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Research article

Existence results for a class of nonlinear singular p-Laplacian Hadamard fractional differential equations

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Abstract: Based on properties of Green's function and the some conditions of $f(t, u)$, we found a minimal and a maximal positive solution by the method of sequence approximation. Moreover, based on the properties of Green's function and fixed point index theorem, the existence of multiple positive solutions for a singular p-Laplacian fractional differential equation with infinite-point boundary conditions was obtained and, at last, an example was given to demonstrate the validity of our main results.

Keywords: Hadamard fractional differential equation; multiple positive solutions; infinite-point; minimal and maximal positive solutions

1. Introduction

Fractional-order differential operator has become one of the most important tools for mathematical modeling of complex mechanics and physical processes because it can describe mechanical and physical processes with historical memory and spatial global correlation succinctly and accurately. Additionally, and the fractional-order derivative modeling is simple, the physical meaning of parameters is clear and the description is accurate. In recent years, fractional derivative has become an important tool to describe all kinds of complex mechanical and physical behaviors, so the study of a positive solutions of fractional differential equations has attracted much attention. For an extensive collection of such literature, readers can refer to [1, 2, 2–12]. Alsaedi et al. [13] investigated the following equation

$$^{\rho}D^{\alpha}_{1^-}(\varphi_P(^{\rho}D^{\beta}_{0^+}u(t))) = v_1 f(t, u(t),^{\rho} D^{\beta}_{0^+}u(t))) + v_2^{\rho} I^{\xi}_{0^+} g(t, u(t),^{\rho} D^{\beta}_{0^+}u(t)))$$

with boundary value condition

$$u(0) = 0, u(1) = \lambda_1 u(\mu),^{\rho} D^{\beta}_{0^+}u(1) = 0, \varphi_p(^{\rho}D^{\beta}_{0^+}u(0)) = \lambda_2\varphi_p(^{\rho}D^{\beta}_{0^+}u(\eta))$$

where $\varphi_p(t) = |t|^{p-2} \cdot t$, $\frac{1}{p} + \frac{1}{q} = 1, p, q > 1, 1 < \alpha, \beta \leq 2, \rho > 0, \zeta > 0, 0 < \mu, \eta < 1, 0 \leq \lambda_1 < \frac{1}{\mu^{\rho(\beta-1)}}, 0 \leq \lambda_2 < \frac{1}{(1-\eta^{\rho})^{\alpha-1}}$, and $^{\rho}D^{\alpha}_{0^+}u$ and $^{\rho}D^{\beta}_{0^+}u$ denote the right and left fractional derivatives of orders α and β with respect to a power function, respectively. The authors proved the uniqueness of positive solutions for the given problem for the cases $1 < p \leq 2$ and $p > 2$ by applying an efficient novel approach together with the Banach contraction mapping principle. Li and Liu [8] considered the fractional differential equation

$$^{C}_{0}D^{\alpha}_{g}u(t) + f(t, u) = 0, 0 < t < 1$$

with boundary value condition

$$x(0) = 0,\ ^{C}_{0}D^{1}_{g}u(0) = 0,\ ^{C}_{0}D^{1}_{g}u(1) = \int_0^1 h(t)^{C}_{0}D^{v}_{g}u(t)g'(t)dt$$

where $2 < \alpha < 3, 1 < v < 2, \alpha - v - 1 > 0$, $f \in C([0, 1] \times \mathbb{R}^+, \mathbb{R}^+)$, $g' > 0$, $h \in C([0, 1], \mathbb{R}^+)$, $\mathbb{R}^+ = [0, +\infty)$, and the existence of multiple solutions for the following system by the fixed point theorem are on cone.

In recent years, more and more scientists have devoted themselves to the study of Hadamard fractional differential systems. For the part of outstanding results of Hadamard's research on fractional differential systems, please refer to [14–16]. Ardjouni [14] studied the following Hadamard fractional differential equations

$$^{H}D^{\alpha}_{1^+}u(t) + \phi(t, u(t)) =^{H} D^{\beta}_{1^+}\varphi(t, u(t)), 1 < t < e$$

with integral boundary conditions

$$u(1) = 0, u(e) = \frac{1}{\Gamma(\alpha - \beta)} \int_1^e (\log \frac{e}{s})^{\alpha-\beta-1} g(s, x(s))\frac{ds}{s}$$

where $1 < \alpha \leq 2, 0 < \beta \leq \alpha - 1$, $g, f : [1, e] \times [0, \infty) \to [0, \infty)$ are given continuous functions, ϕ is not required for any monotone assumption, and φ is nondecreasing on x. The authors get the existence and uniqueness of the positive solution by the method of upper and lower solutions and Schauder and Banach fixed point theorems. In [15], Berhail and Tabouche studied the following fractional differential equation

