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Research article

Alternating series in terms of Riemann zeta function and Dirichlet beta function

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Abstract: By making use of the multisection series method, four classes of alternating infinite series are evaluated, in closed form, by the Riemann zeta function and the Dirichlet beta function.

Keywords: infinite summation; higher derivative; integer function; Riemann zeta function; Dirichlet beta function

1. Introduction and motivation

The Riemann zeta function and the Dirichlet beta function are important functions in number theory that are defined by (cf. [1, Chapter 25])

$$\zeta(\lambda)=\sum_{k=0}^{\infty}\frac{1}{(k+1)^{\lambda}}\quad\text{and}\quad\beta(\lambda)=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)^{\lambda}},$$

provided that $\Re(\lambda)>1$ and $\Re(\lambda)>0$ so that $\zeta(\lambda)$ and $\beta(\lambda)$ are convergent. There are numerous useful formulae for these functions in the literature (cf. [2–5]). For example, we shall frequently utilize the following relation:

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^{\lambda}}=(1-2^{1-\lambda})\zeta(\lambda),\qquad\Re(\lambda)>0.$$

In particular, there exist closed form expressions in π

$$\zeta(2n)=\frac{(-1)^{n-1}(2\pi)^{2n}}{2(2n)!}B_{2n}\quad\text{and}\quad\beta(2n+1)=\frac{(-1)^n\pi^{2n+1}}{4^{n+1}(2n)!}E_{2n},$$

where B_n and E_n are Bernoulli and Euler numbers (cf. [6, §6.5]), whose exponential generating functions are given by

$$\frac{y}{e^y - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} y^n, \quad |y| < 2\pi;$$
$$\frac{2e^y}{1 + e^{2y}} = \sum_{n=0}^{\infty} \frac{E_n}{n!} y^n, \quad |y| < \pi/2.$$

For a real number x, denote by $\lfloor x \rfloor$ and $\lceil x \rceil$ the usual floor and ceiling functions. The aim this paper is to examine, for two natural numbers $m, n \in \mathbb{N}$, the following four variants of the Riemann zeta function:

$$\mathrm{U}_m(n) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+m}{2m} \rfloor}}{(k+1)^{2n}}, \qquad \mathcal{U}_m(n) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+m}{2m} \rfloor}}{(2k+1)^{2n}};$$
$$\mathrm{V}_m(n) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{m} \rfloor}}{(k+1)^{2n-1}}, \qquad \mathcal{V}_m(n) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{m} \rfloor}}{(2k+1)^{2n-1}}.$$

Several closed formulae will be established that express these alternating infinite series in terms of the Riemann zeta function and the Dirichlet beta function.

Throughout the paper, we shall make use of the following polynomial expression for higher derivatives of cotangent function (cf. [7, 8])

$$\frac{d^n \cot x}{dx^n} = P_n(\cot x): \; P_n(x) = (2\mathbf{i})^n (x - \mathbf{i}) \sum_{k=0}^{n} \frac{k!}{2^k} \left\{ {n \atop k} \right\} (\mathbf{i}x - 1)^k,$$

where $\left\{ {n \atop k} \right\}$ are the Stirling numbers of the second kind (cf. [6, §6.1]). The first five polynomials read explicitly as follows:

$$P_1(x) = -x^2 - 1,$$
$$P_2(x) = 2x^3 + 2x,$$
$$P_3(x) = -6x^4 - 8x^2 - 2,$$
$$P_4(x) = 24x^5 + 40x^3 + 16x,$$
$$P_5(x) = -120x^6 - 240x^4 - 136x^2 - 16.$$

By putting these in conjunction with the partial fraction decomposition (see, for example, Stromberg [9, Theorem 5.43] and Higgins [10])

$$\cot(\pi x) = \lim_{n \to \infty} \frac{1}{\pi} \sum_{k=-n}^{n} \frac{1}{k + x} = \frac{1}{\pi x} + \frac{2x}{\pi} \sum_{k=1}^{\infty} \frac{1}{x^2 - k^2},$$

we deduce for $m \in \mathbb{N}$, by computing mth derivative with respect to x, the following compact expression for the bilateral series

$$\sum_{k=-\infty}^{\infty} \frac{1}{(k+x)^{m+1}} = \frac{(-1)^m \pi^{m+1}}{m!} P_m(\cot(\pi x)). \tag{1}$$

2. Evaluation of $\mathrm{U}_m(n)$ and $\mathcal{U}_m(n)$

In this section, we shall examine the series $\mathrm{U}_m(n)$ and $\mathcal{U}_m(n)$ by dividing them into multisection series. Two general summation theorems will be proved and several explicit formulae will be presented as consequences.

