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Research article

Blow-up upper and lower bounds for solutions of a class of higher order nonlinear pseudo-parabolic equations

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Abstract: In this paper, we study the initial boundary value problem for a class of higher-order nonlinear pseudo-parabolic equations with a memory term. First, the blow-up results of the solution when the initial energy is negative or positive are obtained by using concavity analysis, and an upper bound on the blow-up time T^* is given. Second, a lower bound on the blow-up time T^* is obtained by applying differential inequalities when the solutions blow up.

Keywords: higher order pseudo-parabolic equation; memory term; blow-up; upper bound; lower bound

1. Introduction

In this paper, we consider the following initial boundary value problem for higher-order nonlinear viscous parabolic type equations.

$$\begin{cases} u_t + (-\Delta)^L u + (-\Delta)^K u_t - \displaystyle\int_0^t g(t-s)(-\Delta)^L u(s)ds \\ = a|u|^{R-2}u, \quad x \in \Omega, t \geq 0, & (1.1) \\ u(x,0) = u_0(x) \in H_0^L(\Omega), & (1.2) \\ \dfrac{\partial^i u}{\partial \nu^i} = 0, i = 0,1,2,...L-1 \quad x \in \partial\Omega, \quad t \geq 0, & (1.3) \end{cases}$$

where $L, K \geq 1$ is an integer number, $R \geq \max\{2, 2a, 2H\}$ where $a > 0$ is a real number, and $\Omega \subseteq R^N (N \geq 1)$ is a bounded domain with a smooth boundary $\partial\Omega$.

Equation (1.1) includes many important physical models. In the absence of the memory term and dispersive term, and with $L = K = 1$ and $a = 0$, Eq (1.1) becomes the linear pseudo-parabolic equation

$$u_t - \Delta u - \beta u_t = 0. \tag{1.4}$$

Showalter and Ting [1] and Gopala Rao and Ting [2] investigated the initial boundary value problem of the linear Eq (1.4) and proved the existence and uniqueness of solutions. Pseudo-parabolic equations appear in many applications for natural sciences, such as radiation with time delay [3], two-phase porous media flow models with dynamic capillarity or hysteresis [4], phase field-type models for unsaturated porous media flows [5], heat conduction models [6], models to describe lightning [7], and so on. A number of authors (Showalter [8, 9], DiBenedetto and Showalter [10], Cao and Pop [11], Fan and Pop [12], Cuesta and Pop [13], Schweizer [14], Kaikina [15, 16], Matahashi and Tsutsumi [17, 18]) have considered this kind of equation by various methods and made a lot of progress. Not only were the existence, uniqueness, and nonexistence results for pseudo-parabolic equations were obtained, but the asymptotic behavior, regularity, and other properties of solutions were also investigated.

In 1972, Gopala Rao et al. [2, 19] studied the equation $u_t - k\Delta u_t - \Delta u = 0$. They use the principle of maximum value to establish the uniqueness and the existence of solutions. Using the potential well method and the comparison principle, Xu and Su [20] studied the overall existence, nonexistence, and asymptotic behavior of the solution of the equation $u_t - \Delta u_t - \Delta u = u^q$, and they also proved that the solution blows up in finite time when $J(u_0) > d$.
When $L = K = 1$, Eq (1.1) becomes

$$u_t - \Delta u = \int_0^t b(t-\tau)\Delta u(\tau)d\tau + f(u). \tag{1.5}$$

Equation (1.5) originates from various mathematical models in engineering and physical sciences, such as in the study of heat conduction in materials with memory. Yin [21] discussed the problem of initial boundary values of Eq (1.5) and obtained the global existence of classical solutions under one-sided growth conditions. Replacing the memory term $b(\cdot)$ in (1.5) by $-g(\cdot)$, Messaoudi [22] proves the blow-up of the solution with negative and vanishing initial energies. When $f(u) = |u|^{q-2}u$, Messaoudi [23] proved the result of the blow-up of solutions for this equation with positive initial energy under the appropriate conditions of b and q. Sun and Liu [24] studied the equation

$$u_t - \Delta u - \Delta u_t + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = u^{q-2}u. \tag{1.6}$$

They applied the Galerkin method, the concavity method, and the improved potential well method to prove existence of a global solution and the blow-up results of the solution when the initial energy $J(u(0)) \leq d(\infty)$, and Di et al. [25] obtained the blow-up results of the solution of Eq (1.6) when the initial energy is negative or positive and gave some upper bounds on the blow-up time, and they proved lower bounds on the blow up time by applying differential inequalities.

When $m > 1$, Cao and Gu [26] studied the higher order parabolic equations

$$u_t + (-\Delta)^m u = |u|^q u. \tag{1.7}$$

By applying variational theory and the Galerkin method, they obtained existence and uniqueness results for the global solution. When the initial value belongs to the negative index critical space H^{-s,R^s}, $R^s = \dfrac{n\alpha}{w - s\alpha}$, Wang [27, 28] proved the existence and uniqueness of the local and the global solutions of the Cauchy problem of Eq (1.7) by using $L^r - L^q$ estimates. Caristi and Mitidieri [29] applied the method in [30] to prove the existence and nonexistence of the global solution of the initial boundary value problem for higher-order parabolic equations when the initial value decays slowly. Budd et al. [31] studied the self-similar solutions of Eq (1.7) for $n = 1, k > 1$. Ishige et al. [32] proved the existence of solutions to the Cauchy problem for a class of higher-order semilinear parabolic equations by introducing a new majority kernel, and also gave the existence of a local time solution for the initial data and necessary conditions for the solution of the Cauchy problem, and determine the strongest singularity of the initial data for the solutions of the Cauchy problem.

