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Research article

Nonlinear Jordan triple derivable mapping on $*$-type trivial extension algebras

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Abstract: The aim of the paper was to give a description of nonlinear Jordan triple derivable mappings on trivial extension algebras. We proved that every nonlinear Jordan triple derivable mapping on a 2-torsion free $*$-type trivial extension algebra is a sum of an additive derivation and an additive antiderivation. As an application, nonlinear Jordan triple derivable mappings on triangular algebras were characterized.

Keywords: nonlinear Jordan triple derivable mapping; additive Jordan derivation; additive derivation; additive antiderivation

1. Introduction

Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ a unital algebra over $\mathcal{R}$, and $\Delta : \mathcal{A} \longmapsto \mathcal{A}$ be an additive (resp., without assumption of additivity) mapping. For any $X, Y \in \mathcal{A}$, denote the Jordan product of X, Y by $X \circ Y = XY + YX$. Recall that Δ is an additive derivation (resp., nonlinear derivable mapping) if $\Delta(XY) = \Delta(X)Y + X\Delta(Y)$ for all $X, Y \in \mathcal{A}$. It is an additive antiderivation (resp., nonlinear antiderivable mapping) if $\Delta(XY) = \Delta(Y)X + Y\Delta(X)$ for all $X, Y \in \mathcal{A}$. It is said an additive Jordan derivation (resp., nonlinear Jordan derivable mapping) if $\Delta(X\circ Y) = \Delta(Y)\circ X+Y\circ\Delta(X)$ for all $X, Y \in \mathcal{A}$. Obviously, every additive derivation or additive antiderivation is a Jordan derivation. However, the inverse statement is not true in general (see [1]). In the study of additive Jordan derivations, one of the most important problems is the following:

Problem A What conditions can imply additive Jordan derivation is an additive derivation.

In past decades, many mathematicians studied this problem and obtained abundant results. We refer the readers to [1–10] and references therein for more details and the importance of this problem.

Similar to **Problem A**, another important and meaningful problem naturally arises, as follows:

Problem B What conditions can imply a nonlinear Jordan derivable mapping is an additive derivation.

For example, [11, 12] studied nonlinear Jordan derivable mappings.

In this paper, we say Δ is an additive Jordan triple derivation (resp., nonlinear Jordan triple derivable mapping) on $\mathcal{A}$, if Δ is an additive mapping (resp., without assumption of additivity mapping) and satisfies

$$\Delta(X \circ Y \circ Z) = \Delta(X) \circ Y \circ Z + X \circ \Delta(Y) \circ Z + X \circ Y \circ \Delta(Z)$$

for all $X, Y, Z \in \mathcal{A}$.

With the deepening of research, many research achievements have been obtained about additive Jordan triple derivations and nonlinear Jordan triple derivable mappings, for example, [13–17]. Specifically, in [10] Theorem 1.1 we proved that every additive Jordan triple derivation on a 2-torsion free $*$-type trivial extension algebra is a sum of an additive derivation and an additive anti-derivation, and in [18] Theorem 2.1 we proved that every nonlinear local Jordan triple derivable mapping on triangular algebras is an additive derivation. In this article, our main purpose is to further generalize the research conclusions of references [10] and [18]. We obtained that every nonlinear Jordan triple derivable mapping on a 2-torsion free $*$-type trivial extension algebra is a sum of an additive derivation and an additive antiderivation.

For the convenience of reading, we introduce the concepts and properties of trivial extension algebra as follows:

Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ a unital algebra over $\mathcal{R}$, and $\mathcal{M}$ an $\mathcal{A}$-bimodule. The direct product $\mathcal{A} \bigotimes \mathcal{M}$ together with the pairwise addition, scalar product, and the algebra multiplication defined by

$$(a, m)(b, n) = (ab, an + mb)(\forall a, b \in \mathcal{A}, m, n \in \mathcal{M})$$

is an $\mathcal{R}$-algebra with a unity $(1, 0)$, denoted by

$$\mathcal{T} = \mathcal{A} \bigotimes \mathcal{M} = \{(a, m) : a \in \mathcal{A}, m \in \mathcal{M}\}$$

which is called a trivial extension algebra.

An important example of trivial extension algebra is the triangular algebra, which was introduced by Cheung in [19]. Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a commutative ring $\mathcal{R}$ and $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as both a left $\mathcal{A}$-module and a right $\mathcal{B}$-module, then the $\mathcal{R}$-algebra

$$\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ 0 & b \end{pmatrix} : a \in \mathcal{A}, m \in \mathcal{M}, b \in \mathcal{B} \right\}$$

under the usual matrix operations is called a triangular algebra. Basic examples of triangular algebras are upper triangular matrix algebras and nest algebras.

It is well known that every triangular algebra can be viewed as a trivial extension algebra. Indeed, denote by $\mathcal{A} \bigotimes \mathcal{B}$ direct product as an $\mathcal{R}$-algebra, and $\mathcal{M}$ is viewed as an $\mathcal{A} \bigotimes \mathcal{B}$-bimodule with the module action given by $(a, b)m = am$ and $m(a, b) = mb$ for all $(a, b) \in \mathcal{A} \bigotimes \mathcal{B}$ and $m \in \mathcal{M}$. The triangular algebra $\mathcal{U}$ is isomorphic to trivial extensions algebra $\mathcal{T} = (\mathcal{A} \bigotimes \mathcal{B}) \bigotimes \mathcal{M}$. However, a trivial extension algebra dose not necessarily have a triangular algebra. For more details about triangular algebras and trivial extension algebras, we refer the readers to [20–23] and references therein.

The following notations will be used in our paper. Let $\mathcal{R}$ be a commutative ring with identity, $\mathcal{A}$ a unital algebra over $\mathcal{R}$, and $\mathcal{M}$ an $\mathcal{A}$-bimodule. $\mathcal{T} = \mathcal{A} \bigotimes \mathcal{M}$ is a 2-torsion free trivial extension algebra (i.e., for any $X \in \mathcal{T}$, if $2X = \{0\}$ implies $X = 0$); denoted by 1 and 0, they are the unity and zero of $\mathcal{T} = \mathcal{A} \bigotimes \mathcal{M}$, respectivly.

We say $\mathcal{T} = \mathcal{A} \bigotimes \mathcal{M}$ is a $*$-type trivial extension algebra, if $\mathcal{A}$ has a nontrivial idempotent element e and $f = 1 - e$ such that

(i) $e\mathcal{M}f = \mathcal{M}$;

(ii) $exe\mathcal{M} = \{0\}$ implies $exe = 0, \forall x \in \mathcal{A}$;

(iii) $\mathcal{M}fxf = \{0\}$ implies $fxf = 0, \forall x \in \mathcal{A}$;

(iv) $exfye = 0 = fxeyf = 0, \forall x, y \in \mathcal{A}$.

