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Research article

Unconditional well-posedness for the periodic Boussinesq and Kawahara equations

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Abstract: In this article, we obtain new results on the unconditional well-posedness for a pair of periodic nonlinear dispersive equations using an abstract framework introduced by Kishimoto. This framework is based on a normal form reductions argument coupled with a number of crucial multilinear estimates.

Keywords: Boussinesq equation; Kawahara equation; well-posedness; unconditional uniqueness; normal form

1. Introduction

In this paper, we study the periodic Cauchy problems

$$\begin{cases} \partial_t^2 u - \partial_x^2 u + \partial_x^4 u + \partial_x^2(u^2) = 0, & u = u(t,x) : \mathbb{R} \times \mathbb{T} \to \mathbb{R}, \\ u(0,x) = u_0(x),\ \partial_t u(0,x) = u_1(x), & (u_0, u_1) \in H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T}), \end{cases} \tag{1.1}$$

and

$$\begin{cases} \partial_t u + \beta \partial_x^3 u - \partial_x^5 u + \partial_x(u^2) = 0, & u = u(t,x) : \mathbb{R} \times \mathbb{T} \to \mathbb{R}, \\ u(0,x) = u_0(x), \qquad u_0 \in H^s(\mathbb{T}), \end{cases} \tag{1.2}$$

where

$$u(t,0) = u(t,2\pi), \qquad \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}, \quad \text{and} \quad \beta = 0, 1, \text{ or } -1.$$

The first partial differential equation is called *the "good" Boussinesq equation* and it is known to describe electromagnetic waves in nonlinear dielectrics [1]. When the quadratic nonlinearity is replaced by $4u^3 - 6u^5$, the resulting equation was, in fact, derived in the context of shape-memory alloys [2]. The "good" Boussinesq equation can be seen as an improved frequency dispersion version of the original water wave equation derived by Boussinesq [3],

$$\partial_t^2 u - gh\,\partial_x^2 u - gh\,\partial_x^2 \left\{ \frac{3}{2h}u^2 + \frac{h^2}{3}\partial_x^2 u \right\} = 0,$$

where g is the gravitational acceleration, h is undisturbed depth of the channel where the water waves are observed, and $u = u(t,x)$ is the elevation of the water surface. This equation admits solitary wave solutions of the form

$$u(t,x) = a\,\mathrm{sech}^2\left[\sqrt{\frac{3a}{h^3}}\,(x \pm ct) \right],$$

where a and c are suitably chosen constants, and many consider this to be the place where the stability theory of solitary wave solutions started. For a detailed review of various mathematical and physical aspects concerning Boussinesq equations and their generalizations, we refer the reader to the excellent review article by Makhankov [4].

The partial differential equation in the second Cauchy problem is referred to as *the Kawahara equation* and it represents a scalar approximation for the full water wave equations in the shallow water regime, when one also takes into account the surface tension. For the derivation of this type of equations, one uses three non-dimensional parameters: δ and ϵ which quantize the dispersive and nonlinear effects, and μ, also known as the Bond number. Shallow water regime corresponds to $\delta \ll 1$ and, if $\epsilon = \delta^2$ and $\mu \neq 1/3$, then one obtains

$$\pm\,\partial_t u + \left(\frac{1}{6} - \frac{\mu}{2} \right)\partial_x^3 u + \frac{3}{4}\partial_x(u^2) = 0,$$

which is the well-known KdV equation. On the other hand, if $\epsilon = \delta^4$ and $\mu = 1/3 + \nu\epsilon^{1/2}$, then one arrives at the Kawahara equation

$$\pm\,\partial_t u - \frac{\nu}{2}\partial_x^3 u + \frac{1}{90}\partial_x^5 u + \frac{3}{4}\partial_x(u^2) = 0.$$

This equation was first derived by Kakutani and Ono [5] to describe nonlinear hydromagnetic waves in plasmas and by Hasimoto [6] precisely in the context of shallow water waves with surface tension and Bond number close to $1/3$. For an extensive mathematical and physical perspective on the Kawahara equation, we ask the interested reader to consult the impressive monograph [7] by Lannes.

The well-posedness (WP) of (1.1) and (1.2) has received considerable interest, with the Boussinesq problem being the subject of works by Fang and Grillakis [8], Farah and Scialom [9], Oh and Stefanov [10], Kishimoto [11], Geba et al. [12], and Okamoto [13]. For the Kawahara problem, important contributions were made by Hirayama [14], Kato [15], and Okamoto [13]. The articles by Kishimoto [11] and Kato [15] are of particular significance. The former proves that (1.1) is locally WP for $s \geq -1/2$ and is ill-posed for $s < -1/2$, while the latter shows that (1.2) is locally WP for $s \geq -3/2$ and is ill-posed for $s < -3/2$. A parallel, more comprehensive literature addresses the corresponding non-periodic Cauchy problems.