2.1. $\mathrm{U}_m(n)$

Since the series $\mathrm{U}_m(n)$ is absolutely convergent, in order to evaluate it explicitly, we can regroup its terms. For example $\mathrm{U}_1(n)$ and $\mathrm{U}_2(n)$ can be expressed as follows:

$$\begin{aligned}\mathrm{U}_1(n) &= \sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{(k+1)^{2n}} = 1-\frac{1}{2^{2n}}-\frac{1}{3^{2n}}+\frac{1}{4^{2n}}+\frac{1}{5^{2n}}-\frac{1}{6^{2n}}-\frac{1}{7^{2n}}+\frac{1}{8^{2n}}+\cdots\\ &=\Big\{1-\frac{1}{3^{2n}}+\frac{1}{5^{2n}}-\frac{1}{7^{2n}}+\cdots\Big\}-\Big\{\frac{1}{2^{2n}}-\frac{1}{4^{2n}}+\frac{1}{6^{2n}}-\frac{1}{8^{2n}}+\cdots\Big\},\end{aligned}$$

$$\begin{aligned}\mathrm{U}_2(n) &= \sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{k+2}{4}\rfloor}}{(k+1)^{2n}} = 1+\frac{1}{2^{2n}}-\frac{1}{3^{2n}}-\frac{1}{4^{2n}}-\frac{1}{5^{2n}}-\frac{1}{6^{2n}}+\frac{1}{7^{2n}}+\frac{1}{8^{2n}}+\frac{1}{9^{2n}}\\ &\qquad+\frac{1}{10^{2n}}-\frac{1}{11^{2n}}-\frac{1}{12^{2n}}-\frac{1}{13^{2n}}-\frac{1}{14^{2n}}+\frac{1}{15^{2n}}+\frac{1}{16^{2n}}+\cdots\\ &=\Big\{1-\frac{1}{5^{2n}}+\frac{1}{9^{2n}}-\frac{1}{13^{2n}}+\cdots\Big\}+\Big\{\frac{1}{2^{2n}}-\frac{1}{6^{2n}}+\frac{1}{10^{2n}}-\frac{1}{14^{2n}}+\cdots\Big\}\\ &\quad-\Big\{\frac{1}{3^{2n}}-\frac{1}{7^{2n}}+\frac{1}{11^{2n}}-\frac{1}{15^{2n}}+\cdots\Big\}-\Big\{\frac{1}{4^{2n}}-\frac{1}{8^{2n}}+\frac{1}{12^{2n}}-\frac{1}{16^{2n}}+\cdots\Big\}.\end{aligned}$$

In general, by dividing the summation index k according to the residue classes modulo $4m$, i.e., making the replacements $k \to 4km + j - 1$ with $1 \le j \le 4m$, we can express $\mathrm{U}_m(n)$ in terms of multisection series

$$\begin{aligned}\mathrm{U}_m(n) &= \sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{k+m}{2m}\rfloor}}{(k+1)^{2n}} = \sum_{j=1}^{4m}\sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{4km+m+j-1}{2m}\rfloor}}{(4km+j)^{2n}}\\ &=\sum_{j=1}^{m}\sum_{k=0}^{\infty}\Big\{\frac{(-1)^{\lfloor\frac{m+j-1}{2m}\rfloor}}{(4km+j)^{2n}}+\frac{(-1)^{\lfloor\frac{2m+j-1}{2m}\rfloor}}{(4km+m+j)^{2n}}\Big\}\\ &\quad+\sum_{j=1}^{m}\sum_{k=0}^{\infty}\Big\{\frac{(-1)^{\lfloor\frac{3m+j-1}{2m}\rfloor}}{(4km+2m+j)^{2n}}+\frac{(-1)^{\lfloor\frac{4m+j-1}{2m}\rfloor}}{(4km+3m+j)^{2n}}\Big\}.\end{aligned}$$