In order to prove our main conclusion, denoted by $P_1 = (e, 0)$ and $P_2 = (f, 0)$,

$$\mathcal{T}_{ij} = P_i \mathcal{T} P_j (1 \leq i \leq j \leq 2).$$

It is clear that the trivial extension algebra $\mathcal{T}$ may be represented as

$$\mathcal{T} = P_1\mathcal{T}P_1 + P_1\mathcal{T}P_2 + P_2\mathcal{T}P_1 + P_2\mathcal{T}P_2 = \mathcal{T}_{11} + \mathcal{T}_{12} + \mathcal{T}_{21} + \mathcal{T}_{22}.$$

For any element $A \in \mathcal{T}$, A may be represented as $A = A_{11} + A_{12} + A_{21} + A_{22}$, where $A_{ij} \in \mathcal{T}_{ij}(1 \leq i \leq j \leq 2)$.

In order to prove our main conclusion Theorem 2.1, we need to cite Lemma 0.1 from [10] as follows:

Lemma 1.1 Let $\mathcal{T}$ be a $*$-type trivial extension algebra and $1 \leq i \neq j \leq 2$, then

(i) for any $A_{11} \in \mathcal{T}_{11}$, if $A_{11}\mathcal{T}_{12} = 0$, then $A_{11} = 0$;

(ii) for any $A_{22} \in \mathcal{T}_{22}$, if $\mathcal{T}_{12}A_{22} = 0$, then $A_{22} = 0$;

(iii) $A_{ij}B_{ji} = 0, \forall A_{ij} \in \mathcal{T}_{ij}, \forall B_{ji} \in \mathcal{T}_{ji}$.

The main result of this paper is the following Theorem 2.1.

2. Main results

Theorem 2.1 Let $\mathcal{T} = \mathcal{A} \bigotimes \mathcal{M}$ be a 2-torsion free $*$-type trivial extension algebra and Δ be a nonlinear Jordan triple derivable mapping on $\mathcal{T}$, then there exist an additive derivation D and an additive antiderivation φ on $\mathcal{T}$, respectively, such that

$$\Delta(A) = D(A) + \varphi(A)$$

for all $A \in \mathcal{T}$.

In order to prove Theorem 2.1, we introduce Lemmas 2.2–2.5, and prove that Lemmas 2.2–2.5 hold. We assume that $\mathcal{T}$ is a 2-torsion free $*$-type trivial extension algebra and Δ is a nonlinear Jordan triple derivable mapping on $\mathcal{T}$.

Lemma 2.2 For any $A_{ii} \in \mathcal{T}_{ii}, A_{ij} \in \mathcal{T}_{ij}$ $(1 \leq i \neq j \leq 2)$,

(i) $\Delta(0) = 0$;

(ii) $\Delta(P_i) = P_i\Delta(P_i)P_j + P_j\Delta(P_i)P_i$ and $\Delta(P_1) = -\Delta(P_2)$;

(iii) $\Delta(P_1) \circ A_{ij} = 0$ and $\Delta(P_2) \circ A_{ij} = 0$;

(iv) $\Delta(A_{ij}) = P_i\Delta(A_{ij})P_j + P_j\Delta(A_{ij})P_i$;

(v) $P_j\Delta(A_{ii})P_j = 0, P_i\Delta(A_{ii})P_j = A_{ii}\Delta(P_i)$ and $P_j\Delta(A_{ii})P_i = \Delta(P_i)A_{ii}$.

proof (i) For any $X, Y, Z \in \mathcal{T}$, it follows that Δ is a nonlinear Jordan triple derivable mapping on $\mathcal{T}$ where

$$\Delta(X \circ Y \circ Z) = \Delta(X) \circ Y \circ Z + X \circ \Delta(Y) \circ Z + X \circ Y \circ \Delta(Z). \tag{2.1}$$

Therefore, taking $X = Y = Z = 0$ in Eq (2.1), we get $\Delta(0) = 0$.

(*ii*) Taking $X = Y = P_i, Z = P_j$ $(1 \leq i \neq j \leq 2)$ in Eq (2.1), by the property of 2-torsion freeness of $\mathcal{T}$, we have

$$\begin{aligned} 0 &= \Delta(P_i \circ P_i \circ P_j) \\ &= \Delta(P_i) \circ P_i \circ P_j + P_i \circ \Delta(P_i) \circ P_j + P_i \circ P_i \circ \Delta(P_j) \\ &= (\Delta(P_i)P_i + P_i\Delta(P_i)) \circ P_j + (P_i\Delta(P_i) + \Delta(P_i)P_i) \circ P_j + 2P_i\Delta(P_j) + 2\Delta(P_j)P_i \\ &= 2P_i\Delta(P_i)P_j + 2P_j\Delta(P_i)P_i + 2P_i\Delta(P_j) + 2\Delta(P_j)P_i \\ &= P_i\Delta(P_i)P_j + P_j\Delta(P_i)P_i + P_i\Delta(P_j) + \Delta(P_j)P_i. \end{aligned}$$

Multiplying the above equation by P_i from both sides, we obtain $P_i\Delta(P_j)P_i = 0$.

Next, we show that $P_j\Delta(P_j)P_j = 0$ $(1 \leq j \leq 2)$. For any $A_{12} \in \mathcal{T}_{12}$, taking $X = P_1, Y = A_{12}, Z = P_1$ in Eq (2.1), we get from Lemma 1.1 (*iii*) and $P_2\Delta(P_1)P_2 = 0$ that

$$\begin{aligned} \Delta(A_{12}) &= \Delta(P_1 \circ A_{12} \circ P_1) \\ &= \Delta(P_1) \circ A_{12} \circ P_1 + P_1 \circ \Delta(A_{12}) \circ P_1 + P_1 \circ A_{12} \circ \Delta(P_1) \\ &= (\Delta(P_1)A_{12} + A_{12}\Delta(P_1)) \circ P_1 + (P_1\Delta(A_{12}) + \Delta(A_{12})P_1) \circ P_1 \\ &+ A_{12}\Delta(P_1) + \Delta(P_1)A_{12} \\ &= A_{12}\Delta(P_1)P_1 + P_1\Delta(P_1)A_{12} + A_{12}\Delta(P_1) \\ &+ 2P_1\Delta(A_{12})P_1 + \Delta(A_{12})P_1 + P_1\Delta(A_{12}) + A_{12}\Delta(P_1) + \Delta(P_1)A_{12} \\ &= P_1\Delta(P_1)A_{12} + 2P_1\Delta(A_{12})P_1 + \Delta(A_{12})P_1 + P_1\Delta(A_{12}) + \Delta(P_1)A_{12}. \end{aligned} \tag{2.2}$$

Multiplying the above Eq (2.2) from the left by P_1 and from the right by P_2, we have

$$2P_1\Delta(P_1)A_{12} = 0.$$

This yields from the property of 2-torsion freeness of $\mathcal{T}$ that

$$P_1\Delta(P_1)P_1A_{12} = 0.$$

Therefore, by the Lemma 1.1 (*i*), we get $P_1\Delta(P_1)P_1 = 0$. Similarly, we show $P_2\Delta(P_2)P_2 = 0$ holds. Thus, we get $\Delta(P_i) = P_i\Delta(P_i)P_j + P_j\Delta(P_i)P_i$ $(1 \leq i \neq j \leq 2)$.

