u^{\bar{k}})_i = \nu \delta_x^2 u_i^{\bar{k}}, \tag{3.11}$$

$$\delta_t u_i^{\frac{1}{2}} + \sum_{m=0}^{q} C_q^m \frac{(-1)^m}{p+m+2} \psi(R_{(m)}, u^{\frac{1}{2}})_i = \nu \delta_x^2 u_i^{\frac{1}{2}}, \tag{3.12}$$

$$w_{(m)}{}_i^k = g_m(u_i^k), \quad i \in N_{\tilde{m}-1}, \quad k \in N_{\tilde{n}-1}, \tag{3.13}$$

$$u_i^0 = \Psi(x_i), \quad i \in N_{\tilde{m}-1}, \tag{3.14}$$

$$u_0^k = 0, \quad u_{\tilde{m}}^k = 0, \quad k \in N_{\tilde{n}}^0. \tag{3.15}$$

For the above difference scheme, it is conservative.

Theorem 3.5. *Suppose that* $\{u_i^k, w_{(m)}{}_i^k \mid i \in N_{\tilde{m}}^0, k \in N_{\tilde{n}}^0\}$ *represents the solution of (3.11)–(3.15), we get*

$$\frac{1}{2}(\|u^1\|^2 + \|u^0\|^2) + \nu\tau |u^{\frac{1}{2}}|_1^2 = \|u^0\|^2, \tag{3.16}$$

$$\Upsilon^k = \Upsilon^0, \quad k \in N_{\tilde{n}-1}, \tag{3.17}$$

where

$$\Upsilon^k = \frac{1}{2}(\|u^{k+1}\|^2 + \|u^k\|^2) + 2\nu\tau \sum_{s=1}^{k} |u^{\bar{s}}|_1^2, \quad k \in N_{\tilde{n}-1}^0.$$

Proof. The proof of (3.16) and (3.17) is similar to the proof of Theorem 3.1.

Now we prove the L_2-norm and L_∞-norm convergence of (3.11)–(3.15).

Theorem 3.6. *Assume that* $\{u_i^k, w_{(m)}{}_i^k \mid i \in N_{\tilde{m}}^0, k \in N_{\tilde{n}}^0\}$ *is the solution of (3.11)–(3.15) and* $\{U_i^k, W_{(m)}{}_i^k \mid i \in N_{\tilde{m}}^0, k \in N_{\tilde{n}}^0\}$ *is the solution of (1.1)–(1.3), there exists a positive constant* c_2 *such that*

$$\|U^k - u^k\| \le c_2(\tau^2 + h^2), \quad k \in N_{\tilde{n}}^0. \tag{3.18}$$

Proof. Define

$$e_i^k = U_i^k - u_i^k, b_{(m)}{}_i^k = W_{(m)}{}_i^k - w_{(m)}{}_i^k.$$

Since (3.10), we get

$$g_m(U_i^k) = (U_i^k)^{p+m}.$$

Subtracting (3.11)–(3.15) from (2.3), (2.7) and (2.9) follows

$$\delta_t e_i^{\frac{1}{2}} + \sum_{m=0}^{q} \frac{C_q^m(-1)^m}{p+m+2} \psi(R_{(m)}, e^{\frac{1}{2}})_i = \nu \delta_x^2 e_i^{\frac{1}{2}} + P_i^0, \quad i \in N_{\tilde{m}-1}, \tag{3.19}$$

$$\Delta_t e_i^k + \sum_{m=0}^{q} \frac{C_q^m(-1)^m}{p+m+2} [\psi(W_{(m)}^k, U^{\bar{k}})_i - \psi(w_{(m)}^k, u^{\bar{k}})_i] = \nu \delta_x^2 e_i^{\bar{k}} + P_i^k,$$
$$i \in N_{\tilde{m}-1}, \quad k \in N_{\tilde{n}-1}, \tag{3.20}$$

$$b_{(m)}{}_i^k = g_m(U_i^k) - g_m(u_i^k), \quad i \in N_{\tilde{m}}^0, \quad k \in N_{\tilde{n}-1}, \tag{3.21}$$

$$e_i^0 = 0, \quad i \in N_{\tilde{m}-1}, \tag{3.22}$$
$$e_0^k = 0, \quad e_{\tilde{m}}^k = 0, \quad k \in N_{\tilde{n}}^0. \tag{3.23}$$

When $k = 0$, from (3.22) and (3.23), we get

$$\|e^0\| = 0. \tag{3.24}$$

Taking the inner product of (3.19) with $e^{\frac{1}{2}}$, one gets

$$(\delta_t e^{\frac{1}{2}}, e^{\frac{1}{2}}) + \sum_{m=0}^{q} \frac{C_q^m (-1)^m}{p+m+2} (\psi(R_{(m)}, e^{\frac{1}{2}}), e^{\frac{1}{2}}) = \nu(\delta_x^2 e^{\frac{1}{2}}, e^{\frac{1}{2}}) + (P^0, e^{\frac{1}{2}}). \tag{3.25}$$

By Lemmas 2.1 and 2.2, we obtain

$$(\delta_t e^{\frac{1}{2}}, e^{\frac{1}{2}}) = \frac{1}{2\tau}(\|e^1\|^2 - \|e^0\|^2) = \frac{1}{2\tau}\|e^1\|^2, \tag{3.26}$$
$$(\psi(R_{(m)}, e^{\frac{1}{2}}), e^{\frac{1}{2}}) = 0, \tag{3.27}$$
$$(\delta_x^2 e^{\frac{1}{2}}, e^{\frac{1}{2}}) = -\|\delta_x e^{\frac{1}{2}}\|^2. \tag{3.28}$$

Substituting (3.26)–(3.28) into (3.25), we have

$$\begin{aligned} \frac{1}{2\tau}\|e^1\|^2 &= -\|\delta_x e^{\frac{1}{2}}\|^2 + (P^0, e^{\frac{1}{2}}) \\ &\leq (P^0, e^{\frac{1}{2}}) \\ &\leq \frac{1}{2}\|P^0\|^2 + \frac{1}{2}\|e^{\frac{1}{2}}\|^2 \\ &\leq \frac{1}{2}\|P^0\|^2 + \frac{1}{4}\|e^1\|^2. \end{aligned}$$

Thus,

$$(1 - \frac{\tau}{2})\|e^1\|^2 \leq \tau\|P^0\|^2.$$

When $\frac{\tau}{2} \leq \frac{1}{3}$, noticing (2.8), one gets

$$\|e^1\|^2 \leq \|P^0\|^2 \leq L c_1^2 (\tau^2 + h^2)^2.$$

or

$$\|e^1\| \leq \sqrt{L} c_1 (\tau^2 + h^2). \tag{3.29}$$

By (3.10) and Lagrange mean value theorem, one gets

$$|\Delta_x U_i^k| \leq \tilde{c}_1, \tag{3.30}$$
$$|b_{(m)}{}_i^k| \leq \hat{c}_1 |e_i^k|. \tag{3.31}$$