For $1 \le j \le m$, observing that

$$\begin{aligned}(-1)^{\lfloor\frac{m+j-1}{2m}\rfloor} &= (-1)^{\lfloor\frac{4m+j-1}{2m}\rfloor} = 1,\\ (-1)^{\lfloor\frac{2m+j-1}{2m}\rfloor} &= (-1)^{\lfloor\frac{3m+j-1}{2m}\rfloor} = -1;\end{aligned}$$

and then putting the summation terms corresponding to $j = m$ aside, we can reformulate further

$$\begin{aligned}\mathrm{U}_m(n) = &\sum_{k=0}^{\infty}\left\{\frac{(-1)^k}{(2km+m)^{2n}} - \frac{(-1)^k}{(2km+2m)^{2n}}\right\}\\ &+\sum_{j=1}^{m-1}\sum_{k=0}^{\infty}\left\{\frac{1}{(4km+j)^{2n}} - \frac{1}{(4km+m+j)^{2n}}\right\}\\ &+\sum_{j=1}^{m-1}\sum_{k=0}^{\infty}\left\{\frac{1}{(4km+3m+j)^{2n}} - \frac{1}{(4km+2m+j)^{2n}}\right\}.\end{aligned}$$

Writing the first line directly in $\beta(2n)$ and $\zeta(2n)$, then inverting simultaneously the summation indices by $j \to m-j$ and $k \to -k-1$ in the last line, we derive the following bilateral series expression

$$\begin{aligned}\mathrm{U}_m(n) = &\frac{\beta(2n)}{m^{2n}} - \frac{\zeta(2n)}{(2m)^{2n}}\left\{1-\frac{2}{4^n}\right\}\\ &+\sum_{j=1}^{m-1}\sum_{k=-\infty}^{\infty}\left\{\frac{1}{(4km+j)^{2n}} - \frac{1}{(4km+m+j)^{2n}}\right\}.\end{aligned}$$

By applying Eq (1), we find the simplified formula as in the following theorem.

Theorem 1 ($m, n \in \mathbb{N}$)**.**

$$\begin{aligned}\mathrm{U}_m(n) = \sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{k+m}{2m}\rfloor}}{(k+1)^{2n}} = &\frac{\beta(2n)}{m^{2n}} + \frac{\zeta(2n)}{(2m)^{2n}}\left\{\frac{2}{4^n}-1\right\}\\ &+(-1)^n\frac{4n\,\zeta(2n)}{4^{3n}m^{2n}B_{2n}}\sum_{j=1}^{m-1}\left\{P_{2n-1}\Big(\cot\big(\tfrac{j\pi}{4m}\big)\Big) - P_{2n-1}\Big(\cot\big(\tfrac{m+j}{4m}\pi\big)\Big)\right\}.\end{aligned}$$

For small $m, n \in \mathbb{N}$, this theorem can be utilized to compute exact values of $\mathrm{U}_m(n)$. The following interesting formulae are exhibited as examples.

Example 1.

$$\mathrm{U}_1(1) = \sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{(k+1)^2} = \beta(2) - \frac{\zeta(2)}{8},$$

$$\mathrm{U}_1(2) = \sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{(k+1)^4} = \beta(4) - \frac{7\zeta(4)}{128},$$

$$\mathrm{U}_1(3) = \sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{(k+1)^6} = \beta(6) - \frac{31\zeta(6)}{2048},$$

$$\mathrm{U}_1(4) = \sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{(k+1)^8} = \beta(8) - \frac{127\zeta(8)}{32768},$$

$$\mathrm{U}_1(5) = \sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{(k+1)^{10}} = \beta(10) - \frac{511\zeta(10)}{524288}.$$

Example 2.

$$\begin{aligned}
\mathrm{U}_2(1) &= \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+2}{4} \rfloor}}{(k+1)^2} = \frac{\beta(2)}{4} + \frac{3\zeta(2)}{4\sqrt{2}} - \frac{\zeta(2)}{32},\\
\mathrm{U}_2(2) &= \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+2}{4} \rfloor}}{(k+1)^4} = \frac{\beta(4)}{16} + \frac{165\zeta(4)}{128\sqrt{2}} - \frac{7\zeta(4)}{2048},\\
\mathrm{U}_2(3) &= \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+2}{4} \rfloor}}{(k+1)^6} = \frac{\beta(6)}{64} + \frac{22743\zeta(6)}{16384\sqrt{2}} - \frac{31\zeta(6)}{131072},\\
\mathrm{U}_2(4) &= \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+2}{4} \rfloor}}{(k+1)^6} = \frac{\beta(8)}{256} + \frac{369165\zeta(8)}{262144\sqrt{2}} - \frac{127\zeta(8)}{8388608},\\
\mathrm{U}_2(5) &= \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+2}{4} \rfloor}}{(k+1)^{10}} = \frac{\beta(10)}{1024} + \frac{94810353\zeta(10)}{67108864\sqrt{2}} - \frac{511\zeta(10)}{536870912}.
\end{aligned}$$

Example 3.