When studying the WP of (1.1) and (1.2), the natural solution spaces, also called *Hadamard spaces*, are

$$X = C(I; H^s(\mathbb{T})) \cap C^1(I; H^{s-2}(\mathbb{T}))$$

and

$$X = C(I; H^s(\mathbb{T}))$$

respectively, where $I \subseteq \mathbb{R}$ is a time interval with $0 \in I$. Yet, in the previously listed WP references, uniqueness of solutions to either (1.1) or (1.2) is only attained in a proper subset of X. This has the format $X \cap Y$, where Y is an additional functional space. It is then natural to ask under what conditions these solutions become unique in their full Hadamard spaces. This is what is called the studying of *unconditional uniqueness* or, by extension, *unconditional well-posedness* (UWP) for these Cauchy problems. In this direction, our article provides the following results:

Theorem 1.1. *UWP of solutions to the Cauchy problem* (1.1) *holds for data in* $H^s(\mathbb{T}) \times H^{s-2}(\mathbb{T})$ *when* $s > 0$.

Theorem 1.2. *UWP of solutions to the Cauchy problem* (1.2) *holds for data in* $H^s(\mathbb{T})$ *when* $s > 1/4$.

The subject of UWP for nonlinear dispersive equations has produced an impressive amount of research for the last 30 years, arguably starting with Kato's work [16] on nonlinear Schrödinger equations. In the same context, we mention the recent seminal papers of Kwon et al. [17] and Kishimoto [18] addressing fairly generic dispersive problems. Specifically for periodic problems, a selection of notable results consists of the ones obtained by Babin et al. [19], Kwon and Oh [20], Guo et al. [21], Kishimoto [22], and Kato and Tsugawa [23]. For comprehensive discussions on UWP, as well as access to extensive bibliographies on the subject, we refer the reader to [17, 18].

Related to our results, we first recall that UWP in the non-periodic case was previously obtained by Farah [24] for the "good" Boussinesq equation and by Geba and Lin [25] for the Kawahara equation, both for data in $L^2(\mathbb{R})$. To our knowledge, prior to this paper, there are no UWP arguments in the literature for either (1.1) or (1.2). For both of the sharp locally WP results mentioned before (i.e., Kishimoto [11] for (1.1) and Kato [15] for (1.2)), uniqueness of solutions is derived only in proper subsets of their corresponding Hadamard space. Moreover, it is important to note that the question of what the optimal Sobolev index needs to be in order to guarantee UWP is a fairly delicate issue, as discussed in [17, 18]. Current methodologies and previous results on related dispersive equations suggest that $s = 0$ should be the right value for both (1.1) and (1.2).

In proving Theorems 1.1 and 1.2, we apply an abstract framework developed by Kishimoto [18] in the context of nonlinear dispersive equations. This has been successfully used in obtaining UWP for periodic Cauchy problems associated to nonlinear Schrödinger equations [26], the Benjamin-Ono equation [22], and the modified Benjamin-Ono equation [27]. At the heart of this method lies a critical set of multilinear bounds, which are used to derive even more complex multilinear estimates. The latter represent the key elements in an infinite iteration scheme of normal form reductions proving UWP for the Cauchy problem under consideration.

In concluding this section, we want to emphasize that one of the goals of this paper is to advocate for the simplicity and the robustness of Kishimoto's method. These two features allowed us to keep the argument quite concise and, in our opinion, very transparent. Of course, one could argue that, at least for (1.2), our result is likely not optimal, since the Cauchy problem for the KdV equation (which enjoys weaker dispersion when compared to the Kawahara equation) is UWP in $L^2(\mathbb{T})$, as proven in [19] through finitely many normal form reductions. Again, our aim is to present a streamlined approach to UWP questions, which caters to a large audience.

2. Preliminaries

2.1. Basic notational conventions and terminology

First, we write $A \lesssim B$ to stand for $A \leq CB$, where $C > 0$ is a constant varying from line to line and depending on various fixed parameters. Next, we ask that $A \sim B$ denotes that both $A \lesssim B$ and $B \lesssim A$ are valid. We also let $A \ll B$ signify that $A \leq \epsilon B$ for some small absolute constant $\epsilon > 0$.

Secondly, for a function $f = f(x)$ defined on $\mathbb{T}$, its Fourier series is given by

$$f(x) = \sum_{k\in\mathbb{Z}} f_k \, e^{ikx}, \qquad f_k = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ikx} f(x)\, dx, \quad \forall\, k \in \mathbb{Z}. \tag{2.1}$$

In similar fashion, for $v = v(t,x)$ defined on $\mathbb{R}\times\mathbb{T}$, we write

$$v(t,x) = \sum_{k\in\mathbb{Z}} v_k(t)\, e^{ikx}, \qquad v_k(t) = \frac{1}{2\pi}\int_{\mathbb{T}} e^{-ikx} v(t,x)\, dx, \quad \forall\, k \in \mathbb{Z}. \tag{2.2}$$

Finally, for $s \in \mathbb{R}$, we will operate with the space

$$l^2_s = l^2_s(\mathbb{Z}) = \{\omega = (\omega_k)_k; (\langle k\rangle^s \omega_k)_k \in l^2(\mathbb{Z})\}, \quad \|\omega\|^2_{l^2_s} = \sum_{k\in\mathbb{Z}} \langle k\rangle^{2s} |\omega_k|^2,$$

where $\langle k\rangle = (1+|k|^2)^{1/2}$.

2.2. Adapted Kishimoto's methodology

Here, we present a version of Kishimoto's framework specifically suited to be applied to our two Cauchy problems, (1.1) and (1.2).