When $K = L, g = 0$, problem (1.1) becomes the following n-dimensional higher-order proposed parabolic equation

$$u_t(x,t) + (-1)^M \Delta^M u_t(x,t) + (-1)^M \Delta^M u(x,t) = a|u|^{q-1}u. \tag{1.8}$$

Equation (1.8) describes some important physical problems [33] and has attracted the attention of many scholars. Xiao and Li [34] have proved the existence of a non-zero weak solution to the static problem of problem (1.8) by means of the mountain passing theorem, and, additionally, based on the method of potential well theory, they proved the existence of a global weak solution of the development in the equations.

Based on the idea of Li and Tsai [35], this paper discusses the property of the solution of problem (1.1)–(1.3) regarding the solution blow-up in finite time under different initial energies $E(0)$. An upper bound on the blow-up time T^* is established for different initial energies, and, additionally, a lower bound on the blow-up time T^* is established by applying a differential inequality.

2. Preliminaries

To describe the main results of this paper, this section gives some notations, generalizations, and important lemmas. We adopt the usual notations and convention. Let $H^L(\Omega)$ denote the Sobolev space with the usual scalar products and norm, Where $H_0^L(\Omega)$ denotes the closure in $H_0^L(\Omega)$ of $C_0^\infty(\Omega)$. For simplicity of notation, hereafter we denote by $\|.\|_p$ the Lebesgue space $L^p(\Omega)$ norm, and by $\|.\|$ the $L^2(\Omega)$ norm; equivalently we write the norm $\|D^L \cdot\|$ instead of the $H_0^L(\Omega)$ norm $\|.\|_{H_0^L(\Omega)}$, where D denotes the gradient operator, that is, $D\cdot = \nabla\cdot = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \ldots. \frac{\partial}{\partial x_n}\right)$. Moreover, $D^L\cdot = \triangle^j\cdot$ if $L = 2j$, and $D^L\cdot = D\triangle^j\cdot$ if $L = 2j+1$.

$$L^p(\Omega) = L^p, \quad \|u\|_{L^p(\Omega)} = \|u\|_p = \left(\int_\Omega |u|^p dx\right)^{\frac{1}{p}},$$

$$H_0^L(\Omega) = W_0^{L,2}(\Omega) = H_0^L, \quad \|u\|_{H_0^L(\Omega)} = \|u\|_{H_0^L} = \left(\int_\Omega |u|^2 + |D^L u|^2 dx\right)^{\frac{1}{2}}.$$

To justify the main conclusions of this paper, the following assumptions are made on K L, and the relaxation function $g(\cdot)$.

(A_1) $1 \le K < L$ are integers with $2a \le R < +\infty$ if $n < 2L$; $2a \le R \le \frac{2n}{n-2L}$ if $n > 2L$, where $a > 1$

(A_2) $g : R^+ \to R^+$ is a C^1 function, satisfing

$$g(t) \geq 0, g'(t) \leq 0, \frac{2a}{R-2a} < \beta = 1 - \int_0^\infty g(s)\,ds \leq 1 - \int_0^t g(s)\,ds. \tag{2.1}$$

Define the energy functional of problem (1.1) − (1.3) as

$$\begin{aligned} E(t) = & \int_0^t \| u_t \|^2 + \frac{1}{2}\left(1 - \int_0^t g(s)\,ds\right) \| D^L u \|^2 + \frac{1}{2}(g \circ D^L u)(t) \\ & - \frac{a}{R} \| u \|_R^R \end{aligned} \tag{2.2}$$

where $\left(g \circ D^L u\right)(t) = \int_0^t g(t-s) \| D^L u(t) - D^L u(s) \|^2 \ ds.$
Both sides of Eq (1.1) are simultaneously multiplied by u_t and integrated over Ω, and from (A_1) and (2.1) we have that

$$E'(t) = - \| D^K u_t \|^2 + \frac{1}{2}(g' \circ D^L u)(t) - \frac{1}{2} g(t) \| D^L u \|^2 < 0. \tag{2.3}$$

Definition 2.1 We say that $u(x,t)$ is a weak solution of problem (1.1) if $u \in L^\infty([0,T); H_0^L(\Omega)), u_t \in L^2([0,T); H_0^L(\Omega))$, and u satisfies

$$(u_t, v) + (D^L u, D^L v) + (D^K u_t, D^K v) - \int_0^t g(t-\tau)(D^L u(\tau), D^L v)d\tau = (a|u|^{R-2}u, v)$$

for all test functions $v \in H_0^L(\Omega)$ and $t \in [0,T]$.
Theorem 2.1 (Local existence) Suppose that (A1) and (A2) hold. If $(u_0, u_1) \in H_0^L(\Omega) \times L^2(\Omega)$, then there exists $T > 0$ such that problem (1.1) admits a unique local solution $u(t)$ which satisfies

$$u \in L^2([0,T); H_0^L(\Omega)), \quad u_t \in L^2([0,T); L^2(\Omega) \cap L^2([0,T]; H_0^K(\Omega)).$$

Moreover, at least one of the following statements holds true:

$$\int_0^t \|u\|^2 + \|D^L u\|^2 \to +\infty, as\ t \to T, or \quad T = +\infty.$$