Taking $X = P_i, Y = P_j, Z = P_i$ $(1 \leq i \neq j \leq 2)$ in Eq (2.1), we have

$$\begin{aligned} 0 &= \Delta(P_i \circ P_j \circ P_i) \\ &= \Delta(P_i) \circ P_j \circ P_i + P_i \circ \Delta(P_j) \circ P_i + P_i \circ P_j \circ \Delta(P_i) \\ &= (\Delta(P_i)P_j + P_j\Delta(P_i)) \circ P_i + (P_i\Delta(P_j) + \Delta(P_j)P_i) \circ P_j \\ &= P_i\Delta(P_i)P_j + P_j\Delta(P_i)P_i + P_i\Delta(P_j)P_j + P_j\Delta(P_j)P_i \\ &= \Delta(P_i) + \Delta(P_j). \end{aligned}$$

Therefore, $\Delta(P_1) = -\Delta(P_2)$.

(*iii*) For any $A_{12} \in \mathcal{T}_{12}$ and $A_{21} \in \mathcal{T}_{21}$, we get from $\Delta(P_1) = P_1\Delta(P_1)P_2 + P_2\Delta(P_1)P_1$ and Lemma 1.1 (*iii*) that

$$\Delta(P_1) \circ A_{12} = (P_1\Delta(P_1)P_2 + P_2\Delta(P_1)P_1) \circ A_{12}$$

$$
\begin{aligned}
&= (P_1\Delta(P_1)P_2 + P_2\Delta(P_1)P_1)A_{12} + A_{12}(P_1\Delta(P_1)P_2 + P_2\Delta(P_1)P_1) \\
&= (P_2\Delta(P_1)P_1)A_{12} + A_{12}(P_2\Delta(P_1)P_1) \\
&= 0
\end{aligned}
$$

Similarly, we get that $\Delta(P_1) \circ A_{21} = 0$. Furthermore, since $\Delta(P_1) = -\Delta(P_2)$, we get $\Delta(P_2) \circ A_{12} = 0 = \Delta(P_2) \circ A_{21}$.

(iv) For any $A_{ij} \in \mathcal{T}_{ij}$, taking $X = P_i, Y = A_{ij}, Z = P_j$ ($1 \le i \ne j \le 2$) in Eq (2.1), then it follows from Lemma 1.1 (iii) and Lemma 2.2 (ii) $-$ (iii) that

$$
\begin{aligned}
\Delta(A_{ij}) &= \Delta(P_i \circ A_{ij} \circ P_j) \\
&= \Delta(P_i) \circ A_{ij} \circ P_j + P_i \circ \Delta(A_{ij}) \circ P_j + P_i \circ A_{ij} \circ \Delta(P_j) \\
&= P_i \circ \Delta(A_{ij}) \circ P_j \\
&= P_i\Delta(A_{ij})P_j + P_j\Delta(A_{ij})P_i.
\end{aligned}
$$

(v) For any $A_{11} \in \mathcal{T}_{11}$, taking $X = P_2, Y = A_{11}, Z = P_2$ in Eq (2.1), we get from Lemma 2.2 (ii) that

$$
\begin{aligned}
0 &= \Delta(P_2 \circ A_{11} \circ P_2) \\
&= \Delta(P_2) \circ A_{11} \circ P_2 + P_2 \circ \Delta(A_{11}) \circ P_2 + P_2 \circ A_{11} \circ \Delta(P_2) \\
&= (\Delta(P_2)A_{11} + A_{11}\Delta(P_2)) \circ P_2 + (P_2\Delta(A_{11}) + \Delta(A_{11})P_2) \circ P_2 \\
&= P_2\Delta(P_2)A_{11} + A_{11}\Delta(P_2)P_2 + P_2\Delta(A_{11}) + \Delta(A_{11})P_2 + 2P_2\Delta(A_{11})P_2 \\
&= \Delta(P_2)A_{11} + A_{11}\Delta(P_2) + P_2\Delta(A_{11}) + \Delta(A_{11})P_2 + 2P_2\Delta(A_{11})P_2 \\
&= -\Delta(P_1)A_{11} - A_{11}\Delta(P_1) + P_2\Delta(A_{11}) + \Delta(A_{11})P_2 + 2P_2\Delta(A_{11})P_2.
\end{aligned}
$$

This implies that

$$P_2\Delta(A_{11})P_2 = 0, P_1\Delta(A_{11})P_2 = A_{11}\Delta(P_1) \text{ and } P_2\Delta(A_{11})P_1 = \Delta(P_1)A_{11}.$$

Similarly, for any $A_{22} \in \mathcal{T}_{22}$, we get that

$$P_1\Delta(A_{22})P_1 = 0, P_1\Delta(A_{22})P_2 = \Delta(P_2)A_{22} \text{ and } P_2\Delta(A_{22})P_1 = A_{22}\Delta(P_2).$$

The proof is completed.

Lemma 2.3 For all $A_{11}, B_{11} \in \mathcal{T}_{11}, A_{12}, B_{12} \in \mathcal{T}_{12}, A_{21}, B_{21} \in \mathcal{T}_{21}, A_{22}, B_{22} \in \mathcal{T}_{22}$,
(i) $\Delta(A_{11} + A_{12}) = \Delta(A_{11}) + \Delta(A_{12})$;
(ii) $\Delta(A_{12} + A_{22}) = \Delta(A_{12}) + \Delta(A_{22})$;
(iii) $\Delta(A_{11} + A_{21}) = \Delta(A_{11}) + \Delta(A_{21})$;
(iv) $\Delta(A_{21} + A_{22}) = \Delta(A_{21}) + \Delta(A_{22})$;
(v) $\Delta(A_{12} + B_{12}) = \Delta(A_{12}) + \Delta(B_{12})$;
(vi) $\Delta(A_{21} + B_{21}) = \Delta(A_{21}) + \Delta(B_{21})$;
(vii) $\Delta(A_{12} + A_{21}) = \Delta(A_{12}) + \Delta(A_{21})$;
($viii$) $\Delta(A_{11} + B_{11}) = \Delta(A_{11}) + \Delta(B_{11})$;
(ix) $\Delta(A_{22} + B_{22}) = \Delta(A_{22}) + \Delta(B_{22})$.

