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Research article

Dynamical analysis of a heterogeneous spatial diffusion Zika model with vector-bias and environmental transmission

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Abstract: In this study, we formulate a reaction-diffusion Zika model which incorporates vector-bias, environmental transmission and spatial heterogeneity. The main question of this paper is the analysis of the threshold dynamics. For this purpose, we establish the mosquito reproduction number R_1 and basic reproduction number R_0. Then, we analyze the dynamical behaviors in terms of R_1 and R_0. Numerically, we find that the ignorance of the vector-bias effect will underestimate the infection risk of the Zika disease, ignorance of the spatial heterogeneity effect will overestimate the infection risk, and the environmental transmission is indispensable.

Keywords: Zika; reaction-diffusion; vector-bias; environmental transmission; spatial heterogeneity; reproduction number; threshold dynamics

1. Introduction

Zika, caused by the Zika virus, is a mosquito borne disease. It is mainly transmitted to humans through mosquito bites. In March 2015, a large outbreak in Brazil attracted worldwide attention. The World Health Organization stated Zika as a Public Health Emergency of International Concern in February 2016 [1]. Zika is associated with Guillain-Barr*é* syndrome and microcephaly [2]. It poses a major threat to global health in developing countries [3].

Differential equation models are an excellent tool in studying the spread of infectious diseases [4–11]. Recently, reaction-diffusion models play an important role in exploring the effects of spatial heterogeneity on the spread of Zika [12–19]. Recently, researches showed that the Zika virus in a water environment could be transmitted to aquatic mosquitoes via breeding [20]. This suggested that determination of the route of Zika transmission needs to take environmental factors into account. In our previous article [21], the environmental transmission route was introduced into Zika model. We researched dynamical analysis of the system for three incidence functions related to the environmental

transmission rate. Then, we extended [21] to consider sexual transmissions and spatial heterogeneity [22, 23]. But, the above models did not consider the vector-bias. In fact, the vector-bias is important for Zika transmission [24, 25]. However, few Zika models consider the vector-bias, environmental transmission, and spatial heterogeneity simultaneously.

The vector-bias describes how mosquitoes prefer to bite infected people over susceptible ones. To account for the vector-bias in the model, we introduce the parameters p and l. p and l represent the probabilities that an adult mosquito arrives at a human at random and bites the human if he/she is infectious and susceptible, respectively. Let $q = \frac{p}{l}$ ($q \geq 1$). Then q is called a vector-bias parameter. In this study, we modify and add the vector-bias to our previous model in [23]. We assume that the mosquitoes and humans are living in a bounded domain Γ with smooth boundary $\partial\Gamma$. $S_1(t,x)$ and $I_1(t,x)$ indicate densities of susceptible and infectious aquatic mosquitoes at time t and position x, respectively. $S_2(t,x)$ and $I_2(t,x)$ represent densities of susceptible and infectious adult mosquitoes at time t and position x, respectively. The densities of infectious humans at time t and position x are denoted by $I_3(t,x)$. The densities of the Zika virus in the water environment are denoted by $V(t,x)$ at time t and position x. We assume that a susceptible human is unchanging [19, 24]. The density of susceptible humans is denoted by $H_*(x)$. Inspired by article [24], we set the number of newly infectious adult mosquitoes and humans per unit time to $\alpha_2(x)\frac{pI_3}{pI_3+lH_*(x)}S_2$ and $\alpha_3(x)\frac{lH_*(x)}{pI_3+lH_*(x)}I_2$, respectively. $\alpha_2(x)$ denotes the transmission rate from I_3 to S_2. $\alpha_3(x)$ denotes the transmission rate from I_2 to S_3. Then, we propose a reaction-diffusion Zika model as follows

$$
\begin{cases}
\dfrac{\partial S_1}{\partial t} = \tau(x)(S_2+I_2)\left(1-\dfrac{S_1+I_1}{K_1(x)}\right) - \alpha_1(x)\dfrac{V}{K_2(x)+V}S_1 - \omega(x)S_1 - \delta_1(x)S_1, & x\in\Gamma,\\
\dfrac{\partial I_1}{\partial t} = \alpha_1(x)\dfrac{V}{K_2(x)+V}S_1 - \omega(x)I_1 - \delta_1(x)I_1, & x\in\Gamma,\\
\dfrac{\partial S_2}{\partial t} = \omega(x)S_1 - \alpha_2(x)\dfrac{pI_3}{pI_3+lH_*(x)}S_2 - \delta_2(x)S_2 + \nabla\cdot(d_1(x)\nabla S_2), & x\in\Gamma,\\
\dfrac{\partial I_2}{\partial t} = \omega(x)I_1 + \alpha_2(x)\dfrac{pI_3}{pI_3+lH_*(x)}S_2 - \delta_2(x)I_2 + \nabla\cdot(d_1(x)\nabla I_2), & x\in\Gamma,\\
\dfrac{\partial I_3}{\partial t} = \alpha_3(x)\dfrac{lH_*(x)}{pI_3+lH_*(x)}I_2 - \gamma(x)I_3 + \nabla\cdot(d_2(x)\nabla I_3), & x\in\Gamma,\\
\dfrac{\partial V}{\partial t} = \eta(x)I_3 - \varrho(x)V, & x\in\Gamma.\\
\dfrac{\partial S_2}{\partial n} = \dfrac{\partial I_2}{\partial n} = \dfrac{\partial I_3}{\partial n} = 0, & x\in\partial\Gamma,\\
S_1(0,x) = S_{10}(x),\ I_1(0,x) = I_{10}(x),\ S_2(0,x) = S_{20}(x), & x\in\Gamma,\\
I_2(0,x) = I_{20}(x),\ I_3(0,x) = I_{30}(x),\ V(0,x) = V_0(x), & x\in\Gamma,
\end{cases}
\tag{1.1}
$$

for $t > 0$. $\tau(x)$ represents the birth rate of susceptible aquatic mosquitoes. $K_1(x)$ represents the maximal capacity of aquatic mosquitoes in the water environment. $\alpha_1(x)$ denotes the transmission rate from V to S_1. $K_2(x)$ denotes the half-saturation constant, which can cause a 50% chance of catching the Zika virus. $\omega(x)$ denotes the maturity rate of aquatic mosquitoes. $\delta_1(x)$ and $\delta_2(x)$ denote the death rate of aquatic and adult mosquitoes, respectively. $\gamma(x)$ represents the recovery and death rate of infected humans. The rate of excreting the Zika virus for each infected human is denoted by $\eta(x)$. The clearance rate of $V(x,t)$ is denoted by $\varrho(x)$. $d_1(x)$ and $d_2(x)$ represent the diffusion coefficients of adult mosquitoes and

humans. Here we employ the Neumann boundary condition $\frac{\partial U}{\partial n} = 0$, $(U = S_2, I_2, I_3)$, where n represents the outward unit normal vector on $\partial\Gamma$. Assume that $\tau(x)$, $K_1(x)$, $K_2(x)$, $\alpha_1(x)$, $\alpha_2(x)$, $\alpha_3(x)$, $\omega(x)$, $\delta_1(x)$, $\delta_2(x)$, $\gamma(x)$, $\eta(x)$, $\varrho(x)$, $d_1(x)$, $d_2(x)$, and $H_*(x)$ are continuous and positive functions of x.

The remainder of this paper is organized as follows: In the next section, we give the well-posedness of system (1.1). In Section 3, the mosquito reproduction number R_1 and basic reproduction number R_0 will be established. In Section 4, threshold dynamical behaviors are analyzed. In Section 5, we conduct some numerical simulations. This study ends with a brief conclusion.

2. The well-posedness

Let $\mathbb{G} := C(\bar{\Gamma}, \mathbb{R}^6)$ be a Banach space with the supremum norm $\|\cdot\|_{\mathbb{G}}$. Let $\mathbb{G}^+ := C(\bar{\Gamma},\ \mathbb{R}^6_+)$. We then have that $(\mathbb{G},\ \mathbb{G}^+)$ is a strongly ordered Banach space. Denote

$$\mathbb{G}_K := \left\{\theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_6)^T \in \mathbb{G}^+ : 0 \le \theta_1(x) + \theta_2(x) \le K_1(x),\ \forall\ x \in \bar{\Gamma}\right\}.$$

Throughout, we denote $\overline{g} := \max\limits_{x\in\bar{\Omega}} g(x)$, $\underline{g} := \min\limits_{x\in\bar{\Omega}} g(x)$.

Set $\mathbb{F} := C(\bar{\Gamma}, \mathbb{R})$ and $\mathbb{F}^+ := C(\bar{\Gamma}, \mathbb{R}_+)$. Assume that $\Upsilon_2(t)$ and $\Upsilon_3(t) : \mathbb{F} \to \mathbb{F}^+$ are the evolution operators associated with

$$\begin{aligned}
\frac{\partial v_2}{\partial t} &= -\delta_2(x)v_2 + \nabla\cdot(d_1(x)\nabla v_2), && x \in \Gamma,\\
\frac{\partial v_3}{\partial t} &= -\gamma(x)v_3 + \nabla\cdot(d_2(x)\nabla v_3), && x \in \Gamma,\\
\frac{\partial v_2}{\partial n} &= \frac{\partial v_3}{\partial n} = 0, && x \in \partial\Gamma.
\end{aligned}$$

We have that $\Upsilon_2(t)$ and $\Upsilon_3(t)$ are strongly positive and compact (see Chapter II in [26] and Theorems 7.3.1 and 7.4.1 in [27]). According to Subsection 2.1 in [28], one has

$$(\Upsilon_i(t)\psi)(x) = \int_\Gamma F_i(t, x, y)\psi(y)dy,\ i = 2, 3,$$

for $t \ge 0$, $\psi \in \mathbb{F}$. Here, $F_2(t, x, y)$ and $F_3(t, x, y)$ are the Green functions associated with $-\delta_2(x)v_2 + \nabla\cdot(d_1(x)\nabla v_2)$ and $-\gamma(x)v_3 + \nabla\cdot(d_2(x)\nabla v_3)$ subject to the Neumann boundary condition, respectively.

Let a_1 and a_2 be the principle eigenvalue of $-\delta_2(x)v_2 + \nabla\cdot(d_1(x)\nabla v_2)$ and $-\gamma(x)v_3 + \nabla\cdot(d_2(x)\nabla v_3)$ subject to the Neumann boundary condition, respectively. From [28], we obtain that there is $\mathcal{M} > 0$ such that $\|\ \Upsilon_i(t)\ \| \le \mathcal{M}e^{a_i t}\ \forall t \ge 0$, $i = 2, 3$. According to Theorem 2.27 in [29], one has that there exists some $b_2 > 0$ and $b_3 > 0$ such that $F_2(t, x, y) \le b_2 e^{-\underline{\delta_2} t}$ and $F_3(t, x, y) \le b_3 e^{-\underline{\gamma} t}$.