The existence and uniqueness of the local solution for problem (1.1) can be obtained by using Faedo-Galerkin methods and the contraction mapping principle in [30, 36–38].
Lemma 2.1 [39]. Let q be a real number with $2 \leq q \leq +\infty$ if $n \leq 2L$, and $2 \leq q \leq \frac{2n}{n-2L}$ if $n > 2L$. Then there exists a constant B dependent on Ω and q such that

$$\| u \|_q \leq B \| D^L u \|, \quad u \in H_0^L(\Omega). \tag{2.4}$$

Remark 2.1. According to Eqs (1.1) − (1.3) and Lemma 2.1, we get

$$
\begin{aligned}
E(t) &\geq \frac{1}{2}\left(1-\int_0^t g(s)ds\right)\|D^L u\|^2 + \frac{1}{2}(g\circ D^L u)(t) - \frac{a}{R}\|u\|_R^R \\
&\geq \frac{1}{2}\beta\|D^L u\|^2 + \frac{1}{2}(g\circ D^L u)(t) - \frac{aB^R}{R}(\|D^L u\|^2)^{\frac{R}{2}} \\
&\geq \frac{1}{2}[(g\circ D^L u)(t) + \beta\|D^L u\|^2] - \frac{aB^R}{R\beta^{\frac{R}{2}}}[\beta\|D^L u\|^2 + (g\circ D^L u)(t)]^{\frac{R}{2}} \\
&= Q\left([\beta\|D^L u\|^2 + (g\circ D^L u)(t)]^{\frac{1}{2}}\right).
\end{aligned}
\tag{2.5}
$$

Let $Q(\xi) = \frac{1}{2}\xi^2 - \frac{aB^R}{R\beta^{\frac{R}{2}}}\xi^R,\ \xi = \left(\beta\|D^L u\|^2 + (g\circ D^L u)(t)\right)^{\frac{1}{2}} > 0$. A direct calculation yields that $Q'(\xi) = \xi - \frac{aB^R}{\beta^{\frac{R}{2}}}\xi^{R-1}, Q''(\xi) = 1 - \frac{a(R-1)B^R}{\beta^{\frac{R}{2}}}\xi^{R-2}$. From $Q'(\xi) = 0$, we get that $\xi_1 = \left(\frac{\beta}{aB^2}\right)^{\frac{R}{2(R-2)}}$. When $\xi = \xi_1$, direct calculation gives $Q''(\xi) = 2 - R < 0$. Therefore, $Q(\xi)$ is maximum at ξ_1, and its maximum value is

$$
H = Q(\xi_1) = \frac{R-2}{2R}\left(\frac{\beta}{aB^2}\right)^{\frac{R}{(R-2)}} = \frac{R-2}{2R}\xi_1^2. \tag{2.6}
$$

Lemma 2.2. Let conditions $(A_1), (A_2)$ hold, u be a solution of $((1.1-(1.3))$, $E(0) < H$, and $\beta^{\frac{1}{2}}\|D^L u_0\| > \xi_1$. Then there exists $\xi_2 > \xi_1$, such that

$$
\beta\|D^L u\|^2 + (g\circ D^L u)(t) \geq \xi_2^2. \tag{2.7}
$$

Proof. From Remark 2.1, $Q(\xi)$ is increasing on $(0,\xi_1)$ and decreasing on $(\xi_1,+\infty)$. $Q(\xi)\to-\infty, (\xi\to\infty)$. According to $E(0) < H$, there exists ξ_2', ξ_2 such that $\xi_1 \in (\xi_2', \xi_2)$, and $Q(\xi_2') = Q(\xi_2) = E(0)$. To prove Eq (2.7), we use the converse method. Assume that there exists $t_0 > 0$ such that

$$
\beta\|D^L u(t_0)\|^2 + (g\circ D^L u)(t_0) < \xi_2^2. \tag{2.8}
$$

1) If $\xi_2' < \left(\beta\|D^L u(t_0)\|^2 + (g\circ D^L u)(t_0)\right)^{\frac{1}{2}} < \xi_2$, then

$$
Q\left([\beta\|D^L u(t_0)\|^2 + (g\circ D^L u)(t_0)]^{\frac{1}{2}}\right) > Q(\xi_2') = Q(\xi_2) = E(0) > E(t_0).
$$

This contradicts (2.5).

2) If $\left(\beta\|D^L u(t_0)\|^2 + (g\circ D^L u)(t_0)\right)^{\frac{1}{2}} \leq \xi_2'$.

As $\beta^{\frac{1}{2}}\|D^L u_0\| > \xi_1$, according to (2.5), $Q\left(\beta^{\frac{1}{2}}\|D^L u_0\|\right) < E(0) = Q(\xi_2)$, which implies that $\beta^{\frac{1}{2}}\|D^L u_0\| > \xi_2$. Applying the continuity of $\left(\beta\|D^L u(t_0)\|^2 + (g\circ D^L u)(t_0)\right)^{\frac{1}{2}}$, we know that there exists a $t_1 \in (0,t_0)$ such that $\xi_2' < \left(\beta\|D^L u(t_1)\|^2 + (g\circ D^L u)(t_1)\right)^{\frac{1}{2}} < \xi_2$. hence, we have $Q(\left(\beta\|D^L u(t_1)\|^2 + (g\circ D^L u)(t_1)\right)^{\frac{1}{2}}) > E(0) \geq E(t_0)$, which contradicts (2.5).

The following lemma is very important and is similar to the proof of Lemma 4.2 in [35]. Here, we make some appropriate modifications

Lemma 2.3 [40] . Let $\Gamma(t)$ be a nonincreasing function of $[t_0, \infty], t_0 \geq 0$. Satisfying the differential inequality

$$\Gamma'^2(t) \geq \rho + \psi \Gamma(t)^{2+\frac{1}{\varepsilon}}, \quad t \geq t_0 \tag{2.9}$$

where $\rho > 0, \psi < 0$, there exists a positive number T^* such that

$$\lim_{t \to T^*} \Gamma(t) = 0. \tag{2.10}$$

The upper bound for T^* is

$$T^* \leq t_0 + \frac{1}{\sqrt{-\psi}} \ln \frac{\sqrt{\frac{\rho}{-\psi}}}{\sqrt{\frac{\rho}{-\psi}} - \Gamma(t_0)} \tag{2.11}$$

where $\Gamma(t_0) < min\{1, \sqrt{\dfrac{\rho}{-\psi}}\}$, and T_{max} denotes the maximal existence time of the solution

$$T_{max} = \sup\{T > 0 : u(.,t) \in [0, T]\} < +\infty.$$

3. Upper bound on blow-up time

In this section, we will give some blow-up results for solutions with initial energy (i) $E(0) < 0$; (ii) $0 \leq E(0) < \dfrac{w}{R-2}H$; and (iii) $\dfrac{w}{R-2}H \leq E(0) < \dfrac{\|u_0\|^2 + \|D^K u_0\|^2}{\mu}$. Moreover, some upper bounds for blow-up time T^* depending on the sign and size of initial energy $E(0)$ are obtained for problem (1.1)–(1.3).