Taking the inner product of (3.20) with $e^{\bar{k}}$, one gets

$$\begin{aligned} &(\Delta_t e^k, e^{\bar{k}}) + \sum_{m=0}^{q} \frac{C_q^m(-1)^m}{p+m+2}(\psi(W_{(m)}^k, U^{\bar{k}}) - \psi(w_{(m)}^k, u^{\bar{k}}), e^{\bar{k}}) \\ = \;& \nu(\delta_x^2 e^{\bar{k}}, e^{\bar{k}}) + (P^k, e^{\bar{k}}), \quad k \in N_{\tilde{n}-1}. \end{aligned} \tag{3.32}$$

Using Lemma 2.2, one obtains

$$(\Delta_t e^k, e^{\bar{k}}) = \frac{1}{4\tau}(\|e^{k+1}\|^2 - \|e^{k-1}\|^2), \tag{3.33}$$

$$(\delta_x^2 e^{\bar{k}}, e^{\bar{k}}) = -\|\delta_x e^{\bar{k}}\|^2. \tag{3.34}$$

Substituting (3.33) and (3.34) into (3.32), above equality (3.32) becomes

$$\begin{aligned} &\frac{1}{4\tau}(\|e^{k+1}\|^2 - \|e^{k-1}\|^2) + \nu\|\delta_x e^{\bar{k}}\|^2 \\ = \;& -\sum_{m=0}^{q} \frac{C_q^m(-1)^m}{p+m+2}(\psi(W_{(m)}^k, U^{\bar{k}}) - \psi(w_{(m)}^k, u^{\bar{k}}), e^{\bar{k}}) + (P^k, e^{\bar{k}}) \\ \leq \;& \sum_{m=0}^{q} \frac{a_0}{p+2}|-(\psi(W_{(m)}^k, U^{\bar{k}}) - \psi(w_{(m)}^k, u^{\bar{k}}), e^{\bar{k}})| + |(P^k, e^{\bar{k}})|, \end{aligned} \tag{3.35}$$

where $a_0 = \max_{0\leq m\leq q} C_q^m$.

Noticing that

$$\begin{aligned} &\psi(W_{(m)}^k, U^{\bar{k}})_i - \psi(w_{(m)}^k, u^{\bar{k}})_i \\ = \;& \psi(W_{(m)}^k, U^{\bar{k}})_i - \psi(W_{(m)}^k - b_{(m)}^k, U^{\bar{k}} - e^{\bar{k}})_i \\ = \;& \psi(W_{(m)}^k, e^{\bar{k}})_i + \psi(b_{(m)}^k, U^{\bar{k}})_i - \psi(b_{(m)}^k, e^{\bar{k}})_i. \end{aligned}$$

Thus, by Lemma 2.1, we have

$$\begin{aligned} &-(\psi(W_{(m)}^k, U^{\bar{k}}) - \psi(w_{(m)}^k, u^{\bar{k}}), e^{\bar{k}}) \\ = \;& -(\psi(b_{(m)}^k, U^{\bar{k}}), e^{\bar{k}}) \\ = \;& -h\sum_{i=1}^{\tilde{m}-1}[b_{(m)}{}_i^k \Delta_x U_i^{\bar{k}} + \Delta_x(b_{(m)}{}^k U^{\bar{k}})_i]e_i^{\bar{k}} \\ = \;& -h\sum_{i=1}^{\tilde{m}-1} b_{(m)}{}_i^k e_i^{\bar{k}} \Delta_x U_i^{\bar{k}} + h\sum_{i=1}^{\tilde{m}-1} b_{(m)}{}_i^k U_i^{\bar{k}} \Delta_x e_i^{\bar{k}}. \end{aligned} \tag{3.36}$$

Noticing (3.30), (3.31) and (3.10), we have

$$\begin{aligned} &|-(\psi(W_{(m)}^k, U^{\bar{k}}) - \psi(w_{(m)}^k, u^{\bar{k}}), e^{\bar{k}})| \\ \leq \;& h\sum_{i=1}^{\tilde{m}-1} \tilde{c}_1\hat{c}_1|e_i^k||e_i^{\bar{k}}| + h\sum_{i=1}^{\tilde{m}-1} M\hat{c}_1|e_i^k||\Delta_x e_i^{\bar{k}}| \end{aligned}$$

$$\begin{aligned} &\leq \tilde{c}_1\hat{c}_1\|e^k\|\cdot\|e^{\bar{k}}\| + M\hat{c}_1\|e^k\|\cdot\|\Delta_x e^{\bar{k}}\| \\ &\leq \tilde{c}_1\hat{c}_1\|e^k\|\cdot\|e^{\bar{k}}\| + M\hat{c}_1\|e^k\|\cdot\|\delta_x e^{\bar{k}}\|. \end{aligned} \tag{3.37}$$

Substituting (3.37) into (3.35), (3.35) yields

$$\begin{aligned} &\frac{1}{4\tau}(\|e^{k+1}\|^2 - \|e^{k-1}\|^2) + \nu\|\delta_x e^{\bar{k}}\|^2 \\ \leq\ & \sum_{m=0}^{q}\frac{a_0}{p+2}(\tilde{c}_1\hat{c}_1\|e^k\|\cdot\|e^{\bar{k}}\| + M\hat{c}_1\|e^k\|\cdot\|\delta_x e^{\bar{k}}\|) + \frac{1}{2}\|P^k\|^2 + \frac{1}{2}\|e^{\bar{k}}\|^2 \\ \leq\ & \frac{a_0(1+q)}{p+2}(\frac{\tilde{c}_1\hat{c}_1}{2}\|e^k\|^2 + \frac{\tilde{c}_1\hat{c}_1}{2}\|e^{\bar{k}}\|^2 + \frac{(p+2)\nu}{a_0(1+q)}\|\delta_x e^{\bar{k}}\|^2 + \frac{a_0(1+q)M^2\hat{c}_1^2}{4(p+2)\nu}\|e^k\|^2) \\ & + \frac{1}{2}\|P^k\|^2 + \frac{1}{2}\|e^{\bar{k}}\|^2 \\ =\ & [\frac{a_0(1+q)\tilde{c}_1\hat{c}_1}{2(p+2)} + \frac{a_0^2(1+q)^2M^2\hat{c}_1^2}{4(p+2)^2\nu}]\|e^k\|^2 + [\frac{a_0(1+q)\tilde{c}_1\hat{c}_1}{2(p+2)} + \frac{1}{2}]\|e^{\bar{k}}\|^2 \\ & + \nu\|\delta_x e^{\bar{k}}\|^2 + \frac{1}{2}\|P^k\|^2. \end{aligned} \tag{3.38}$$

Combining (2.4), above equality (3.38) becomes

$$\begin{aligned} &\frac{1}{4\tau}(\|e^{k+1}\|^2 - \|e^{k-1}\|^2) \\ \leq\ & c_3\|e^k\|^2 + 2c_4\|e^{\bar{k}}\|^2 + \frac{1}{2}Lc_1^2(\tau^2+h^2)^2 \\ \leq\ & c_3\|e^k\|^2 + c_4\|e^{k+1}\|^2 + c_4\|e^{k-1}\|^2 + \frac{1}{2}Lc_1^2(\tau^2+h^2)^2, \end{aligned} \tag{3.39}$$

where $c_3 = \frac{a_0(1+q)\tilde{c}_1\hat{c}_1}{2(p+2)} + \frac{a_0^2(1+q)^2M^2\hat{c}_1^2}{4(p+2)^2\nu}$ and $c_4 = \frac{a_0(1+q)\tilde{c}_1\hat{c}_1}{4(p+2)} + \frac{1}{4}$ are two positive constants.