Define $\Upsilon_1(t)$ and $\Upsilon_4(t) : \mathbb{F} \to \mathbb{F}^+$ by

$$\Upsilon_1(t)\psi(x) = e^{-(\omega(x)+\delta_1(x))t}\psi(x),\ \Upsilon_4(t)\psi(x) = e^{-\varrho(x)t}\psi(x).$$

Denote $\Upsilon = \mathrm{diag}(\Upsilon_1(t), \Upsilon_1(t), \Upsilon_2(t), \Upsilon_2(t), \Upsilon_3(t), \Upsilon_4(t))$. Define $\mathcal{H} = (\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3, \mathcal{H}_4, \mathcal{H}_5, \mathcal{H}_6) : \mathbb{G}_K \to \mathbb{G}$ by

$$\begin{aligned}
\mathcal{H}_1(\theta) &= \tau(\cdot)(\theta_3 + \theta_4)\left(1 - \frac{\theta_1 + \theta_2}{K_1(\cdot)}\right) - \alpha_1(\cdot)\frac{\theta_6}{K_2(\cdot) + \theta_6}\theta_1,\\
\mathcal{H}_2(\theta) &= \alpha_1(\cdot)\frac{\theta_6}{K_2(\cdot) + \theta_6}\theta_1,\\
\mathcal{H}_3(\theta) &= \omega(\cdot)\theta_1 - \frac{\alpha_2(\cdot)p\theta_5\theta_3}{p\theta_5 + lH_*(\cdot)},\\
\mathcal{H}_4(\theta) &= \omega(\cdot)\theta_2 + \frac{\alpha_2(\cdot)p\theta_5\theta_3}{p\theta_5 + lH_*(\cdot)},\\
\mathcal{H}_5(\theta) &= \frac{\alpha_3(\cdot)lH_*(\cdot)\theta_4}{p\theta_5 + lH_*(\cdot)},\\
\mathcal{H}_6(\theta) &= \eta(\cdot)\theta_5,
\end{aligned}$$

for $t \geq 0$, $x \in \bar{\Gamma}$, and $\theta = (\theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6) \in \mathbb{G}_K$. Then, rewrite system (1.1) as

$$\begin{cases} \dfrac{\mathrm{d}w}{\mathrm{d}t} = \mathcal{B}w + \mathcal{H}(w), t > 0, \\ w(0) = \theta \in \mathbb{G}_K, \end{cases} \tag{2.1}$$

where $\mathcal{B} = \mathrm{diag}\,(\mathcal{B}_1, \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4)$ and $\mathcal{B}_j (j = 1, 2, 3, 4)$ are given by

$$\begin{aligned}
&D(\mathcal{B}_i) = \left\{\theta \in C^2(\bar{\Gamma})\right\},\ i = 1, 4,\\
&D(\mathcal{B}_j) = \left\{\theta \in C^2(\bar{\Gamma}) : \frac{\partial\theta}{\partial n} = 0 \text{ on } \partial\Gamma\right\},\ j = 2, 3,\\
&\mathcal{B}_1\theta = -(\omega(x) + \delta_1(x))\theta, && \theta \in D(\mathcal{B}_1),\\
&\mathcal{B}_2\theta = -\delta_2(x)\theta + \nabla\cdot(d_1(x)\nabla\theta), && \theta \in D(\mathcal{B}_2),\\
&\mathcal{B}_3\theta = -\gamma(x)\theta + \nabla\cdot(d_2(x)\nabla\theta), && \theta \in D(\mathcal{B}_3),\\
&\mathcal{B}_4\theta = -\varrho(x)\theta, && \theta \in D(\mathcal{B}_4).
\end{aligned}$$

System (1.1) is also equivalent to an integral equation as follows:

$$\begin{cases} w(t, \theta) = \Upsilon(t)\theta + \displaystyle\int_0^t \Upsilon(t - s)\mathcal{H}(w(s, \cdot))\mathrm{d}s, t > 0, \\ w(0) = \theta \in \mathbb{G}_K, \end{cases}$$

where $w := (S_1, I_1, S_2, I_2, I_3, V)$.

Lemma 2.1. *For every $\theta \in \mathbb{G}_K$, system (1.1) admits a unique nonnegative solution $w(t,\cdot,\theta) \in \mathbb{G}_K$ on $[0,\ \infty)$ with $w(0,\cdot,\theta) = \theta$. Moreover, the solution is uniformly bounded and ultimately bounded.*

Proof. For any $\theta \in \mathbb{G}_K$ and $c > 0$, then one has

$$\begin{aligned}
\theta(x) + c\mathcal{H}(\theta)(x) &= \begin{pmatrix}
\theta_1(x) + c\tau(x)(\theta_3(x)+\theta_4(x))\left(1 - \dfrac{\theta_1(x)+\theta_2(x)}{K_1(x)}\right) - c\alpha_1(x)\dfrac{\theta_6(x)}{K_2(x)+\theta_6(x)}\theta_1(x) \\
\theta_2(x) + c\alpha_1(x)\dfrac{\theta_6(x)}{K_2(x)+\theta_6(x)}\theta_1(x) \\
\theta_3(x) + c\omega(x)\theta_1(x) - c\dfrac{\alpha_2(x)p\theta_5(x)\theta_3(x)}{p\theta_5(x)+lH_*(x)} \\
\theta_4(x) + \omega(x)\theta_2(x) + c\dfrac{\alpha_2(x)p\theta_5(x)\theta_3(x)}{p\theta_5(x)+lH_*(x)} \\
\theta_5(x) + c\dfrac{\alpha_3(x)lH_*(x)\theta_4(x)}{p\theta_5(x)+lH_*(x)} \\
\theta_6(x) + c\eta(x)\theta_5(x)
\end{pmatrix} \\
&\geq \begin{pmatrix}
\theta_1(x)\left(1 - c\dfrac{\overline{\alpha_1}}{\underline{K_2}}\theta_6(x)\right) \\
\theta_2(x) \\
\theta_3(x)\left(1 - c\dfrac{\overline{\alpha_2}p}{\underline{lH_*}}\theta_5(x)\right) \\
\theta_4(x) \\
\theta_5(x) \\
\theta_6(x)
\end{pmatrix},
\end{aligned}$$

and

$$\begin{aligned}
&K_1(x) - (\theta_1(x) + c\mathcal{H}_1(\theta)(x) + \theta_2(x) + c\mathcal{H}_2(\theta)(x) \\
&= K_1(x) - \left(\theta_1(x) + \theta_2(x) + c\tau(x)(\theta_3(x)+\theta_4(x))\left(1 - \frac{\theta_1(x)+\theta_2(x)}{K_1(x)}\right)\right) \\
&= (K_1(x) - (\theta_1(x)+\theta_2(x))\left(1 - c\frac{\tau(x)(\theta_3(x)+\theta_4(x))}{K_1(x)}\right).
\end{aligned} \tag{2.2}$$

So, for small enough $c > 0$, one has $\theta + c\mathcal{H}(\theta) \in \mathbb{G}_K$, and

$$\lim_{c\to 0^+} \frac{1}{c} dist\,(\theta + c\mathcal{H}(\theta), \mathbb{G}_K) = 0,\ \forall\ \theta \in \mathbb{G}_K.$$

By Corollary 4 in [30], we can obtain that for any t in its maximal existence interval $[0, t_\theta)$ with $t_\theta \leq \infty$, system (1.1) has a unique mild solution $w(t,\cdot,\theta)$ with $w(0,\cdot,\theta) = \theta$, and $w(0,\cdot,\theta) \in \mathbb{G}_K$. Moreover, $w(t,\cdot,\theta) \in \mathbb{G}_K\ \forall\ t \in [0, t_\theta)$. $w(t,\cdot,\theta)$ is a classical solution.

Let $M_1(t,x) = S_1(t,x) + I_1(t,x)$, $M_2(t,x) = S_2(t,x) + I_2(t,x)$. From (1.1), one has that $(M_1(t,x), M_2(t,x))$ satisfies

$$\begin{cases} \dfrac{\partial M_1}{\partial t} = \tau(x)M_2\left(1 - \dfrac{M_1}{K_1(x)}\right) - (\omega(x) + \delta_1(x))M_1, & x \in \Gamma, \\ \dfrac{\partial M_2}{\partial t} = \omega(x)M_1 - \delta_2(x)M_2 + \nabla \cdot (d_1(x)\nabla M_2), & x \in \Gamma, \\ \dfrac{\partial M_2}{\partial n} = 0, & x \in \partial\Gamma, \end{cases} \tag{2.3}$$

for $t > 0$. Since $w(t,\cdot,\theta) \in \mathbb{G}_K \ \forall\ t \in [0, t_\theta)$, we have $M_1(t,x) = S_1(t,x) + I_1(t,x) \le K_1(x)$ for $t \in [0, t_\theta)$, $x \in \bar{\Gamma}$. That is, $S_1(t,\cdot)$ and $I_1(t,\cdot)$ are bounded on $[0, t_\theta)$. From the second equation of (2.3), we have

$$\begin{cases} \dfrac{\partial M_2}{\partial t} \le \omega(x)K_1(x) - \delta_2(x)M_2 + \nabla \cdot (d_1(x)\nabla M_2), \\ \qquad\quad \le \overline{\omega}\overline{K_1} - \underline{\delta_2}M_2 + \nabla \cdot (d_1(x)\nabla M_2), & x \in \Gamma, \\ \dfrac{\partial M_2}{\partial n} = 0, & x \in \partial\Gamma. \end{cases} \tag{2.4}$$

Consider a comparison system as follows

$$\begin{cases} \dfrac{\partial v}{\partial t} = \overline{\omega}\overline{K_1} - \underline{\delta_2}v + \nabla \cdot (d_1(x)\nabla v), & x \in \Gamma, \\ \dfrac{\partial v}{\partial n} = 0, & x \in \partial\Gamma. \end{cases} \tag{2.5}$$

Let $\iota = \frac{\overline{\omega}\overline{K_1}}{\underline{\delta_2}}$. We then have that $\frac{\partial \iota}{\partial t} - \nabla \cdot (d_1(x)\nabla \iota) - (\overline{\omega}\overline{K_1} - \underline{\delta_2}\iota) \ge 0$. Therefore, ι is an upper solution of (2.5). By the comparison principle, we have $M_2(t,x) \le \iota$ for $t \in [0,\ t_\theta)$, $x \in \bar{\Gamma}$. That is, $S_2(t,\cdot)$ and $I_2(t,\cdot)$ are bounded on $[0, t_\theta)$.

From the fifth equation of (1.1), one has

$$\begin{cases} \dfrac{\partial I_3}{\partial t} \le \overline{\alpha_3}\iota - \underline{\gamma}I_3 + \nabla \cdot (d_2(x)\nabla I_3), & x \in \Gamma, \\ \dfrac{\partial I_2}{\partial n} = 0, & x \in \partial\Gamma. \end{cases}$$

It follows from the comparison principle that $I_3(t,x)$ is bounded on $[0, t_\theta)$. So, there exists a positive constant D and along with the sixth equation of (1.1), we have $\frac{\partial V}{\partial t} \le \overline{\eta}D - \underline{\varrho}V$. By the comparison principle, we can obtain that $V(t,x)$ is bounded on $[0, t_\theta)$. Therefore, for any initial value $\theta \in \mathbb{G}_K$, solutions of system (1.1) exist globally on $[0, +\infty)$.

Since $S_1(t,x) + I_1(t,x) \le K_1(x)$ for $x \in \bar{\Gamma},\ t \ge 0$, we have that $S_1(t,x)$ and $I_1(t,x)$ are ultimately bounded. From the third equation of (1.1), we have

$$\begin{cases} \dfrac{\partial S_2}{\partial t} \le \overline{\omega}\overline{K_1} - \underline{\delta_2}S_2 + \nabla \cdot (d_1(x)\nabla S_2), & x \in \Gamma, \\ \dfrac{\partial S_2}{\partial n} = 0, & x \in \partial\Gamma. \end{cases}$$

According to the comparison principle, $S_2(t,x)$ is ultimately bounded. So, there exists t_0 such that $S_2(t,x) \le 2\frac{\overline{\omega}\overline{K_1}}{\underline{\delta_2}} = 2\iota$ for $t > t_0$, $x \in \bar{\Gamma}$.

From the fourth equation of (1.1), one has

$$\begin{cases} \dfrac{\partial I_2}{\partial t} \le \overline{\omega}\overline{K_1} + 2\iota\overline{\alpha_2} - \underline{\delta_2} I_2 + \nabla \cdot (d_1(x)\nabla I_2), & x \in \Gamma, \\ \dfrac{\partial I_2}{\partial n} = 0, & x \in \partial\Gamma. \end{cases}$$

So, $I_2(t,x)$ is ultimately bounded. There exists t_{01} such that $I_2(t,x) \le 2\frac{\overline{\omega}\overline{K_1}+2\iota\overline{\alpha_2}}{\underline{\delta_2}}$ for $t > t_{01}$, $x \in \bar{\Gamma}$.