Define the functionals

$$\Phi(t) = \int_0^t \|u\|^2 \, ds + \int_0^t \|D^K u\|^2 \, ds, \tag{3.1}$$

$$\Gamma(t) = \left[\Phi(t) + (T_0 - t)(\|u_0\|^2 + \|D^K u_0\|^2)\right]^{-\varepsilon} \tag{3.2}$$

where $\dfrac{1}{\beta} \leq \varepsilon \leq \dfrac{R-2a}{2a}$, and T_0 is positive .

Lemma 3.1. Let X, Y, and ϕ be positive, with $p, q \geq 1, \dfrac{1}{p} + \dfrac{1}{q} = 1$. Then,

$$XY \leq \frac{\phi^p X^p}{p} + \frac{Y^q}{q\phi^q}. \tag{3.3}$$

Lemma 3.2. Let $(A_1), (A_2)$ hold, $u_0 \in H_0^L(\Omega)$, and u be a solution of (1.1) – (1.3). Then, we have

$$\Phi''(t) - 4(1+\varepsilon) \int_0^t \|u_t\|^2 \, ds \geq \Pi(t) \tag{3.4}$$

where $\Pi(t) = -4(1+\varepsilon)E(0) + w[\beta\|D^L u\|^2 + (g \circ D^L u)],\ w = 2\varepsilon - \dfrac{1}{2\beta} > 0.$

Proof. From (3.1), a direct calculation yields that

$$\begin{aligned}\Phi'(t) &= \|u\|^2 + \|D^K u\|^2 \\ &= 2\int_0^t \int_\Omega u u_t \, dx \, d\tau + \|u_0\|^2 + 2\int_0^t \int_\Omega D^K u D^K u_t \, dx \, ds + \|D^K u_0\|^2, \end{aligned} \tag{3.5}$$

$$\begin{aligned}\Phi''(t) &= 2\int_\Omega u u_t \, dx + 2\int_\Omega D^K u D^K u_t \, dx \\ &= 2\int_\Omega u \left[-(-\triangle)^L u - (-\triangle)^K u_t + \int_0^t g(t-s)(-\triangle)^L u(s) \, ds + a|u|^{R-2} u \right] dx \\ &+ 2\int_\Omega D^K u D^K u_t \, dx \\ &= -2\|D^L u\|^2 - \frac{d}{dt}\|D^K u\|^2 + 2\int_0^t \int_\Omega g(t-s) D^L u(s) D^L u(t) \, dx \, ds \\ &+ 2a\|u\|_R^R + \frac{d}{dt}\|D^k u\|^2. \end{aligned} \tag{3.6}$$

We infer from (2.2), (2.3), and (3.6) that

$$\begin{aligned}&\Phi''(t) - 4(1+\varepsilon)\int_0^t \|u_t\|^2 \, ds \\ &= \Phi''(t) - 4(1+\varepsilon)E(t) + (2+2\varepsilon)\left(1 - \int_0^t g(s) \, ds\right) \| D^L u \|^2 \\ &+ (2+2\varepsilon)(g \circ D^L u) - \frac{4a(1+\varepsilon)}{R} \| u \|_R^R \\ &\geq -4(1+\varepsilon)E(0) + 2\varepsilon \| D^L u \|^2 + (2+2\varepsilon)(g \circ D^L u) + \left[2 - \frac{4a(1+\varepsilon)}{R}\right] \| u \|_R^R \\ &- (2+2\varepsilon)\int_0^t g(s) \, ds \|D^L u\|^2 + 2\int_0^t \int_\Omega g(t-s) D^L u(s) D^L u(t) \, dx \, ds. \end{aligned} \tag{3.7}$$

Applying Lemma 3.1 yields

$$\begin{aligned}&\int_0^t \int_\Omega g(t-s) D^L u(t) D^L u(s) \, dx \, ds \\ &= \int_0^t \int_\Omega g(t-s) D^L u(t) [D^L u(s) - D^L u(t)] \, dx \, ds + \int_0^t \int_\Omega g(t-s) D^L u(t) D^l u(t) \, dx \, ds \\ &\geq -(g \circ D^L u)(t) + \frac{3}{4}\int_0^t g(s) \, ds \|D^L u(t)\|^2. \end{aligned} \tag{3.8}$$

Combining (3.7) and (3.8), we get

$$\begin{aligned}
&\Phi''(t) - 4(1+\varepsilon)\int_0^t \|u_t\|^2\,ds\\
&\geq -4(1+\varepsilon)E(0) + 2\varepsilon\|D^L u\|^2 + 2\varepsilon(g\circ D^L u) - \Big(\frac{1}{2}+2\varepsilon\Big)\int_0^t g(s)\,ds\|D^L u\|^2\\
&> -4(1+\varepsilon)E(0) + 2\varepsilon\|D^L u\|^2 + 2\varepsilon(g\circ D^L u) + \Big(\frac{1}{2}+2\varepsilon\Big)(\beta-1)\|D^L u\|^2\\
&> -4(1+\varepsilon)E(0) + w[\beta\|D^L u\|^2 + (g\circ D^L u)(t)] \qquad (3.9)
\end{aligned}$$

where $w = 2\varepsilon - \dfrac{1}{2\beta}$.