In the case of (1.2), we consider the generic Cauchy problem

$$\begin{cases} \partial_t w_k(t) = \displaystyle\sum_{k_1+k_2=k} e^{it\varphi(k,k_1,k_2)} m(k,k_1,k_2)\, w_{k_1}(t)\, w_{k_2}(t), \\ w(0) = w(0) \in l^2_s, \end{cases} \tag{2.3}$$

and we work with the following version of Theorem 1.1 in [18]:

Theorem 2.1. *The Cauchy problem* (2.3) *has at most one solution*

$$w \in C([0,T]; l^2_s)$$

if, for some $0 < \delta < 1/2$*,*

$$\left\| \sum_{k_1+k_2=k} \frac{|m|}{\langle\varphi\rangle^{1/2}}\, y_{k_1}\, z_{k_2} \right\|_{l^2_s} \lesssim \|y\|_{l^2_s}\|z\|_{l^2_s}, \tag{2.4}$$

$$\left\| \sum_{k_1+k_2=k} \frac{|m|}{\langle\varphi\rangle^{1-\delta}}\, y_{k_1}\, z_{k_2} \right\|_{l^2_{s-2}} \lesssim \min\{\|y\|_{l^2_{s-2}}\|z\|_{l^2_s},\, \|y\|_{l^2_s}\|z\|_{l^2_{s-2}}\}, \tag{2.5}$$

$$\left\| \sum_{k_1+k_2=k} |m|\, y_{k_1}\, z_{k_2} \right\|_{l^2_{s-2}} \lesssim \|y\|_{l^2_s}\|z\|_{l^2_s} \tag{2.6}$$

hold true.

For (1.1), we look at the generic Cauchy problem

$$
\begin{cases}
\partial_t v_k(t) = \displaystyle\sum_{k_1+k_2=k} e^{it\varphi_{11}(k,k_1,k_2)} m_{11}(k,k_1,k_2)\, v_{k_1}(t)\, v_{k_2}(t) \\
\qquad + \displaystyle\sum_{k_1+k_2=k} e^{it\varphi_{12}(k,k_1,k_2)} m_{12}(k,k_1,k_2)\, v_{k_1}(t)\, w_{k_2}(t) \\
\qquad + \displaystyle\sum_{k_1+k_2=k} e^{it\varphi_{13}(k,k_1,k_2)} m_{13}(k,k_1,k_2)\, w_{k_1}(t)\, v_{k_2}(t) \\
\qquad + \displaystyle\sum_{k_1+k_2=k} e^{it\varphi_{14}(k,k_1,k_2)} m_{14}(k,k_1,k_2)\, w_{k_1}(t)\, w_{k_2}(t) \\
\qquad + R_1[v,w]_k, \\
\partial_t w_k(t) = \displaystyle\sum_{k_1+k_2=k} e^{it\varphi_{21}(k,k_1,k_2)} m_{21}(k,k_1,k_2)\, v_{k_1}(t)\, v_{k_2}(t) \\
\qquad + \displaystyle\sum_{k_1+k_2=k} e^{it\varphi_{22}(k,k_1,k_2)} m_{22}(k,k_1,k_2)\, v_{k_1}(t)\, w_{k_2}(t) \\
\qquad + \displaystyle\sum_{k_1+k_2=k} e^{it\varphi_{23}(k,k_1,k_2)} m_{23}(k,k_1,k_2)\, w_{k_1}(t)\, v_{k_2}(t) \\
\qquad + \displaystyle\sum_{k_1+k_2=k} e^{it\varphi_{24}(k,k_1,k_2)} m_{24}(k,k_1,k_2)\, w_{k_1}(t)\, w_{k_2}(t) \\
\qquad + R_2[v,w]_k, \\
(v(0),w(0)) = (v_0,w_0) \in l^2_s \times l^2_s.
\end{cases}
\tag{2.7}
$$

In Section 2.5 of [18], Kishimoto discusses how his scheme can be adapted to abstract systems like the one above, and we take as a working version the following result:

Theorem 2.2. *The Cauchy problem* (2.7) *has at most one solution*

$$(v,w) \in C([0,T]; l^2_s \times l^2_s)$$

if

$$\|R_1[y,z]\|_{C([0,T];l^2_s)} + \|R_2[y,z]\|_{C([0,T];l^2_s)} \le C(\|y,z\|_{C([0,T];l^2_s)}), \tag{2.8}$$

$$
\begin{aligned}
&\|R_1[y,z] - R_1[\tilde{y},\tilde{z}]\|_{C([0,T];l^2_s)} + \|R_2[y,z] - R_2[\tilde{y},\tilde{z}]\|_{C([0,T];l^2_s)} \\
&\qquad \le C(\|y,z,\tilde{y},\tilde{z}\|_{C([0,T];l^2_s)})(\|y-\tilde{y}\|_{C([0,T];l^2_s)} + \|z-\tilde{z}\|_{C([0,T];l^2_s)}),
\end{aligned}
\tag{2.9}
$$

and, for some $0 < \delta < 1/2$*,*

$$\left\| \sum_{k_1+k_2=k} \frac{|m_{ij}|}{\langle \varphi_{ij} \rangle^{1/2}} y_{k_1} z_{k_2} \right\|_{l^2_s} \lesssim \|y\|_{l^2_s} \|z\|_{l^2_s}, \tag{2.10}$$