Rearranging (3.39) to yield

$$\begin{aligned} (1-4c_4\tau)\|e^{k+1}\|^2 \leq\ & 4c_3\tau\|e^k\|^2 + (1+4c_4\tau)\|e^{k-1}\|^2 \\ & + 2Lc_1^2\tau(\tau^2+h^2)^2, \quad k \in N_{\tilde{n}-1}. \end{aligned} \tag{3.40}$$

For $k \in N_{\tilde{n}-1}$, when $4c_4\tau \leq \frac{1}{3}$, (3.40) yields

$$\|e^{k+1}\|^2 \leq 6c_3\tau\|e^k\|^2 + (1+12c_4\tau)\|e^{k-1}\|^2 + 3Lc_1^2\tau(\tau^2+h^2)^2. \tag{3.41}$$

Therefore,

$$\begin{aligned} \max\{\|e^{k+1}\|^2, \|e^k\|^2\} \leq\ & [1+6(c_3+2c_4)\tau]\max\{\|e^{k-1}\|^2, \|e^k\|^2\} \\ & + 3Lc_1^2\tau(\tau^2+h^2)^2. \end{aligned} \tag{3.42}$$

According to Gronwall's inequality, we obtain

$$\max\{\|e^{k+1}\|^2, \|e^k\|^2\}$$

$$\leq \quad e^{6(c_3+2c_4)T} \cdot [\max\{\|e^1\|^2, \|e^0\|^2\} + \frac{Lc_1^2}{2(c_3+2c_4)}(\tau^2+h^2)^2].$$

Noticing (3.24) and (3.29), one gets

$$\begin{aligned}\|e^k\|^2 &\leq e^{6(c_3+2c_4)T} \cdot [\|e^1\|^2 + \frac{Lc_1^2}{2(c_3+2c_4)}(\tau^2+h^2)^2] \\ &= e^{6(c_3+2c_4)T} \cdot [Lc_1^2 + \frac{Lc_1^2}{2(c_3+2c_4)}](\tau^2+h^2)^2 \\ &\equiv c_2^2(\tau^2+h^2)^2, \quad k \in N_{\tilde{n}},\end{aligned}$$

where $c_2 = e^{6(c_3+2c_4)T} \cdot [Lc_1^2 + \frac{Lc_1^2}{2(c_3+2c_4)}]^{\frac{1}{2}}$.

Namely,

$$\|e^k\| \leq c_2(\tau^2+h^2).$$

Theorem 3.7. *Assume that* $\{u_i^k, w_{(m)}{}_i^k \mid i \in N_{\tilde{m}}^0, k \in N_{\tilde{n}}^0\}$ *is the solution of (3.11)–(3.15) and* $\{U_i^k, W_{(m)}{}_i^k \mid i \in N_{\tilde{m}}^0, k \in N_{\tilde{n}}^0\}$ *is the solution of (1.1)–(1.3), there exists positive constants* c_7 *and* c_8 *such that*

$$|U^k - u^k|_1 \leq c_7(\tau^2+h^2), \quad k \in N_{\tilde{n}}^0, \tag{3.43}$$

$$\|U^k - u^k\|_\infty \leq c_8(\tau^2+h^2), \quad k \in N_{\tilde{n}}^0. \tag{3.44}$$

Proof. We will use the mathematical induction to prove the result. When $k = 0$, from (3.22) and (3.23), we get

$$|e^0|_1 = 0, \quad \|e^0\|_\infty = 0. \tag{3.45}$$

Therefore, the conclusion is valid for $k = 0$.

1) Taking the inner product of (3.19) with $\delta_t e^{\frac{1}{2}}$, one gets

$$\|\delta_t e^{\frac{1}{2}}\|^2 + \sum_{m=0}^{q} \frac{C_q^m(-1)^m}{p+m+2}(\psi(R_{(m)}, e^{\frac{1}{2}}), \delta_t e^{\frac{1}{2}}) = \nu(\delta_x^2 e^{\frac{1}{2}}, \delta_t e^{\frac{1}{2}}) + (P^0, \delta_t e^{\frac{1}{2}}). \tag{3.46}$$

Noticing that

$$e_i^0 = 0, \quad i \in N_{\tilde{m}}^0,$$

then (3.46) becomes

$$\frac{1}{\tau^2}\|e^1\|^2 + \frac{1}{2\tau}\sum_{m=0}^{q} \frac{C_q^m(-1)^m}{p+m+2}(\psi(R_{(m)}, e^1), e^1) = \frac{\nu}{2\tau}(\delta_x^2 e^1, e^1) + \frac{1}{\tau}(P^0, e^1). \tag{3.47}$$

Using Lemmas 2.1 and 2.2, we have

$$\frac{1}{\tau^2}\|e^1\|^2 + \frac{\nu}{2\tau}|e^1|_1^2 = \frac{1}{\tau}(P^0, e^1) \leq \frac{1}{\tau^2}\|e^1\|^2 + \frac{1}{4}\|P^0\|^2. \tag{3.48}$$

From (2.8), we get

$$|e^1|_1^2 \le \frac{2\tau}{\nu} \cdot \frac{1}{4}\|P^0\|^2 \le \frac{\tau}{2\nu} L c_1^2(\tau^2 + h^2)^2.$$

When $\tau \le 2\nu$, one gets

$$|e^1|_1^2 \le L c_1^2(\tau^2 + h^2)^2,$$

or

$$|e^1|_1 \le \sqrt{L} c_1(\tau^2 + h^2). \tag{3.49}$$

2) Taking the inner product of (3.20) with $\Delta_t e^k$, one gets

$$\begin{aligned} &\|\Delta_t e^k\|^2 + \sum_{m=0}^{q} \frac{C_q^m(-1)^m}{p+m+2}(\psi(W_{(m)}^k, U^{\bar{k}}) - \psi(w_{(m)}^k, u^{\bar{k}}), \Delta_t e^k) \\ = \;& \nu(\delta_x^2 e^{\bar{k}}, \Delta_t e^k) + (P^k, \Delta_t e^k), \quad k \in N_{\tilde{n}-1}. \end{aligned} \tag{3.50}$$

Suppose (3.43) and (3.44) hold for $0 \le k \le s$ $(1 \le s \le \tilde{n} - 1)$.
From (3.10) and Lemma 2.2, one gets

$$|U^k|_1 \le \sqrt{L}\tilde{c}_1, \quad \|U^k\|_\infty \le \frac{L}{2}\tilde{c}_1, \quad k \in N_{\tilde{n}}^0. \tag{3.51}$$