From the fifth equation of (1.1), one has

$$\begin{cases} \dfrac{\partial I_3}{\partial t} \le \overline{\alpha_3}\iota_1 - \underline{\gamma} I_3 + \nabla \cdot (d_2(x)\nabla I_3), & x \in \Gamma, \\ \dfrac{\partial I_2}{\partial n} = 0, & x \in \partial\Gamma, \end{cases}$$

where $\iota_1 = 2\frac{\overline{\omega}\overline{K_1}+2\iota\overline{\alpha_2}}{\underline{\delta_2}}$. According to the comparison principle, $I_3(t,x)$ is ultimately bounded. So, there exists t_{02} such that $I_3(t,x) \le 2\frac{\overline{\alpha_3}\iota_1}{\underline{\gamma}}$ for $t > t_{02}$, $x \in \bar{\Gamma}$. Similarly, we can get that $V(t,x)$ is ultimately bounded. Thus, the solution $w(t,x)$ is ultimately bounded.

Next, we give the proof of the uniform boundedness of solutions for system (1.1).

For any $\theta \in \mathbb{G}_K$, one has $\theta_1(x)+\theta_2(x) \le K_1(x)$, $\forall x \in \bar{\Gamma}$, and the solution of system (1.1) $w(t,\cdot,\theta) \in \mathbb{G}_K$. Thus, $w_1(t,\cdot,\theta) + w_2(t,\cdot,\theta) \le K_1(\cdot)$. That is, $S_1(t,x)$ and $I_1(t,x)$ are uniformly bounded.

From the third equation of (1.1), we have

$$\begin{cases} \dfrac{\partial S_2}{\partial t} \le \omega(x)K_1(x) - \delta_2(x)S_2 + \nabla \cdot (d_1(x)\nabla S_2), & x \in \Gamma, \\ \dfrac{\partial S_2}{\partial n} = 0, & x \in \partial\Gamma. \end{cases}$$

Then, for any initial value $\theta \in \mathbb{G}_K$, we can obtain

$$\begin{aligned} S_2(t,x) &\le \Upsilon_2(t)\theta_3(x) + \int_0^t \Upsilon_2(t-s)\omega(x)K_1(x)ds \\ &\le \mathcal{M}e^{a_2 t} \parallel \theta_3 \parallel_{\mathbb{F}} + \int_0^t \int_\Gamma F_2(t-s,x,y)\omega(y)K_1(y)dyds \\ &\le \mathcal{M}e^{a_2 t} \parallel \theta_3 \parallel_{\mathbb{F}} + \int_0^t b_2 e^{-\underline{\delta_2}(t-s)}\overline{\omega}\overline{K_1} \mid \Gamma \mid ds \\ &\le \mathcal{M} \parallel \theta_3 \parallel_{\mathbb{F}} + \frac{b_2\overline{\omega}\overline{K_1} \mid \Gamma \mid}{\underline{\delta_2}}. \end{aligned}$$

For any $c_1 > 0$, choose $\mathcal{N}(c_1) = \mathcal{M}c_1 + \frac{b_2\overline{\omega}\overline{K_1}|\Gamma|}{\underline{\delta_2}} > 0$. Then for any initial value $\parallel \theta \parallel_{\mathbb{G}} \le c_1$ and $\forall t \ge 0$, we have $S_2(t,x) \le \mathcal{N}(c_1)$. So, $S_2(t,x)$ is uniformly bounded. Similarly, for any $c_2 > 0$, choose $\mathcal{N}(c_2) = \mathcal{M}c_2 + \frac{b_2(\overline{\omega}\overline{K_1}+\overline{\alpha_2}\mathcal{N}(c_1))|\Gamma|}{\underline{\delta_2}} > 0$, then, for any initial value $\parallel \theta \parallel_{\mathbb{G}} \le c_2$ and $\forall t \ge 0$, we have

$I_2(t,x) \le \mathcal{N}(c_2)$. So, $I_2(t,x)$ is uniformly bounded. From the fifth equation of (1.1), for any $c_3 > 0$, choose $\mathcal{N}(c_3) = \mathcal{M}c_3 + \frac{b_3\overline{\alpha_3}\mathcal{N}(c_2)|\Gamma|}{\underline{\gamma}} > 0$, then, for any initial value $\| \theta \|_{\mathbb{G}} \le c_3$ and $\forall t \ge 0$, we have $I_3(t,x) \le \mathcal{N}(c_3)$. So, we can obtain that $I_3(t,x)$ is uniformly bounded. From the last equation of (1.1), we can obtain $V(t,x) \le e^{-\underline{\rho}t} \| \theta_6 \|_{\mathbb{F}} + \frac{\overline{\eta}\mathcal{N}(c_3)|\Gamma|}{\underline{\rho}}(1 - e^{-\underline{\rho}t}) \le \| \theta_6 \|_{\mathbb{F}} + \frac{\overline{\eta}\mathcal{N}(c_3)|\Gamma|}{\underline{\rho}}$. for any $c_4 > 0$, we choose $\mathcal{N}(c_4) = c_4 + \frac{\overline{\eta}\mathcal{N}(c_3)|\Gamma|}{\underline{\rho}}$ such that $V(t,x) \le \mathcal{N}(c_4)$ for any initial value $\| \theta \|_{\mathbb{G}} \le c_4$ and $\forall t \ge 0$. Thus, $V(t,x)$ is uniformly bounded. In short, the solution $w(t,x)$ of system (1.1) is uniformly bounded. This completes the proof of Lemma 2.1. □

3. Reproduction number

In this section, the mosquito reproduction number R_1 and basic reproduction number R_0 will be established though applying the theorem in article [31].

3.1. Mosquito reproduction number R_1

Linearizing system (2.3) at $(0,0)$, we can get

$$\begin{cases} \dfrac{\partial M_1}{\partial t} = \tau(x)M_2 + (\omega(x) + \delta_1(x))M_1, & x \in \Gamma, t > 0, \\ \dfrac{\partial M_2}{\partial t} = \omega(x)M_1 - \delta_2(x)M_2 + \nabla \cdot (d_1(x)\nabla M_2), & x \in \Gamma, t > 0, \\ \dfrac{\partial M_2}{\partial n} = 0, & x \in \partial\Gamma. \end{cases} \tag{3.1}$$

Define P_1 as

$$P_1(x) = \begin{pmatrix} -(\omega(x) + \delta_1(x)) & \tau(x) \\ \omega(x) & \delta_2(x) + \nabla \cdot (d_1(x)\nabla) \end{pmatrix}. \tag{3.2}$$

In addition, the eigenvalue problem of system (3.1) is given by

$$\begin{cases} \tau(x)\chi_2 + (\omega(x) + \delta_1(x))\chi_1 = \lambda_1\chi_1, & x \in \Gamma, t > 0, \\ \omega(x)\chi_1 - \delta_2(x)\chi_2 + \nabla \cdot (d_1(x)\nabla\chi_2) = \lambda_1\chi_2, & x \in \Gamma, t > 0, \\ \dfrac{\partial \chi_2}{\partial n} = 0, & x \in \partial\Gamma. \end{cases} \tag{3.3}$$

According to Lemma 2.2 in [32] and Theorem 7.6.1 in [27], the following Lemma can be obtained:

Lemma 3.1. *Let $\nu_{p1}^* := s(P_1)$, where s represents the spectral bound. If $\nu_{p1}^* \ge 0$, then ν_{p1}^* is a principal eigenvalue of eigenvalue problem (3.3) with a strongly positive eigenfunction.*

Define

$$\mathbb{F}_1(\cdot) = \begin{pmatrix} 0 & \tau(\cdot) \\ 0 & 0 \end{pmatrix}, -\mathbb{V}_1(\cdot) = \begin{pmatrix} -(\omega(\cdot) + \delta_1(\cdot)) & 0 \\ \omega(\cdot) & -\delta_2(\cdot) \end{pmatrix}.$$

Assume that $\mathbb{A}_1(t) : C(\bar{\Gamma}, \mathbb{R}^2) \to C(\bar{\Gamma}, \mathbb{R}^2)$ is the C_0-semigroup associated with the following linear system

$$\begin{cases} \begin{pmatrix} \frac{\partial \bar{w}_1}{\partial t} \\ \frac{\partial \bar{w}_2}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 \\ \nabla \cdot (d_1(x)\nabla \bar{w}_2) \end{pmatrix} + \mathbb{V}_1(x) \begin{pmatrix} \bar{w}_1 \\ \bar{w}_2 \end{pmatrix}, \ x \in \Gamma, t > 0, \\ \dfrac{\partial \bar{w}_2}{\partial n} = 0, \ x \in \partial\Gamma. \end{cases}$$

Let $\phi_m(x) \in C(\bar{\Gamma}, \mathbb{R}^2)$ be the density of initial fertile mosquitoes, and $\mathbb{L}_1 : C(\bar{\Gamma}, \mathbb{R}^2) \to C(\bar{\Gamma}, \mathbb{R}^2)$ be defined by

$$\mathbb{L}_1(\phi_m)(\cdot) := \int_0^\infty \mathbb{F}_1(\cdot)\mathbb{A}_1(t)\phi_m(\cdot)\mathrm{d}t. \tag{3.4}$$

Here, $\mathbb{L}_1(\phi_m)(\cdot)$ denotes the distribution of the total new aquatic/adult mosquitoes generated by initial density ϕ_m. So, the spectral radius of $\mathbb{L}_1$ is R_1, that is,

$$R_1 := r(\mathbb{L}_1). \tag{3.5}$$

When system (1.1) is spatially homogeneous, we can give an explicit representation of R_1 as follows:

$$R_1 = \frac{\tau\omega}{\delta_2(\omega + \delta_1)}.$$

We can then get the following Lemma according to [31]:

Lemma 3.2. *$R_1 - 1$ has the same sign as v^*_{p1}.*

We can then get the following Lemma to hold applying to Lemma 2.5 in [32]:

Lemma 3.3. *Let $\mathbb{B} := \{(M_{10}, M_{20})^T \in C(\bar{\Gamma}, \mathbb{R}^2_+) : 0 \le M_{10}(x) \le K_1(x)), \forall\ x \in \bar{\Gamma}\}$. Assume that $R_1 > 1$. Then, we can obtain*

$$\lim_{t\to\infty}(M_1(t,x), M_2(t,x)) = (M^*_1(x), M^*_2(x)), uniformly\ for\ x \in \bar{\Gamma},$$

with $(M_{10}, M_{20}) \in \mathbb{B}\backslash\{(0,0)\}$. Moreover, $0 < M^*_1(x) < K_1(x)$, and $M^*_2(x) > 0\ \forall\ x \in \bar{\Gamma}$.