$$\begin{aligned}
\mathrm{U}_3(1) &= \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+3}{6} \rfloor}}{(k+1)^2} = \frac{\beta(2)}{9} + \frac{\zeta(2)}{\sqrt{3}} + \frac{7\zeta(2)}{72},\\
\mathrm{U}_3(2) &= \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+3}{6} \rfloor}}{(k+1)^4} = \frac{\beta(4)}{81} + \frac{115\zeta(4)}{72\sqrt{3}} + \frac{553\zeta(4)}{10368},\\
\mathrm{U}_3(3) &= \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+3}{6} \rfloor}}{(k+1)^6} = \frac{\beta(6)}{729} + \frac{11767\zeta(6)}{6912\sqrt{3}} + \frac{22537\zeta(6)}{1492992},\\
\mathrm{U}_3(4) &= \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+3}{6} \rfloor}}{(k+1)^8} = \frac{\beta(8)}{6561} + \frac{1287715\zeta(8)}{746496\sqrt{3}} + \frac{832993\zeta(8)}{214990848},\\
\mathrm{U}_3(5) &= \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+3}{6} \rfloor}}{(k+1)^{10}} = \frac{\beta(10)}{59049} + \frac{744010091\zeta(10)}{429981696\sqrt{3}} + \frac{30173017\zeta(10)}{30958682112}.
\end{aligned}$$

2.2. $\mathcal{U}_m(n)$

Performing the replacement $k \to 4km + j - 1$ with $1 \le j \le 4m$, we can express $\mathcal{U}_m(n)$ as the following multisection series

$$\begin{aligned}
\mathcal{U}_m(n) &= \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+m}{2m} \rfloor}}{(2k+1)^{2n}} = \sum_{k=0}^{\infty} \sum_{j=1}^{4m} \frac{(-1)^{\lfloor \frac{m+j-1}{2m} \rfloor}}{(8km+2j-1)^{2n}}\\
&= \sum_{k=0}^{\infty} \left\{ \sum_{j=1}^{m} \frac{(-1)^{\lfloor \frac{m+j-1}{2m} \rfloor}}{(8km+2j-1)^{2n}} + \sum_{j=1}^{m} \frac{(-1)^{\lfloor \frac{2m+j-1}{2m} \rfloor}}{(8km+2m+2j-1)^{2n}} \right.\\
&\quad \left. + \sum_{j=1}^{m} \frac{(-1)^{\lfloor \frac{3m+j-1}{2m} \rfloor}}{(8km+4m+2j-1)^{2n}} + \sum_{j=1}^{m} \frac{(-1)^{\lfloor \frac{4m+j-1}{2m} \rfloor}}{(8km+6m+2j-1)^{2n}} \right\}.
\end{aligned}$$

The above expression can be further reduced as follows:

$$\begin{aligned}\mathcal{U}_m(n) &= \sum_{k=0}^{\infty}\sum_{j=1}^{m}\left\{\frac{1}{(8km+2j-1)^{2n}}-\frac{1}{(8km+2m+2j-1)^{2n}}\right\}\\ &+\sum_{k=0}^{\infty}\sum_{j=1}^{m}\left\{\frac{1}{(8km+6m+2j-1)^{2n}}-\frac{1}{(8km+4m+2j-1)^{2n}}\right\}.\end{aligned}$$

For the two sums on the ultimate line, making the changes on summation indices by $j \to 1+m-j$ and $k \to -k-1$

$$\sum_{k=-\infty}^{-1}\sum_{j=1}^{m}\left\{\frac{1}{(8km+2j-1)^{2n}}-\frac{1}{(8km+2m+2j-1)^{2n}}\right\}$$

and then unifying this resulting expression to the first line, we find the bilateral series expression

$$\begin{aligned}\mathcal{U}_m(n) &= \sum_{j=1}^{m}\sum_{k=-\infty}^{\infty}\left\{\frac{1}{(8km+2j-1)^{2n}}-\frac{1}{(8km+2m+2j-1)^{2n}}\right\}\\ &= \sum_{j=1}^{m}\frac{1}{(8m)^{2n}}\sum_{k=-\infty}^{\infty}\left\{\frac{1}{(k+\frac{2j-1}{8m})^{2n}}-\frac{1}{(k+\frac{2m+2j-1}{8m})^{2n}}\right\}.\end{aligned}$$

By invoking Eq (1), we establish the following summation formula.