When $c_7(\tau^2 + h^2) \le 1$, one gets

$$\begin{aligned} &|u^k|_1 \le |U^k|_1 + |e^k|_1 \le \sqrt{L}\tilde{c}_1 + 1, \quad 1 \le k \le s, \\ &\|u^k\|_\infty \le \frac{\sqrt{L}}{2}(\sqrt{L}\tilde{c}_1 + 1), \quad 1 \le k \le s. \end{aligned} \tag{3.52}$$

Using Lemma 2.2, above equality (3.50) becomes

$$\begin{aligned} &\|\Delta_t e^k\|^2 + \sum_{m=0}^{q} \frac{C_q^m(-1)^m}{p+m+2}(\psi(W_{(m)}^k, U^{\bar{k}}) - \psi(w_{(m)}^k, u^{\bar{k}}), \Delta_t e^k) \\ = \;& -\frac{\nu}{4\tau}(|e^{k+1}|_1^2 - |e^{k-1}|_1^2) + (P^k, \Delta_t e^k). \end{aligned} \tag{3.53}$$

Noticing that

$$\begin{aligned} &\psi(W_{(m)}^k, U^{\bar{k}})_i - \psi(w_{(m)}^k, u^{\bar{k}})_i \\ = \;& \psi(b_{(m)}^k, U^{\bar{k}})_i + \psi(w_{(m)}^k, e^{\bar{k}})_i \\ = \;& b_{(m)}{}_i^k \Delta_x U_i^{\bar{k}} + \Delta_x(b_{(m)}^k U^{\bar{k}})_i + w_{(m)}{}_i^k \Delta_x e_i^{\bar{k}} + \Delta_x(w_{(m)}^k e^{\bar{k}})_i \\ = \;& 2b_{(m)}{}_i^k \Delta_x U_i^{\bar{k}} + \frac{1}{2}(\delta_x b_{(m)}{}_{i+\frac{1}{2}}^k) U_{i+1}^{\bar{k}} + \frac{1}{2}(\delta_x b_{(m)}{}_{i-\frac{1}{2}}^k) U_{i-1}^{\bar{k}} \\ &+ 2w_{(m)}{}_i^k \Delta_x e_i^{\bar{k}} + \frac{1}{2}(\delta_x w_{(m)}{}_{i+\frac{1}{2}}^k) e_{i+1}^{\bar{k}} + \frac{1}{2}(\delta_x w_{(m)}{}_{i-\frac{1}{2}}^k) e_{i-1}^{\bar{k}}. \end{aligned} \tag{3.54}$$

By Lagrange mean value theorem and the Lemma 2.3, we have

$$\begin{aligned}
&|b_{(m)}{}_i^k| \le \hat{c}_1|e_i^k|,\\
&|\delta_x b_{(m)}{}_{i+\frac{1}{2}}^k| \le \hat{c}_1|\delta_x e_{i+\frac{1}{2}}^k| + \tilde{c}_1\hat{c}_2[\rho|e_{i+1}^k| + (1-\rho)|e_i^k|],\\
&|\delta_x b_{(m)}{}_{i-\frac{1}{2}}^k| \le \hat{c}_1|\delta_x e_{i-\frac{1}{2}}^k| + \tilde{c}_1\hat{c}_2[\rho|e_i^k| + (1-\rho)|e_{i-1}^k|].
\end{aligned} \tag{3.55}$$

Thus, combining (3.54) and (3.55) yields

$$\begin{aligned}
&|\psi(W_{(m)}^k, U^{\bar{k}})_i - \psi(w_{(m)}^k, u^{\bar{k}})_i|\\
\le\ & 2\hat{c}_1|e_i^k|\cdot|\Delta_x U_i^{\bar{k}}| + \frac{1}{2}[\hat{c}_1|\delta_x e_{i+\frac{1}{2}}^k| + \rho\tilde{c}_1\hat{c}_2|e_{i+1}^k| + (1-\rho)\tilde{c}_1\hat{c}_2|e_i^k|]|U_{i+1}^{\bar{k}}|\\
&+\frac{1}{2}[\hat{c}_1|\delta_x e_{i-\frac{1}{2}}^k| + \rho\tilde{c}_1\hat{c}_2|e_i^k| + (1-\rho)\tilde{c}_1\hat{c}_2|e_{i-1}^k|]|U_{i-1}^{\bar{k}}|\\
&+2\hat{c}_0|\Delta_x e_i^{\bar{k}}| + \frac{1}{2}\hat{c}_1|\delta_x u_{i+\frac{1}{2}}^k|\cdot|e_{i+1}^{\bar{k}}| + \frac{1}{2}\hat{c}_1|\delta_x u_{i-\frac{1}{2}}^k|\cdot|e_{i-1}^{\bar{k}}|.
\end{aligned} \tag{3.56}$$

Using Lemma 2.2, combining (3.51), (3.52) and (3.56), it is easy to get

$$\begin{aligned}
&-(\psi(W_{(m)}^k, U^{\bar{k}}) - \psi(w_{(m)}^k, u^{\bar{k}}), \Delta_t e^k)\\
\le\ & [2\hat{c}_1\|e^k\|_\infty|U^{\bar{k}}|_1 + \|U^{\bar{k}}\|_\infty(\hat{c}_1|e^k|_1 + \tilde{c}_1\hat{c}_2\|e^k\|)]\cdot\|\Delta_t e^k\|\\
&+(2\hat{c}_0|e^{\bar{k}}|_1 + \hat{c}_1|u^k|_1\|e^{\bar{k}}\|_\infty)\cdot\|\Delta_t e^k\|\\
\le\ & (2\sqrt{L}\hat{c}_1\tilde{c}_1\|e^k\|_\infty + \frac{L}{2}\hat{c}_1\tilde{c}_1|e^k|_1 + \frac{L}{2}\tilde{c}_1^2\hat{c}_2\|e^k\|)\cdot\|\Delta_t e^k\|\\
&+[2\hat{c}_0|e^{\bar{k}}|_1 + \hat{c}_1(\sqrt{L}\tilde{c}_1 + 1)\|e^{\bar{k}}\|_\infty]\cdot\|\Delta_t e^k\|\\
\le\ & (2\sqrt{L}\hat{c}_1\tilde{c}_1\cdot\frac{\sqrt{L}}{2}|e^k|_1 + \frac{L}{2}\hat{c}_1\tilde{c}_1|e^k|_1 + \frac{L}{2}\tilde{c}_1^2\hat{c}_2\cdot\frac{L}{\sqrt{6}}|e^k|_1)\cdot\|\Delta_t e^k\|\\
&+[2\hat{c}_0|e^{\bar{k}}|_1 + \hat{c}_1(\sqrt{L}\tilde{c}_1 + 1)\cdot\frac{\sqrt{L}}{2}|e^{\bar{k}}|_1]\cdot\|\Delta_t e^k\|\\
=\ & (L\hat{c}_1\tilde{c}_1 + \frac{1}{2}L\hat{c}_1\tilde{c}_1 + \frac{1}{2\sqrt{6}}L^2\hat{c}_2\tilde{c}_1^2)|e^k|_1\cdot\|\Delta_t e^k\|\\
&+[2\hat{c}_0 + \frac{1}{2}\sqrt{L}\hat{c}_1(\sqrt{L}\tilde{c}_1 + 1)]|e^{\bar{k}}|_1\cdot\|\Delta_t e^k\|\\
=\ & c_9|e^k|_1\cdot\|\Delta_t e^k\| + c_{10}|e^{\bar{k}}|_1\cdot\|\Delta_t e^k\|\\
\le\ & \frac{p+2}{4a_0(1+q)}\|\Delta_t e^k\|^2 + \frac{a_0(1+q)c_9^2}{p+2}|e^k|_1^2 + \frac{p+2}{4a_0(1+q)}\|\Delta_t e^k\|^2\\
&+\frac{a_0(1+q)c_{10}^2}{p+2}|e^{\bar{k}}|_1^2,
\end{aligned} \tag{3.57}$$