$$\left\| \sum_{k_1+k_2=k} \frac{|m_{ij}|}{\langle \varphi_{ij} \rangle^{1-\delta}} y_{k_1} z_{k_2} \right\|_{l^\infty} \lesssim \min\{\|y\|_{l^\infty} \|z\|_{l^2_s},\ \|y\|_{l^2_s} \|z\|_{l^\infty}\}, \tag{2.11}$$

$$\left\| \sum_{k_1+k_2=k} |m_{ij}|\, y_{k_1} z_{k_2} \right\|_{l^\infty} \lesssim \|y\|_{l^2_s} \|z\|_{l^2_s} \tag{2.12}$$

hold true for all $1 \le i \le 2$ *and* $1 \le j \le 4$.

3. Proof of Theorem 1.1

We begin this section by reformulating the Boussinesq Cauchy problem (1.1) in such a way that we can implement the methodology described before. The first step consists in rewriting it as a Cauchy problem for a Schrödinger equation. As in Kishimoto and Tsugawa [28], if we take

$$(v, v_0) := (u - i(1-\partial_x^2)^{-1}u_t, u_0 - i(1-\partial_x^2)^{-1}u_1),$$

then (1.1) gets transformed into

$$\begin{cases} i\partial_t v - \partial_x^2 v = \frac{\overline{v}-v}{2} + \omega(\partial_x)\left(\frac{v+\overline{v}}{2}\right)^2, \quad v = v(t,x) : \mathbb{R}\times\mathbb{T} \to \mathbb{C}, \\ v(0,x) = v_0(x), \end{cases} \tag{3.1}$$

where

$$\omega(\partial_x) := -\partial_x^2(1-\partial_x^2)^{-1}.$$

Conversely, if v and v_0 satisfy this Cauchy problem, then, by letting

$$(u, u_0, u_1) = \left(\frac{v+\overline{v}}{2}, \frac{v_0+\overline{v_0}}{2}, (1-\partial_x^2)\left(\frac{\overline{v_0}-v_0}{2i}\right)\right),$$

it is easy to check that u, u_0, and u_1 are all real-valued and solve (1.1). It is equally important to notice that, for an arbitrary $T > 0$,

$$U = (C([0,T], H^s) \cap C^1([0,T], H^{s-2})) \times H^s \times H^{s-2}, \quad V = C([0,T]; H^s) \times H^s,$$

the maps

$$(u, u_0, u_1) \in U \mapsto (v, v_0) \in V$$

and

$$(v, v_0) \in V \mapsto (u, u_0, u_1) \in U$$

are both Lipschitz continuous. Thus, UWP for (1.1) with data in $H^s \times H^{s-2}$ becomes equivalent to UWP for (3.1) in H^s.

Next, by introducing the Fourier series coefficients for v_0 and v according to (2.1) and (2.2), respectively, it follows that (3.1) can be turned into the infinite coupled system of ordinary differential equations

$$\begin{cases} i\partial_t v_k + k^2 v_k = \dfrac{1}{2}(\overline{v_{-k}} - v_k) + \dfrac{\omega(k)}{4} \displaystyle\sum_{\substack{k_1,k_2\in\mathbb{Z} \\ k_1+k_2=k}} (v_{k_1} + \overline{v_{-k_1}})(v_{k_2} + \overline{v_{-k_2}}), \\ v_k(0) = v_{0k}, \end{cases}$$

where

$$\omega(k) := \frac{k^2}{1+k^2}. \tag{3.2}$$

If we take

$$u_k^+(t) := e^{-itk^2} v_k(t), \qquad u_k^-(t) := e^{itk^2}\overline{v_{-k}(t)},$$

then the previous system comes to be

$$\begin{cases} \partial_t u_k^+ = \frac{i}{2}(u_k^+ - e^{-2itk^2} u_k^-) \\ \qquad - \frac{i\omega(k)}{4} \sum_{\substack{k_1,k_2\in\mathbb{Z} \\ k_1+k_2=k}} e^{-itk^2}(e^{itk_1^2}u_{k_1}^+ + e^{-itk_1^2}u_{k_1}^-)(e^{itk_2^2}u_{k_2}^+ + e^{-itk_2^2}u_{k_2}^-), \\ \partial_t u_k^- = \frac{i}{2}(e^{2itk^2}u_k^+ - u_k^-) \\ \qquad + \frac{i\omega(k)}{4} \sum_{\substack{k_1,k_2\in\mathbb{Z} \\ k_1+k_2=k}} e^{itk^2}(e^{itk_1^2}u_{k_1}^+ + e^{-itk_1^2}u_{k_1}^-)(e^{itk_2^2}u_{k_2}^+ + e^{-itk_2^2}u_{k_2}^-), \\ (u_k^+(0), u_k^-(0)) = (v_{0k}, \overline{v_{0,-k}}). \end{cases}$$