3.2. Basic reproduction number R_0

System (1.1) has two infection-free steady states $E_0(x) = (0,0,0,0,0,0)$ and $E_1(x) = (M^*_1(x), 0, M^*_2(x), 0, 0, 0)$. When $R_1 > 1$, $E_1(x)$ exists. Linearizing system (1.1) at $E_1(x)$ and considering infection compartments, we can then get

$$\begin{cases} \dfrac{\partial I_1}{\partial t} = \dfrac{\alpha_1(x)M^*_1(x)}{K_2(x)}V - (\omega(x) + \delta_1(x))I_1, & x \in \Gamma, \\ \dfrac{\partial I_2}{\partial t} = \omega(x)I_1 + \dfrac{p\alpha_2(x)M^*_2(x)}{lH_*(x)}I_3 - \delta_2(x)I_2 + \nabla\cdot(d_1(x)\nabla I_2), & x \in \Gamma, \\ \dfrac{\partial I_3}{\partial t} = \alpha_3(x)I_2 - \gamma(x)I_3 + \nabla\cdot(d_2(x)\nabla I_3), & x \in \Gamma, \\ \dfrac{\partial V}{\partial t} = \eta(x)I_3 - \varrho(x)V, & x \in \Gamma. \\ \dfrac{\partial I_2}{\partial n} = \dfrac{\partial I_3}{\partial n} = 0, & x \in \partial\Gamma, \end{cases} \tag{3.6}$$

for $t > 0$. Denote P_2 as

$$P_2(x) = \begin{pmatrix} -(\omega(x) + \delta_1(x)) & 0 & 0 & \frac{\alpha_1(x)M^*_1(x)}{K_2(x)} \\ \omega(x) & -\delta_2(x) + \nabla\cdot(d_1(x)\nabla) & \frac{p\alpha_2(x)M^*_2(x)}{lH_*(x)} & 0 \\ 0 & \alpha_3(x) & -\gamma(x) + \nabla\cdot(d_2(x)\nabla) & 0 \\ 0 & 0 & \eta(x) & -\varrho(x) \end{pmatrix}. \tag{3.7}$$

In addition, the eigenvalue problem of system (3.6) is given by

$$\begin{cases} \dfrac{\alpha_1(x)M_1^*(x)}{K_2(x)}\bar{\chi}_4 - (\omega(x)+\delta_1(x))\bar{\chi}_1 = \lambda_2\bar{\chi}_1, & x\in\Gamma,\\ \omega(x)\bar{\chi}_1 + \dfrac{p\alpha_2(x)M_2^*(x)}{lH_*(x)}\bar{\chi}_3 - \delta_2(x)\bar{\chi}_2 + \nabla\cdot(d_1(x)\nabla\bar{\chi}_2) = \lambda_2\bar{\chi}_2, & x\in\Gamma,\\ \alpha_3(x)\bar{\chi}_2 - \gamma(x)\bar{\chi}_3 + \nabla\cdot(d_2(x)\nabla\bar{\chi}_3) = \lambda_2\bar{\chi}_3, & x\in\Gamma,\\ \eta(x)\bar{\chi}_3 - \varrho(x)\bar{\chi}_4 = \lambda_2\bar{\chi}_4, & x\in\Gamma,\\ \dfrac{\partial\bar{\chi}_2}{\partial n} = \dfrac{\partial\bar{\chi}_3}{\partial n} = 0, & x\in\partial\Gamma, \end{cases} \tag{3.8}$$

for $t > 0$. According to Theorem 7.6.1 in [27], the following Lemma can be obtained:

Lemma 3.4. *Let $\nu_{p2}^* := s(P_2)$, where s represents the spectral bound. If $\nu_{p2}^* \geq 0$, then ν_{p2}^* is a principal eigenvalue of eigenvalue problem (3.8) with a strongly positive eigenfunction.*

Define

$$\mathbb{F}_2(\cdot) = \begin{pmatrix} 0 & 0 & 0 & \frac{\alpha_1(\cdot)M_1^*(\cdot)}{K_2(\cdot)} \\ 0 & 0 & \frac{p\alpha_2(\cdot)M_2^*(\cdot)}{lH_*(\cdot)} & 0 \\ 0 & \alpha_3(\cdot) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, -\mathbb{V}_2(\cdot) = \begin{pmatrix} -(\omega(\cdot)+\delta_1(\cdot)) & 0 & 0 & 0 \\ \omega(\cdot) & -\delta_2(\cdot) & 0 & 0 \\ 0 & 0 & -\gamma(\cdot) & 0 \\ 0 & 0 & \eta(\cdot) & -\varrho(\cdot) \end{pmatrix}.$$

Let $\mathbf{u} = (I_1, I_2, I_3, V)^T$, $\nabla\cdot(d(x)\nabla\mathbf{u}) = (0, \nabla\cdot(d_1(x)\nabla I_2), \nabla\cdot(d_2(x)\nabla I_3), 0)^T$. Assume that $\mathbb{A}_2(t) : C(\bar{\Gamma},\mathbb{R}^4) \to C(\bar{\Gamma},\mathbb{R}^4)$ is the C_0-semigroup associated with the linear system

$$\begin{cases} \dfrac{\partial\mathbf{u}}{\partial t} = -\mathbb{V}_2(x)\mathbf{u} + \nabla\cdot(d(x)\nabla\mathbf{u}), & x\in\Gamma,\\ \dfrac{\partial I_2}{\partial n} = \dfrac{\partial I_3}{\partial n} = 0, & x\in\partial\Gamma, \end{cases}$$

for $t > 0$. Let $\phi_{m2}(x) \in C(\bar{\Gamma},\mathbb{R}^4)$ be the density of initial infectious individuals, and $\mathbb{L}_2 : C(\bar{\Gamma},\mathbb{R}^4) \to C(\bar{\Gamma},\mathbb{R}^4)$ be defined by

$$\mathbb{L}_2(\phi_{m2})(\cdot) := \int_0^\infty \mathbb{F}_2(\cdot)\mathbb{A}_2(t)\phi_{m2}(\cdot)\mathrm{d}t. \tag{3.9}$$

Here, $\mathbb{L}_2(\phi_{m2})(\cdot)$ denotes the distribution of new productive infected individuals generated by initial density ϕ_{m2}. So, the spectral radius of $\mathbb{L}_2$ is R_0, that is,

$$R_0 := r(\mathbb{L}_2). \tag{3.10}$$

When all parameters of system (1.1) are constants, we can give the actual formula of R_0 by

$$R_0 = \sqrt{\frac{p\alpha_2\alpha_3M_2^*}{l\gamma\delta_2H^*} + \frac{\eta\omega\alpha_1\alpha_3M_1^*}{\gamma K_2\rho\delta_2(\omega+\delta_1)}}.$$

We can then get the following Lemma according to [31]:

Lemma 3.5. *$R_0 - 1$ has the same sign as ν_{p2}^*.*

4. Threshold dynamics

According to Theorem 4.1 in [23], we know that $E_0(x)$ is globally attractive when $R_1 < 1$. Biologically, the mosquito population will vanish. Under this assumption, it is pointless to study the spread of the Zika disease. Thus, in this study, we just consider the case $R_1 > 1$.

Theorem 4.1. *Assume $R_1 > 1$ and $R_0 < 1$. Then, the disease free state $E_1(x) = (M_1^*(x), 0, M_2^*(x), 0, 0, 0)$ is globally attractive.*

Proof. Since $R_0 < 1$, it follows from Lemma 3.5 that $\nu_{p2}^* < 0$. Then, there exists a sufficiently small $\varsigma > 0$ such that $\nu_{p2\varsigma}^* < 0$, where $\nu_{p2\varsigma}^*$ is the principal eigenvalue of the following eigenvalue problem

$$\begin{cases} \dfrac{\alpha_1(x)(M_1^*(x)+\varsigma)}{K_2(x)}\breve{\chi}_4 - (\omega(x)+\delta_1(x))\breve{\chi}_1 = \lambda\breve{\chi}_1, & x \in \Gamma, \\ \omega(x)\breve{\chi}_1 + \dfrac{p\alpha_2(x)(M_2^*(x)+\varsigma)}{lH_*(x)}\breve{\chi}_3 - \delta_2(x)\breve{\chi}_2 + \nabla\cdot(d_1(x)\nabla\breve{\chi}_2) = \lambda\breve{\chi}_2, & x \in \Gamma, \\ \alpha_3(x)\breve{\chi}_2 - \gamma(x)\breve{\chi}_3 + \nabla\cdot(d_2(x)\nabla\breve{\chi}_3) = \lambda\breve{\chi}_3, & x \in \Gamma, \\ \eta(x)\breve{\chi}_3 - \varrho(x)\breve{\chi}_4 = \lambda\breve{\chi}_4, & x \in \Gamma, \\ \dfrac{\partial\breve{\chi}_2}{\partial n} = \dfrac{\partial\breve{\chi}_3}{\partial n} = 0, & x \in \partial\Gamma, \end{cases} \tag{4.1}$$

with a strongly positive eigenfunction $(\breve{\chi}_1, \breve{\chi}_2, \breve{\chi}_3, \breve{\chi}_4)$.

According to Lemma 3.3, when $R_0^M > 1$, for the above ς, there exists $t_1 > 0$ such that $0 < S_1(t,x) \le M_1^*(x) + \varsigma$, $0 < S_2(t,x) \le M_2^*(x) + \varsigma$, $\forall\ x \in \bar{\Gamma}$, $t \ge t_1$. So, for $t \ge t_1$, we can get

$$\begin{cases} \dfrac{\partial I_1}{\partial t} \le \dfrac{\alpha_1(x)(M_1^*(x)+\varsigma)}{K_2(x)}V - (\omega(x)+\delta_1(x))I_1, & x \in \Gamma, \\ \dfrac{\partial I_2}{\partial t} \le \omega(x)I_1 + \dfrac{p\alpha_2(x)(M_2^*(x)+\varsigma)}{lH_*(x)}I_3 - \delta_2(x)I_2 + \nabla\cdot(d_1(x)\nabla I_2), & x \in \Gamma, \\ \dfrac{\partial I_3}{\partial t} \le \alpha_3(x)I_2 - \gamma(x)I_3 + \nabla\cdot(d_2(x)\nabla I_3), & x \in \Gamma, \\ \dfrac{\partial V}{\partial t} \le \eta(x)I_3 - \varrho(x)V, & x \in \Gamma, \\ \dfrac{\partial I_2}{\partial n} = \dfrac{\partial I_3}{\partial n} = 0, & x \in \partial\Gamma. \end{cases}$$

For any initial value $\theta \in \mathbb{G}_K$, there exists some $h_1 > 0$ such that $(I_1(t_1,x,\theta), I_2(t_1,x,\theta), I_3(t_1,x,\theta), V(t_1,x,\theta)) \le h_1(\breve{\chi}_1, \breve{\chi}_2, \breve{\chi}_3, \breve{\chi}_4)$, $\forall\ x \in \bar{\Gamma}$.

Next, consider a comparison system as follows:

$$\begin{cases} \dfrac{\partial \bar{u}_1}{\partial t} = \dfrac{\alpha_1(x)(M_1^*(x)+\varsigma)}{K_2(x)}\bar{u}_4 - (\omega(x)+\delta_1(x))\bar{u}_1, & x \in \Gamma, \\ \dfrac{\partial \bar{u}_2}{\partial t} = \omega(x)\bar{u}_1 + \dfrac{p\alpha_2(x)(M_2^*(x)+\varsigma)}{lH_*(x)}\bar{u}_3 - \delta_2(x)\bar{u}_2 + \nabla\cdot(d_1(x)\nabla\bar{u}_2), & x \in \Gamma, \\ \dfrac{\partial \bar{u}_3}{\partial t} = \alpha_3(x)\bar{u}_2 - \gamma(x)\bar{u}_3 + \nabla\cdot(d_2(x)\nabla\bar{u}_3), & x \in \Gamma, \\ \dfrac{\partial \bar{u}_4}{\partial t} = \eta(x)\bar{u}_3 - \varrho(x)\bar{u}_4, & x \in \Gamma, \\ \dfrac{\partial \bar{u}_2}{\partial n} = \dfrac{\partial \bar{u}_3}{\partial n} = 0, & x \in \partial\Gamma, \end{cases} \tag{4.2}$$

for $t \geq t_1$. Then, system (4.2) admits a solution $h_1 e^{\mathcal{V}^*_{p2\varsigma}(t-t_1)}(\check{\chi}_1, \check{\chi}_2, \check{\chi}_3, \check{\chi}_4)$, $\forall\ t \geq t_1$. According to the comparison principle, we can obtain

$$(I_1(t,x), I_2(t,x), I_3(t,x), V(t,x)) \leq h_1 e^{\mathcal{V}^*_{p2\varsigma}(t-t_1)}(\check{\chi}_1, \check{\chi}_2, \check{\chi}_3, \check{\chi}_4), \forall\ t \geq t_1, x \in \bar{\Gamma}.$$

Hence, $\lim\limits_{t\to\infty}(I_1(t,x), I_2(t,x), I_3(t,x), V(t,x)) = (0,0,0,0)$, uniformly for $x \in \bar{\Gamma}$. According to Lemma 3.3, we have $\lim\limits_{t\to\infty} S_1(t,x) = M_1^*(x)$ and $\lim\limits_{t\to\infty} S_2(t,x) = M_2^*(x)$, uniformly for $x \in \bar{\Gamma}$. □

Remark 1 Biologically, when $R_1 > 1$ and $R_0 < 1$, Theorem 4.1 shows that the Zika disease will eventually disappear.