where $c_9 = (L\hat{c}_1\tilde{c}_1 + \frac{1}{2}L\hat{c}_1\tilde{c}_1 + \frac{1}{2\sqrt{6}}L^2\hat{c}_2\tilde{c}_1^2)$ and $c_{10} = 2\hat{c}_0 + \frac{1}{2}\sqrt{L}\hat{c}_1(\sqrt{L}\tilde{c}_1 + 1)$.

Thus, (3.53) becomes

$$\|\Delta_t e^k\|^2 + \frac{\nu}{4\tau}(|e^{k+1}|_1^2 - |e^{k-1}|_1^2)$$

$$
\begin{aligned}
&= -\sum_{m=0}^{q}\frac{C_q^m(-1)^m}{p+m+2}(\psi(W_{(m)}^k,U^{\bar{k}})-\psi(w_{(m)}^k,u^{\bar{k}}),\Delta_t e^k)+(P^k,\Delta_t e^k)\\
&\le \sum_{m=0}^{q}\frac{a_0}{p+2}|-(\psi(W_{(m)}^k,U^{\bar{k}})-\psi(w_{(m)}^k,u^{\bar{k}}),\Delta_t e^k)|+|(P^k,\Delta_t e^k)|\\
&\le \sum_{m=0}^{q}\frac{a_0}{p+2}(\frac{p+2}{4a_0(1+q)}||\Delta_t e^k||^2+\frac{a_0(1+q)c_9^2}{p+2}|e^k|_1^2+\frac{p+2}{4a_0(1+q)}||\Delta_t e^k||^2\\
&\quad +\frac{a_0(1+q)c_{10}^2}{p+2}|e^{\bar{k}}|_1^2)+\frac{1}{2}||P^k||^2+\frac{1}{2}||\Delta_t e^k||^2\\
&= \frac{1}{4}||\Delta_t e^k||^2+\frac{a_0^2(q+1)^2c_9^2}{(p+2)^2}|e^k|_1^2+\frac{1}{4}||\Delta_t e^k||^2+\frac{a_0^2(q+1)^2c_{10}^2}{(p+2)^2}|e^{\bar{k}}|_1^2\\
&\quad +\frac{1}{2}||P^k||^2+\frac{1}{2}||\Delta_t e^k||^2, \quad 1\le k\le s,
\end{aligned} \tag{3.58}
$$

where $a_0=\max_{0\le m\le q}C_q^m$.

Noticing (2.4), (3.58) becomes

$$
\begin{aligned}
&\frac{\nu}{4\tau}(|e^{k+1}|_1^2-|e^{k-1}|_1^2)\\
\le\ & \frac{a_0^2(q+1)^2c_9^2}{(p+2)^2}|e^k|_1^2+\frac{a_0^2(q+1)^2c_{10}^2}{(p+2)^2}|e^{\bar{k}}|_1^2+\frac{1}{2}||P^k||^2\\
\le\ & c_5|e^k|_1^2+c_6\frac{|e^{k+1}|_1^2+|e^{k-1}|_1^2}{2}+\frac{1}{2}Lc_1^2(\tau^2+h^2)^2, \quad 1\le k\le s,
\end{aligned} \tag{3.59}
$$

where $c_5=\frac{a_0^2(q+1)^2c_9^2}{(p+2)^2}$ and $c_6=\frac{a_0^2(q+1)^2c_{10}^2}{(p+2)^2}$ are two positive constants.

For $1\le k\le s$, rearranging (3.59) to yield

$$
(1-\frac{2c_6\tau}{\nu})|e^{k+1}|_1^2\le\frac{4c_5\tau}{\nu}|e^k|_1^2+(1+\frac{2c_6\tau}{\nu})|e^{k-1}|_1^2+\frac{2Lc_1^2}{\nu}\tau(\tau^2+h^2)^2. \tag{3.60}
$$

When $\frac{2c_6\tau}{\nu}\le\frac{1}{3}$, (3.60) yields

$$
|e^{k+1}|_1^2 \le \frac{6c_5\tau}{\nu}|e^k|_1^2+(1+\frac{6c_6\tau}{\nu})|e^{k-1}|_1^2+\frac{3Lc_1^2}{\nu}\tau(\tau^2+h^2)^2.
$$

Therefore,

$$
\begin{aligned}
\max\{|e^{k+1}|_1^2,|e^k|_1^2\} &\le (1+\frac{6c_5+6c_6}{\nu}\tau)\max\{|e^{k-1}|_1^2,|e^k|_1^2\}\\
&\quad +\frac{3Lc_1^2}{\nu}\tau(\tau^2+h^2)^2, \quad 1\le k\le s.
\end{aligned} \tag{3.61}
$$

According to Gronwall's inequality, (3.61) yields

$$
\max\{|e^{k+1}|_1^2,|e^k|_1^2\}
$$

$$\leq \quad e^{\frac{6c_5+6c_6}{\nu}T} \cdot [\max\{|e^0|_1^2, |e^1|_1^2\} + \frac{Lc_1^2}{2(c_5+c_6)}(\tau^2+h^2)^2], \quad 1 \leq k \leq s.$$

Noticing (3.45) and (3.49), one gets

$$\begin{aligned} |e^{s+1}|_1^2 &\leq e^{\frac{6c_5+6c_6}{\nu}T} \cdot [\max\{|e^0|_1^2, |e^1|_1^2\} + \frac{Lc_1^2}{2(c_5+c_6)}(\tau^2+h^2)^2] \\ &= e^{\frac{6c_5+6c_6}{\nu}T} \cdot [Lc_1^2 + \frac{Lc_1^2}{2(c_5+c_6)}](\tau^2+h^2)^2 \\ &\equiv c_7^2(\tau^2+h^2)^2, \end{aligned}$$

where $c_7 = e^{\frac{3c_5+3c_6}{\nu}T} \cdot [Lc_1^2 + \frac{Lc_1^2}{2(c_5+c_6)}]^{\frac{1}{2}}$.

Namely,

$$|e^{s+1}|_1 \leq c_7(\tau^2+h^2).$$

Consequently, (3.43) holds for $k = s+1$.