Proof (i) For any $A_{11} \in \mathcal{T}_{11}, A_{12} \in \mathcal{T}_{12}$, since $P_1 \circ (A_{11} + A_{12}) \circ P_2 = A_{12}$, taking $X = P_1, Y = A_{11} + A_{12}, Z = P_2$ in Eq (2.1), then by Lemma 1.1 (iii) and Lemma 2.2, we get

$$\Delta(A_{12}) = \Delta(P_1 \circ (A_{11} + A_{12}) \circ P_2)$$

$$
\begin{aligned}
&= \Delta(P_1) \circ (A_{11} + A_{12}) \circ P_2 + P_1 \circ \Delta(A_{11} + A_{12}) \circ P_2 + P_1 \circ (A_{11} + A_{12}) \circ \Delta(P_2) \\
&= \Delta(P_1) \circ A_{11} \circ P_2 + P_1 \circ \Delta(A_{11} + A_{12}) \circ P_2 + 2A_{11} \circ \Delta(P_2) \\
&= (\Delta(P_1)A_{11} + A_{11}\Delta(P_1)) \circ P_2 + P_1\Delta(A_{11} + A_{12})P_2 + P_2\Delta(A_{11} + A_{12})P_1 \\
&+ 2A_{11}\Delta(P_2) + 2\Delta(P_2)A_{11} \\
&= P_1\Delta(A_{11})P_2 + P_2\Delta(A_{11})P_1 + P_1\Delta(A_{11} + A_{12})P_2 + P_2\Delta(A_{11} + A_{12})P_1 \\
&- 2P_1\Delta(A_{11})P_2 - 2P_2\Delta(A_{11})P_1 \\
&= P_1\Delta(A_{11} + A_{12})P_2 + P_2\Delta(A_{11} + A_{12})P_1 - P_1\Delta(A_{11})P_2 - P_2\Delta(A_{11})P_1.
\end{aligned}
$$

Therefore, we get

$$P_1\Delta(A_{11} + A_{12})P_2 + P_2\Delta(A_{11} + A_{12})P_1 = \Delta(A_{12}) + P_1\Delta(A_{11})P_2 + P_2\Delta(A_{11})P_1. \tag{2.3}$$

Next, we show $P_2\Delta(A_{11} + A_{12})P_2 = 0$ and $P_1\Delta(A_{11} + A_{12})P_1 = P_1\Delta(A_{11})P_1$.

Indeed, taking $X = P_2, Y = A_{11} + A_{12}, Z = P_2$ in Eq (2.1), then by Lemma 1.1 (*iii*) and Lemma 2.2, we get

$$
\begin{aligned}
\Delta(A_{12}) &= \Delta(P_2 \circ (A_{11} + A_{12}) \circ P_2) \\
&= \Delta(P_2) \circ (A_{11} + A_{12}) \circ P_2 + P_2 \circ \Delta(A_{11} + A_{12}) \circ P_2 + P_2 \circ (A_{11} + A_{12}) \circ \Delta(P_2) \\
&= \Delta(P_2) \circ A_{11} \circ P_2 + P_2 \circ \Delta(A_{11} + A_{12}) \circ P_2 + A_{12} \circ \Delta(P_2) \\
&= (\Delta(P_2)A_{11} + A_{11}\Delta(P_2)) \circ P_2 + P_2\Delta(A_{11} + A_{12}) \\
&+ \Delta(A_{11} + A_{12})P_2 + 2P_2\Delta(A_{11} + A_{12})P_2 \\
&= -P_1\Delta(A_{11})P_2 - P_2\Delta(A_{11})P_1 + P_2\Delta(A_{11} + A_{12}) \\
&+ \Delta(A_{11} + A_{12})P_2 + 2P_2\Delta(A_{11} + A_{12})P_2.
\end{aligned}
$$

It follows from Lemma 2.2 (*iv*) and the property of 2-torsion freeness of $\mathcal{T}$ that, we get

$$P_2\Delta(A_{11} + A_{12})P_2 = 0. \tag{2.4}$$

On the one hand, for any $B_{12} \in \mathcal{T}_{12}$, taking $X = A_{11} + A_{12}, Y = B_{12}, Z = P_1$ in Eq (2.1), then by Lemma 1.1 (*iii*), Lemma 2.2, and $P_2\Delta(A_{11} + A_{12})P_2 = 0$, we get

$$
\begin{aligned}
\Delta(A_{11}B_{12}) &= \Delta((A_{11} + A_{12}) \circ B_{12} \circ P_1) \\
&= \Delta(A_{11} + A_{12}) \circ B_{12} \circ P_1 + (A_{11} + A_{12}) \circ \Delta(B_{12}) \circ P_1 \\
&+ (A_{11} + A_{12}) \circ B_{12} \circ \Delta(P_1) \\
&= (\Delta(A_{11} + A_{12})B_{12} + B_{12}\Delta(A_{11} + A_{12})) \circ P_1 \\
&+ (A_{11}\Delta(B_{12}) + \Delta(B_{12})A_{11}) \circ P_1 + A_{11}B_{12} \circ \Delta(P_1) \\
&= P_1\Delta(A_{11} + A_{12})B_{12} + B_{12}\Delta(A_{11} + A_{12}) + A_{11}\Delta(B_{12}) + \Delta(B_{12})A_{11} \\
&= P_1\Delta(A_{11} + A_{12})B_{12} + A_{11}\Delta(B_{12}) + \Delta(B_{12})A_{11}.
\end{aligned}
$$

On the other hand, taking $X = A_{11}, Y = B_{12}, Z = P_2$ in Eq (2.1), then by Lemma 1.1 (*iii*) and Lemma 2.2, we get

$$\Delta(A_{11}B_{12}) = \Delta(A_{11} \circ B_{12} \circ P_2)$$

$$
\begin{aligned}
&= \Delta(A_{11}) \circ B_{12} \circ P_2 + A_{11} \circ \Delta(B_{12}) \circ P_2 + A_{11} \circ B_{12} \circ \Delta(P_2) \\
&= (\Delta(A_{11})B_{12} + B_{12}\Delta(A_{11})) \circ P_2 + (A_{11}\Delta(B_{12}) + \Delta(B_{12})A_{11}) \circ P_2 \\
&+ A_{11}B_{12} \circ \Delta(P_2) \\
&= \Delta(A_{11})B_{12} \circ P_2 + (A_{11}\Delta(B_{12}) + \Delta(B_{12})A_{11}) \circ P_2 \\
&= \Delta(A_{11})B_{12} + A_{11}\Delta(B_{12}) + \Delta(B_{12})A_{11}. \qquad (2.5)
\end{aligned}
$$

Comparing the above two equations, we get $(P_1\Delta(A_{11} + A_{12})P_1 - P_1\Delta(A_{11})P_1)B_{12} = 0$; thus, by Lemma 1.1 (i), we get

$$
P_1\Delta(A_{11} + A_{12})P_1 = P_1\Delta(A_{11})P_1. \qquad (2.6)
$$

Therefore, by Eqs (2.3) and (2.4), (2.6), and Lemma 2.2 $(iv) - (v)$, we get

$$
\begin{aligned}
\Delta(A_{11} + A_{12}) &= P_1\Delta(A_{11} + A_{12})P_1 + P_1\Delta(A_{11} + A_{12})P_2 \\
&+ P_2\Delta(A_{11} + A_{12})P_1 + P_2\Delta(A_{11} + A_{12})P_2 \\
&= P_1\Delta(A_{11})P_1 + P_1\Delta(A_{11})P_2 + P_2\Delta(A_{11})P_1 + \Delta(A_{12}) \\
&= \Delta(A_{11}) + \Delta(A_{12}).
\end{aligned}
$$

Similarly, we show that $(ii) - (iv)$ hold.

