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Research article

Second main theorem for holomorphic curves on annuli with arbitrary families of hypersurfaces

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Abstract: The aim of this paper is to establish the second main theorem for holomorphic curves from the annulus into a complex projective variety intersecting an arbitrary family of hypersurfaces. This is done by using the notion of "Distributive Constant" for a family of hypersurfaces with respect to a complex projective variety developed by Quang. We also give an explicit estimate for the level of truncation.

Keywords: Nevanlinna theory; second main theorem; holomorphic curves; hypersurfaces

1. Introduction

The main result in Nevanlinna theory is called the second fundamental theorem. In 1925, Nevanlinna [1] established the second main theorem for meromorphic functions on the complex plane $\mathbb{C}$. In 1933, H. Cartan [2] proved the second main theorem for holomorphic curves with targets in the form of hyperplanes in the general position in the complex projective spaces $\mathbf{P}^n(\mathbb{C})$. In 1983, E. I. Nochka [3] proved the second main theorem in the case of hyperplanes in the N-subgeneral position in $\mathbf{P}^n(\mathbb{C})$ with ramification. In 2004, M. Ru [4] established the second main theorem for holomorphic curves with targets in the form of hypersurfaces in the general position in $\mathbf{P}^n(\mathbb{C})$ without ramification. In 2009, Ru [5] made further extension to algebraically nondegenerate holomorphic curves into an arbitrary smooth complex projective variety. Since that time, the problem of investigation of the characteristics of holomorphic maps has attracted the attention of numerous authors.

In this paper, we mainly consider the case for holomorphic curves from the doubly connected domain into $\mathbf{P}^n(\mathbb{C})$. By the doubly connected mapping theorem [6], each doubly connected domain in $\mathbb{C}$ is conformally equivalent to the annulus $\mathbb{A}(r,R) = \{z : r < |z| < R\}$, $0 \le r < R \le +\infty$. We need only consider two cases: $r = 0, R = +\infty$ simultaneously and $0 < r < R < +\infty$. In the latter case the homothety $z \mapsto \frac{z}{\sqrt{rR}}$ reduces the given domain to the annulus $\left\{z : \frac{1}{R_0} < |z| < R_0\right\}$ with $R_0 = \sqrt{\frac{R}{r}}$.

Thus, in the two cases every annulus is invariant with respect to the inversion $z \mapsto \frac{1}{z}$. Observing the above facts, Khrystiyanyn and Kondratyuk [7, 8] indicated the way to extend the value distribution of Nevanlinna theory to meromorphic functions in annuli.

Let $R_0 < +\infty$ be a fixed positive real number or $+\infty$ and let

$$\mathbb{A} = \left\{ z \in \mathbb{C} : \ \frac{1}{R_0} < |z| < R_0 \right\}$$

be an annuli in **C**. Moreover, for any real number r such that $1 < r < R_0$, we denote

$$\mathbb{A}_r = \left\{ z \in \mathbb{C} : \ \frac{1}{r} < |z| < r \right\},$$

$$\mathbb{A}_{1,r} = \left\{ z \in \mathbb{C} : \ \frac{1}{r} < |z| \leq 1 \right\},$$

and

$$\mathbb{A}_{2,r} = \{ z \in \mathbb{C} : \ 1 < |z| < r \}.$$

Let $f = (f_0 : \ldots : f_{n+1})$ be a holomorphic curve from the annuli $\mathbb{A}$ into the complex projective space $\mathbf{P}^n(\mathbb{C})$. For $1 < r < R_0$, the characteristic function of f is defined by

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log \|f(r^{-1}e^{i\theta})\| d\theta.$$

where

$$\|f(z)\| = \max\{|f_0(z)|, \ldots, |f_n(z)|\}.$$

Remark 1. *The above definition is independent, up to an additive constant, of the choice of the reduced representation of f.*

Let D be a hypersurface in $\mathbf{P}^n(\mathbb{C})$ of degree d. Let Q be the homogeneous polynomial of degree d, defining D. The proximity function of f is defined by

$$m_f(r, D) := \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\left\| f\left(re^{i\theta}\right) \right\|^d}{|Q \circ f\left(re^{i\theta}\right)|} d\theta + \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\left\| f\left(r^{-1}e^{i\theta}\right) \right\|^d}{\left|Q \circ f\left(r^{-1}e^{i\theta}\right)\right|} d\theta.$$

To be within an additive constant, this definition is independent of the choice of the reduced representation of f and the choice of the defining polynomial Q.

Further, for $j = 1, 2$, by $n_{j,f}(r, D)$ we denote the number of zeros of $Q \circ f$ in $\mathbb{A}_{j,r}$, counting multiplicity, and let $n^M_{j,f}(r, D)$ be the number of zeros of $Q \circ f$ in the disk $\mathbb{A}_{j,r}$, where any zero of multiplicity greater than M is "truncated "and counted as if it only had multiplicity M. We set

$$N_{1,f}(r, D) := \int_{r^{-1}}^1 \frac{n_{1,f}(t, D)}{t} dt, \quad N_{2,f}(r, D) := \int_1^r \frac{n_{2,f}(t, D)}{t} dt,$$

$$N^{[M]}_{1,f}(r, D) := \int_{r^{-1}}^1 \frac{n^M_{1,f}(t, D)}{t} dt, \quad N^{[M]}_{2,f}(r, D) := \int_1^r \frac{n^M_{2,f}(t, D)}{t} dt.$$

The integrated counting and truncated counting functions are defined by

$$N_f(r,D) := N_{1,f}(r,D) + N_{2,f}(r,D),$$
$$N_f^{[M]}(r,D) := N_{1,f}^{[M]}(r,D) + N_{2,f}^{[M]}(r,D).$$

When we want to emphasize Q, we sometimes also write $N_f(r,D)$ as $N_f(r,Q)$ and $N_f^{[M]}(r,D)$ as $N_f^{[M]}(r,Q)$.

In the present paper, we set the small error term by

$$O_f(r) = \begin{cases} O(\log r + \log T_f(r)), & \text{if } R_0 = +\infty, \\ O\left(\log \frac{1}{R_0 - r} + \log T_f(r)\right), & \text{if } R_0 < +\infty. \end{cases}$$

In 2015, H. T. Phuong and N. V. Thin [9] considered the extension of the second main theorem for holomorphic curves from $\mathbb{A}$ into $\mathbf{P}^n(\mathbb{C})$ crossing a finite set of fixed hyperplanes in general position.