Let $w(t,x) := (S_1(t,x), I_1(t,x), S_2(t,x), I_2(t,x), I_3(t,x), V(t,x))$ and $\theta(x) := w(0,x)$. Define

$$\begin{aligned} &\mathbb{P} = \{\theta \in \mathbb{G}_K : \theta_2(\cdot) \not\equiv 0, \theta_4(\cdot) \not\equiv 0, \theta_5(\cdot) \not\equiv 0, \theta_6(\cdot) \not\equiv 0\}, \\ &\partial\mathbb{P} := \mathbb{G}_K \backslash \mathbb{P} = \{\theta \in \mathbb{G}_K : \theta_2(\cdot) \equiv 0, \textit{or } \theta_4(\cdot) \equiv 0, \textit{or } \theta_5(\cdot) \equiv 0, \textit{or } \theta_6(\cdot) \equiv 0\}. \end{aligned}$$

Define the solution semiflow of system (1.1) as $\Pi(t) : \mathbb{G}_K \to \mathbb{G}_K$ and $\Pi(t)\theta = w(t,\cdot,\theta)$ for any $t \geq 0$. Applying the method described in [33–35], we can obtain the following Lemma:

Lemma 4.1. *Assume that $R_1 > 1$, and $\theta_2 \not\equiv 0$, and $\theta_4 \not\equiv 0$. If there is a positive constant κ_1 such that $\liminf\limits_{t\to+\infty} w_5(t,x,\theta) \geq \kappa_1$, uniformly for all $x \in \bar{\Gamma}$, then there is a positive constant κ_2 such that*

$$\liminf_{t\to+\infty}(w_1(t,x), w_2(t,x), w_3(t,x), w_4(t,x), w_5(t,x), w_6(t,x),) \geq (\kappa_2, \kappa_2, \kappa_2, \kappa_2, \kappa_2, \kappa_2), \tag{4.3}$$

uniformly for all $x \in \bar{\Gamma}$.

Lemma 4.2. *Let $P_\partial := \{\theta \in \partial\mathbb{P} : \Pi(t)\theta \in \partial\mathbb{P}, \forall\ t \geq 0\}$. Define $\varpi(\theta)$ as the omega limit set of $\{\Pi(t)\theta : t \geq 0\}$ and $Q = \{E_0\} \cup \{E_1\}$. Then, $\bigcup_{\theta\in P_\partial} \varpi(\theta) = Q$.*

Proof. Since $\theta \in P_\partial$, we know $\Pi(t)\theta \in \partial\mathbb{P}$ for any $t \geq 0$. That is, $w_2(t,\cdot,\theta) \equiv 0$, $w_4(t,\cdot,\theta) \equiv 0$, $w_5(t,\cdot,\theta) \equiv 0$, or $w_6(t,\cdot,\theta) \equiv 0$ for any $t \geq 0$. In the case where $w_2(t,\cdot,\theta) \equiv 0$ for any $t \geq 0$, it follows from the second equation of system (1.1) that $w_1(t,\cdot,\theta) \equiv 0$ or $w_6(t,\cdot,\theta) \equiv 0$ for any $t \geq 0$.

Assume $w_1(t,\cdot,\theta)\equiv 0$. From the first equation of system (1.1), one has $w_3(t,\cdot,\theta)\equiv 0$ and $w_4(t,\cdot,\theta)\equiv 0$. $w_5(t,\cdot,\theta)$ satisfies

$$\begin{cases}\dfrac{\partial w_5}{\partial t}=-\gamma(x)w_5+\nabla\cdot(d_2(x)\nabla w_5), & x\in\Gamma,\\ \dfrac{\partial w_5}{\partial n}=0, & x\in\partial\Gamma.\end{cases}$$

Thus, $\lim\limits_{t\to\infty} w_5(t,x,\theta)=0$ uniformly for $x\in\bar{\Gamma}$. From the sixth equation of (1.1), one has $\lim\limits_{t\to\infty} w_6(t,x,\theta)=0$ uniformly for $x\in\bar{\Gamma}$. In other words,

$$\lim_{t\to\infty}(w_1(t,x),w_2(t,x),w_3(t,x),w_4(t,x),w_5(t,x),w_6(t,x))=E_0(x),\ uniformly\ for\ x\in\bar{\Gamma}.$$

If $w_1(t,\cdot,\theta)\equiv 0$ does not hold, then $w_6(t,\cdot,\theta)\equiv 0$ holds. From the sixth equation of (1.1), one has $w_5(t,\cdot,\theta)\equiv 0$ for any $t\geq 0$. So, $w_4(t,\cdot,\theta)\equiv 0$ for $t\geq 0$. In this case, assume $w_3(t,\cdot,\theta)\equiv 0$ for any $t\geq 0$, and then $w_1(t,\cdot,\theta)\equiv 0$ for any $t\geq 0$. This contradicts our assumption. Thus, $w_3(t_3,\cdot,\theta)\not\equiv 0$ for some t_3. From Lemma 3.3, we can obtain $\lim\limits_{t\to\infty}(w_1(t,x,\theta),\ w_1(t,x,\theta))=(M_1^*(x),M_2^*(x))$ uniformly for $x\in\bar{\Gamma}$. Thus,

$$\lim_{t\to\infty}(w_1(t,x),w_2(t,x),w_3(t,x),w_4(t,x),w_5(t,x),w_6(t,x))=E_1(x),\ uniformly\ for\ x\in\bar{\Gamma}.$$

Assume that there exists $t_4>0$ such that $w_2(t_4,\cdot,\theta)\not\equiv 0$. From Lemma 3.3, one has $w_2(t,\cdot,\theta)>0$ for any $t>t_4$. So, $w_4(t,\cdot,\theta)\equiv 0$, $w_5(t,\cdot,\theta)\equiv 0$, $w_6(t,\cdot,\theta)\equiv 0$ for $t>t_4$. Assume $w_4(t,\cdot,\theta)\equiv 0$ for $t>t_4$. Thus, $w_5(t,\cdot,\theta)\equiv 0$ and $w_6(t,\cdot,\theta)\equiv 0$ for $t>t_4$. So, $w_2(t,\cdot,\theta)\equiv 0$ for $t>t_4$. This contradicts our assumption. Similarly, if $w_5(t,\cdot,\theta)\equiv 0$ or $w_6(t,\cdot,\theta)\equiv 0$ for $t>t_4$, then $w_2(t,\cdot,\theta)\equiv 0$ for $t>t_4$, which contradicts our assumption. Thus, $\bigcup_{\theta\in P_\partial}\varpi(\theta)=Q$. □

Lemma 4.3. *Assume that $R_1>1$. Then, $E_0(x)$ is a uniform weak repeller for $\mathbb{P}$ in the sense that there exists $\mu_1>0$ such that*

$$\limsup_{t\to+\infty}\|\Pi(t)\theta-E_0(\cdot)\|_{\mathbb{G}}\geq\mu_1, \tag{4.4}$$

with initial value $\theta\in\mathbb{P}$.

Proof. First, we consider a linear system as follows:

$$\begin{cases}\dfrac{\partial\hat{v}}{\partial t}=P_1(\mu_1)\hat{v}, & x\in\Gamma,\\ \dfrac{\partial\hat{v}_2}{\partial n}=0, & x\in\partial\Gamma.\end{cases} \tag{4.5}$$

Here, $\hat{v}=(\hat{v}_1,\hat{v}_2)^T$, and

$$P_1(\mu_1)=\begin{pmatrix}-(\frac{\alpha_1(x)}{K_2(x)}\mu_1+\omega(x)+\delta_1(x)) & \tau(x)\left(1-\frac{2\mu_1}{K_1(x)}\right)\\ \omega(x) & \frac{\alpha_2(x)p}{lH_*(x)}\mu_1+\delta_2(x)+\nabla\cdot(d_1(x)\nabla)\end{pmatrix}.$$

Clearly, $P_1(0)=P_1$. By Lemma 3.1, it follows from $R_1>1$ that $\nu_{p1}^*=s(P_1)>0$. Since $P_1(\mu_1)$ is a continuous for small enough μ_1, one has $s(P_1(\mu_1))>0$ for small enough μ_1. Let $\nu_{p1\mu_1}^*:=s(P_1(\mu_1))$.

Then, $\nu^*_{p1\mu_1} > 0$. Denote $(\hat{\phi}_1, \hat{\phi}_2)$ as the positive eigenfunction corresponding to $\nu^*_{p1\mu_1}$. Then, system (4.5) has a solution $(\hat{v}(t,x), \hat{v}(t,x)) = e^{\nu^*_{p1\mu_1} t}(\hat{\phi}_1, \hat{\phi}_2)$.

Next, assume (4.7) does not hold. That is,

$$\limsup_{t\to+\infty} \|\Pi(t)\hat{\theta} - E_0(\cdot)\|_{\mathbb{G}} < \mu_1, \tag{4.6}$$

for some $\hat{\theta} \in \mathbb{P}$. Then, there is a constant $t_5 > 0$ such that $0 < w_1(t,x,\hat{\theta}), w_2(t,x,\hat{\theta}), w_3(t,x,\hat{\theta}), w_4(t,x,\hat{\theta}), w_5(t,x,\hat{\theta}), w_6(t,x,\hat{\theta}) < \mu_1$, $x \in \bar{\Gamma}$. Then, $w_1(t,x,\hat{\theta})$ and $w_3(t,x,\hat{\theta})$ satisfy

$$\begin{cases} \dfrac{\partial w_1}{\partial t} = \tau(x)\left(1 - \dfrac{2\mu_1}{K_1(x)}\right) w_2 + (\omega(x) + \delta_1(x)) w_1, & x \in \Gamma, t > t_4, \\ \dfrac{\partial w_2}{\partial t} = \omega(x) w_1 - \left(\dfrac{\alpha_2(x)p}{lH_*(x)}\mu_1 + \delta_2(x)\right) w_2 + \nabla \cdot (d_1(x)\nabla w_2), & x \in \Gamma, t > t_5, \\ \dfrac{\partial w_2}{\partial n} = 0, & x \in \partial\Gamma. \end{cases}$$

Since $w_1(t,x,\hat{\theta}) > 0$ and $w_3(t,x,\hat{\theta}) > 0$ for $t \geq t_5$, there exists $\hat{v}_1 > 0$ such that $(w_1(t_5,x,\hat{\theta}), w_1(t_5,x,\hat{\theta})) \geq \hat{v}_1(\hat{\phi}_1, \hat{\phi}_2)$. Applying the comparison principle, one has

$$(w_1(t,x,\hat{\theta}), w_1(t,x,\hat{\theta})) \geq \hat{v}_1 e^{\nu^*_{p1\mu_1}(t-t_5)}(\hat{\phi}_1, \hat{\phi}_2), \ for\ \forall\ t \geq t_5,\ x \in \bar{\Gamma}.$$

Due to $\nu^*_{p1\mu_1} > 0$, one has $w_1(\cdot,x,\hat{\theta}) \to +\infty$ and $w_3(\cdot,x,\hat{\theta}) \to +\infty$ as $t \to \infty$. This contradicts our assumption. It implies that $\{E_0(x)\}$ is an isolated invariant set in $\mathbb{P}$, and $W^s(\{E_0(x)\}) \cap \mathbb{P} = \emptyset$. □

Similar to the proof method of Lemma 4.3, we can draw the following conclusion:

Lemma 4.4. *Assume that $R_0 > 1$. Then, $E_1(x)$ is a uniform weak repeller for $\mathbb{P}$ in the sense that there exists $\mu_2 > 0$ such that*

$$\limsup_{t\to+\infty} \|\Pi(t)\theta - E_1(\cdot)\|_{\mathbb{G}} \geq \mu_2, \tag{4.7}$$

with initial value $\theta \in \mathbb{P}$.