Theorem 2 ($m, n \in \mathbb{N}$).

$$\begin{aligned}\mathcal{U}_m(n) = \sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{k+m}{2m}\rfloor}}{(2k+1)^{2n}} &= (-1)^n\frac{4n\,\zeta(2n)}{4^{4n}m^{2n}B_{2n}}\\ &\times\sum_{j=1}^{m}\left\{P_{2n-1}\Big(\cot\big(\tfrac{2j-1}{8m}\pi\big)\Big)-P_{2n-1}\Big(\cot\big(\tfrac{2m+2j-1}{8m}\pi\big)\Big)\right\}.\end{aligned}$$

For $m = 1, 2, 3$, we record the following closed formulae as applications.

Example 4 ($m = 1$).

$$\begin{aligned}\mathcal{U}_1(1) &= \sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{(2k+1)^{2}} = \frac{3\zeta(2)}{4\sqrt{2}},\\ \mathcal{U}_1(2) &= \sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{(2k+1)^{4}} = \frac{165\zeta(4)}{128\sqrt{2}},\\ \mathcal{U}_1(3) &= \sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{(2k+1)^{6}} = \frac{22743\zeta(6)}{16384\sqrt{2}},\\ \mathcal{U}_1(4) &= \sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{(2k+1)^{8}} = \frac{369165\zeta(8)}{262144\sqrt{2}},\\ \mathcal{U}_1(5) &= \sum_{k=0}^{\infty}\frac{(-1)^{\lfloor\frac{k+1}{2}\rfloor}}{(2k+1)^{10}} = \frac{94810353\zeta(10)}{67108864\sqrt{2}}.\end{aligned}$$

Example 5 ($m = 2$)**.**

$$\mathcal{U}_2(1) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+2}{4} \rfloor}}{(2k+1)^2} = \frac{3\zeta(2)}{16} \sqrt{10 + \sqrt{2}},$$

$$\mathcal{U}_2(2) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+2}{4} \rfloor}}{(2k+1)^4} = \frac{15\zeta(4)}{2048} \sqrt{15898 + 241\sqrt{2}},$$

$$\mathcal{U}_2(3) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+2}{4} \rfloor}}{(2k+1)^6} = \frac{63\zeta(6)}{1048576} \sqrt{267655210 + 494881\sqrt{2}},$$

$$\mathcal{U}_2(4) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+2}{4} \rfloor}}{(2k+1)^8} = \frac{15\zeta(8)}{67108864} \sqrt{19853759178298 + 4207474321\sqrt{2}},$$

$$\mathcal{U}_2(5) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+2}{4} \rfloor}}{(2k+1)^{10}} = \frac{33\zeta(10)}{68719476736} \sqrt{4327811705145000010 + 103023433654081\sqrt{2}}.$$

Example 6 ($m = 3$)**.**

$$\mathcal{U}_3(1) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+3}{6} \rfloor}}{(2k+1)^2} = \frac{\zeta(2)}{24} \sqrt{218 + 24\sqrt{3}},$$

$$\mathcal{U}_3(2) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+3}{6} \rfloor}}{(2k+1)^4} = \frac{\zeta(4)}{6912} \sqrt{1635146 + 22968\sqrt{3}},$$

$$\mathcal{U}_3(3) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+3}{6} \rfloor}}{(2k+1)^6} = \frac{\zeta(6)}{2654208} \sqrt{138927209978 + 219727704\sqrt{3}},$$

$$\mathcal{U}_3(4) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+3}{6} \rfloor}}{(2k+1)^8} = \frac{\zeta(8)}{1146617856} \sqrt{52163307935186666 + 9179017200888\sqrt{3}},$$

$$\mathcal{U}_3(5) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k+3}{6} \rfloor}}{(2k+1)^{10}} = \frac{\zeta(10)}{2641807540224} \sqrt{57564351813037136458778 + 1125649024023711384\sqrt{3}}.$$

3. Evaluation of $\mathrm{V}_m(n)$ and $\mathcal{V}_m(n)$

This section will be devoted to the two remaining series $\mathrm{V}_m(n)$ and $\mathcal{V}_m(n)$. We shall establish two general summation theorems. Several explicit formulae in terms of the Riemann zeta function and the Dirichlet beta function will be deduced as particular cases.