Theorem A.([9]) *Let $f : \mathbb{A} \to \mathbf{P}^n(\mathbb{C})$ be a linearly nondegenerate holomorphic curve. Let $H_1, \ldots, H_q$ be hyperplanes in $\mathbf{P}^n(\mathbb{C})$, located in general position, then we have*

$$\| \; (q - n - 1)T_f(r) \leq \sum_{j=1}^{q} N_f^{[n]}(r, H_j) + O_f(r).$$

Here and in the following, the notation " $\| \; \mathcal{P}$ "means that if $R_0 = +\infty$, then the assertion $\mathcal{P}$ holds for all $r \in (1, +\infty)$ outside a set $\mathbb{A}'_r$ with $\int_{\mathbb{A}'_r} r^{\lambda-1} dr < +\infty$. At the same time, if $R_0 < +\infty$, then the assertion $\mathcal{P}$ holds for all $r \in (1, R_0)$ outside a set $\mathbb{A}'_r$ with $\int_{\mathbb{A}'_r} \frac{1}{(R_0 - r)^{\lambda+1}} dr < +\infty$, where $\lambda \geq 0$.

Recently, J.L. Chen and T.B. Cao [10] obtained the second main theorem for holomorphic curves on annuli crossing a finite set of moving hyperplanes in sub-general position in $\mathbf{P}^n(\mathbb{C})$. It is well known that all known second main theorems and uniqueness results hold under the conditions that the hyperplanes or hypersurfaces are located in general position (or in sub-general position). More recently, S.D. Quang [11] introduced the notion of "distributive constant"Δ_V of a family of moving hypersurfaces with respect to a subvariety V of $\mathbf{P}^n(\mathbb{C})$ and generalized some results to the case of meromorphic mappings into a projective subvariety V and an arbitrary family of moving hypersurfaces. Motivated by this new notion, we show the second main theorem for holomorphic curves from the annulus into the complex projective space which is ramified over an arbitrary family of hypersurfaces. We also give an explicit estimate for the level of truncation.

For the purpose of this article, we recall some definitions. For a subvariety V and an analytic subset S of $\mathbf{P}^n(\mathbb{C})$, the codimension of S in V is defined by

$$\operatorname{codim}_V S := \dim V - \dim(V \cap S).$$

According to Quang [11], we give the following definition.

Definition 1. *Let $\left\{D_j\right\}_{j=1}^q$ be the hypersurfaces in $\mathbf{P}^n(\mathbb{C})$. Denote by Q the index set $\{1, \ldots, q\}$. Let V be a subvariety of $\mathbf{P}^n(\mathbb{C})$ of dimension k. Assume that V is not contained in any D_j ($j \in Q$). We define the distributive constant of $\left\{D_j\right\}_{j=1}^q$ with respect to V by*

$$\Delta_V := \max_{\Gamma \subset Q} \frac{\sharp\Gamma}{\operatorname{codim}_V \left(\bigcap_{j\in\Gamma} D_j\right)}.$$

Definition 2. *Let $\left\{D_j\right\}_{j=1}^{q}$ be the hypersurfaces in $\mathbf{P}^n(\mathbb{C})$. Denote by Q the index set $\{1,\dots,q\}$. Let $N \geq n$ and $q \geq N+1$. The family $\left\{D_j\right\}_{j=1}^{q}$ is said to be in N-subgeneral position with respect to V if for every subset $R \subset Q$ with the cardinality $\sharp R = N+1$, then*

$$\bigcap_{j\in R} D_j \cap V = \emptyset.$$

If $N = \dim V$ then we say that $\left\{D_j\right\}_{j=1}^{q}$ is in general position with respect to V.

Remark 2. *If the family $\left\{D_j\right\}_{j=1}^{q}$ is in N-subgeneral position with respect to V, then $\Delta_V \leq N - \dim V + 1$ (see [11]).*

In this paper, we establish the following second main theorem for holomorphic curves from the annulus into a complex projective variety intersecting an arbitrary family of hypersurfaces. The proof of our result follows from the paper [12, 13].

Theorem 1. *Let V be a projective subvariety of $\mathbf{P}^n(\mathbb{C})$ of dimension k. Let $f : \mathbb{A} \to V$ be an algebraically nondegenerate holomorphic curve with $0 < R_0 \leq +\infty$. Let $\{D_j\}_{j=1}^{q}$ be q hypersurfaces in $\mathbf{P}^n(\mathbb{C})$ with $\deg D_j = d_j$ $(1 \leq j \leq q)$. Let d be the least common multiple of d_j's, i.e., $d = \mathrm{lcm}(d_1,\dots,d_q)$. Let Δ_V be the distributive constant of $\{D_j\}_{j=1}^{q}$ with respect to V, then for any $\varepsilon > 0$,*

$$\| \; (q - \Delta_V(k+1) - \varepsilon) T_f(r) \;\; \leq \;\; \sum_{j=1}^{q} \frac{1}{d_j} N_f^{[M_\varepsilon]}(r, D_j) + O_f(r),$$

where

$$M_\varepsilon = \left\lfloor \deg(V)^{k+1} e^k d^{k^2+k} \Delta_V^k (2k+4)^k l^k \varepsilon^{-k} \right\rfloor$$

with $l = (k+1)q!$.

Here and in the following, $\lfloor x \rfloor$ denotes the greatest integer less than or equal to the real number x.

By Remark 2, we have an immediate corollary.

Corollary 1. *Let V be a projective subvariety of $\mathbf{P}^n(\mathbb{C})$ of dimension k. Let $f : \mathbb{A} \to V$ be an algebraically nondegenerate holomorphic curve with $0 < R_0 \leq +\infty$. Let $D_1,\dots,D_q$ be the hypersurfaces in $\mathbf{P}^n(\mathbb{C})$, located in N-subgeneral position with respect to V with $d_j := \deg D_j (1 \leq j \leq q)$. Let d be the least common multiple of d_j's, i.e., $d = \mathrm{lcm}(d_1,\dots,d_q)$. Then for any $\varepsilon > 0$,*

$$\| \; (q - (N-k+1)(k+1) - \varepsilon) T_f(r) \;\; \leq \;\; \sum_{j=1}^{q} \frac{1}{d_j} N_f^{[\widetilde{M}_\varepsilon]}(r, D_j) + O_f(r).$$

where

$$\widetilde{M}_\varepsilon = \left\lfloor \deg(V)^{k+1} e^k d^{k^2+k} (N-k+1)^k (2k+4)^k l^k \varepsilon^{-k} \right\rfloor$$

with $l = (k+1)q!$.