$$^{H}D^{\alpha}_{1^+}u(t) = f(t, u(t), u'(t)), 1 < t < T, T < e$$

with integral boundary conditions

$$\begin{aligned} &u''(1) = 0, u^{(3)}(1) = 0, \\ &u(1) + au'(1) = \int_1^T g_1(s)u(s)ds, \quad u(T) - bu'(T) = \int_1^T g_2(s)u(s)ds \end{aligned}$$

where $3 < \alpha \leq 4$, $g_1, g_2 \in C([1,T],[0,+\infty))$, $a, b > 0$, and ${}^HD^{\alpha}_{1^+}$ denotes the Hadamard factional order of α. Based on the properties of Green's function, the authors get the existence of a positive solution for the equation in [15] by the Avery-Peterson fixed point theorem. In [16], the authors studied the existence of a positive solution and stability analysis of the following equation

$$ {}^HD^{\beta}_{1^+}(\phi_p(D^{\alpha}_{1^+}x))(t) = f(t,x), 1 < t < e $$

with integral boundary conditions

$$ \begin{aligned} &x(1) = {}^H D^{\alpha}_{1^+}x(1) = u'(1) = u'(e) = 0, \\ &\phi_p({}^HD^{\alpha}_{1^+}u(e)) = \mu \int_1^e \phi_p({}^HD^{\alpha}_{1^+}x(t))\frac{dt}{t} \end{aligned} $$

where α,β, and μ are three positive real numbers with $\alpha \in (2,3]$,$\beta \in (1,2]$, and $\mu \in [0,\beta)$, $\phi_p(s) = |s|^{p-2}s$ is the P-Laplacian for $p > 1$, $s \in \mathbb{R}$, f is a continuous function on $[1,e] \times \mathbb{R}$, and ${}^HD^{\alpha}_{1^+}$,${}^H D^{\beta}_{1^+}$ is the Hadamard fractional differential equation.The authors get the positive solutions by using the fixed point methods.

Motivated by the excellent results above, in this paper, we will devote to considering the following infinite-point singular p-Laplacian Hadamard fractional differential equation:

$$ {}^HD^{\alpha}_{1^+}\left(\varphi_p\left({}^HD^{\gamma}_{1^+}u\right)\right)(t) + f(t,u(t)) = 0, 1 < t < e \tag{1.1} $$

with infinite-point boundary condition

$$ \begin{aligned} &u^{(j)}(1) = 0, j = 0,1,2,\ldots,n-2; {}^H D^{r_1}_{1^+}u(e) = \sum_{j=1}^{\infty} \eta_j {}^HD^{r_2}_{1^+}u(\xi_j), \\ &{}^HD^{\gamma}_{1^+}u(1) = 0; \varphi_p({}^HD^{\gamma}_{1^+}u(e)) = \sum_{i=1}^{\infty} \zeta_j \varphi_p({}^HD^{\gamma}_{1^+}u(\xi_j)) \end{aligned} \tag{1.2} $$

where $\alpha, \gamma \in \mathbb{R}^+ = [0,+\infty)$, $1 < \alpha \leq 2$, $n-1 < \gamma \leq n (n \geq 3)$, $r_1, r_2 \in [2, n-2]$, $r_2 \leq r_1$, p-Laplacian operator φ_p is defined as $\varphi_p(s) = |s|^{p-2}s$, p, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, and $0 < \eta_i, \zeta_i < 1, 1 < \xi_i < e (i = 1,2,\ldots,\infty)$, $f \in C([1,e] \times \mathbb{R}^1_+, \mathbb{R}^1_+))(\mathbb{R}^1_+ = [0,+\infty)$, and ${}^HD^{\alpha}_{1^+}u$,${}^H D^{\gamma}_{1^+}u$, ${}^HD^{r_i}_{1^+}u (i = 1,2)$ are the standard Hadamard fractional-order derivatives.

In this paper, we investigate the existence of positive solutions for a singular infinite-point p-Laplacian boundary value problem. Compared with [15], the equation in this paper is a p-Laplacian fractional differential equation and the method in which we used is a fixed point index and sequence approximation. Compared with [16, 17], value at infinite points are involved in the boundary conditions of the boundary value problem (1.1,1.2) and the minimal positive solution and maximal positive solution are obtained in this paper.

2. Preliminaries and lemmas

For some basic definitions and lemmas about the theory of Hadamard fractional calculus, the reader can refer to the recent literature such as [6, 9, 18].