(v) For any $A_{12}, B_{12} \in \mathcal{T}_{12}$, since $(P_1 + A_{12}) \circ (B_{12} + P_2) \circ P_2 = A_{12} + B_{12}$, taking $X = P_1 + A_{12}, Y = B_{12} + P_2, Z = P_2$ in Eq (2.1), then by Lemma 1.1 (iii), Lemma 2.2, and Lemma 2.3 (i), we get

$$
\begin{aligned}
\Delta(A_{12} + B_{12}) &= \Delta((P_1 + A_{12}) \circ (B_{12} + P_2) \circ P_2) \\
&= \Delta(P_1 + A_{12}) \circ (B_{12} + P_2) \circ P_2 + (P_1 + A_{12}) \circ \Delta(B_{12} + P_2) \circ P_2 \\
&+ (P_1 + A_{12}) \circ (B_{12} + P_2) \circ \Delta(P_2) \\
&= (\Delta(P_1) + \Delta(A_{12})) \circ (B_{12} + P_2) \circ P_2 + (P_1 + A_{12}) \circ (\Delta(B_{12}) + \Delta(P_2)) \circ P_2 \\
&+ (A_{12} + B_{12}) \circ \Delta(P_2) \\
&= (\Delta(P_1) + \Delta(A_{12})) \circ P_2 + (\Delta(B_{12}) + \Delta(P_2)) \circ P_2 \\
&= \Delta(P_1) + \Delta(A_{12}) + \Delta(B_{12}) + \Delta(P_2) \\
&= \Delta(A_{12}) + \Delta(B_{12}).
\end{aligned}
$$

Similarly, we can show that (vi) holds.

(vii) For any $A_{12} \in \mathcal{T}_{12}, A_{21} \in \mathcal{T}_{21}$, by Lemma 1.1 (iii), we have $(P_1+A_{12})\circ(A_{21}+P_2)\circ P_2 = A_{12}+A_{21}$. Taking $X = P_1 + A_{12}, Y = A_{21} + P_2, Z = P_2$ in Eq (2.1), then by Lemma 1.1 (iii), Lemma 2.2, and Lemma 2.3 $(i), (iv)$, we get

$$
\begin{aligned}
\Delta(A_{12} + A_{21}) &= \Delta((P_1 + A_{12}) \circ (A_{21} + P_2) \circ P_2) \\
&= \Delta(P_1 + A_{12}) \circ (A_{21} + P_2) \circ P_2 + (P_1 + A_{12}) \circ \Delta(A_{21} + P_2) \circ P_2 \\
&+ (P_1 + A_{12}) \circ (A_{21} + P_2) \circ \Delta(P_2) \\
&= (\Delta(P_1) + \Delta(A_{12})) \circ (A_{21} + P_2) \circ P_2 + (P_1 + A_{12}) \circ (\Delta(A_{21}) + \Delta(P_2)) \circ P_2 \\
&+ (A_{12} + A_{21}) \circ \Delta(P_2) \\
&= (\Delta(P_1) + \Delta(A_{12})) \circ P_2 + (\Delta(A_{21}) + \Delta(P_2)) \circ P_2
\end{aligned}
$$

$$
\begin{aligned}
&= \Delta(P_1) + \Delta(A_{12}) + \Delta(A_{21}) + \Delta(P_2) \\
&= \Delta(A_{12}) + \Delta(A_{21}).
\end{aligned}
$$

($viii$) For all $A_{11}, B_{11} \in \mathcal{T}_{11}, B_{12} \in \mathcal{T}_{12}$, then by Lemma 2.2 ($v$), we get

$$
\begin{aligned}
\Delta(A_{11} + B_{11}) &= P_1\Delta(A_{11} + B_{11})P_1 + (A_{11} + B_{11})\Delta(P_1) + \Delta(P_1)(A_{11} + B_{11}) \\
&= P_1\Delta(A_{11} + A_{11})P_1 + (A_{11}\Delta(P_1) + B_{11}\Delta(P_1)) + (\Delta(P_1)A_{11} + \Delta(P_1)B_{11}) \\
&= P_1\Delta(A_{11} + A_{11})P_1 + P_1\Delta(A_{11})P_2 + P_2\Delta(A_{11})P_1 \\
&+ P_1\Delta(B_{11})P_2 + P_2\Delta(B_{11})P_1. \qquad (2.7)
\end{aligned}
$$

Next, we show $P_1\Delta(A_{11} + B_{11})P_1 = P_1\Delta(A_{11})P_1 + P_1\Delta(B_{11})P_1$.

Indeed, for any $Y_{12} \in \mathcal{T}_{12}$, it follows from Eq (2.5) that

$$
\Delta((A_{11} + B_{11})Y_{12}) = \Delta(A_{11} + B_{11})Y_{12} + (A_{11} + B_{11})\Delta(Y_{12}) + \Delta(Y_{12})(A_{11} + B_{11}).
$$

On the other hand, by Eq (2.5) and Lemma 2.3 (v), we get

$$
\begin{aligned}
\Delta((A_{11} + B_{11})Y_{12}) &= \Delta(A_{11}Y_{12} + B_{11}Y_{12}) \\
&= \Delta(A_{11}Y_{12}) + \Delta(B_{11}Y_{12}) \\
&= \Delta(A_{11})Y_{12} + A_{11}\Delta(Y_{12}) + \Delta(Y_{12})A_{11} \\
&+ \Delta(B_{11})Y_{12} + B_{11}\Delta(Y_{12}) + \Delta(Y_{12})B_{11}.
\end{aligned}
$$

Comparing the above two equations, we get $(\Delta(A_{11} + B_{11}) - \Delta(A_{11}) - \Delta(B_{11}))Y_{12} = 0$, which yields from Lemma 1.1 (i) that

$$
P_1\Delta(A_{11} + B_{11})P_1 = P_1\Delta(A_{11})P_1 + P_1\Delta(B_{11})P_1. \qquad (2.8)
$$

Therefore, by the Lemma 2.2 (v) and Eqs (2.7) and (2.8), we have $\Delta(A_{11} + B_{11}) = \Delta(A_{11}) + \Delta(B_{11})$. Similarly, we can show that (ix) holds. The proof is completed.

Lemma 2.4 For all $A_{11} \in \mathcal{T}_{11}, A_{12} \in \mathcal{T}_{12}, A_{21} \in \mathcal{T}_{21}, A_{22} \in \mathcal{T}_{22}$,

(i) $\Delta(A_{11} + A_{12} + A_{21}) = \Delta(A_{11}) + \Delta(A_{12}) + \Delta(A_{21})$;

(ii) $\Delta(A_{12} + A_{21} + A_{22}) = \Delta(A_{12}) + \Delta(A_{21}) + \Delta(A_{22})$.