2. Notation and auxiliary results

2.1. Some facts on holomorphic curves

To prove our result, we need the following second main theorem for holomorphic curves on the annulus (see [10, 14]).

Lemma 1. *(A general form of the second main theorem) Let $f : \mathbb{A} \to \mathbf{P}^n(\mathbb{C})$ be a linearly nondegenerate holomorphic curve (i.e. its image is not contained in any proper subspace of $\mathbf{P}^n(\mathbb{C})$). Let $H_1, \dots, H_q$ (or linear forms $\mathbf{a}_1, \dots, \mathbf{a}_q$) be arbitrary hyperplanes in $\mathbf{P}^n(\mathbb{C})$, then*

$$\| \int_0^{2\pi} \max_K \sum_{j\in K} \log \frac{\left\| f\left(re^{i\theta}\right)\right\|}{\left\|\langle f\left(re^{i\theta}\right); H_j\rangle\right\|}\frac{d\theta}{2\pi} + \int_0^{2\pi} \max_K \sum_{j\in K} \log \frac{\left\| f\left(r^{-1}e^{i\theta}\right)\right\|}{\left\|\langle f\left(r^{-1}e^{i\theta}\right); H_j\rangle\right\|}\frac{d\theta}{2\pi}$$
$$\leq (n+1)T_f(r) - N_W(r,0) + O_f(r).$$

Here, the maximum is taken over all subsets K of $\{1, \dots, q\}$ such that the linear forms $\mathbf{a}_j$, $j \in K$, are linearly independent.

We also need the following lemmas.

Lemma 2. *([11, 15]) Let V be a projective subvariety of $\mathbf{P}^n(\mathbb{C})$ of dimension k. Let $D_0, \dots, D_p$ be $p+1$ hypersurfaces in $\mathbf{P}^n(\mathbb{C})$ of the same degree $d \geq 1$, such that $\bigcap_{i=0}^{p} D_i \cap V = \emptyset$ and*

$$\dim\left(\bigcap_{i=0}^{s} D_i\right) \cap V = k - l, \quad \forall t_{l-1} \leq s < t_l, 1 \leq l \leq k$$

where $t_0, t_1, \dots, t_k$ integers with $0 = t_0 < t_1 < \cdots < t_k = p$, then there exist $k+1$ hypersurfaces $P_0, \dots, P_k$ in $\mathbf{P}^n(\mathbb{C})$ of the forms

$$P_l = \sum_{j=0}^{t_l} c_{lj} D_j, \quad c_{lj} \in \mathbb{C}, \quad l = 0, \dots, k$$

such that $\left(\bigcap_{l=0}^{k} P_l\right) \cap V = \emptyset$.

Lemma 3. *([16]) Let $\{Q_i\}_{i\in R}$ be a set of hypersurfaces in $\mathbf{P}^n(\mathbb{C})$ of the common degree d, let V be a projective subvariety of $\mathbf{P}^n(\mathbb{C})$ and let f be a meromorphic mapping of $\mathbb{C}^m$ into V. Assume that $\bigcap_{i\in R} Q_i \cap V = \emptyset$, then there exist positive constants α and β such that*

$$\alpha \|f\|^d \leq \max_{i\in R} |Q_i(f)| \leq \beta \|f\|^d.$$

Lemma 4. *([11]) Let $t_0, t_1, \dots, t_n$ be $n+1$ integers such that $1 = t_0 < t_1 < \cdots < t_n$, and let $\Delta = \max_{1\leq s\leq n} \frac{t_s - t_0}{s}$, then for every n real numbers $a_0, a_1, \dots, a_{n-1}$ with $a_0 \geq a_1 \geq \cdots \geq a_{n-1} \geq 1$, we have*

$$a_0^{t_1-t_0} a_1^{t_2-t_1} \cdots a_{n-1}^{t_n-t_{n-1}} \leq (a_0 a_1 \cdots a_{n-1})^{\Delta}.$$

2.2. Chow weights and Hilbert weights

We recall the notion of Chow weights and Hilbert weights from [5] (see also [17]). Let $X \subset \mathbf{P}^n(\mathbb{C})$ be a projective variety of dimension k and degree δ. The Chow form of X is the unique polynomial, up to a constant scalar,

$$F_X(\mathbf{u}_0, \ldots, \mathbf{u}_k) = F_X(u_{00}, \ldots, u_{0n}; \ldots; u_{k0}, \ldots, u_{kn})$$

in $n+1$ blocks of variables $\mathbf{u}_i = (u_{i0}, \ldots, u_{in})$, $i = 0, \ldots, k$ with the following properties:

(i) F_X is irreducible in $k[u_{00}, \ldots, u_{kn}]$;

(ii) F_X is homogeneous of degree δ in each block $\mathbf{u}_i$, $i = 0, \ldots, k$;

(iii) $F_X(\mathbf{u}_0, \ldots, \mathbf{u}_k) = 0$, if and only if, $X \cap H_{\mathbf{u}_0} \cap \cdots \cap H_{\mathbf{u}_k} \neq \emptyset$, where $H_{\mathbf{u}_i}, i = 0, \ldots, k$, are the hyperplanes given by

$$u_{i0}x_0 + \cdots + u_{in}x_n = 0.$$

Let $\mathbf{c} = (c_0, \ldots, c_n)$ be a tuple of real numbers and t be an auxiliary variable. We consider the decomposition

$$F_X(t^{c_0}u_{00}, \ldots, t^{c_N}u_{0n}; \ldots; t^{c_0}u_{k0}, \ldots, t^{c_n}u_{kn})$$

$$= t^{e_0}G_0(\mathbf{u}_0, \ldots, \mathbf{u}_n) + \cdots + t^{e_r}G_r(\mathbf{u}_0, \ldots, \mathbf{u}_n)$$