Next, based on Lemmas 4.2, 4.3, and 4.4, we can obtain the following result:

Theorem 4.2. *If $R_1 > 1$ and $R_0 > 1$, then there exists $\varepsilon > 0$ such that, for every initial state $\theta \in \mathbb{P}$, the solution of system (1.1) $w(t,\cdot,\theta)$ satisfies*

$$\liminf_{t\to+\infty}(w_1(t,\cdot,\theta), w_2(t,\cdot,\theta), w_3(t,\cdot,\theta), w_4(t,\cdot,\theta), w_5(t,\cdot,\theta), w_6(t,\cdot,\theta)) \geq (\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon,\varepsilon). \tag{4.8}$$

Proof. By Lemmas 4.3 and 4.4, we can get that Q is an isolated invariant set for Π in $\mathbb{P}$, and $W^s(Q) \cap \mathbb{P} = \emptyset$, where $W^s(Q)$ is the stable set of Q for Π. From Lemma 4.2, we know that any orbit of $\Pi(t)$ in P_∂ converges to Q as $t \to \infty$. So, no subset of Q forms a cycle in $\partial\mathbb{P}$. By the acyclicity theorem on uniform persistence for maps (see Theorem 1.3.1 and Remark 1.3.1 in [36]), we conclude that Π is uniformly persistent with respect to $(\mathbb{P}, \partial\mathbb{P})$. In addition, the uniform boundedness of the solution of (1.1) implies that $\Pi(t)$ is point dissipative. According to Theorem 3.7 and Remark 3.10 in [35], we have that $\Pi(t)$

admits a global attractor $\mathcal{A}$ in $\mathbb{P}$. It follows from $\mathcal{A} = \Pi(t)\mathcal{A}$ that $\theta_2(\cdot) > 0, \theta_4(\cdot) > 0, \theta_5(\cdot) > 0$, and $\theta_6(\cdot) > 0$ for all $\theta \in \mathcal{A}$. Let $\mathcal{E} = \bigcup_{t\geq 0} \Pi(t)\mathcal{A}$. Then, $\mathcal{E} \subset \mathbb{P}$ and $\lim\limits_{t\to+\infty} d(\Pi(t)\theta, \mathcal{E}) = 0$ for all $\theta \in \mathbb{P}$.

Define a continuous function $\mathbf{Y}$: $\mathbb{G}_K \to [0, +\infty)$ by

$$\mathbf{Y}(\theta) = \min\left\{\min_{x\in\bar{\Gamma}} \theta_2(x), \min_{x\in\bar{\Gamma}} \theta_4(x), \min_{x\in\bar{\Gamma}} \theta_5(x), \min_{x\in\bar{\Gamma}} \theta_6(x)\right\}, \ \forall\ \theta \in \mathbb{G}_K.$$

Clearly, $\mathbf{Y}(\theta) > 0$ for all $\theta \in \mathcal{E}$. Since $\mathcal{E}$ is a compact subset of $\mathbb{P}$, we can get $\inf\limits_{\theta\in\mathcal{E}} \mathbf{Y}(\theta) = \min\limits_{\theta\in\mathcal{E}} \mathbf{Y}(\theta) > 0$. It follows from the attractiveness of $\mathcal{E}$ that there is a positive constant $\bar{\kappa}_1$ such that

$$\liminf_{t\to+\infty} \mathbf{Y}(\Pi(t)\theta) = \liminf_{t\to+\infty} \min_{x\in\bar{\Gamma}}(w_2(\cdot, x, \theta), w_4(\cdot, x, \theta), w_5(\cdot, x, \theta), w_6(\cdot, x, \theta)) \geq (\bar{\kappa}_1, \bar{\kappa}_1, \bar{\kappa}_1, \bar{\kappa}_1).$$

From Lemma 4.1, there exists some $\varepsilon > 0$ such that

$$\liminf_{t\to+\infty}(w_1(t, \cdot, \theta), w_2(t, \cdot, \theta), w_3(t, \cdot, \theta), w_4(t, \cdot, \theta), w_5(t, \cdot, \theta), w_6(t, \cdot, \theta)) \geq (\varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon, \varepsilon).$$

□

Remark 2 Biologically, when $R_1 > 1$ and $R_0 > 1$, Theorem 4.2 shows that the disease will persists.

5. Numerical simulations

In this section, we give numerical simulations for the dynamical behaviors of the solutions and the effect of some key factors on the transmission of the Zika disease. For this purpose, some parameters are selected with the following values: $\Gamma = (0, \pi)$, $\tau(x) = 1$, $K_1(x) = 500$, $K_2(x) = 16$, $\omega(x) = 0.05$, $\delta_1(x) = 0.15$, $\delta_2(x) = 0.05$, $\eta(x) = 0.1$, $\rho(x) = 0.3$, $H_*(x) = 100$, $\gamma(x) = 0.14$, $d_1(x) = 0.001$, $d_2(x) = 0.008$, $p = 0.6$, $l = 0.3$. Under this set of parameters, we calculate the mosquito reproduction number $R_1 = 5 > 1$.

5.1. Numerical simulation for the dynamical behaviors of the solutions

Initial values are selected as $S_1(0, x) = 30(1 + \sin(2x))$, $I_1(0, x) = 0.3(1 + \sin(2x))$, $S_2(0, x) = 10(1 + \sin(2x))$, $I_2(0, x) = 0.1(1 + \sin(2x))$, $I_3(0, x) = 1 + \sin(2x)$, $V(0, x) = 0.03(1 + \sin(2x))$. To simulate the result of Theorem 4.1, we choose $\alpha_1(x) = 0.001(1 - \cos(2x))$, $\alpha_2(x) = 0.004(1 - \cos(2x))$, $\alpha_3(x) = 0.005(1 - \cos(2x))$. We calculate the basic reproduction number to be $R_0 = 0.0343 < 1$. Figure 1 shows that the Zika disease will be eliminated.

To simulate to result of Theorem 4.2, we set the transmission rate functions as follows:

$$\alpha_1(x) = 0.05(1 - \cos(2x)), \alpha_2(x) = 0.14(1 - \cos(2x)), \alpha_3(x) = 0.158(1 - \cos(2x)). \tag{5.1}$$

Then, we calculate $R_0 = 1.0832 > 1$. From Theorem 4.2, we know that system (1.1) is uniformly persistent, which is shown in Figure 2.

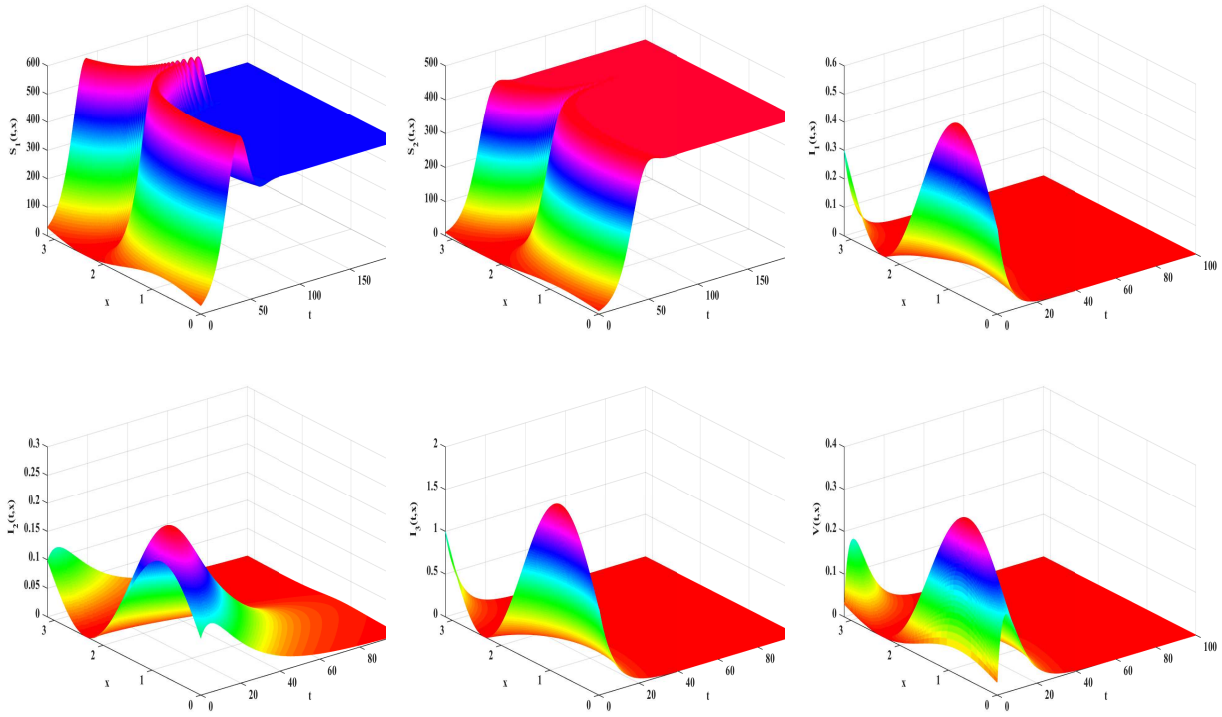


Figure 1. The dynamical behaviors of the solutions for system (1.1) with $R_0 = 0.0343 < 1$.

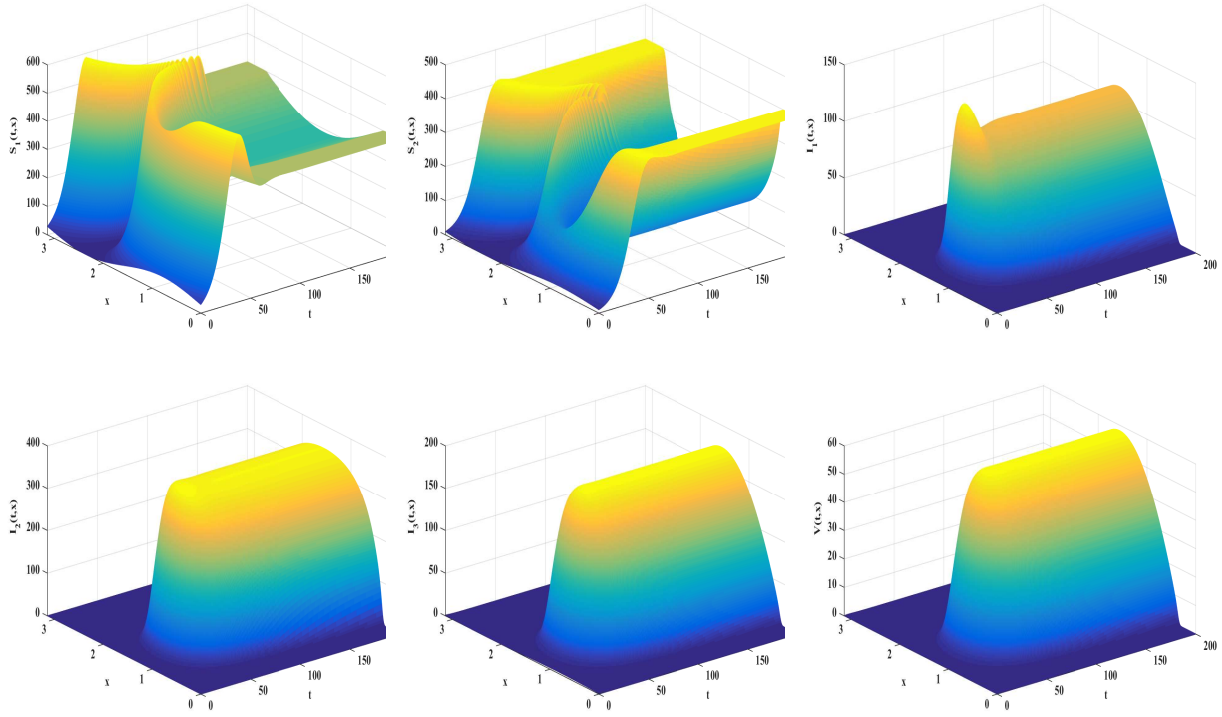


Figure 2. The dynamical behaviors of the solutions for system (1.1) with $R_0 = 1.0832 > 1$.