Proof (i) For any $A_{11} \in \mathcal{T}_{11}, A_{12} \in \mathcal{T}_{12}, A_{21} \in \mathcal{T}_{21}$, since $P_1 \circ (A_{11} + A_{12} + A_{21}) \circ P_2 = A_{12} + A_{21}$, taking $X = P_1, Y = A_{11} + A_{12} + A_{21}, Z = P_2$ in Eq (2.1), then by Lemma 1.1 (iii), Lemma 2.2, and Lemma 2.3 (vii), we have

$$
\begin{aligned}
\Delta(A_{12}) + \Delta(A_{21}) &= \Delta(A_{12} + A_{21}) \\
&= \Delta(P_1 \circ (A_{11} + A_{12} + A_{21}) \circ P_2) \\
&= \Delta(P_1) \circ (A_{11} + A_{12} + A_{21}) \circ P_2 + P_1 \circ \Delta(A_{11} + A_{12} + A_{21}) \circ P_2 \\
&+ P_1 \circ (A_{11} + A_{12} + A_{21}) \circ \Delta(P_2) \\
&= \Delta(P_1) \circ A_{11} \circ P_2 + P_1\Delta(A_{11} + A_{12} + A_{21})P_2 \\
&+ P_2\Delta(A_{11} + A_{12} + A_{21})P_1 + (2A_{11} + A_{12} + A_{21}) \circ \Delta(P_2) \\
&= \Delta(P_1)A_{11} + A_{11}\Delta(P_1) + P_1\Delta(A_{11} + A_{12} + A_{21})P_2
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad P_2\Delta(A_{11}+A_{12}+A_{21})P_1 + 2A_{11}\Delta(P_2) + 2\Delta(P_2)A_{11} \\
&= \quad P_1\Delta(A_{11}+A_{12}+A_{21})P_2 + P_2\Delta(A_{11}+A_{12}+A_{21})P_1 \\
&- \quad P_1\Delta(A_{11})P_2 - P_2\Delta(A_{11})P_1.
\end{aligned}
$$

Therefore, we get

$$
\begin{aligned}
\Delta(A_{11}+A_{12}+A_{21}) &= P_1\Delta(A_{11}+A_{12}+A_{21})P_1 + P_1\Delta(A_{11}+A_{12}+A_{21})P_2 \\
&+ P_2\Delta(A_{11}+A_{12}+A_{21})P_1 + P_2\Delta(A_{11}+A_{12}+A_{21})P_2 \\
&= P_1\Delta(A_{11}+A_{12}+A_{21})P_1 + P_2\Delta(A_{11}+A_{12}+A_{21})P_2 \\
&+ P_1\Delta(A_{11})P_2 + P_2\Delta(A_{11})P_1 + \Delta(A_{12}) + \Delta(A_{21}). \qquad (2.9)
\end{aligned}
$$

Next, we show $P_2\Delta(A_{11}+A_{12}+A_{21})P_2 = 0$ and $P_1\Delta(A_{11}+A_{12}+A_{21})P_1 = P_1\Delta(A_{11})P_1$.

Since $P_2 \circ (A_{11}+A_{12}+A_{21}) \circ P_2 = A_{12}+A_{21}$, taking $X = P_2, Y = A_{11}+A_{12}+A_{21}, Z = P_2$ in Eq (2.1), then by Lemma 1.1 (*iii*), we get

$$
\begin{aligned}
\Delta(A_{12}) + \Delta(A_{21}) &= \Delta(A_{12}+A_{21}) \\
&= \Delta(P_2 \circ (A_{11}+A_{12}+A_{21}) \circ P_2) \\
&= \Delta(P_2) \circ (A_{11}+A_{12}+A_{21}) \circ P_2 + P_2 \circ \Delta(A_{11}+A_{12}+A_{21}) \circ P_2 \\
&+ P_2 \circ (A_{11}+A_{12}+A_{21}) \circ \Delta(P_2) \\
&= \Delta(P_2) \circ A_{11} \circ P_2 + 2P_2\Delta(A_{11}+A_{12}+A_{21})P_2 + P_2\Delta(A_{11}+A_{12}+A_{21}) \\
&+ \Delta(A_{11}+A_{12}+A_{21})P_2 + (A_{12}+A_{21}) \circ \Delta(P_2) \\
&= \Delta(P_2)A_{11} + A_{11}\Delta(P_2) + 2P_2\Delta(A_{11}+A_{12}+A_{21})P_2 \\
&+ P_2\Delta(A_{11}+A_{12}+A_{21}) + \Delta(A_{11}+A_{12}+A_{21})P_2.
\end{aligned}
$$

Multiplying the above equation by P_2 from both sides and by Lemma 2.2 (*iv*), we obtain $4P_2\Delta(A_{11}+A_{12}+A_{21})P_2 = 0$. Therefore, by the property of 2-torsion freeness of $\mathcal{T}$, we get

$$
P_2\Delta(A_{11}+A_{12}+A_{21})P_2 = 0. \qquad (2.10)
$$

Following, we show $P_1\Delta(A_{11}+A_{12}+A_{21})P_1 = P_1\Delta(A_{11})P_1$. Indeed, for any $Y_{12} \in \mathcal{T}_{12}$, it follows from Lemma 1.1 (*iii*) that $(A_{11}+A_{12}+A_{21}) \circ Y_{12} \circ P_2 = A_{11}Y_{12}$. Taking $X = A_{11}+A_{12}+A_{21}, Y = Y_{12}, Z = P_2$ in Eq (2.1), then by Lemma 1.1 (*iii*) and $P_2\Delta(A_{11}+A_{12}+A_{21})P_2 = 0$, we get

$$
\begin{aligned}
\Delta(A_{11}Y_{12}) &= \Delta((A_{11}+A_{12}+A_{21}) \circ Y_{12} \circ P_2) \\
&= \Delta(A_{11}+A_{12}+A_{21}) \circ Y_{12} \circ P_2 + (A_{11}+A_{12}+A_{21}) \circ \Delta(Y_{12}) \circ P_2 \\
&+ (A_{11}+A_{12}+A_{21}) \circ Y_{12} \circ \Delta(P_2) \\
&= (\Delta(A_{11}+A_{12}+A_{21})Y_{12} + Y_{12}\Delta(A_{11}+A_{12}+A_{21})) \circ P_2 \\
&+ (A_{11}\Delta(Y_{12}) + \Delta(Y_{12})A_{11}) \circ P_2 + A_{11}Y_{12} \circ \Delta(P_2) \\
&= P_2\Delta(A_{11}+A_{12}+A_{21})Y_{12} + \Delta(A_{11}+A_{12}+A_{21})Y_{12} \\
&+ Y_{12}\Delta(A_{11}+A_{12}+A_{21})P_2 + (A_{11}\Delta(Y_{12}) + \Delta(Y_{12})A_{11}) \\
&= \Delta(A_{11}+A_{12}+A_{21})Y_{12} + Y_{12}\Delta(A_{11}+A_{12}+A_{21})P_2 + A_{11}\Delta(Y_{12}) + \Delta(Y_{12})A_{11} \\
&= \Delta(A_{11}+A_{12}+A_{21})Y_{12} + A_{11}\Delta(Y_{12}) + \Delta(Y_{12})A_{11}.
\end{aligned}
$$