with $G_0, \ldots, G_r \in \mathbb{C}[u_{00}, \ldots, u_{0n}; \ldots; u_{k0}, \ldots, u_{kn}]$ and $e_0 > e_1 > \cdots > e_r$. The Chow weight of X with respect to $\mathbf{c}$ is defined by

$$e_X(\mathbf{c}) := e_0$$

For each subset $J = \{j_0, \ldots, j_k\}$ of $\{0, \ldots, n\}$ with $j_0 < j_1 < \cdots < j_k$, we define the bracket

$$[J] = [J](\mathbf{u}_0, \ldots, \mathbf{u}_k) := \det\left(u_{ij_t}\right), i, t = 0, \ldots, k,$$

where $\mathbf{u}_i = (u_{i0}, \ldots, u_{in})(1 \leq i \leq k)$ denote the blocks of $n+1$ variables. Let $J_1, \ldots, J_\beta$ with $\beta = \begin{pmatrix} n+1 \\ k+1 \end{pmatrix}$ be all subsets of $\{0, \ldots, n\}$ of cardinality $k+1$.

Therefore, F_X can be written as a homogeneous polynomial of degree δ in $[J_1], \ldots, \left[J_\beta\right]$. We may see that for $\mathbf{c} = (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$ and for any J among $J_1, \ldots, J_\beta$,

$$\begin{aligned}&(t^{c_0}u_{00}, \ldots, t^{c_n}u_{0n}, \ldots, t^{c_0}u_{k0}, \ldots, t^{c_n}u_{kn})\\ &= t\sum_{j\in J} c_j[J](u_{00}, \ldots, u_{0n}, \ldots, u_{k0}, \ldots, u_{kn})\end{aligned}$$

For $\mathbf{a} = (a_0, \ldots, a_n) \in \mathbb{Z}^{n+1}$ we write $\mathbf{x}^{\mathbf{a}}$ for the monomial $x_0^{a_0} \cdots x_n^{a_n}$. Denote by $\mathbb{C}[x_0, \ldots, x_n]_u$ the vector space of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_n]$ of degree u (including 0). For an ideal I in $\mathbb{C}[x_0, \ldots, x_n]$, we put

$$I_u := \mathbb{C}[x_0, \ldots, x_n]_u \cap I.$$

Let $I(X)$ be the prime ideal in $\mathbb{C}[x_0, \ldots, x_n]$ defining X. The Hilbert function H_X of X is defined by, for $u = 1, 2, \ldots,$

$$H_X(u) := \dim\left(\mathbb{C}[x_0, \ldots, x_n]_u / I(X)_u\right)$$

By the usual theory of Hilbert polynomials,

$$H_X(u) = \delta \cdot \frac{u^n}{n!} + O\left(u^{n-1}\right).$$

The u-th Hilbert weight $S_X(u, \mathbf{c})$ of X with respect to the tuple $\mathbf{c} = (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}$ is defined by

$$S_X(u, \mathbf{c}) := \max \left(\sum_{i=1}^{H_X(u)} \mathbf{a}_i \cdot \mathbf{c} \right)$$

where the maximum is taken over all sets of monomials $\mathbf{x}^{\mathbf{a}_1}, \ldots, \mathbf{x}^{\mathbf{a}_{H_X(u)}}$ whose residue classes modulo I form a basis of $\mathbb{C}[x_0, \ldots, x_n]_u / I_u$. The following theorems are due to J. Evertse and R. Ferretti [18].

Lemma 5. *Let $X \subset \mathbf{P}^n(\mathbb{C})$ be an algebraic variety of dimension k and degree δ. Let $u > \delta$ be an integer and let $\mathbf{c} = (c_0, \ldots, c_n) \in \mathbb{R}^{n+1}_{\geq 0}$, then*

$$\frac{1}{uH_X(u)} S_X(u, \mathbf{c}) \geq \frac{1}{(k+1)\delta} e_X(\mathbf{c}) - \frac{(2k+1)\delta}{u} \cdot \left(\max_{i=0,\ldots,n} c_i \right).$$

Lemma 6. *Let $Y \subset \mathbf{P}^n(\mathbb{C})$ be an algebraic variety of dimension k and degree δ. Let $\mathbf{c} = \left(c_1, \ldots, c_q\right)$ be a tuple of positive reals. Let $\{i_0, \ldots, i_n\}$ be a subset of $\{1, \ldots, q\}$ such that*

$$Y \cap \{y_{i_0} = \cdots = y_{i_k} = 0\} = \emptyset,$$

then

$$e_Y(\mathbf{c}) \geq (c_{i_0} + \cdots + c_{i_k})\, \delta.$$

3. Proof of Theorem 1

Proof. Assume that V is a projective subvariety of $\mathbf{P}^n(\mathbb{C})$ of dimension k. If there exists $i_0 \in Q = \{1, 2, \ldots, q\}$ such that $\cap_{j \in Q \setminus \{i_0\}} D_j \bigcap V \neq \emptyset$, then it follows from the definition that

$$\Delta_V \geq \frac{q-1}{k - \dim(\cap_{j \in Q \setminus \{i_0\}} D_j \bigcap V)} \geq \frac{q-1}{k} > \frac{q}{k+1}.$$

Hence, $q < \Delta_V(k+1)$, which implies the conclusion of Theorem 1 is trivial. Therefore, we only need to consider the case that for each $i \in Q$, the set $\cap_{j \in Q \setminus \{i\}} D_j \bigcap V = \emptyset$.

Replacing D_j by D_j^{d/d_j} if necessary, without loss of generality, we may assume that all hypersurfaces $D_1, \ldots, D_q$ are of the same degree d. We denote by $\{\sigma_i\}_{i \in I}$ the set of all permutations of the set $\{1, \ldots, q\}$, where $I = \{1, 2, \cdots, n_0\}$ and $n_0 = q!$. For each $i \in I$, since $\bigcap_{j=1}^{q-1} D_{\sigma_i(j)} \cap V = \emptyset$, there exist $k+1$ integers $t_{i,0}, t_{i,1}, \ldots, t_{i,k}$ with $1 = t_{i,0} < \cdots < t_{i,k} = p_i$, where $p_i \leq q-1$ such that $\bigcap_{j=1}^{p_i} D_{\sigma_i(j)} \cap V = \emptyset$ and

$$\dim \left(\bigcap_{j=1}^{s} D_{\sigma_i(j)} \cap V \right) = k - l \quad \forall t_{i,l-1} \leq s < t_{i,l}, 1 \leq l \leq k.$$

For each $i \in I$, we denote by $P_{i,0}, \ldots, P_{i,k}$ the hypersurfaces obtained in Lemma 2 with respect to the hypersurfaces $D_{\sigma_i(1)}, \ldots, D_{\sigma_i(p_i)}$.