Finally, we rely on the notation

$$\begin{cases} \phi^{++}(k,k_1,k_2) := -k^2 + k_1^2 + k_2^2, \\ \phi^{+-}(k,k_1,k_2) := -k^2 + k_1^2 - k_2^2, \\ \phi^{-+}(k,k_1,k_2) := -k^2 - k_1^2 + k_2^2, \\ \phi^{--}(k,k_1,k_2) := -k^2 - k_1^2 - k_2^2, \end{cases} \tag{3.3}$$

and

$$\begin{cases} R^+[y,z]_k := \frac{i}{2}(y_k - e^{-2itk^2} z_k), \\ R^-[y,z]_k := \frac{i}{2}(e^{2itk^2} y_k - z_k), \end{cases} \tag{3.4}$$

to arrive at the following working version of the original Cauchy problem (1.1):

$$\begin{cases} \partial_t u_k^\pm = \mp \frac{i\omega(k)}{4} \sum_{\substack{k_1,k_2\in\mathbb{Z} \\ k_1+k_2=k}} \Big(e^{\pm it\phi^{++}} u_{k_1}^+ u_{k_2}^+ + e^{\pm it\phi^{+-}} u_{k_1}^+ u_{k_2}^- \\ \qquad\qquad + e^{\pm it\phi^{-+}} u_{k_1}^- u_{k_2}^+ + e^{\pm it\phi^{--}} u_{k_1}^- u_{k_2}^- \Big) \\ \qquad + R^\pm[u^+, u^-]_k, \\ (u_k^+(0), u_k^-(0)) = (v_{0k}, \overline{v_{0,-k}}). \end{cases} \tag{3.5}$$

It is for this system that we verify the validity of Theorem 2.2 when $s > 0$, thus proving Theorem 1.1.

Proposition 3.1. *Theorem 2.2 holds true for the Cauchy problem* (3.5) *when* $s > 0$.

Proof. We start the argument by recognizing that, in our setting,

$$\varphi_{ij} \in \{\phi^{++}, \phi^{+-}, \phi^{-+}, \phi^{--}\},\ m_{ij} \in \left\{ \frac{i\omega(k)}{4}, -\frac{i\omega(k)}{4} \right\},\ \{R_1, R_2\} = \{R^+, R^-\}.$$

We deduce directly from (3.4) that

$$\begin{aligned} |R^+[y,z]_k| + |R^-[y,z]_k| &\le |y_k| + |z_k|, \\ |R^+[y,z]_k - R^+[\tilde{y},\tilde{z}]_k| + |R^-[y,z]_k - R^-[\tilde{y},\tilde{z}]_k| &\le |y_k - \tilde{y}_k| + |z_k - \tilde{z}_k|, \end{aligned}$$

which immediately implies (2.8) and (2.9).

From (3.2), we derive

$$\Big(m_{ij} = 0 \Longleftrightarrow k = 0\Big) \quad \text{and} \quad \left(|m_{ij}| < \frac{1}{4}\right). \tag{3.6}$$

Using this fact, the Cauchy-Schwarz inequality, and $s \geq 0$, we can easily prove (2.12) as follows:

$$\Big\| \sum_{k_1+k_2=k} |m_{ij}|\, y_{k_1}\, z_{k_2} \Big\|_{l^\infty} \lesssim \Big\| \sum_{k_1+k_2=k} |y_{k_1}|\, |z_{k_2}| \Big\|_{l^\infty} \leq \|y\|_{l^2} \|z\|_{l^2} \leq \|y\|_{l^2_s} \|z\|_{l^2_s}.$$

Next, we turn to the argument for (2.10), which can be reduced, with the help of the Cauchy-Schwarz inequality, to proving

$$\sup_k \left(\sum_{k_1+k_2=k} \frac{|m_{ij}|^2}{\langle \varphi_{ij} \rangle} \frac{\langle k \rangle^{2s}}{\langle k_1 \rangle^{2s} \langle k_2 \rangle^{2s}} \right) \lesssim 1.$$

By relying on (3.6), we can further simplify the previous estimate down to

$$\sup_{k \neq 0} \left(\sum_{k_1+k_2=k} \frac{1}{\langle \varphi_{ij} \rangle} \frac{\langle k \rangle^{2s}}{\langle k_1 \rangle^{2s} \langle k_2 \rangle^{2s}} \right) \lesssim 1. \tag{3.7}$$

Now, we introduce the notation

$$|k_{\max}| = \max\{|k_1|, |k_2|\}, \qquad |k_{\min}| = \min\{|k_1|, |k_2|\}, \tag{3.8}$$

and, based on the triangle inequality, we have

$$|k| = |k_1 + k_2| \lesssim |k_{\max}|. \tag{3.9}$$

Jointly with (3.3), this bound yields that the size of $|\varphi_{ij}|$ is similar to one of

$$\{|k_{\max}|\, |k_{\min}|, |k|\, |k_{\max}|, |k|\, |k_{\min}|, |k_{\max}|^2\}.$$

If $k_{\min} = 0$, then $k_{\max} = k$ and (3.7) follows at once. If $k_{\min} \neq 0$, then we split the analysis into two complementary scenarios:

$$1 \leq |k| \lesssim |k_{\max}| \sim |k_{\min}|, \qquad 1 \leq |k_{\min}| \ll |k| \sim |k_{\max}|. \tag{3.10}$$