5.2. *The effect of some key factors on the transmission of Zika disease*

5.2.1. The effect of the vector-bias $q = \frac{p}{l}$ on R_0 and $I_2(t, x)$ and $I_3(t, x)$

Here, we choose $\alpha_2(x) = 0.14$ and $d_2(x) = 0.011$. Other parameter values remain unchanged, and other transmission rate functions are selected as in (5.1). Figure 3 shows that the basic reproduction number R_0 is an increasing function with q. When $q = 1$, i.e., $p = l$, this case indicates no vector-bias, and one has $R_0 = 0.9908 < 1$. The Zika disease will disappear from Theorem 4.1. Then, R_0 increases as q increases. When $q > q^*$ (here $q^* = 2.9089$), we know $R_0 > 1$. The Zika disease persists from Theorem 4.2. It implies that the vector-bias can cause disease outbreaks. Figure 3 indicates that neglecting the vector-bias will underestimate the risk of disease.

Next, we demonstrate the effect of the vector-bias $q = \frac{p}{l}$ on the infected adult mosquitoes $I_2(t, x)$ and infected humans $I_3(t, x)$. In Figure 4, the red solid curve represents $q = 1$. It indicates no vector-bias. The blue solid curve represents $q = 5$. It indicates that there is a vector-bias. We find that the presence of the vector-bias leads to an increased peak value of $I_2(t, x)$ and $I_3(t, x)$. In addition, as time goes on, the vector-bias has an increasingly strong effect on the peak of the distribution of infected adult mosquitoes and infected humans.

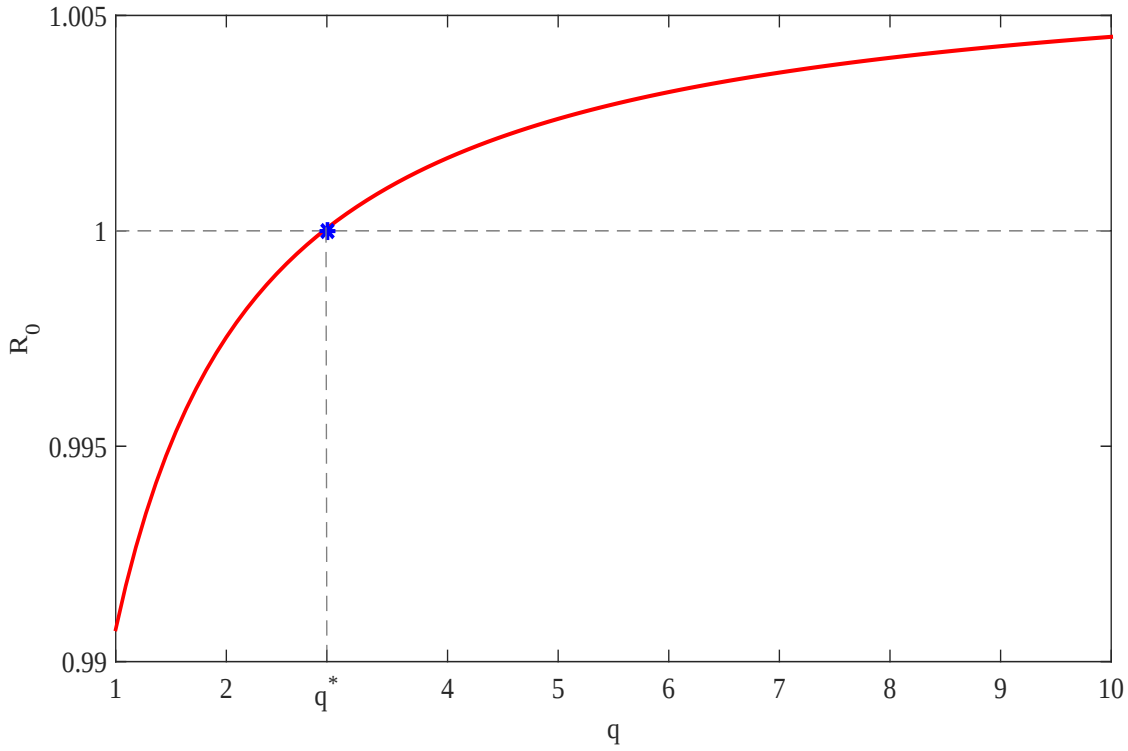
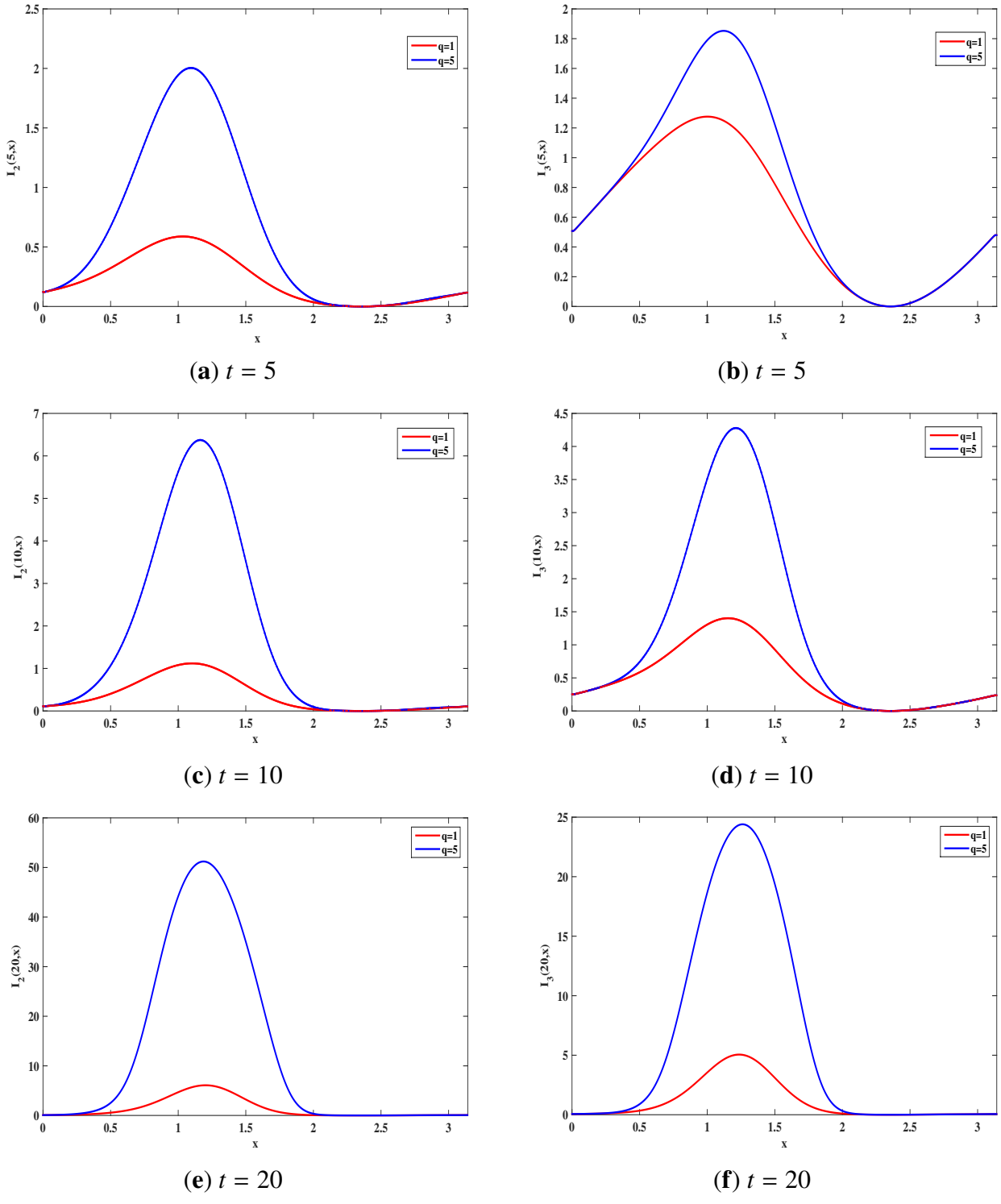


Figure 3. Plot of contours of R_0 versus the vector-bias $q = \frac{p}{l}$.



(**a**) $t = 5$ (**b**) $t = 5$

(**c**) $t = 10$ (**d**) $t = 10$

(**e**) $t = 20$ (**f**) $t = 20$

Figure 4. Distribution of $I_2(t, x)$ and $I_3(t, x)$ prevalence with $t = 5,\ 10,\ 20$ for various q.

5.2.2. The effect of the environmental transmission rate α_1 on R_0 and $I_1(t, x)$ and $V(t, x)$

Here, we choose $d_2(x) = 0.01$. Other parameter values remain unchanged and the transmission rate functions are selected as in (5.1). Figure 5 gives that the basic reproduction number R_0 is an increasing function with α_1. When $\alpha_1 < \alpha_1^*$, we have $R_0 < 1$. From Theorem 4.1, the Zika disease will disappear. R_0 increases as α_1 increases. When $\alpha_1 > \alpha_1^*$ (here $\alpha_1^* = 0.2818$), one has $R_0 > 1$. The Zika disease persists according to Theorem 4.2. Figure 5 indicates that the environmental transmission is important and indispensable.

Next, we will present the effect of the environmental transmission rate α_1 on the distribution of $I_1(t, x)$ and $V(t, x)$. We change the value of α_1, and other parameter values remain unchanged. We consider three different values: $\alpha_1 = 0.1,\ 0.2,\ 0.3$. From Figure 6, when the time is fixed, the peak value of infected aquatic mosquitoes increases significantly with the increase of the environmental transmission rate α_1. As time goes on, the environmental transmission rate has an increasingly strong effect on the peak of the distribution of infected aquatic mosquitoes and the densities of the Zika virus in the water environment.