From Lemma 2.2, it is easy to get

$$\|e^k\|_\infty \leq \frac{\sqrt{L}}{2}|e^k|_1 \leq \frac{\sqrt{L}}{2}c_7(\tau^2+h^2) \equiv c_8(\tau^2+h^2), \quad k \in N_{\tilde{n}}^0.$$

Corollary 3.8. *Let $\{u_i^k, w_{(m)}{}_i^k \,|\, i \in N_{\tilde{m}}^0, k \in N_{\tilde{n}}^0\}$ be the solution of (3.11)–(3.15). When $c_7(\tau^2+h^2) \leq 1$, there exists two constants c_{11} and c_{12} such that*

$$|u^k|_1 \leq c_{11}, \quad \|u^k\|_\infty \leq c_{12}, \quad k \in N_{\tilde{n}}^0.$$

Proof. When $c_7(\tau^2+h^2) \leq 1$, one has

$$\begin{aligned} |u^k|_1 &\leq |U^k|_1 + |e^k|_1 \\ &\leq \sqrt{L}\tilde{c}_1 + c_7(\tau^2+h^2) \leq c_{11}, \quad k \in N_{\tilde{n}}^0. \end{aligned}$$

By Lemma 2.2, we get $\|u^k\|_\infty \leq \frac{\sqrt{L}}{2}c_{10} \equiv c_{12}$.

This means the solution of (3.11)–(3.15) is bounded.

In the end, for the proposed scheme (2.10)–(2.14), we can obtain the following convergence.

Corollary 3.9. *Let $\{u_i^k, w_{(m)}{}_i^k \,|\, i \in N_{\tilde{m}}^0, k \in N_{\tilde{n}}^0\}$ be the solution of (2.10)–(2.14) and $\{U_i^k, W_{(m)}{}_i^k \,|\, i \in N_{\tilde{m}}^0, k \in N_{\tilde{n}}^0\}$ be the solution of (1.1)–(1.3). When $c_8(\tau^2+h^2) \leq 1$, one has*

$$\begin{aligned} &\|U^k - u^k\| \leq c_{13}(\tau^2+h^2), \quad k \in N_{\tilde{n}}^0, \\ &\|U^k - u^k\|_\infty \leq c_{13}(\tau^2+h^2), \quad k \in N_{\tilde{n}}^0, \end{aligned}$$

where c_{13} is a constant.

Proof. Let $\{\hat{u}_i^k \,|\, i \in N_{\tilde{m}}^0, k \in N_{\tilde{n}}^0\}$ be the solution of (3.11)–(3.15). When $c_8(\tau^2+h^2) \leq 1$, one has

$$\begin{aligned} |\hat{u}_i^k| &\leq |U_i^k| + |U_i^k - \hat{u}_i^k| \\ &\leq M + c_8(\tau^2+h^2) \leq M+1, \quad k \in N_{\tilde{n}}^0. \end{aligned}$$

Thus, $g_m(\hat{u}_i^k) = (\hat{u}_i^k)^{p+m}$.

This means the difference scheme (2.10)–(2.14) is equivalent to (3.11)–(3.15). According to Theorems 3.6 and 3.7, we finish the proof of Corollary 3.9.

4. Numerical test

A numerical example is given to verify theoretical conclusions of the three-level difference scheme for supergeneralized viscous Burgers' equation.

Example 4.1. We consider (1.1)–(1.3) with $T = L = 1$, $\nu = 1$, $\Psi(x) = \sin(\pi x)$, and p, q take some different integer values, respectively.

To describe the numerical errors in L_∞-norm for the computed solution and corresponding convergence orders, we denote

$$E^1_\infty(h,\tau) = \max_{0\le i\le \tilde{m}} \max_{0\le k\le \tilde{n}} |u^k_i(h,\tau) - u^{2k}_i(h,\frac{\tau}{2})|, \quad Order1 = \log_2 \frac{E^1_\infty(h,2\tau)}{E^1_\infty(h,\tau)},$$

$$E^2_\infty(h,\tau) = \max_{0\le i\le \tilde{m}} \max_{0\le k\le \tilde{n}} |u^k_i(h,\tau) - u^{k}_{2i}(\frac{h}{2},\tau)|, \quad Order2 = \log_2 \frac{E^2_\infty(2h,\tau)}{E^2_\infty(h,\tau)},$$

and

$$E^3_\infty(h,\tau) = \max_{0\le i\le \tilde{m}} \max_{0\le k\le \tilde{n}} |u^k_i(h,\tau) - u^{2k}_{2i}(\frac{h}{2},\frac{\tau}{2})|, \quad Order3 = \log_2 \frac{E^3_\infty(2h,2\tau)}{E^3_\infty(h,\tau)},$$

where h and τ are sufficiently small.

Table 1 lists the temporal convergence orders with $h = \frac{1}{64}$. We compute the spatial convergence orders with $\tau = \frac{1}{64}$ in Table 2. Table 3 presents the temporal and spatial errors and convergence orders with $\tau = h$. The corresponding error and convergence orders are presented in Figures 1–6. The results demonstrate (2.10)–(2.14) is convergent with the convergence order of two both in space and in time.

In Figures 7–9, we compute Υ^k in Theorem 3.1 to verify the conservativity of the difference scheme (2.10)–(2.14). The results demonstrate that difference scheme (2.10)–(2.14) is conservative.

Table 1. The temporal convergence orders with $h = \frac{1}{64}$.

τ	$p=2, q=1$		$p=2, q=3$		$p=3, q=4$	
	$E^1_\infty(h,\tau)$	Order1	$E^1_\infty(h,\tau)$	Order1	$E^1_\infty(h,\tau)$	Order1
1/20	2.6456e-02	-	2.5871e-02	-	2.5872e-02	-
1/40	5.8120e-03	2.1865	5.8182e-03	2.1527	5.7895e-03	2.1599
1/80	1.4154e-03	2.0379	1.4109e-03	2.0440	1.4108e-03	2.0369
1/160	3.5049e-04	2.0137	3.5054e-04	2.0090	3.5051e-04	2.0090
1/320	8.7491e-05	2.0022	8.7498e-05	2.0022	8.7490e-05	2.0022
1/640	2.1864e-05	2.0006	2.1866e-05	2.0006	2.1864e-05	2.0006

5. Conclusions

In this paper, a three-level linearized conservative scheme approximating supergeneralized viscous Burgers' equation is studied. We construct the discretization of the nonlinear term by a second-order operator in supergeneralized viscous Burgers' equation and prove the three-level scheme is uniquely solvable based on the mathematical induction. At last, the L_2-norm and L_∞-norm convergence are proved with separate and different ways.