On the other hand, by Eq (2.5), we get

$$\Delta(A_{11}Y_{12}) = \Delta(A_{11})Y_{12} + A_{11}\Delta(Y_{12}) + \Delta(Y_{12})A_{11}.$$

Comparing the above two equations, we get $(\Delta(A_{11}+A_{12}+A_{21})-\Delta(A_{11}))Y_{12} = 0$. Thus, by Lemma 1.1 ($i$), we get

$$P_1\Delta(A_{11}+A_{12}+A_{21})P_1 = P_1\Delta(A_{11})P_1. \tag{2.11}$$

Therefore, we obtain from Eqs (2.9)–(2.11) and Lemma 2.2 (v) that $\Delta(A_{11}+A_{12}+A_{21}) = \Delta(A_{11}) + \Delta(A_{12}) + \Delta(A_{21})$. Similarly, we get ($ii$). The proof is completed.

Lemma 2.5 For all $A_{11} \in \mathcal{T}_{11}, A_{12} \in \mathcal{T}_{12}, A_{21} \in \mathcal{T}_{21}, A_{22} \in \mathcal{T}_{22}$, we get $\Delta(A_{11}+A_{12}+A_{21}+A_{22}) = \Delta(A_{11}) + \Delta(A_{12}) + \Delta(A_{21}) + \Delta(A_{22})$.

Proof For any $A_{11} \in \mathcal{T}_{11}, A_{12} \in \mathcal{T}_{12}, A_{21} \in \mathcal{T}_{21}, A_{22} \in \mathcal{T}_{22}$, since $P_1 \circ (A_{11}+A_{12}+A_{21}+A_{22}) \circ P_1 = 4A_{11}+A_{12}+A_{21}$, taking $X = P_1, Y = A_{11}+A_{12}+A_{21}+A_{22}, Z = P_1$ in Eq (2.1), then by Lemma 1.1 (iii) and Lemma 2.2, we have

$$\begin{aligned}
\Delta(4A_{11}+A_{12}+A_{21}) &= \Delta(P_1 \circ (A_{11}+A_{12}+A_{21}+A_{22}) \circ P_1) \\
&= \Delta(P_1) \circ (A_{11}+A_{12}+A_{21}+A_{22}) \circ P_1 \\
&+ P_1 \circ \Delta(A_{11}+A_{12}+A_{21}+A_{22}) \circ P_1 \\
&+ P_1 \circ (A_{11}+A_{12}+A_{21}+A_{22}) \circ \Delta(P_1) \\
&= \Delta(P_1) \circ (A_{11}+A_{22}) \circ P_1 + 2P_1\Delta(A_{11}+A_{12}+A_{21}+A_{22})P_1 \\
&+ P_1\Delta(A_{11}+A_{12}+A_{21}+A_{22}) + \Delta(A_{11}+A_{12}+A_{21}+A_{22})P_1 \\
&+ (2A_{11}+A_{12}+A_{21}) \circ \Delta(P_1) \\
&= \Delta(P_1)A_{11} + A_{11}\Delta(P_1) + \Delta(P_1)A_{22} + A_{22}\Delta(P_1) \\
&+ 4P_1\Delta(A_{11}+A_{12}+A_{21}+A_{22})P_1 + P_1\Delta(A_{11}+A_{12}+A_{21}+A_{22})P_2 \\
&+ P_2\Delta(A_{11}+A_{12}+A_{21}+A_{22})P_1 + 2A_{11}\Delta(P_1) + 2\Delta(P_1)A_{11} \\
&= 3P_1\Delta(A_{11})P_2 + 3P_2\Delta(A_{11})P_1 - P_1\Delta(A_{22})P_2 - P_2\Delta(A_{22})P_1 \\
&+ 4P_1\Delta(A_{11}+A_{12}+A_{21}+A_{22})P_1 + P_1\Delta(A_{11}+A_{12}+A_{21}+A_{22})P_2 \\
&+ P_2\Delta(A_{11}+A_{12}+A_{21}+A_{22})P_1.
\end{aligned}$$

On the other hand, by Lemma 2.3 ($viii$) and Lemma 2.4 (i), we have

$$\begin{aligned}
\Delta(4A_{11}+A_{12}+A_{21}) &= \Delta(4A_{11}) + \Delta(A_{12}) + \Delta(A_{21}) \\
&= 4\Delta(A_{11}) + \Delta(A_{12}) + \Delta(A_{21}) \\
&= 4P_1\Delta(A_{11})P_1 + 4P_1\Delta(A_{11})P_2 + 4P_2\Delta(A_{11})P_1 \\
&+ P_1\Delta(A_{12})P_2 + P_2\Delta(A_{12})P_1 + P_1\Delta(A_{21})P_2 + P_2\Delta(A_{21})P_1.
\end{aligned}$$

Comparing the above two equations, we get

$$P_1\Delta(A_{11}+A_{12}+A_{21}+A_{22})P_1 = P_1\Delta(A_{11})P_1, \tag{2.12}$$

and

$$P_1\Delta(A_{11}+A_{12}+A_{21}+A_{22})P_2 + P_2\Delta(A_{11}+A_{12}+A_{21}+A_{22})P_1$$

$$
\begin{aligned}
&= P_1\Delta(A_{11})P_2 + P_2\Delta(A_{11})P_1 \\
&+P_1\Delta(A_{12})P_2 + P_2\Delta(A_{12})P_1 \\
&+P_1\Delta(A_{21})P_2 + P_2\Delta(A_{21})P_1 \\
&+P_1\Delta(A_{22})P_2 + P_2\Delta(A_{22})P_1.
\end{aligned} \tag{2.13}
$$

Following, we show $P_2\Delta(A_{11} + A_{12} + A_{21} + A_{22})P_2 = P_2\Delta(A_{22})P_2$.