Therom 3.1. Let assumptions (A_1) and (A_2) hold, and $T_0 < \dfrac{1}{\|u_0\|^2 + \|D^K u_0\|^2}$. In addition, it is assumed that one of the following conditions holds true:

$$\begin{aligned}
&(1)E(0) < 0;\\
&(2)0 \leq E(0) < \frac{w}{R-2H}, \quad \beta^{\frac{1}{2}}\|D^L u_0\| > \xi_1;\\
&(3)0 < \frac{w}{R-2H} < E(0) < \frac{\|u_0\|^2 + \|D^K u_0\|^2}{\mu}.
\end{aligned}$$

Then, the solution of problem (1.1) – (1.3) blows up in finite time, which means the maximum time T^* of u is finite and

$$\lim_{t\to T^*}\left(\int_0^t \|u\|^2\,ds + \int_0^t \|D^K u\|^2\,ds\right) = +\infty. \qquad (3.10)$$

Case (1). if $E(0) < 0$, an upper bound on the blow-up time T^* can also be estimated according to the sign and size of energy $E(0)$. Then,

$$T^* \leq \sqrt{\frac{-(2\varepsilon+1)}{8\varepsilon^2(\varepsilon+1)E(0)}}\cdot \ln\frac{1}{1-\sqrt{T_0(\|u_0\|^2 + \|D^K u_0\|^2)}}.$$

Case (2). if $0 < E(0) < \dfrac{w}{R-2}H$, and $\xi_1 < \beta^{\frac{1}{2}}\|D^L u_0\|$, then

$$T^* \leq \sqrt{\frac{2\varepsilon+1}{8\varepsilon^2(\varepsilon+1)[\frac{w}{R-2}H - E(0)]}}.$$

Case (3). if $\dfrac{w}{R-2}H \leq E(0) < \dfrac{\|u_0\|^2 + \|D^K u_0\|^2}{\mu}$, then

$$T^* \leq \sqrt{\frac{2\varepsilon+1}{2\varepsilon^2\Lambda\chi(0)}}\cdot \ln\frac{1}{1-\sqrt{T_0(\|u_0\|^2 + \|D^K u_0\|^2)}}$$

where $\chi(0) = \|u_0\|^2 + \|D^K u_0\|^2 - \mu E(0) = \Phi'(0) - \mu E(0)$, $\mu = \dfrac{4(1+\delta)}{\Lambda}$, $\Lambda = w\beta\dfrac{1}{B}$.

Case (1). if $E(0) < 0$, from (3.9) we infer that

$$\Phi''(t) \geq -4(1+\varepsilon)E(0) + w[\beta\|D^L u\|^2 + (g \circ D^L u(t))] +$$
$$4(1+\varepsilon)\int_0^t \|u_t\|^2 ds > 0 \quad , t \geq 0. \tag{3.11}$$

Thus, it follows that $\Phi'(t)$ is monotonically increasing. Therefore, $\Phi'(t) > \Phi'(0) = \|u_0\|^2 + \|D^K u_0\|^2$ and the second derivative of Eq (3.2) gives

$$\Gamma'(t) = -\varepsilon\Gamma(t)^{1+\frac{1}{\varepsilon}}[\Phi'(t) - \|u_0\|^2], \tag{3.12}$$

$$\begin{aligned}\Gamma''(t) &= -\varepsilon\Gamma(t)^{1+\frac{1}{\varepsilon}}\{\Phi''(t)[\Phi(t) + (T_0 - t)(\|u_0\|^2 + \|D^K u_0\|^2)] \\ &\quad - (1+\varepsilon)[\Phi'(t) - \|u_0\|^2 - \|D^K u_0\|^2]^2\} \\ &= -\varepsilon\Gamma(t)^{1+\frac{2}{\varepsilon}} V(t)\end{aligned} \tag{3.13}$$

where

$$V(t) = \Phi''(t)[\Phi(t) + (T_0 - t)(\|u_0\|^2 + \|D^K u_0\|^2)] - (1+\varepsilon)[\Phi'(t) - \|u_0\|^2 - \|D^K u_0\|^2]^2.$$

From Lemma 3.2,we have

$$\begin{aligned}&\Phi''(t)[\Phi(t) + (T_0 - t)(\|u_0\|^2 + \|D^K u_0\|^2)] \\ &\geq \left[\Pi(t) + 4(1+\varepsilon)\int_0^t \|u_t\|^2\, ds\right] \\ &\left[\int_0^t \|u\|^2\, ds + \int_0^t \|D^K u\|^2\, ds + (T_0 - t)(\|u_0\|^2 + D^K u_0\|^2)\right] \\ &\geq \Pi(t)\Gamma(t)^{-\frac{1}{\varepsilon}} + 4(1+\varepsilon)\int_0^t \|u_t\|^2\, ds \int_0^t \|u\|^2\, ds \\ &+ 4(1+\varepsilon)\int_0^t \|u_t\|^2\, ds \int_0^t \|D^K u\|^2\, ds.\end{aligned} \tag{3.14}$$

Therefore,

$$\begin{aligned}&[\Phi'(t) - \|u_0\|^2 - \|D^K u_0\|^2]^2 \\ &= 4\left(\int_0^t \int_\Omega u u_t\, dx\, ds\right)^2 + 4\left(\int_0^t \int_\Omega D^K u D^K u_t\, dx\, ds\right)^2 \\ &+ 8\int_0^t \int_\Omega u u_t\, dx\, ds \int_0^t \int_\Omega D^K u D^K u_t\, dx\, ds.\end{aligned} \tag{3.15}$$