Table 2. The spatial convergence orders with $\tau = \frac{1}{64}$.

h	$p = 2, q = 1$		$p = 2, q = 3$		$p = 3, q = 4$	
	$E^2_\infty(h,\tau)$	Order2	$E^2_\infty(h,\tau)$	Order2	$E^2_\infty(h,\tau)$	Order2
1/20	5.7545e-04	-	5.7530e-04	-	5.7521e-04	-
1/40	1.4410e-04	1.9977	1.4381e-04	2.0001	1.4379e-04	2.0001
1/80	3.6024e-05	2.0000	3.5952e-05	2.0000	3.5947e-05	2.0000
1/160	9.0074e-06	1.9998	8.9879e-06	2.0000	8.9866e-06	2.0000
1/320	2.2519e-06	2.0000	2.2470e-06	2.0000	2.2466e-06	2.0000
1/640	5.6296e-07	2.0000	5.6174e-07	2.0000	5.6166e-07	2.0000

Table 3. The temporal and spatial errors and convergence orders with $\tau = h$.

h	τ	$p = 2, q = 1$		$p = 2, q = 3$		$p = 3, q = 4$	
		$E^3_\infty(h,\tau)$	Order3	$E^3_\infty(h,\tau)$	Order3	$E^3_\infty(h,\tau)$	Order3
1/20	1/20	2.5727e-02	-	2.5170e-02	-	2.5171e-02	-
1/40	1/40	5.6650e-03	2.1831	5.6706e-03	2.1501	5.6420e-03	2.1575
1/80	1/80	1.3805e-03	2.0369	1.3756e-03	2.0435	1.3755e-03	2.0363
1/160	1/160	3.4173e-04	2.0142	3.4189e-04	2.0084	3.4176e-04	2.0089
1/320	1/320	8.5308e-05	2.0021	8.5337e-05	2.0023	8.5308e-05	2.0022
1/640	1/640	2.1319e-05	2.0005	2.1326e-05	2.0006	2.1319e-05	2.0005

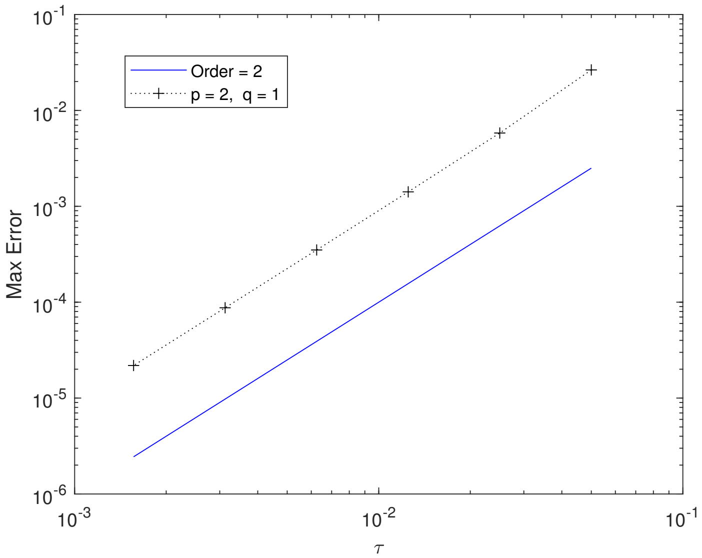

Figure 1. The convergence orders of time when $h = \frac{1}{64}$ for $p = 2, q = 1$.

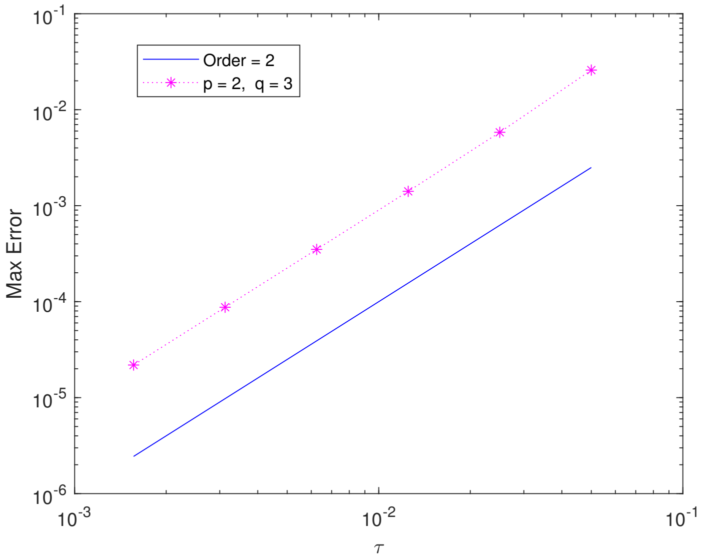

Figure 2. The convergence orders of time when $h = \frac{1}{64}$ for $p = 2, q = 3$.

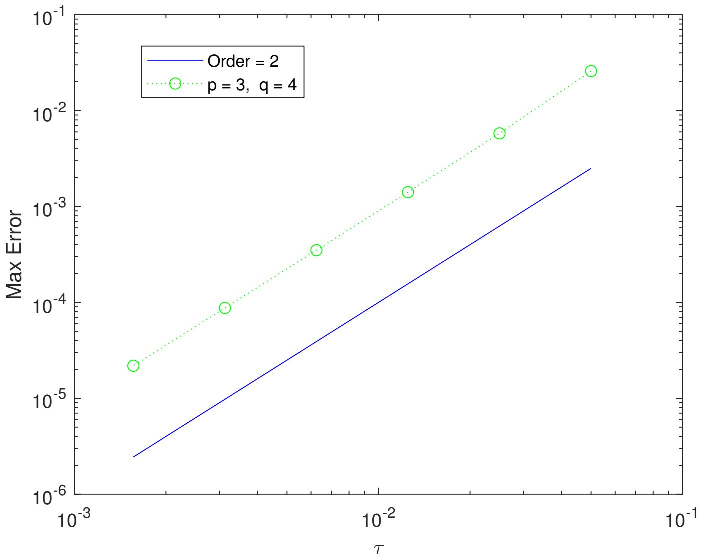

Figure 3. The convergence orders of time when $h = \frac{1}{64}$ for $p = 3, q = 4$.

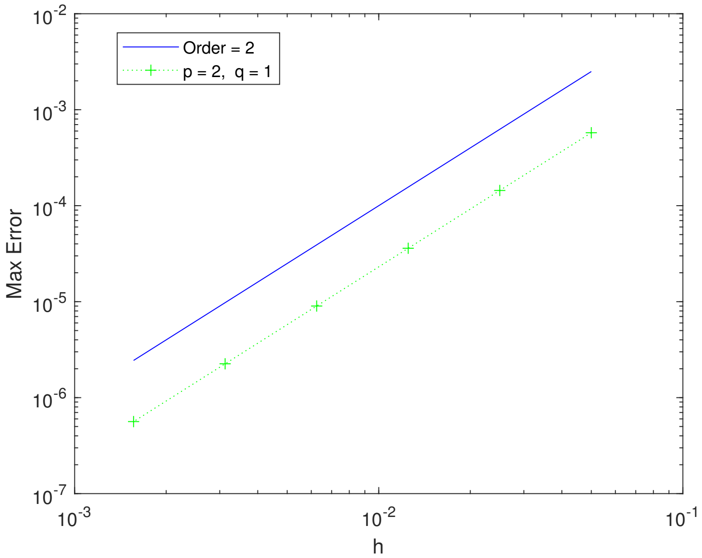

Figure 4. The spatial convergence orders when $\tau = \frac{1}{64}$ for $p = 2, q = 1$.