We consider the mapping Φ from V into $\mathbf{P}^l(\mathbb{C})$ $(l = n_0(k+1) - 1)$, which maps a point $x \in V$ into the point $\Phi(x) \in \mathbf{P}^l(\mathbb{C})$ given by

$$\Phi(x) = (P_{1,0}(x) : \cdots : P_{1,k}(x) : P_{2,0}(x) : \cdots : P_{2,k}(x) : \cdots : P_{n_0,0}(x) \\ : \cdots : P_{n_0,k}(x))$$

Let $Y := \Phi(V)$. Since $V \cap \left(\bigcap_{j=0}^{k} P_{1,j}\right) = \emptyset, \Phi$ is a finite morphism on V and Y is a complex projective subvariety of $\mathbb{P}^l(\mathbf{C})$ with $\dim Y = k$ and

$$\delta := \deg Y \leq d^k \cdot \deg V.$$

For every

$$\mathbf{a} = (a_{1,0}, \ldots, a_{1,k}, a_{2,0}, \ldots, a_{2,k}, \ldots, a_{n_0,0}, \ldots, a_{n_0,k}) \in \mathbb{Z}_{\geq 0}^{l+1}$$

and

$$\mathbf{y} = (y_{1,0}, \ldots, y_{1,k}, y_{2,0}, \ldots, y_{2,k}, \ldots, y_{n_0,0}, \ldots, y_{n_0,k})$$

we denote $\mathbf{y}^{\mathbf{a}} = y_{1,0}^{a_{1,0}} \cdots y_{1,k}^{a_{1,k}} \cdots y_{n_0,0}^{a_{n_0,0}} \cdots y_{n_0,k}^{a_{n_0,k}}$. Let u be a positive integer. We set

$$n_u := H_Y(u) - 1, \quad m_u := \binom{m+u-1}{u} - 1$$

and define the space

$$Y_u = \mathbb{C}\left[y_0, \ldots, y_l\right]_u / (I_Y)_u,$$

which is a vector space of dimension $H_Y(u)$. We fix a basis $\{v_1, \ldots, v_{H_Y(u)}\}$ of Y_u and consider the meromorphic mapping F with a reduced representation

$$\tilde{F} = \left(v_1(\Phi \circ \tilde{f}), \ldots, v_{H_Y(u)}(\Phi \circ \tilde{f})\right) : \mathbb{A} \to \mathbf{C}^{n_u+1}.$$

Since f is algebraically nondegenerate, the holomorphic curve $F : \mathbb{A} \to \mathbf{P}^{n_u}(\mathbb{C})$ is linearly nondegenerate (i.e., its image is not contained in any hyperplanes in $\mathbf{P}^{n_u}(\mathbb{C})$).

By Lemma 3, there exists a constant $A > 0$, which is chosen common for all $i \in I$, such that

$$\|\tilde{f}(z)\|^d \leq A \max_{0 \leq j \leq p_i} \left|D_{\sigma_i(j)}(\tilde{f}(z))\right|.$$

According to the definition of $P_{i,j}$, we may choose a positive constant $B \geq 1$, commonly for all $i \in I$, such that

$$\left|P_{i,j}(\mathbf{x})\right| \leq B \max_{1 \leq s \leq t_{i,j}} \left|D_{\sigma_i(s)}(\mathbf{x})\right|,$$

for all $\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$ and for all $0 \leq j \leq k$. It is easily seen that, there exists a positive constant C, such that

$$|P_{i,j}(\mathbf{x})| \leq C\|\mathbf{x}\|^d$$

for all $\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{C}^{n+1}$, $1 \leq i \leq n_0$, and $0 \leq j \leq k$.

Fix an element $i \in I$. Denote by $S(i)$ the set of all points

$$z \in \triangle(R) \backslash \left\{ \bigcup_{i=1}^{q} D_i(\tilde{f}(z))^{-1}(\{0\}) \cup \bigcup_{\substack{0 \leq j \leq k \\ i \in I}} P_{i,j}(\tilde{f}(z))^{-1}(\{0\}) \right\}$$

such that

$$\left|D_{\sigma_i(1)}(\tilde{f}(z))\right| \leq \left|D_{\sigma_i(2)}(\tilde{f}(z))\right| \leq \cdots \leq \left|D_{\sigma_i(q)}(\tilde{f}(z))\right|.$$

Therefore, for each $z \in S(i)$, by Lemma 4 we have

$$\begin{aligned}
\prod_{i=1}^{q} \frac{\|\tilde{f}(z)\|^d}{\left|D_i(\tilde{f}(z))\right|} &\leq A^{q-p_i} \prod_{j=1}^{p_i} \frac{\|\tilde{f}(z)\|^d}{\left|D_{\sigma_i(j)}(\tilde{f}(z))\right|} \\
&\leq A^{q-p_i} \prod_{j=1}^{k} \left(\frac{\|\tilde{f}(z)\|^d}{\left|D_{\sigma_i(t_{j-1})}(\tilde{f}(z))\right|} \right)^{t_{i,j}-t_{i,j-1}} \\
&\leq A^{q-p_i} \prod_{j=1}^{k} \left(\frac{\|\tilde{f}(z)\|^d}{\left|D_{\sigma_i(t_{j-1})}(\tilde{f}(z))\right|} \right)^{\Delta_V} \\
&\leq A^{q-p_i} B^{k\Delta_V} \prod_{j=0}^{k-1} \left(\frac{\|\tilde{f}(z)\|^d}{\left|P_{i,j}(\tilde{f}(z))\right|} \right)^{\Delta_V} \\
&\leq A^{q-p_i} B^{k\Delta_V} C^{\Delta_V} \prod_{j=0}^{k} \left(\frac{\|\tilde{f}(z)\|^d}{\left|P_{i,j}(\tilde{f}(z))\right|} \right)^{\Delta_V},
\end{aligned}$$

which implies that

$$\log \prod_{i=1}^{q} \frac{\|\tilde{f}(z)\|^d}{\left|D_i(\tilde{f}(z))\right|} \leq \log(A^{q-p_i} B^{k\Delta_V} C^{\Delta_V}) + \Delta_V \log \prod_{j=0}^{k} \left(\frac{\|\tilde{f}(z)\|^d}{\left|P_{i,j}(\tilde{f}(z))\right|} \right). \tag{3.1}$$