Applying Holder's inequality, Lemma 3.1 yields

$$4\left(\int_0^t \int_\Omega u u_t\, dx\, ds\right)^2 \leq 4\int_0^t \|u\|^2\, ds \cdot \int_0^t \|u_t\|^2\, ds, \tag{3.16}$$

$$4\left(\int_0^t \int_\Omega D^K u \cdot D^K u_t\, dx\, ds\right)^2 \leq 4\int_0^t \|D^K u\|^2\, ds \cdot \int_0^t \|D^K u_t\|^2\, ds, \tag{3.17}$$

$$
\begin{aligned}
&8\int_0^t\int_\Omega uu_t\,dx\,ds\cdot\int_0^t\int_\Omega D^K u\cdot D^K u_t\,dx\,ds\\
&\le 4\int_0^t\|u\|^2\,ds\cdot\int_0^t\|D^K u_t\|^2\,ds+4\int_0^t\|D^K u\|^2\,ds\cdot\int_0^t\|u_t\|^2\,ds. \qquad (3.18)
\end{aligned}
$$

Substituting (3.14) – (3.18) into (3.13) yields

$$
\begin{aligned}
V(t) &> \Pi(t)\Gamma(t)^{-\frac{1}{\varepsilon}}+4(1+\varepsilon)\int_0^t\|u_t\|^2\,ds\int_0^t\|u\|^2\,ds\\
&+4(1+\varepsilon)\int_0^t\|u_t\|^2\,ds\int_0^t\|D^K u\|^2\,ds\\
&-4(1+\varepsilon)\int_0^t\|D^K u\|^2\,ds\int_0^t\|D^K u_t\|^2\,ds-4(1+\varepsilon)\int_0^t\|u\|^2\,ds\int_0^t\|u_t\|^2\,ds\\
&-4(1+\varepsilon)\int_0^t\|u\|^2\,ds\int_0^t\|D^K u_t\|^2\,ds-4(1+\varepsilon)\int_0^t\|D^K u\|^2\,ds\int_0^t\|u_t\|^2\,ds\\
&> \Pi(t)\Gamma(t)^{-\frac{1}{\varepsilon}}. \qquad (3.19)
\end{aligned}
$$

From the definitions of (3.12), (3.19), and $\Pi(t)$, it follows that

$$
\Gamma''(t)\le-\varepsilon\Pi(t)\Gamma(t)^{1+\frac{1}{\varepsilon}}\le\varepsilon(1+\varepsilon)E(0)\Gamma(t)^{1+\frac{1}{\varepsilon}}. \qquad (3.20)
$$

From $\Phi'(t)>\Phi'(0)=\|u_0\|^2+\|D^K u_0\|^2>0$ and (3.12), we get $\Gamma'(t)<0$, $\Gamma'(0)=0$. (3.20) multiplied by $\Gamma'(t)$ and integrated over $(0,t)$ gives

$$
\begin{aligned}
\Gamma'(t)^2 &\ge -\frac{8\varepsilon^2(1+\varepsilon)}{1+2\varepsilon}E(0)\Gamma(0)^{2+\frac{1}{\varepsilon}}+\frac{8\varepsilon^2(1+\varepsilon)}{1+2\varepsilon}E(0)\Gamma(0)^{2+\frac{1}{\varepsilon}}\\
&=\rho+\psi\Gamma(t)^{2+\frac{1}{\varepsilon}} \qquad (3.21)
\end{aligned}
$$

where

$$
\rho=-\frac{8\varepsilon^2(1+\varepsilon)}{1+2\varepsilon}E(0)\Gamma(0)^{2+\frac{1}{\varepsilon}}>0, \qquad (3.22)
$$

$$
\psi=\frac{8\varepsilon^2(1+\varepsilon)}{1+2\varepsilon}E(0)<0 \qquad (3.23)
$$

where $\Gamma(0)=\left[T_0(\|u_0\|^2+\|D^K u_0\|^2)\right]^{-\varepsilon}$.
Combining (3.21) – (3.23) and Lemma 2.3 shows that there exists T^* such that $\lim_{t\to T^*}\Gamma(t)=0$. I.e.

$$
\lim_{t\to T^*}\left(\int_0^t\|u\|^2\,ds+\|D^K u\|^2\,ds\right)=+\infty.
$$

Furthermore, according to Lemma 2.3, the upper bound on the blow-up is given by

$$
T^*\le\sqrt{\frac{-(2\varepsilon+1)}{8\varepsilon^2(1+\varepsilon)E(0}}\ln\frac{1}{1-\sqrt{T_0(\|u_0\|^2+\|D^K u_0\|^2)}}. \qquad (3.24)
$$

Case (2). if $0 < E(0) < \frac{w}{R-2}H$, and $\beta^{\frac{1}{2}}\|D^L u_0\| > \xi_1$, by Lemma 2.2 and the definition of ξ_1

$$\begin{aligned}
\Pi(t) &= -4(1+\varepsilon)E(0) + w[\beta\|D^L u\|^2 + (g \circ D^L u)(t)] \\
&\geq -4(1+\varepsilon)E(0) + w\xi_2^2 \\
&\geq -4(1+\varepsilon)E(0) + w\xi_1^2 \\
&> -4(1+\varepsilon)E(0) + w\frac{4(1+\varepsilon)}{r-2}H \\
&= 4(1+\varepsilon)\left[\frac{w}{r-2}H - E(0)\right] > 0.
\end{aligned}$$

Substituting (3.25) into (3.9) yields

$$\begin{aligned}
\Phi''(t) &\geq \Pi(t) + (4+4\varepsilon)\int_0^t \|u_t\|^2\, ds \\
&> (4+4\varepsilon)\left[\frac{w}{R-2}H - E(0)\right] + (4+4\varepsilon)\int_0^t \|u_t\|^2\, ds > 0. \qquad (3.25)
\end{aligned}$$