3.1. $\mathrm{V}_m(n)$

Making the replacement $k \to km + j - 1$ with $1 \le j \le m$, we can rewrite $\mathrm{V}_m(n)$ as

$$\mathrm{V}_m(n) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{m} \rfloor}}{(k+1)^{2n-1}} = \sum_{k=0}^{\infty} \sum_{j=1}^{m} \frac{(-1)^{\lfloor \frac{km+j-1}{m} \rfloor}}{(km+j)^{2n-1}}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{km+m-1}{m} \rfloor}}{(km+m)^{2n-1}} + \sum_{k=0}^{\infty} \sum_{j=1}^{m-1} \frac{(-1)^{\lfloor \frac{km+j-1}{m} \rfloor}}{(km+j)^{2n-1}}.$$

We remark that the above equalities are valid for $n \geq 1$. When $n = 1$, the corresponding formula in Theorem 3 can be shown analogously by examining the truncated sum below

$$\mathrm{V}_m(1) = \lim_{n\to\infty}\sum_{k=0}^{n}\frac{(-1)^{\lfloor \frac{k}{m}\rfloor}}{k+1}.$$

The first series in the last line can be evaluated directly as follows:

$$\sum_{k=0}^{\infty}\frac{(-1)^{\lfloor \frac{km+m-1}{m}\rfloor}}{(km+m)^{2n-1}} = \frac{1}{m^{2n-1}}\sum_{k=0}^{\infty}\frac{(-1)^k}{(k+1)^{2n-1}} = \begin{cases}\dfrac{\ln 2}{m}, & n=1;\\ \dfrac{2^{2n-1}-2}{(2m)^{2n-1}}\zeta(2n-1), & n>1.\end{cases}$$

According to the parity of k, the double series can be split into two

$$\begin{aligned}\sum_{k=0}^{\infty}\sum_{j=1}^{m-1}\frac{(-1)^{\lfloor \frac{km+j-1}{m}\rfloor}}{(km+j)^{2n-1}} &= \sum_{k=0}^{\infty}\sum_{j=1}^{m-1}\frac{(-1)^k}{(km+j)^{2n-1}}\\ &= \sum_{k=0}^{\infty}\sum_{j=1}^{m-1}\frac{1}{(2km+j)^{2n-1}} - \sum_{k=0}^{\infty}\sum_{j=1}^{m-1}\frac{1}{(2km+m+j)^{2n-1}}.\end{aligned}$$

Now making the replacements on summation indices by $j \to m-j$ and $k \to -k-1$ in the latter double sum and then unifying it to the former, we can simplify the resulting expression into the following bilateral series:

$$\sum_{k=0}^{\infty}\sum_{j=1}^{m-1}\frac{(-1)^{\lfloor \frac{km+j-1}{m}\rfloor}}{(km+j)^{2n-1}} = \sum_{j=1}^{m-1}\frac{1}{(2m)^{2n-1}}\sum_{k=-\infty}^{\infty}\frac{1}{(k+\frac{j}{2m})^{2n-1}}.$$

Finally, by making use of (1), we can express the rightmost bilateral series in terms of Dirichlet beta function:

$$\begin{aligned}\sum_{k=-\infty}^{\infty}\frac{1}{(k+\frac{j}{2m})^{2n-1}} &= \frac{\pi^{2n-1}}{(2n-2)!}P_{2n-2}\Big(\cot\Big(\frac{j\pi}{2m}\Big)\Big)\\ &= (-1)^{n-1}\frac{4^n\beta(2n-1)}{E_{2n-2}}P_{2n-2}\Big(\cot\Big(\frac{j\pi}{2m}\Big)\Big).\end{aligned}$$

Summing up, we have proved the following theorem.

Theorem 3 ($m, n \in \mathbb{N}$).

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^{\lfloor \frac{k}{m}\rfloor}}{(k+1)^{2n-1}} &= (-1)^{n-1}\frac{2\beta(2n-1)}{m^{2n-1}E_{2n-2}}\sum_{j=1}^{m-1}P_{2n-2}\Big(\cot\Big(\frac{j\pi}{2m}\Big)\Big)\\ &\quad+\begin{cases}\dfrac{\ln 2}{m}, & n=1;\\ \dfrac{2^{2n-1}-2}{(2m)^{2n-1}}\zeta(2n-1), & n\geq 2.\end{cases}\end{aligned}$$

When $m = 1$, it is trivial to check that

$$\mathrm{V}_1(n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k+1)^{2n-1}} = \begin{cases} \ln 2, & n = 1; \\ (1 - 4^{1-n})\zeta(2n-1), & n \geq 2. \end{cases}$$

For $m = 2, 3$ and $1 \leq n \leq 5$, the closed formulae are highlighted as examples.