Now, we fix an index $i \in I$ and a point $z \in S(i)$ and define

$$\mathbf{c}_z = (c_{1,0,z}, \ldots, c_{1,k,z}, c_{2,0,z}, \ldots, c_{2,k,z}, \ldots, c_{n_0,0,z}, \ldots, c_{n_0,k,z}) \in \mathbb{R}_{\geq 0}^{l+1},$$

where $c_{i,j,z} := \log \frac{\|\tilde{f}(z)\|^d \|P_{i,j}\|}{\left|P_{i,j}(\tilde{f}(z))\right|}$ for $i = 1, \ldots, n_0$ and $j = 0, \ldots, k$. By the definition of the Hilbert weight, there are $\mathbf{a}_1, \ldots, \mathbf{a}_{H_Y(u)} \in \mathbb{Z}_{\geq 0}^{l+1}$ with

$$\mathbf{a}_{i,z} = (a_{i,1,0,z}, \ldots, a_{i,1,k,z}, \ldots, a_{i,n_0,,z}, \ldots, a_{i,n_0,k,z}), \quad a_{i,j,s,z} \in \{1, \ldots, m_u + 1\},$$

such that the residue classes modulo $(I_Y)_u$ of $\mathbf{y}^{\mathbf{a}_{1,z}}, \ldots, \mathbf{y}^{\mathbf{a}_{H_Y(u),z}}$ forms a basis of $\mathbf{C}[y_0, \ldots, y_l]_u / (I_Y)_u$ and

$$S_Y(u, \mathbf{c}_z) = \sum_{i=1}^{H_Y(u)} \mathbf{a}_{i,z} \cdot \mathbf{c}_z.$$

Since $\mathbf{y}^{\mathbf{a}_{i,z}} \in Y_u$ (modulo $(I_Y)_u$), we may write

$$\mathbf{y}^{\mathbf{a}_{i,z}} = L_{i,z}(v_1, \ldots, v_{H_Y(u)}),$$

where L_i $(1 \leq i \leq H_Y(u))$ are independent linear forms. We see at once that

$$\log \prod_{i=1}^{H_Y(u)} \left|L_{i,z}(\tilde{F}(z))\right| = \log \prod_{i=1}^{H_Y(u)} \prod_{\substack{1 \leq t \leq n_0 \\ 0 \leq j \leq k}} \left|P_{tj}(\tilde{f}(z))\right|^{a_{i,t,j,z}}$$

$$= \quad -S_Y(u,\mathbf{c}_z) + duH_Y(u)\log\|\tilde{f}(z)\| + O(uH_Y(u)).$$

This implies that

$$\log\prod_{i=1}^{H_Y(u)}\frac{\|\tilde{F}(z)\|\cdot\|L_{i,z}\|}{|L_{i,z}(\tilde{F}(z))|} = S_Y(u,\mathbf{c}_z) - duH_Y(u)\log\|\tilde{f}(z)\| + H_Y(u)\log\|\tilde{F}(z)\| + O(uH_Y(u)).$$

Here, we note that $L_{i,z}$ depends on i and z, but the number of these linear forms is finite. We denote by $\mathcal{L}$ the set of all $L_{i,z}$ occurring in the above equalities, then we have

$$\begin{aligned} S_Y(u,\mathbf{c}_z) \quad \leq \quad & \max_{\mathcal{J}\subset\mathcal{L}}\log\prod_{L\in\mathcal{J}}\frac{\|\tilde{F}(z)\|\cdot\|L\|}{|L(\tilde{F}(z))|} + duH_Y(u)\log\|\tilde{f}(z)\| \\ & -H_Y(u)\log\|\tilde{F}(z)\| + O(uH_Y(u)), \end{aligned} \tag{3.2}$$

where the maximum is taken over all subsets $\mathcal{J}\subset\mathcal{L}$ with $\sharp\mathcal{J} = H_Y(u)$ and where $\{L; L\in\mathcal{J}\}$ is linearly independent. From Lemma 5, we have

$$S_Y(u,\mathbf{c}_z) \geq \frac{uH_Y(u)}{(k+1)\delta}e_Y(\mathbf{c}_z) - (2k+1)\delta H_Y(u)\left(\max_{\substack{1\leq j\leq k+1\\ 1\leq i\leq n_0}} c_{i,j,z}\right). \tag{3.3}$$

We choose an index i_0 such that $z\in S(i_0)$. Since $P_{i_0,1},\ldots,P_{i_0,k+1}$ are in general with respect to V, by Lemma 6, we have

$$e_Y(\mathbf{c}_z) \geq (c_{i_0,0,z}+\cdots+c_{i_0,k,z})\cdot\delta = \left(\log\prod_{0\leq j\leq k}\frac{\|\tilde{f}(z)\|^d\,\|P_{i_0,j}\|}{|P_{i_0,j}(\tilde{f})(z)|}\right)\cdot\delta \tag{3.4}$$

By combining (3.2), (3.3), and (3.4), we get

$$\begin{aligned} & \log\prod_{0\leq j\leq k}\frac{\|\tilde{f}(z)\|^d\,\|P_{i_0,j}\|}{|P_{i_0,j}(\tilde{f})(z)|} \\ \leq \quad & \frac{k+1}{uH_Y(u)}\left(\max_{\mathcal{J}\subset\mathcal{L}}\log\prod_{L\in\mathcal{J}}\frac{\|\tilde{F}(z)\|\cdot\|L\|}{|L(\tilde{F}(z))|} - H_Y(u)\log\|\tilde{F}(z)\|\right) \\ & +d(k+1)\log\|\tilde{f}(z)\| + \frac{(2k+1)(k+1)\delta}{u}\left(\max_{\substack{1\leq j\leq k+1\\ 1\leq i\leq n_0}} c_{i,j,z}\right). \end{aligned} \tag{3.5}$$