Hence, $\Phi'(t) > \Phi'(0) = \|u_0\|^2 + \|D^K u_0\|^2 \geq 0$.
Similar to case (1), we get

$$\Gamma''(t) = -\varepsilon\Gamma(t)^{1+\frac{2}{\varepsilon}} \cdot V(t), V(t) \geq \Pi(t)\Gamma(t)^{-\frac{1}{\varepsilon}}. \qquad (3.26)$$

From (3.25) and (3.26), we get

$$\begin{aligned}
\Gamma''(t) &\leq -\varepsilon\Pi(t)\Gamma(t)^{1+\frac{1}{\varepsilon}} \\
&\leq -4\varepsilon(1+\varepsilon)\left[\frac{w}{R-2}H - E(0)\right]\Gamma(t)^{1+\frac{1}{\varepsilon}}, \quad t \geq 0. \qquad (3.27)
\end{aligned}$$

Similar to case (1) , we have $\Gamma'(t) < 0, \Gamma(0) = 0$. (3.27) Multiply by $\Gamma'(t)$ and integratig over $(0, t)$ gives

$$\begin{aligned}
\Gamma'(t)^2 &\geq \frac{8\varepsilon^2(\varepsilon+1)}{2\varepsilon+1}\left[\frac{w}{R-2}H - E(0)\right]\left[H(0)^{2+\frac{1}{\varepsilon}} - H(t)^{2+\frac{1}{\varepsilon}}\right] \\
&= \rho_1 + \psi_1\Gamma(t)^{2+\frac{1}{\varepsilon}} \qquad (3.28)
\end{aligned}$$

where

$$\rho_1 = \frac{8\varepsilon^2(\varepsilon+1)}{2\varepsilon+1}\left[\frac{w}{R-2}H - E(0)\right]\Gamma(0)^{2+\frac{1}{\varepsilon}} > 0, \qquad (3.29)$$

$$\psi_1 = \frac{8\varepsilon^2(\varepsilon+1)}{2\varepsilon+1}\left[\frac{w}{R-2}H - E(0)\right] > 0. \qquad (3.30)$$

By Lemma 2.3 and (3.28) − (3.30), there exists T^* such that

$$\lim_{t\to T^*}\Gamma(t) = 0,$$

$$\lim_{t\to T^*}\left(\int_0^t \|u\|^2 ds + \int_0^t \|D^K u\|^2\, ds\right) = +\infty$$

and

$$T^* \leq \sqrt{\frac{2\varepsilon+1}{8\varepsilon^2(\varepsilon+1)[\frac{w}{R-2}H-E(0)]}}\ln\frac{1}{1-\sqrt{T_0(||u_0||^2+||D^K u_0||^2)}}. \tag{3.31}$$

Case (3) : $\dfrac{w}{R-2}H \leq E(0) < \dfrac{||u_0||^2+||D^K u_0||^2}{\mu}$.
Define

$$\chi(t) = ||u||^2 + ||D^K u||^2 - \mu E(0) = \Phi'(t) - \mu E(0) \tag{3.32}$$

where $\mu = \dfrac{4(1+\varepsilon)}{\Lambda},\ \Lambda = w\beta\dfrac{1}{B}$.

$$\begin{aligned}
\frac{d}{dt}\chi(t) &= \Phi''(t)\\
&\geq -4(1+\varepsilon)E(0) + w[\beta||D^L u||^2 + (g\circ D^L u)(t)] + 4(1+\varepsilon)\int_0^t ||u_t||^2\,ds\\
&- 4(1+\varepsilon)E(0) + w\beta\frac{1}{\beta}||u||^2 + 4(1+\varepsilon)\int_0^t ||u_t||^2\,ds\\
&= \frac{w\beta}{\beta}[||u||^2 - \frac{4(1+\varepsilon)B}{w\beta}E(0)] + 4(1+\varepsilon)\int_0^t ||u_t||^2\,ds\\
&> \Lambda[||u||^2 + ||D^K u||^2 - \mu E(0)] + 4(1+\delta)\int_0^t ||u_t||^2\,ds\\
&= \Lambda\chi(t) + 4(1+\varepsilon)\int_0^t ||u_t||^2\,ds.
\end{aligned} \tag{3.33}$$

According to (3.31) and

$$||u_0||^2 + ||D^K u||^2 - \mu E(0) = \Phi'(0) - \mu E(0) = \chi(0) > 0 \tag{3.34}$$

we have $\dfrac{d}{dt}\chi(t) \geq \Lambda\chi(t)$, i.e., $\chi(t) \geq \chi(0)e^{\Lambda t}$. Thereby, we have

$$\chi(t) = \Phi'(t) - \mu E(0) \geq \chi(0)e^{\Lambda t} \geq \chi(0) > 0, \quad t \geq 0. \tag{3.35}$$

By (3.33) − (3.35), we obtain

$$\frac{d}{dt}\chi(t) = \Phi''(t) \geq \Lambda\chi(t) \geq \Lambda\chi(0) > 0. \tag{3.36}$$

Thus, we get

$$\Phi'(t) > \Phi'(0) = ||u_0||^2 + ||D^K u_0||^2 > 0, \quad t > 0.$$

Similar to the process in case (1), it is possible to derive

$$\Gamma''(t) \leq -\varepsilon\Pi(t)\Gamma(t)^{1+\frac{1}{\varepsilon}}, \quad t \geq 0. \tag{3.37}$$

By (3.33) − (3.35),we conclude that

$$\Pi(t) \geq \Lambda\chi(t) \geq \Lambda\chi(0).$$

Consequently,

$$\Gamma''(t) \leq -\varepsilon\Pi(t)\Gamma(t)^{1+\frac{1}{\varepsilon}} \leq -\varepsilon\Lambda\chi(0)\Gamma(t)^{1+\frac{1}{\varepsilon}}, \quad t > 0. \tag{3.38}$$