If $1 \leq |k| \lesssim |k_{\max}| \sim |k_{\min}|$, then the previous fact about the size of $|\varphi_{ij}|$ and $s > 0$ lead to

$$\begin{aligned} \sup_{k \neq 0} \left(\sum_{k_1+k_2=k} \frac{1}{\langle \varphi_{ij} \rangle} \frac{\langle k \rangle^{2s}}{\langle k_1 \rangle^{2s} \langle k_2 \rangle^{2s}} \right) &\lesssim \sup_{k \neq 0} \left(\sum_{|k| \lesssim |k_{\max}|} \frac{1}{|k|\, |k_{\max}|} \frac{|k|^{2s}}{|k_{\max}|^{4s}} \right) \\ &\lesssim \sup_{k \neq 0} \left(\frac{1}{|k|} \sum_{|k| \lesssim |k_{\max}|} \frac{1}{|k_{\max}|^{1+2s}} \right) \lesssim 1. \end{aligned}$$

If we are in the second scenario, then, using again the information about the size of $|\varphi_{ij}|$ and $s \geq 0$, we obtain

$$\sup_{k\neq 0}\left(\sum_{k_1+k_2=k}\frac{1}{\langle\varphi_{ij}\rangle}\frac{\langle k\rangle^{2s}}{\langle k_1\rangle^{2s}\langle k_2\rangle^{2s}}\right) \lesssim \sup_{k\neq 0}\left(\sum_{|k_{\min}|\ll|k|}\frac{1}{|k|\,|k_{\min}|}\frac{1}{|k_{\min}|^{2s}}\right)$$
$$\lesssim \sup_{k\neq 0}\left(\sum_{|k_{\min}|\ll|k|}\frac{1}{|k_{\min}|^{2}}\right) \lesssim 1.$$

This completes the proof of (3.7) and, hence, of (2.10).

Finally, we come to the argument for (2.11). First, we use (3.6), the Cauchy-Schwarz inequality, and $s \geq 0$ to infer

$$\left\|\sum_{k_1+k_2=k}\frac{|m_{ij}|}{\langle\varphi_{ij}\rangle^{1-\delta}}\,y_{k_1}\,z_{k_2}\right\|_{l^\infty} \lesssim \left\|\sum_{k_1+k_2=k\neq 0}\frac{1}{\langle\varphi_{ij}\rangle^{1-\delta}}\,|z_{k_2}|\right\|_{l^\infty}\|y\|_{l^\infty}$$
$$\lesssim \sup_{k\neq 0}\left(\sum_{k_1+k_2=k}\frac{1}{\langle\varphi_{ij}\rangle^{2-2\delta}}\right)^{1/2}\|y\|_{l^\infty}\|z\|_{l^2}$$
$$\lesssim \sup_{k\neq 0}\left(\sum_{k_1+k_2=k}\frac{1}{\langle\varphi_{ij}\rangle^{2-2\delta}}\right)^{1/2}\|y\|_{l^\infty}\|z\|_{l^2_s}.$$

This effectively reduces the proof of (2.11) to the one for

$$\sup_{k\neq 0}\left(\sum_{k_1+k_2=k}\frac{1}{\langle\varphi_{ij}\rangle^{2-2\delta}}\right) \lesssim 1.$$

Here, we reason in identical fashion with the argument for (3.7), involving the notation (3.8), the information about the size of $|\varphi_{ij}|$, and the complementary scenarios (3.10). The case $k_{\min} = 0$ can be easily dispensed with. Otherwise, if $1 \leq |k| \lesssim |k_{\max}| \sim |k_{\min}|$, then

$$\sup_{k\neq 0}\left(\sum_{k_1+k_2=k}\frac{1}{\langle\varphi_{ij}\rangle^{2-2\delta}}\right) \lesssim \sup_{k\neq 0}\left(\frac{1}{|k|^{2-2\delta}}\sum_{|k|\lesssim|k_{\max}|}\frac{1}{|k_{\max}|^{2-2\delta}}\right) \lesssim 1,$$

since $\delta < 1/2$. If instead $1 \leq |k_{\min}| \ll |k| \sim |k_{\max}|$ is valid, then

$$\sup_{k\neq 0}\left(\sum_{k_1+k_2=k}\frac{1}{\langle\varphi_{ij}\rangle^{2-2\delta}}\right) \lesssim \sup_{k\neq 0}\left(\frac{1}{|k|^{2-2\delta}}\sum_{|k_{\min}|\ll|k|}\frac{1}{|k_{\min}|^{2-2\delta}}\right) \lesssim 1,$$

again due to $\delta < 1/2$. This concludes the argument for (2.11) and the whole proof.

Remark 3.2. *It is important to recognize that we used $s > 0$ only in the argument for* (2.10)*, while the proofs for* (2.8), (2.9), (2.11)*, and* (2.12) *required only that $s \geq 0$.*

4. Proof of Theorem 1.2

For the Kawahara Cauchy problem (1.2), we proceed in similar fashion to the previous section and we reformulate it in a form amenable to the framework we want to implement. First, we notice that its smooth solutions satisfy

$$\int_{\mathbb{T}} \partial_t u(t,x)\,dx = 0,$$

which implies

$$\int_{\mathbb{T}} u(t,x)\,dx = \int_{\mathbb{T}} u(0,x)\,dx = \int_{\mathbb{T}} u_0(x)\,dx := \mu. \tag{4.1}$$

If we take

$$(v, v_0) := (u-\mu, u_0-\mu),$$

then (1.2) becomes

$$\begin{cases} \partial_t v + \beta\partial_x^3 v - \partial_x^5 v + 2\mu\partial_x v = -\partial_x(v^2), \\ v(0,x) = v_0(x), \end{cases} \tag{4.2}$$

and now, due to (4.1), one has

$$\int_{\mathbb{T}} v(t,x)\,dx = 0. \tag{4.3}$$

This will turn out to be important moving forward.