Example 7 ($m = 2$)**.**

$$\mathrm{V}_2(1) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{2} \rfloor}}{k+1} = \beta(1) + \frac{\ln 2}{2},$$
$$\mathrm{V}_2(2) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{2} \rfloor}}{(k+1)^3} = \beta(3) + \frac{3}{32}\zeta(3),$$
$$\mathrm{V}_2(3) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{2} \rfloor}}{(k+1)^5} = \beta(5) + \frac{15}{512}\zeta(5),$$
$$\mathrm{V}_2(4) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{2} \rfloor}}{(k+1)^7} = \beta(7) + \frac{63}{8192}\zeta(7),$$
$$\mathrm{V}_2(5) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{2} \rfloor}}{(k+1)^9} = \beta(9) + \frac{255}{131072}\zeta(7).$$

Example 8 ($m = 3$)**.**

$$\mathrm{V}_3(1) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{3} \rfloor}}{k+1} = \frac{8\beta(1)}{3\sqrt{3}} + \frac{\ln 2}{3},$$
$$\mathrm{V}_3(2) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{3} \rfloor}}{(k+1)^3} = \frac{160\beta(3)}{81\sqrt{3}} + \frac{\zeta(3)}{36},$$
$$\mathrm{V}_3(3) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{3} \rfloor}}{(k+1)^5} = \frac{2176\beta(5)}{1215\sqrt{3}} + \frac{5\zeta(5)}{1296},$$
$$\mathrm{V}_3(4) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{3} \rfloor}}{(k+1)^7} = \frac{232960\beta(7)}{133407\sqrt{3}} + \frac{7\zeta(7)}{15552},$$
$$\mathrm{V}_3(5) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{3} \rfloor}}{(k+1)^9} = \frac{425805824\beta(9)}{245348595\sqrt{3}} + \frac{85\zeta(9)}{1679616}.$$

3.2. $\mathcal{V}_m(n)$

Under the replacements $k \to 2km + j - 1$ with $1 \leq j \leq 2m$, for $n > 1$, we can rewrite $\mathcal{V}_m(n)$ as the following multisection series

$$
\begin{aligned}
\mathcal{V}_m(n) &= \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{m} \rfloor}}{(2k+1)^{2n-1}} = \sum_{k=0}^{\infty} \sum_{j=1}^{2m} \frac{(-1)^{\lfloor \frac{j-1}{m} \rfloor}}{(4km+2j-1)^{2n-1}} \\
&= \sum_{k=0}^{\infty} \left\{ \sum_{j=1}^{m} \frac{(-1)^{\lfloor \frac{j-1}{m} \rfloor}}{(4km+2j-1)^{2n-1}} + \sum_{j=1}^{m} \frac{(-1)^{\lfloor \frac{m+j-1}{m} \rfloor}}{(4km+2m+2j-1)^{2n-1}} \right\} \\
&= \sum_{j=1}^{m} \sum_{k=0}^{\infty} \frac{1}{(4km+2j-1)^{2n-1}} - \sum_{j=1}^{m} \sum_{k=0}^{\infty} \frac{1}{(4km+2m+2j-1)^{2n-1}}.
\end{aligned}
$$

When $n = 1$, the corresponding formula in Theorem 4 can be confirmed by examining the following truncated sum

$$
\mathcal{V}_m(1) = \lim_{\ell \to \infty} \sum_{k=0}^{\ell} \frac{(-1)^{\lfloor \frac{k}{m} \rfloor}}{2k+1}.
$$

Now, making the replacements $j \to m+1-j$ and $k \to -k-1$ simultaneously for the latter double sums and then unifying it to the former one, we obtain the bilateral series expression below

$$
\begin{aligned}
\mathcal{V}_m(n) &= \sum_{k=0}^{\infty} \sum_{j=1}^{m} \frac{1}{(4km+2j-1)^{2n-1}} + \sum_{k=-\infty}^{-1} \sum_{j=1}^{m} \frac{1}{(4km+2j-1)^{2n-1}} \\
&= \sum_{j=1}^{m} \sum_{k=-\infty}^{\infty} \frac{1}{(4km+2j-1)^{2n-1}} = \sum_{j=1}^{m} \frac{1}{(4m)^{2n-1}} \sum_{k=-\infty}^{\infty} \frac{1}{(k+\frac{2j-1}{4m})^{2n-1}}.
\end{aligned}
$$

Keeping in mind (1) and writing the rightmost sum in terms of the Dirichlet beta function

$$
\begin{aligned}
\sum_{k=-\infty}^{\infty} \frac{1}{(k+\frac{2j-1}{4m})^{2n-1}} &= \frac{\pi^{2n-1}}{(2n-2)!} P_{2n-2}\Big(\cot\Big(\frac{2j-1}{4m}\pi\Big)\Big) \\
&= \frac{(-1)^{n-1} 4^n \beta(2n-2)}{E_{2n-2}} P_{2n-2}\Big(\cot\Big(\frac{2j-1}{4m}\pi\Big)\Big),
\end{aligned}
$$

we arrive at the compact expression as in the following theorem.