Multiplying both sides of (3.38) by $\Gamma(t)$, and integrating over $[0, t]$, we have

$$\begin{aligned}\Gamma'(t)^2 &\geq \frac{2\varepsilon^2\Gamma\chi(0)}{2\varepsilon+1}\left[\Gamma_0^{2+\frac{1}{\varepsilon}} - \Gamma(t)^{2+\frac{1}{\varepsilon}}\right]\\ &= \rho_2 + \psi_2\Gamma(t)^{2+\frac{1}{\varepsilon}}, \end{aligned}\tag{3.39}$$

$$\rho_2 = \frac{2\varepsilon^2\Gamma\chi(0)}{2\varepsilon+1}\Gamma(0)^{2+\frac{1}{\varepsilon}} > 0, \psi_2 = \frac{2\varepsilon^2\Gamma\chi(0)}{2\varepsilon+1} > 0. \tag{3.40}$$

By Lemma 2.3 and (3.39) − (3.40), there exists a time T^* such that

$$\lim_{t\to T^*}\left(\int_0^t \|u\|^2\,ds + \int_0^t \|D^K u\|^2\,ds\right) = +\infty$$

and

$$T^* \leq \sqrt{\frac{2\varepsilon+1}{2\varepsilon^2\Lambda\chi(0)}}\ln\frac{1}{1 - \sqrt{T_0(\|u_0\|^2 + \|D^K u_0\|^2)}}.$$

4. Lower bound on blow-up time

This section investigates a lower bound on the blow-up time T^* when the solution of Eqs (1.1)−(1.3) occurs in finite time.

Theorem 4.1. Let A_1 and A_2 hold, $u_0 \in H_0^L(\Omega)$, and u be a solution of Eqs (1.1) − (1.3). If u blows up in the sense of $H_0^L(\Omega)$, then the lower bound T^* of the blow-up can be estimated as

$$T^* \geq \int_{R(0)}^{+\infty} \frac{1}{K_1 + \frac{a(4+2R)}{r}B^R K_1^{\frac{R}{2}} + 4E(0)}\,dK_1.$$

Proof. Let

$$R(t) = \|u\|^2 + \|D^L u\|^2. \tag{4.1}$$

Differentiating (4.1) with respect to t, we know from (1.1) that

$$\begin{aligned}R'(t) &= 2\int_\Omega u \cdot u_t\,dx + \frac{d}{dt}\|D^L u\|^2\\ &= 2\int_\Omega u[-(-\Delta)^L u - (-\Delta)^K u_t]\\ &+ 2\int_\Omega u[\int_0^t g(t-s)(-\Delta)^L u(s)\,ds + a|u|^{R-2}u + \frac{d}{dt}\|D^L u\|^2]\end{aligned}\tag{4.2}$$

$$= -2\|D^L u\|^2 + 2\iint_\Omega g(t-s)D^L u(s)D^L u(t)\,dx\,ds \tag{4.3}$$

$$+ 2a\|u\|_R^R - \frac{d}{dt}\|D^K u\|^2 + \frac{d}{dt}\|D^L u\|^2. \tag{4.4}$$

By Lemma 3.1, we have

$$\begin{aligned}
&2\int_0^t \int_\Omega g(t-s)D^L u(s)D^L u(t)\,dx\,ds \\
&\le 2(g \circ D^L u\|^2 + \frac{1}{2}\int_0^t g(s)\,ds\|D^L u\|^2 + 2\int_0^t g(s)\|D^L u\|^2 \\
&= 2(g \circ D^L u\|^2 + \frac{5}{2}\int_0^t g(s)\,ds\|D^L u\|^2.
\end{aligned} \tag{4.5}$$

Substituting (4.5) into (4.4) yields

$$\begin{aligned}
R'(t) \le& \left[\frac{5}{2}\int_0^t g(s)\,ds - 1\right]\|D^L u\|^2 + \frac{1}{2}\|D^L\|^2 + 2(g \circ D^L u)(t) + 2a\|u\|_R^R \\
&- \frac{d}{dt}\|D^K u\|^2 + \frac{d}{dt}\|D^L u\|^2 \\
<& \frac{1}{2}\|D^L\|^2 + 4\int_0^t \|u_t\|^2\,ds + 2\left(1 - \int_0^t g(s)\,ds\right)\|D^L u\|^2 + 2(g \circ D^L u)(t) \\
&- \frac{4a}{R}\|u\|_R^R + \Big(\frac{4a}{R} + 2a\Big)\|u\|_R^R \\
<& R(t) + 4E(0) + a\Big(\frac{4}{R} + 2\Big)B^R\|D^L u\|^R \\
<& R(t) + 4E(0) + a\Big(\frac{4}{R} + 2\Big)B^R R(t)^{\frac{R}{2}}.
\end{aligned} \tag{4.6}$$

Integrating (4.6) over $[0, t]$ yields

$$\int_{R(0)}^{R(t)} \frac{1}{K_1 + a\frac{4+2R}{R}B^R K_1^{\frac{R}{2}} + 4E(0)}\,dK_1 \le t. \tag{4.7}$$

If u blows up with H_0^L, then T^* has a lower bound

$$T^* \ge \int_{R(0)}^{+\infty} \frac{1}{K_1 + a\frac{4+2R}{R}B^R K_1^{\frac{R}{2}} + 4E(0)}, \tag{4.8}$$

which thereby completes the proof of Theorem 4.1.

5. Conclusions

By using concavity analysis, we get the blow-up results of the solution when the initial energy is negative or positive and an upper bound on the blow-up time T^*. In addition, a lower bound on the blow-up time T^* is obtained by applying differential inequalities in the case where the solution has a blow-up.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there is no conflict of interest.

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