Next, by turning to the Fourier series coefficients (2.1) and (2.2) for v_0 and v, respectively, we obtain the equivalent form of (4.2) as an infinite coupled system of ordinary differential equations:

$$\begin{cases} \partial_t v_k + i(-\beta k^3 - k^5 + 2\mu k)v_k = -ik \sum\limits_{\substack{k_1,k_2\in\mathbb{Z} \\ k_1+k_2=k}} v_{k_1}v_{k_2} \\ v_k(0) = v_{0k}. \end{cases}$$

In the end, we let

$$w_k(t) := e^{it(-\beta k^3 - k^5 + 2\mu k)} v_k(t)$$

to rewrite the previous system as

$$\begin{cases} \partial_t w_k = -ik \sum\limits_{\substack{k_1,k_2\in\mathbb{Z} \\ k_1+k_2=k}} e^{it\phi} w_{k_1} w_{k_2}, \\ w_k(0) = v_{0k}, \end{cases}$$

where

$$\phi = \phi(k,k_1,k_2) := -\beta k^3 - k^5 + 2\mu k + \beta k_1^3 + k_1^5 - 2\mu k_1 + \beta k_2^3 + k_2^5 - 2\mu k.$$

When $k = k_1 + k_2$, we have the following important factorization:

$$\phi = -kk_1k_2(5(k_1^2 + k_2^2 + k_1k_2) + 3\beta). \tag{4.4}$$

At this moment, we observe that (4.3) implies $v_0(t) \equiv 0$ and, hence, $w_0(t) \equiv 0$. This allows us to remove from the system the equation corresponding to $k = 0$. Moreover, when $k \neq 0$, we see that (4.4) implies

$$\phi = 0 \Longleftrightarrow k_1 = 0 \text{ or } k_2 = 0.$$

Thus, a working version for the Cauchy problem (1.2) is represented by

$$\begin{cases} \partial_t w_k = -ik \sum\limits_{\substack{k_1+k_2=k \\ \phi\neq 0}} e^{it\phi} w_{k_1} w_{k_2}, & \forall\, k \neq 0, \\ w_k(0) = v_{0k}, \end{cases} \tag{4.5}$$

for which we argue that Theorem 2.1 holds true when $s > 1/4$. This clearly proves Theorem 1.2.

Proposition 4.1. *Theorem 2.1 is valid for the Cauchy problem* (4.5) *when* $s > 1/4$.

Proof. All three estimates in Theorem 2.1 (i.e., (2.4)–(2.6)) to be verified share the generic profile

$$\left\| \sum_{k_1+k_2=k} a(k, k_1, k_2)\, y_{k_1}\, z_{k_2} \right\|_{l^2_{s_3}} \lesssim \|y\|_{l^2_{s_1}} \|z\|_{l^2_{s_2}}.$$

Using duality, this bound can be seen to be the consequence of

$$\sup_k \left(\sum_{k_1+k_2=k} \frac{a^2(k, k_1, k_2)\langle k\rangle^{2s_3}}{\langle k_1\rangle^{2s_1}\langle k_2\rangle^{2s_2}} \right) \lesssim 1. \tag{4.6}$$

It is important to note that, due to (4.4), $\beta \in \{-1, 0, 1\}$, and (4.5), we can further assume above that, in our context, none of k, k_1, and k_2 are equal to 0.

Here, as in the argument for Proposition 3.1, we rely on the notation (3.8), the bound (3.9), and the complementary scenarios (3.10). We start by proving (2.4), for which

$$a(k, k_1, k_2) = \frac{|k|}{\langle\phi\rangle^{1/2}}, \quad s_1 = s_2 = s_3 = s.$$

Since $s \geq 0$, the estimate (3.9) implies

$$\frac{\langle k\rangle^{2s}}{\langle k_1\rangle^{2s}\langle k_2\rangle^{2s}} \lesssim 1$$

and, hence, (4.6) holds true if we show

$$\sup_k \left(\sum_{\substack{k_1+k_2=k \\ \phi\neq 0}} \frac{|k|^2}{\langle\phi\rangle} \right) \lesssim 1. \tag{4.7}$$

If $1 \leq |k| \lesssim |k_{\max}| \sim |k_{\min}|$, then we deduce from (4.4) that $|\phi| \sim |k||k_{\max}|^4$ and, hence,

$$\sum_{\substack{k_1+k_2=k \\ \phi\neq 0}} \frac{|k|^2}{\langle\phi\rangle} \lesssim \sum_{|k|\leq|k_{\max}|} \frac{1}{|k_{\max}|^3} \lesssim 1,$$

which yields (4.7).