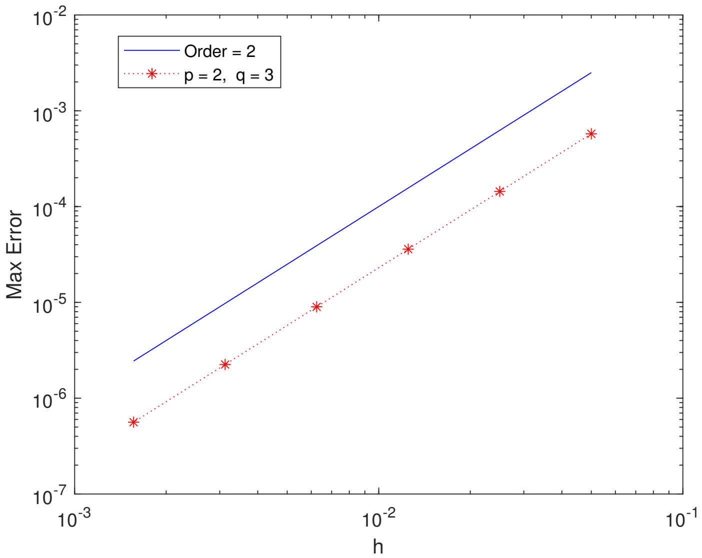

Figure 5. The spatial convergence orders when $\tau = \frac{1}{64}$ for $p = 2, q = 3$.

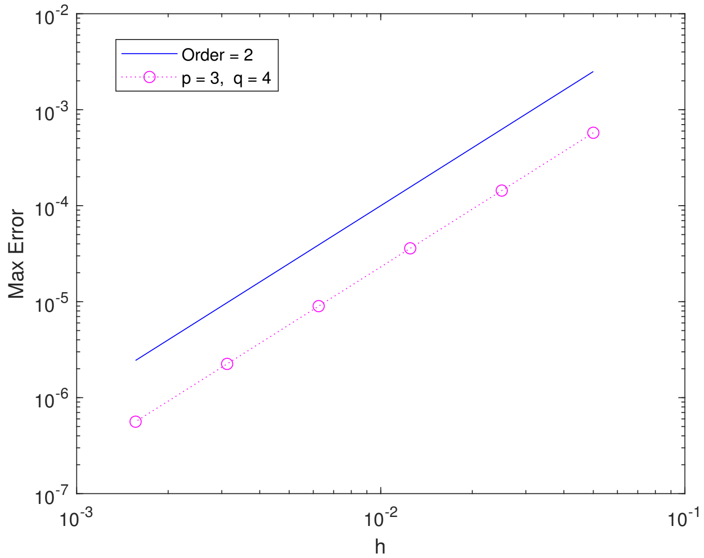

Figure 6. The spatial convergence orders when $\tau = \frac{1}{64}$ for $p = 3, q = 4$.

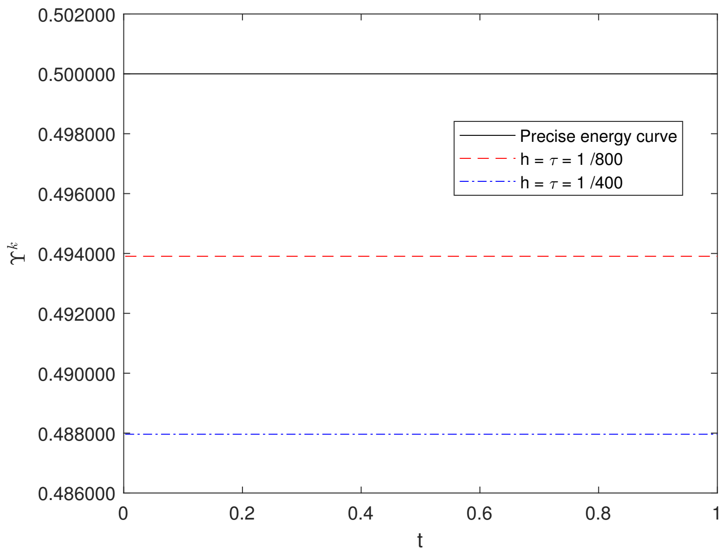

Figure 7. Conservative invariant Υ^k of the scheme (2.10)–(2.14) with $p = 2$ and $q = 1$.

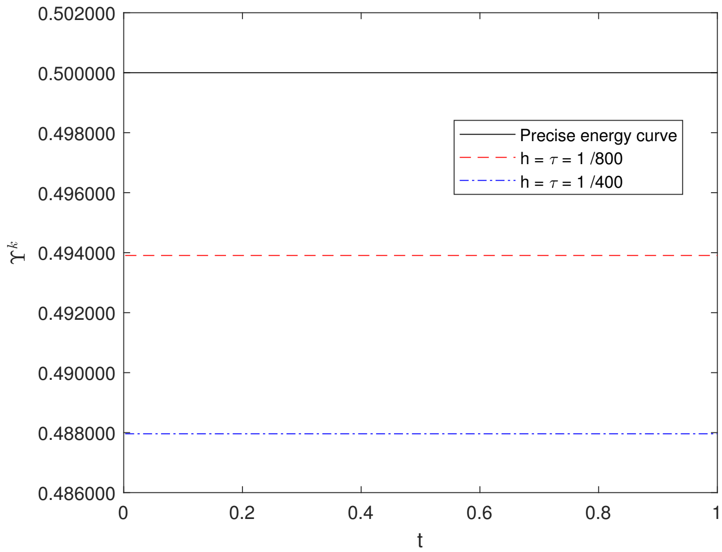

Figure 8. Conservative invariant Υ^k of the scheme (2.10)–(2.14) with $p = 2$ and $q = 3$.

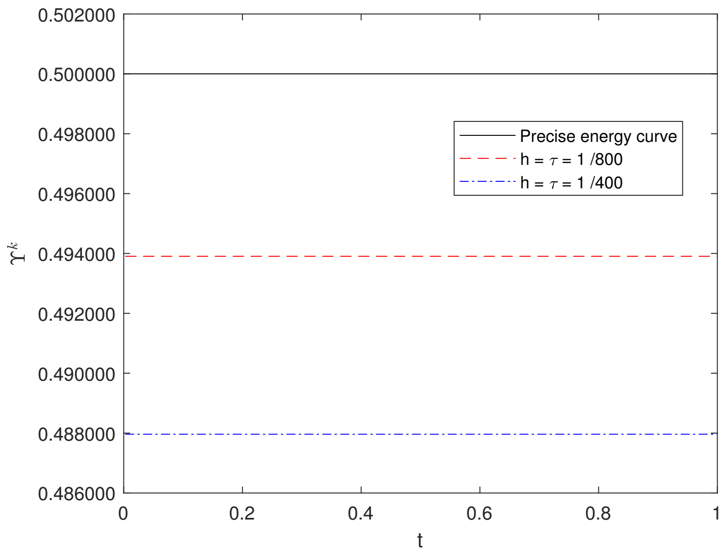

Figure 9. Conservative invariant Υ^k of the scheme (2.10)–(2.14) with $p = 3$ and $q = 4$.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflicts of interest

The authors declare no conflict of interest.

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