From (3.1) and (3.5), we have

$$\begin{aligned} & \frac{1}{\Delta_V}\log\prod_{i=1}^{q}\frac{\|\tilde{f}(z)\|^d}{|D_i(\tilde{f})(z)|} \\ \leq \quad & \frac{k+1}{uH_Y(u)}\left(\max_{\mathcal{J}\subset\mathcal{L}}\log\prod_{L\in\mathcal{J}}\frac{\|\tilde{F}(z)\|\cdot\|L\|}{|L(\tilde{F}(z))|} - H_Y(u)\log\|\tilde{F}(z)\|\right) \end{aligned}$$

$$+d(k+1)\log\|\tilde{f}(z)\| + \frac{(2k+1)(k+1)\delta}{u}\sum_{\substack{0\le j\le k\\1\le i\le n_0}}\log\frac{\|\tilde{f}(z)\|^d\left\|P_{i,j}\right\|}{\left|P_{i,j}(\tilde{f})(z)\right|} + O(1),$$

where the term $O(1)$ does not depend on z. Integrating both sides of the above inequality, we then obtain

$$\begin{aligned}
&\frac{1}{\Delta_V}\sum_{i=1}^{q} m_f\left(r, D_i\right)\\
\le\ & \frac{k+1}{uH_Y(u)}\int_0^{2\pi}\max_{\mathcal{J}\subset\mathcal{L}}\log\prod_{L\in\mathcal{J}}\frac{\|\tilde{F}(re^{i\theta})\|\cdot\|L\|}{|L(\tilde{F}(re^{i\theta}))|}\frac{d\theta}{2\pi}\\
&+\frac{k+1}{uH_Y(u)}\int_0^{2\pi}\max_{\mathcal{J}\subset\mathcal{L}}\log\prod_{L\in\mathcal{J}}\frac{\|\tilde{F}(r^{-1}e^{i\theta})\|\cdot\|L\|}{|L(\tilde{F}(r^{-1}e^{i\theta}))|}\frac{d\theta}{2\pi}-\frac{k+1}{u}T_F(r)\\
&+d(k+1)T_f(r)+\frac{(2k+1)(k+1)\delta}{u}\sum_{\substack{0\le j\le k\\1\le i\le n_0}} m_f\left(r, P_{i,j}\right).
\end{aligned}\tag{3.6}$$

Applying Lemma 1 with $\epsilon' > 0$ (which will be chosen later) to the holomorphic curve F and linear forms $L_i\,(1 \le i \le H_Y(u))$, we obtain that

$$\begin{aligned}
\| \quad & \int_0^{2\pi}\max_{\mathcal{J}\subset\mathcal{L}}\log\prod_{L\in\mathcal{J}}\frac{\|\tilde{F}(re^{i\theta})\|\cdot\|L\|}{|L(\tilde{F}(re^{i\theta}))|}\frac{d\theta}{2\pi}+\int_0^{2\pi}\max_{\mathcal{J}\subset\mathcal{L}}\log\prod_{L\in\mathcal{J}}\frac{\|\tilde{F}(r^{-1}e^{i\theta})\|\cdot\|L\|}{|L(\tilde{F}(r^{-1}e^{i\theta}))|}\frac{d\theta}{2\pi}\\
\le\ & H_Y(u)T_F(r)-N_{W(\tilde{F})}(r)+O_F(r).
\end{aligned}$$

Combining this inequality with (3.6), we have

$$\begin{aligned}
\| \quad & (q-\Delta_V(k+1))T_f(r)\\
\le\ & \sum_{i=1}^{q}\frac{1}{d}N_f(r,D_i)+\frac{\Delta_V(2k+1)(k+1)\delta}{du}\sum_{\substack{0\le j\le k\\1\le i\le n_0}} m_f\left(r, P_{i,j}\right)\\
&-\frac{\Delta_V(k+1)}{duH_Y(u)}N_{W(\tilde{F})}(r)+O_f(r).
\end{aligned}\tag{3.7}$$

We now estimate the quantity $N_{W(\tilde{F})}(r)$. In this case, we define

$$\mathbf{c} = (c_{1,0},\ldots,c_{1,k},c_{2,0},\ldots,c_{2,k},\ldots,c_{n_0,0},\ldots,c_{n_0,k}) \in \mathbb{Z}^{l+1}_{\ge 0},$$

where $c_{i,j} := \max\{\nu_{P_{i,j}(f)}(z) - n_u, 0\}$ for $i = 1,\ldots,n_0$ and $j = 0,\ldots,k$. By the definition of the Hilbert weight, there are $\mathbf{a}_1,\ldots,\mathbf{a}_{H_Y(u)} \in \mathbb{Z}^{l+1}_{\ge 0}$ with

$$\mathbf{a}_i = (a_{i,1,0},\ldots,a_{i,1,k},\ldots,a_{i,n_0,0},\ldots,a_{i,n_0,k}), \quad a_{i,j,s} \in \{1,\ldots,m_u+1\},$$

such that the residue classes modulo $(I_Y)_u$ of $\mathbf{y}^{\mathbf{a}_1},\ldots,\mathbf{y}^{\mathbf{a}_{H_Y(u)}}$ forms a basis of $\mathbb{C}\left[y_0,\ldots,y_l\right]_u / (I_Y)_u$ and

$$S_Y(u,\mathbf{c}) = \sum_{i=1}^{H_Y(u)}\mathbf{a}_i\cdot\mathbf{c}.$$

Again, there exist independent linear forms $\widehat{L_i}$ $(1 \le i \le H_Y(u))$ such that

$$\mathbf{y}^{\mathbf{a}_i} = \widehat{L_i}\left(v_1, \ldots, v_{H_Y(u)}\right) \quad (1 \le i \le H_Y(u)).$$