If $1 \le |k_{\min}| \ll |k| \sim |k_{\max}|$, then (4.4) implies $|\phi| \sim |k_{\min}||k|^4$ and, thus,

$$\sum_{\substack{k_1+k_2=k \\ \phi\neq 0}} \frac{|k|^2}{\langle \phi\rangle} \lesssim \sum_{1\le |k_{\min}|\ll |k|} \frac{1}{|k_{\min}||k|^2} \lesssim \frac{\ln|k|}{|k|^2} \lesssim 1,$$

which also yields (4.7).

Next, we turn to the argument for (2.5), which corresponds to (4.6) with

$$a(k,k_1,k_2) = \frac{|k|}{\langle \phi\rangle^{1-\delta}}, \quad s_1 = s_3 = s-2, \quad s_2 = s.$$

Like above, on the basis of $s \ge 0$ and (3.9), we derive

$$\frac{|k|^2\langle k\rangle^{2s-4}}{\langle k_1\rangle^{2s-4}\langle k_2\rangle^{2s}} \lesssim \frac{\langle k_{\max}\rangle^4}{\langle k\rangle^2},$$

and, hence, reduce the proof of (4.6) to the one for

$$\sup_k \left(\sum_{\substack{k_1+k_2=k \\ \phi\neq 0}} \frac{\langle k_{\max}\rangle^4}{\langle k\rangle^2\langle \phi\rangle^{2-2\delta}} \right) \lesssim 1. \tag{4.8}$$

If $1 \le |k| \lesssim |k_{\max}| \sim |k_{\min}|$, then $|\phi| \sim |k||k_{\max}|^4$ and, by taking $\delta < 3/8$, we deduce

$$\sum_{\substack{k_1+k_2=k \\ \phi\neq 0}} \frac{\langle k_{\max}\rangle^4}{\langle k\rangle^2\langle \phi\rangle^{2-2\delta}} \lesssim \sum_{|k|\le |k_{\max}|} \frac{1}{|k_{\max}|^{4-8\delta}|k|^{4-2\delta}} \lesssim \frac{1}{|k|^{4-2\delta}} \lesssim 1,$$

which yields (4.8).

If $1 \le |k_{\min}| \ll |k| \sim |k_{\max}|$, then $|\phi| \sim |k_{\min}||k|^4$ and, now using only that $\delta < 1/2$, we infer

$$\sum_{\substack{k_1+k_2=k \\ \phi\neq 0}} \frac{\langle k_{\max}\rangle^4}{\langle k\rangle^2\langle \phi\rangle^{2-2\delta}} \lesssim \sum_{1\le |k_{\min}|\ll |k|} \frac{1}{|k_{\min}|^{2-2\delta}|k|^{6-8\delta}} \lesssim \frac{1}{|k|^{6-8\delta}} \lesssim 1,$$

which also yields (4.8).

Finally, we get to prove (2.6), for which

$$a(k,k_1,k_2) = |k|, \quad s_3 = s-2, \quad s_1 = s_2 = s.$$

Thus, due to the symmetry in this case with respect to the indices 1 and 2, (4.6) is valid if we show

$$\sup_k \left(\sum_{\substack{k_1+k_2=k \\ \phi\neq 0}} \frac{|k|^2\langle k\rangle^{2s-4}}{\langle k_{\max}\rangle^{2s}\langle k_{\min}\rangle^{2s}} \right) \lesssim 1. \tag{4.9}$$

If $1 \le |k| \lesssim |k_{\max}| \sim |k_{\min}|$, then, as $s > 1/4$, we deduce

$$\sum_{\substack{k_1+k_2=k \\ \phi\neq 0}} \frac{|k|^2\langle k\rangle^{2s-4}}{\langle k_{\max}\rangle^{2s}\langle k_{\min}\rangle^{2s}} \lesssim \sum_{|k|\le |k_{\max}|} \frac{1}{|k_{\max}|^{2s+1/2}|k|^{3/2}} \lesssim \frac{1}{|k|^{3/2}} \lesssim 1,$$

which implies (4.9).

If $1 \leq |k_{\min}| \ll |k| \sim |k_{\max}|$, then, since $s \geq 0$, we infer

$$\sum_{\substack{k_1+k_2=k \\ \phi\neq 0}} \frac{|k|^2 \langle k \rangle^{2s-4}}{\langle k_{\max} \rangle^{2s} \langle k_{\min} \rangle^{2s}} \lesssim \sum_{1 \leq |k_{\min}| \ll |k|} \frac{1}{|k_{\min}|^{2s+2}} \lesssim 1,$$

which also implies (4.9).

Remark 4.2. *It is important to note that in the previous proposition we used* $s > 1/4$ *only in the proof of* (2.6)*, when* $1 \leq |k| \lesssim |k_{max}| \sim |k_{min}|$*. All the other arguments simply required that* $s \geq 0$*.*

Remark 4.3. *It seems that the restriction* $s > 1/4$ *is essential in the sense that, by switching from* $s-2$ *to* $s-\sigma$ *for some other* $\sigma > 0$*, one encounters it again in the same scenario. As suggested by one of the referees, this is likely one of the shortcomings of this methodology, as* (2.6) *is particularly not responsive to the presence of dispersion in the original equation.*

Use of AI tools declaration

The author declares he has not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

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