Similarly, for any $A_{11} \in \mathcal{T}_{11}, A_{12} \in \mathcal{T}_{12}, A_{21} \in \mathcal{T}_{21}, A_{22} \in \mathcal{T}_{22}$, since $P_2 \circ (A_{11} + A_{12} + A_{21} + A_{22}) \circ P_2 = A_{12} + A_{21} + 4A_{22}$, taking $X = P_2, Y = A_{11} + A_{12} + A_{21} + A_{22}, Z = P_2$ in Eq (2.1), then by Lemma 1.1 (*iii*), Lemma 2.2, Lemma 2.3 (*ix*), and Lemma 2.4 (*ii*), we have

$$
P_2\Delta(A_{11} + A_{12} + A_{21} + A_{22})P_2 = P_2\Delta(A_{22})P_2. \tag{2.14}
$$

Therefore, by Eqs (2.12)–(2.14) and Lemma 2.2, we have

$$
\begin{aligned}
\Delta(A_{11} + A_{12} + A_{21} + A_{22}) &= P_1\Delta(A_{11} + A_{12} + A_{21} + A_{22})P_1 + P_1\Delta(A_{11} + A_{12} + A_{21} + A_{22})P_2 \\
&= P_2\Delta(A_{11} + A_{12} + A_{21} + A_{22})P_1 + P_2\Delta(A_{11} + A_{12} + A_{21} + A_{22})P_2 \\
&= P_1\Delta(A_{11})P_1 + P_1\Delta(A_{11})P_2 + P_2\Delta(A_{11})P_1 \\
&+ P_1\Delta(A_{12})P_2 + P_2\Delta(A_{12})P_1 \\
&+ P_1\Delta(A_{21})P_2 + P_2\Delta(A_{21})P_1 \\
&+ P_1\Delta(A_{22})P_2 + P_2\Delta(A_{22})P_1 + P_2\Delta(A_{22})P_2 \\
&= \Delta(A_{11}) + \Delta(A_{12}) + \Delta(A_{21}) + \Delta(A_{22}).
\end{aligned}
$$

The proof is completed.

Next, we show that Theorem 2.1 holds.

Proof of Theorem 2.1 For any $A, B \in \mathcal{T}$, let $A = A_{11} + A_{12} + A_{21} + A_{22}$ and $B = B_{11} + B_{12} + B_{21} + B_{22}$, where $A_{ij}, B_{ij} \in \mathcal{T}_{ij}$ $(1 \leq i, j \leq 2)$, by Lemma 2.3 (*i*) – (*ii*), (*viii*) – (*ix*), and Lemma 2.5, we obtain that

$$
\begin{aligned}
\Delta(A + B) &= \Delta((A_{11} + A_{12} + A_{21} + A_{22}) + (B_{11} + B_{12} + B_{21} + B_{22})) \\
&= \Delta((A_{11} + B_{11}) + (A_{12} + B_{12}) + (A_{21} + B_{21}) + (A_{22} + B_{22})) \\
&= \Delta(A_{11} + B_{11}) + \Delta(A_{12} + B_{12}) + \Delta(A_{21} + B_{21}) + \Delta(A_{22} + B_{22}) \\
&= \Delta(A_{11}) + \Delta(B_{11}) + \Delta(A_{12}) + \Delta(B_{12}) + \Delta(A_{21}) + \Delta(B_{21}) + \Delta(A_{22}) + \Delta(B_{22}) \\
&= \Delta(A_{11} + A_{12} + A_{21} + A_{22}) + \Delta(B_{11} + B_{12} + B_{21} + B_{22}) \\
&= \Delta(A) + \Delta(B).
\end{aligned}
$$

Therefore, Δ is an additive mapping on $\mathcal{T}$, and Δ is an additive Jordan triple derivation on $\mathcal{T}$. By reference [10] Theorem 1.1, we get that there exist an additive derivation D and an additive antiderivation φ on $\mathcal{T}$, respectively, such that

$$
\Delta(A) = D(A) + \varphi(A)
$$

for all $A \in \mathcal{T}$. The proof is completed.

In the following, we will provide applications of Theorem 2.1.

Because triangular algebra is a special type of $*$-type trivial extension algebra, and if triangular algebra $\mathcal{U}$ is a 2-torsion free algebra, then by reference [10] Corollary 1.1, we get the following Corollary 2.6.

Corollary 2.6 Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over a commutative ring $\mathcal{R}$ and $\mathcal{M}$ be a unital $(\mathcal{A}, \mathcal{B})$-bimodule, which is faithful as both a left $\mathcal{A}$-module and a right $\mathcal{B}$-module. Let $\mathcal{U} = \mathrm{Tri}(\mathcal{A}, \mathcal{M}, \mathcal{B})$ be the 2-torsion free triangular algebra, and Δ be a nonlinear Jordan triple derivable mapping on $\mathcal{U}$, then Δ is an additive derivation.

Next, we give an application of Corollary 2.6 to certain special classes of triangular algebras, such as block upper triangular matrix algebras and nest algebras.

Let $\mathcal{R}$ be a commutative ring with identity and let $M_{n\times k}(\mathcal{R})$ be the set of all $n \times k$ matrices over $\mathcal{R}$. For $n \geq 2$ and $m \leq n$, the block upper triangular matrix algebra $T_n^{\bar{k}}(\mathcal{R})$ is a subalgebra of $M_n(\mathcal{R})$ with the form

$$\begin{pmatrix} M_{k_1}(\mathcal{R}) & M_{k_1\times k_2}(\mathcal{R}) & \cdots & M_{k_1\times k_m}(\mathcal{R}) \\ 0 & M_{k_2}(\mathcal{R}) & \cdots & M_{k_2\times k_m}(\mathcal{R}) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & M_{k_m}(\mathcal{R}) \end{pmatrix},$$

where $\bar{k} = (k_1, k_2, \cdots, k_m)$ is an ordered m-vector of positive integers such that $k_1 + k_2 + \cdots + k_m = n$.

A nest of a complex Hilbert space $\mathcal{H}$ is a chain $\mathcal{N}$ of closed subspaces of $\mathcal{H}$ containing $\{0\}$, which is closed under arbitrary intersections and closed linear span. Denote by $\mathcal{B}(\mathcal{H})$ the algebra of all bounded linear operators on $\mathcal{H}$. The nest algebra associated to $\mathcal{N}$ is the algebra

$$\mathrm{Alg}\mathcal{N} = \{T \in \mathcal{B}(\mathcal{H}) : TN \subseteq \mathcal{N} \text{ for all } N \in \mathcal{N}\}.$$

A nest $\mathcal{N}$ is called trivial if $\mathcal{N} = \{0, \mathcal{H}\}$. It is clear that every nontrivial nest algebra is a triangular algebra and every finite dimensional nest algebra is isomorphic to a complex block upper triangular matrix algebra.

Corollary 2.7 Let $T_n^{\bar{k}}(\mathcal{R})$ be a 2-torsion free block upper triangular matrix algebra, and Δ be a nonlinear Jordan triple derivable mapping on $T_n^{\bar{k}}(\mathcal{R})$, then Δ is an additive derivation.

Corollary 2.8 Let $\mathcal{N}$ be a nontrivial nest of a complex Hilbert space $\mathcal{H}$, $\mathrm{Alg}\mathcal{N}$ be a nest algebra, and Δ be a nonlinear Jordan triple derivable mapping on $\mathrm{Alg}\mathcal{N}$, then Δ is an additive derivation.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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