We also easily see that

$$\max\{\nu_{\widehat{L_i}(F)}(z) - n_u, 0\} \ge \sum_{0\le s\le k} \sum_{1\le j\le n_0} a_{i,j,s} \max\{\nu_{P_{j,s}(f)}(z) - n_u, 0\} = \mathbf{a}_i \cdot \mathbf{c}$$

and hence

$$\nu_{W(\tilde{F})}(z) \ge \sum_{i=1}^{H_Y(u)} \max\{\nu_{\widehat{L_i}(F)}(z) - n_u, 0\} \ge \sum_{i=1}^{H_Y(u)} \mathbf{a}_i \cdot \mathbf{c} = S_Y(u, \mathbf{c}). \tag{3.8}$$

For the fixed point $z \in \triangle(R)$, without lose of generality, we may assume that

$$\nu_{D_1(f)}(z) \ge \cdots \ge \nu_{D_q(f)}(z)$$

and $\sigma_1 = (1, 2, \ldots, q)$.Since $P_{1,0}, \ldots, P_{1,k}$ are in general position with respect to V, by Lemma 6 we have

$$e_Y(\mathbf{c}) \ge \delta \cdot \sum_{j=0}^{k} c_{1,j} = \delta \cdot \sum_{j=0}^{k} \max\{\nu_{P_{1,j}(f)}(z) - n_u, 0\}.$$

This, together with Lemma 2, gives that

$$\begin{aligned} S_Y(u, \mathbf{c}) \quad \ge \quad & \frac{uH_Y(u)}{k+1} \sum_{j=0}^{k} \max\{\nu_{P_{1,j}(f)}(z) - n_u, 0\} \\ & -(2k+1)\delta H_Y(u) \max_{\substack{1\le i\le n_0 \\ 0\le j\le k}} \nu_{P_{i,j}(f)}(z). \end{aligned} \tag{3.9}$$

Note the definition of $P_{1,j}(0 \le j \le k)$. We then have

$$\begin{aligned} \Delta_V \sum_{j=0}^{k} \max\{\nu_{P_{1,j}(f)}(z) - n_u, 0\} &\ge \Delta_V \sum_{j=0}^{k} \max\{\nu_{D_{t_j}(f)}(z) - n_u, 0\} \\ &\ge \sum_{j=0}^{k} \left(t_{1,j} - t_{1,j-1}\right) \max\{\nu_{D_{t_j}(f)}(z) - n_u, 0\} \\ &\ge \sum_{i=1}^{p_1} \max\{\nu_{D_i(f)}(z) - n_u, 0\} \\ &= \sum_{i=1}^{q} \max\{\nu_{D_i(f)}(z) - n_u, 0\}. \end{aligned}$$

Again, we set $t_{1,-1} = 0$. Thus, we derive from (3.9) that

$$S_Y(u, \mathbf{c}) \quad \ge \quad \frac{uH_Y(u)}{\Delta_V(k+1)} \sum_{i=1}^{q} \max\{\nu_{D_i(f)}(z) - n_u, 0\}$$

$$-(2k+1)\delta H_Y(u)\max_{\substack{1\le i\le n_0\\0\le j\le k}}\nu_{P_{i,j}(f)}(z). \tag{3.10}$$

Therefore, we derive from (3.8) and (3.10) that

$$\begin{aligned}\nu_{W(\tilde{F})}(z) &\ge \frac{uH_Y(u)}{\Delta_V(k+1)}\sum_{i=1}^{q}\max\{\nu_{D_i(f)}(z)-n_u,0\}\\&\quad -(2k+1)\delta H_Y(u)\max_{\substack{1\le i\le n_0\\0\le j\le k}}\nu_{P_{i,j}(f)}(z)\\&\ge \frac{uH_Y(u)}{\Delta_V(k+1)}\sum_{i=1}^{q}\left(\nu_{D_i(f)}(z)-\min\{\nu_{D_i(f)}(z),n_u\}\right)\\&\quad -(2k+1)\delta H_Y(u)\max_{\substack{1\le i\le n_0\\0\le j\le k}}\nu_{P_{i,j}(f)}(z).\end{aligned} \tag{3.11}$$

Integrating both sides of this inequality, we obtain

$$\begin{aligned}N_{W(\tilde{F})}(r) &\ge \frac{uH_Y(u)}{\Delta_V(k+1)}\sum_{i=1}^{q}\left(N_f(r,D_i)-N_f^{[n_u]}(r,D_i)\right)\\&\quad -(2k+1)\delta H_Y(u)\max_{\substack{1\le i\le n_0\\0\le j\le k}}N_f(r,P_{i,j}).\end{aligned} \tag{3.12}$$

Combining inequalities (3.7) and (3.12), we get

$$\begin{aligned}\| \quad (q-\Delta_V(k+1))T_f(r) &\le \sum_{i=1}^{q}\frac{1}{d}N_f^{[n_u]}(r,D_i)\\&+\frac{\Delta_V(2k+1)(k+1)\delta}{du}\sum_{\substack{0\le j\le k\\1\le i\le n_0}}\left(m_f(r,P_{i,j})+N_f\left(r,P_{i,j}\right)\right)+O_f(r).\end{aligned} \tag{3.13}$$

For each $\varepsilon>0$, we now choose u as the biggest integer such that

$$u>\frac{\Delta_V(2k+1)(k+1)^2n_0\delta}{\varepsilon}. \tag{3.14}$$

From (3.13) we have

$$\| \ (q-\Delta_V(k+1)-\varepsilon)T_f(r)\le\sum_{i=1}^{q}\frac{1}{d}N_f^{[n_u]}(r,D_i)+O_f(r).$$

Note that $\deg Y=\delta\le d^k\deg(V)$,

$$\begin{aligned}n_u=H_Y(u)-1&\le\delta\binom{k+u}{k}\le d^k\deg(V)e^k\left(1+\frac{u}{k}\right)^k\\&<d^k\deg(V)e^k\left(\Delta_V(2k+4)\delta l\varepsilon^{-1}\right)^k\\&\le\left\lfloor\deg(V)^{k+1}e^kd^{k^2+k}\Delta_V^k(2k+4)^kl^k\varepsilon^{-k}\right\rfloor=M_\varepsilon.\end{aligned}$$

Thus, it follows from (3.15) that

$$\| \; (q - \Delta_V(k+1) - \varepsilon)T_f(r) \leq \sum_{i=1}^{q} \frac{1}{d} N_f^{[M_\varepsilon]}(r, D_i) + O_f(r).$$

The proof of the theorem is finally completed. □

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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