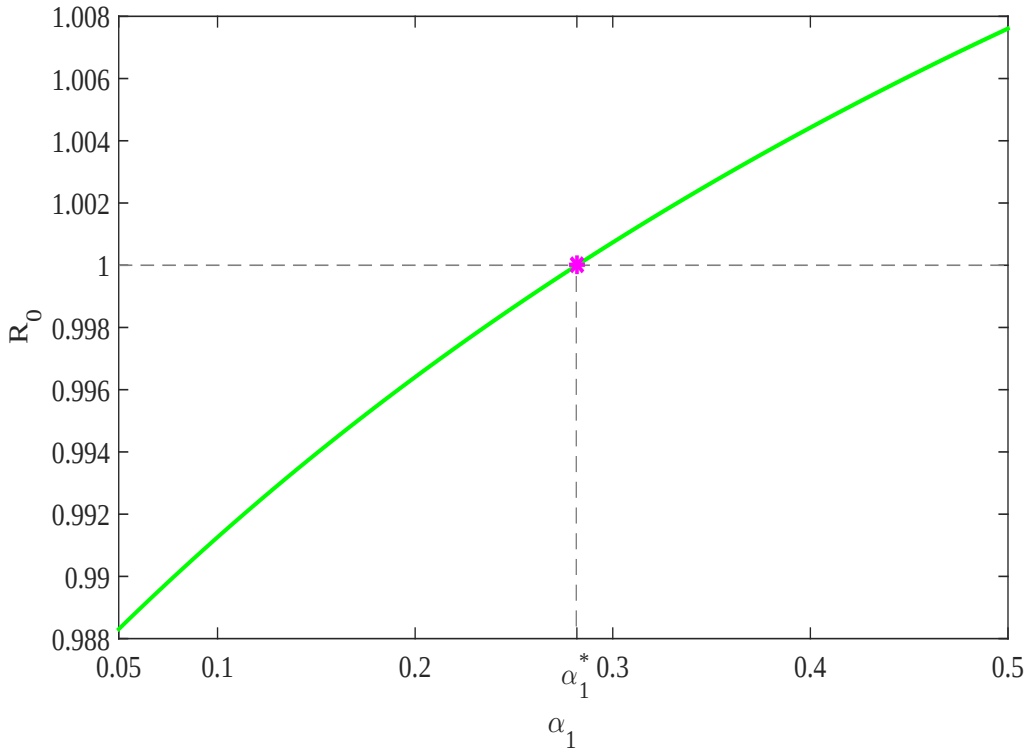
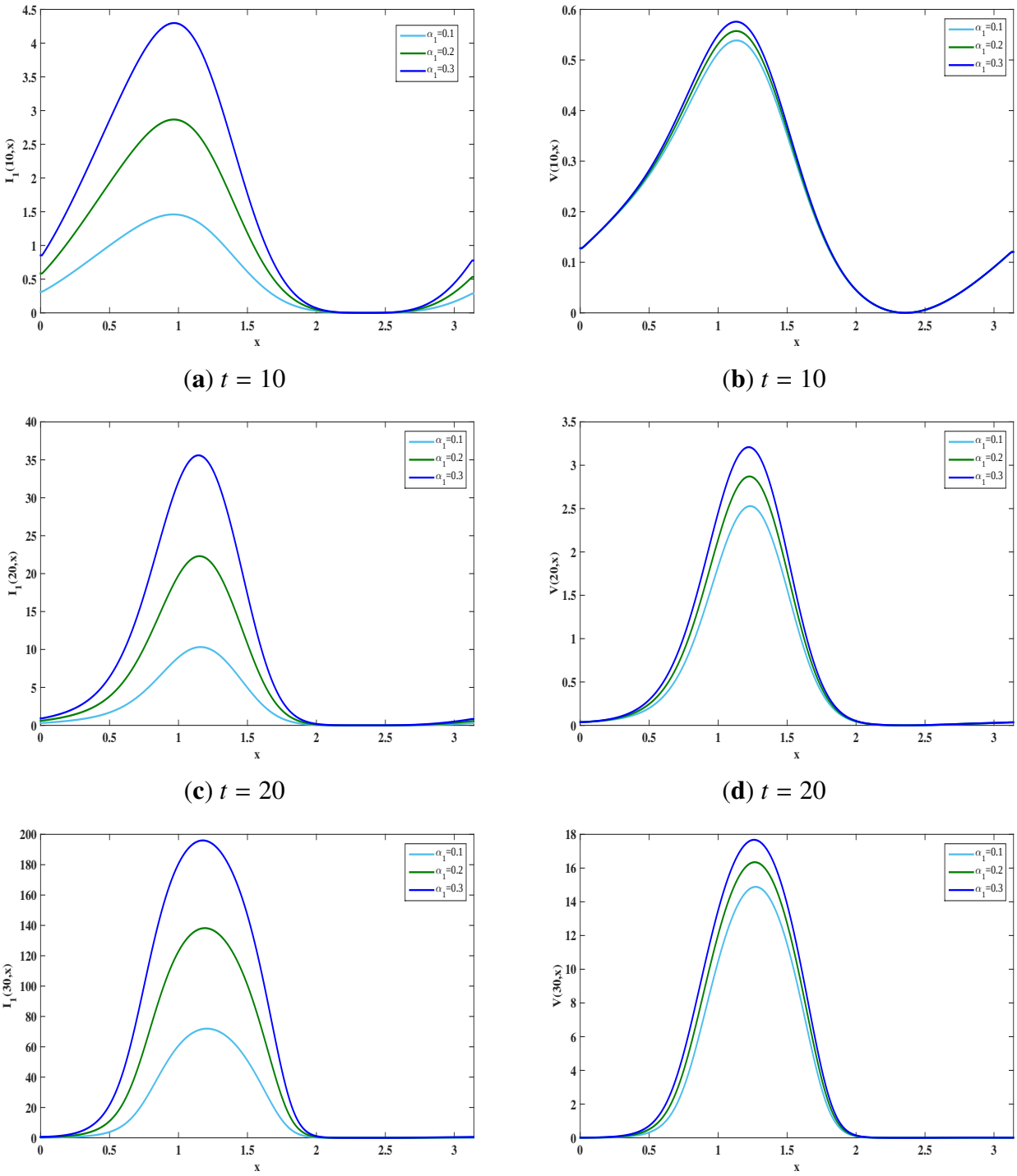


Figure 5. Plot of contours of R_0 versus the environmental transmission rate α_1.



(**a**) $t = 10$ (**b**) $t = 10$

(**c**) $t = 20$ (**d**) $t = 20$

(**e**) $t = 30$ (**f**) $t = 30$

Figure 6. Distribution of $I_1(t, x)$ and $V(t, x)$ prevalence with $t = 10,\ 20,\ 30$ for various α_1.

5.2.3. The effect of the diffusion rate on R_0

Here, we just consider the effect of the human diffusion rate d_2 on R_0. The transmission rate functions are selected as in (5.1). Figure 7 gives that the basic reproduction number R_0 is a decreasing function with d_2, which is consistent with the results obtained in [37, 38]. When $d_2 = 0$, this case indicates no diffusion in human population, and one has $R_0 = 2.2571 > 1$. The Zika disease persists according to Theorem 4.2. As d_2 increases, R_0 decreases. When $d_2 > d_2^*$ (here $d_2^* = 0.0098$), $R_0 < 1$. From Theorem 4.1, the Zika disease will disappear. Figure 7 shows that neglecting the human diffusion will overestimate the risk of the Zika disease.

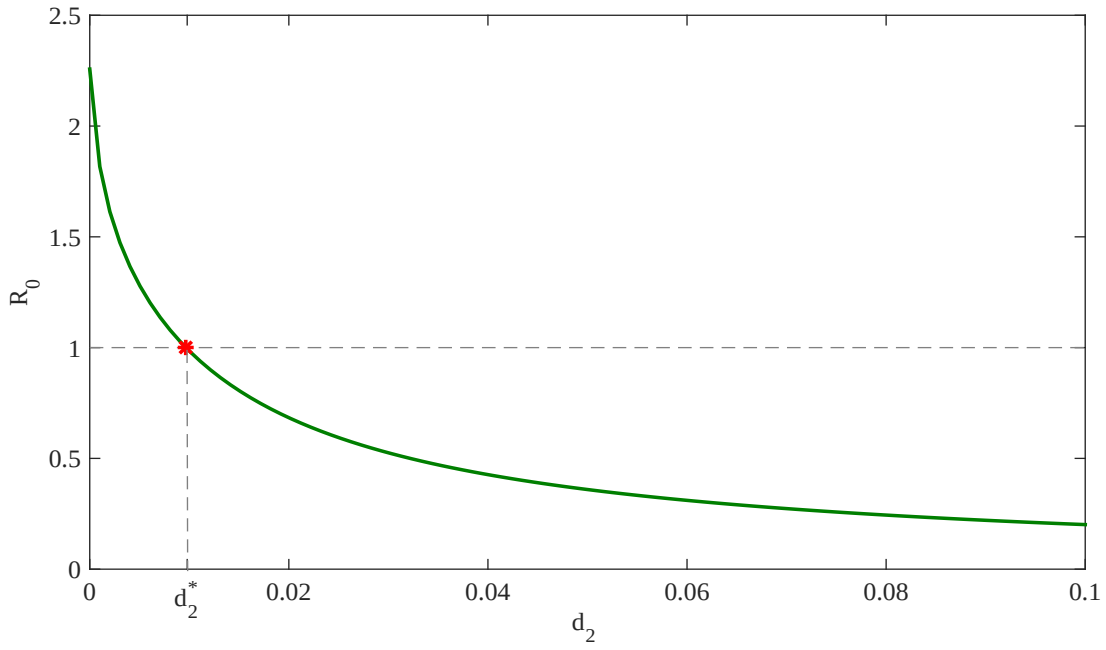


Figure 7. Plot of contours of R_0 versus the human diffusion rate d_2.

5.3. A numerical application to Brazil

From the Brazil Ministry of Health [39], we collect the weekly reported accumulated Zika cases in Brazil from March 25, 2016 to April 14, 2018. We use model (1.1) to fit the real-world data and take a week as the time unit. Some other parameters can be selected from previous literature [21]. $\tau(x) = 1$, $\omega(x) = 0.05 \times 7$, $\delta_1(x) = 0.15 \times 7$, $\gamma(x) = 0.1 \times 7$, $\delta_2(x) = 0.05 \times 7$, $\alpha_2(x) = 0.025 \times 7$, $\alpha_3(x) = 0.028 \times 7$. $H_*(x) = 2.05 \times 10^8$ [40]. Other parameters will be estimated by applying the least-squares estimation method. $\alpha_1(x) = 0.0001 \times 7$, $\eta(x) = 0.1 \times 7$, $\rho(x) = 0.3 \times 7$, $d_1(x) = 0.001$, $d_2(x) = 0.008$, $p = 0.4$, $l = 0.3$. $K_1(x) = 1.02 \times 10^9$, $K_2(x) = 3.28 \times 10^7$. We can get $R_0 < 1$, and from Theorem 4.1, the Zika disease will disappear in Brazil. The fitting result for the accumulated cases is given in Figure 8.

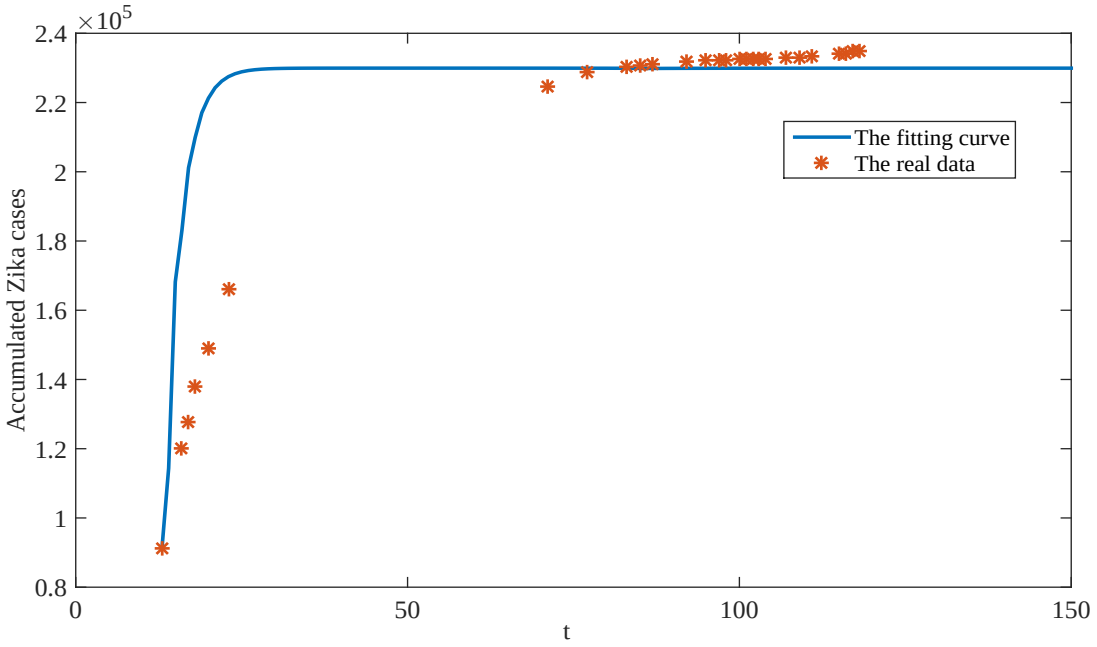


Figure 8. Simulation of the reported accumulated Zika cases in Brazil from March 25, 2016 to April 14, 2018.

6. Conclusions

This study investigated a reaction-diffusion Zika model based on our previous model in [23]. We introduced the vector-bias in the model in article [23]. In this study, we considered the combined effects of vector-bias, environmental transmission, and spatial heterogeneity on spread of Zika disease. We defined two threshold indexes: the mosquito reproduction number R_1 and basic reproduction number R_0. Dynamical behaviors in terms of R_1 and R_0 were analyzed. Finally, we simulated the effects of the vector-bias $q = \frac{p}{l}$, the environmental transmission rate α_1, and the human diffusion rate d_2 on R_0. We found that the ignorance of the vector-bias effect will underestimate the infection risk of the Zika disease and the ignorance of the human diffusion rate effect will overestimate the infection risk.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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