Theorem 4 ($m, n \in \mathbb{N}$)**.**

$$
\sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{m} \rfloor}}{(2k+1)^{2n-1}} = \frac{\beta(2n-1)}{m^{2n-1} E_{2n-2}} \Big(\frac{-1}{4}\Big)^{n-1} \sum_{j=1}^{m} P_{2n-2}\Big(\cot\Big(\frac{2j-1}{4m}\pi\Big)\Big).
$$

When $m = 1$, it is obvious to see that

$$
\mathcal{V}_1(n) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n-1}} = \beta(2n-1).
$$

For $m = 2, 3$, the corresponding closed formulae are displayed as follows.

Example 9 ($m = 2$)**.**

$$\mathcal{V}_2(1) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{2} \rfloor}}{2k+1} = \sqrt{2}\beta(1),$$

$$\mathcal{V}_2(2) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{2} \rfloor}}{(2k+1)^3} = \frac{3\beta(3)}{2\sqrt{2}},$$

$$\mathcal{V}_2(3) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{2} \rfloor}}{(2k+1)^5} = \frac{57\beta(5)}{40\sqrt{2}},$$

$$\mathcal{V}_2(4) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{2} \rfloor}}{(2k+1)^7} = \frac{2763\beta(7)}{1952\sqrt{2}},$$

$$\mathcal{V}_2(5) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{2} \rfloor}}{(2k+1)^9} = \frac{250737\beta(9)}{177280\sqrt{2}}.$$

Example 10 ($m = 3$)**.**

$$\mathcal{V}_3(1) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{3} \rfloor}}{2k+1} = \frac{5\beta(1)}{3},$$

$$\mathcal{V}_3(2) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{3} \rfloor}}{(2k+1)^3} = \frac{29\beta(3)}{27},$$

$$\mathcal{V}_3(3) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{3} \rfloor}}{(2k+1)^5} = \frac{245\beta(5)}{243},$$

$$\mathcal{V}_3(4) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{3} \rfloor}}{(2k+1)^7} = \frac{2189\beta(7)}{2187},$$

$$\mathcal{V}_3(5) = \sum_{k=0}^{\infty} \frac{(-1)^{\lfloor \frac{k}{3} \rfloor}}{(2k+1)^9} = \frac{19685\beta(9)}{19683}.$$

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

1. T. M. Apostol, Zeta and related functions, in *NIST Handbook of Mathematical Functions*, Cambridge University Press, Cambridge, (2010), 601–616.
2. G. Bretti, C. Cesarano, P. E. Ricci, Laguerre-type exponentials and generalized Appell polynomials, *Comput. Math. Appl.*, **48** (2004), 833–839. https://doi.org/10.1016/j.camwa.2003.09.031

3. E. Guariglia, Fractional calculus, zeta functions and Shannon entropy, *Open Math.* **19** (2021), 87–100. https://doi.org/10.1515/math-2021-0010
4. S. D. Lin, H. M. Srivastava, Some families of the Hurwitz-Lerch Zeta functions and associated fractional derivative and other integral representations, *Appl. Math. Comput.*, **154** (2004), 725–733. https://doi.org/10.1016/S0096-3003(03)00746-X
5. W. Ramírez, D. Bedoya, A. Urieles, C. cesarano, M. Ortega, New U-Bernoulli, U-Euler and U-Genocchi polynomials and their matrices, *Carpathian Math. Publ.*, **15** (2023), 449–467. https://doi.org/10.15330/cmp.15.2.449-467
6. R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete mathematics*, Addison-Wesley Professional, Massachusetts, 1989.
7. K. N. Boyadzhiev, Derivative polynomials for tanh, tan, sech and sec in explicit form, *Fibonacci Quart.*, **45** (2007), 291–303.
8. W. Chu, C. Wang, Convolution formulae for Bernoulli numbers, *Integr. Transf. Spec. Funct.*, **21** (2010), 437–457. https://doi.org/10.1080/10652460903360861
9. K. Stromberg, *An Introduction to Classical Real Analysis*, American Mathematical Society, California, 1981.
10. J. R. Higgins, Two basic formulae of Euler and their equivalence to Tschakalov's sampling theorem, *Sampl. Theory Signal Image Process.*, **2** (2003), 259–270. https://doi.org/10.1007/BF03549398

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