Definition 2.1([18,19]). The Hadamard fractional integral of $\alpha(\alpha > 0)$ order of a function $\hbar : (0, \infty) \rightarrow \mathbb{R}_+^1$ is given by

$$^{H}I_{1^+}^{\alpha}\hbar(t) = \frac{1}{\Gamma(\alpha)} \int_1^t (\ln \frac{t}{s})^{\alpha-1} \frac{\hbar(s)}{s} ds$$

Definition 2.2([18, 19]). The Hadamard fractional derivative of $\alpha(\alpha > 0)$ order of a continuous function $\hbar : (0, \infty) \rightarrow \mathbb{R}_+^1$ is given by

$$^{H}D_{1^+}^{\alpha}\hbar(t) = \frac{1}{\Gamma(n-\alpha)} \left(t\frac{d}{dt}\right)^n \int_1^t \frac{\hbar(s)}{s\left(\ln \frac{t}{s}\right)^{\alpha-n+1}} ds$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the number α, provided that the righthand side is pointwise defined on $(0, \infty)$.
Lemma 2.1([18,19]). If $\alpha, \beta > 0$, then

$$^{H}I_{1^+}^{\alpha}(\ln x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)} (\ln x)^{\beta+\alpha-1}, ^{H}D_{1^+}^{\alpha}(\ln x)^{\beta-1} = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)} (\ln x)^{\beta-\alpha-1}$$

Lemma 2.2([19]). Suppose that $\alpha > 0$ and $\hbar \in C[0, \infty) \cap L^1[0, \infty)$, then the solution of Hadamard fractional differential equation $^{H}D_{1^+}^{\alpha}\hbar(t) = 0$ is

$$\hbar(t) = c_1(\ln t)^{\alpha-1} + c_2(\ln t)^{\alpha-2} + \cdots + c_n(\ln t)^{\alpha-n}, c_i \in R(i = 0, 1, \cdots, n), n = [\alpha] + 1$$

Lemma 2.3([19]). Suppose that $\alpha > 0, \alpha$ is not a natural number and $\hbar \in C[1, \infty) \cap L^1[1, \infty)$, then

$$\hbar(t) =^{H} I_{1^+}^{\alpha}\, ^{H}D_{1^+}^{\alpha}\hbar(t) + \sum_{k=1}^{n} c_k(\ln t)^{\alpha-k}$$

for $t \in (1, e]$ where $c_k \in R(k = 1, 2, \cdots, n)$, and $n = [\alpha] + 1$.

Lemma 2.4. *Let $y \in L^1(1, e) \cap C(1, e)$, then the equation of the BVPs*

$$-^{H} D_{1^+}^{\gamma}u(t) = y(t), \ 1 < t < e \tag{2.1}$$

with boundary condition $u^{(j)}(1) = 0(j = 0, 1, 2, \ldots, n-2),^{H} D_{1^+}^{r_1}u(e) = \sum_{j=1}^{\infty} \eta_j^{H}D_{1^+}^{r_2}u(\xi_j)$ has integral representation

$$u(t) = \int_1^e \Psi(t, s)y(s)\frac{ds}{s} \tag{2.2}$$

where

$$\Psi(t, s) = \frac{1}{\Delta\Gamma(\gamma)} \begin{cases} \Gamma(\gamma)(\ln t)^{\gamma-1}\Xi(s)(\ln e - \ln s)^{\gamma-r_1-1} - \Delta(\ln t - \ln s)^{\gamma-1}, \\ 1 \leq s \leq t \leq e, \\ \Gamma(\gamma)(\ln t)^{\gamma-1}\Xi(s)(\ln e - \ln s)^{\gamma-r_1-1}, \ 1 \leq t \leq s \leq e \end{cases} \tag{2.3}$$

in which

$$\Xi(s) = \frac{1}{\Gamma(\gamma-r_1)} - \frac{1}{\Gamma(\gamma-r_2)} \sum_{s \leq \xi_j} \eta_j \left(\frac{\ln \xi_j - \ln s}{\ln e - \ln s}\right)^{\gamma-r_2-1} (\ln e - \ln s)^{r_1-r_2},$$

$$\Delta = \frac{\Gamma(\gamma)}{\Gamma(\gamma - r_1)} - \frac{\Gamma(\gamma)}{\Gamma(\gamma - r_2)} \sum_{j=1}^{\infty} \eta_j \ln \xi_j^{\gamma - r_2 - 1} \neq 0 \tag{2.4}$$

Proof. By means of the Lemma 2.3, we reduce (2.1) to an equivalent integral equation

$$u(t) = -{}^H I_{1^+}^{\gamma} y(t) + C_1(\ln t)^{\gamma-1} + C_2(\ln t)^{\gamma-2} + \cdots + C_n(\ln t)^{\gamma-n}$$

for $C_i (i = 1, 2, \cdots, n) \in \mathbb{R}$. From $u(0) = 0$, we have $C_n = 0$, then taking the first derivative, we have

$$u'(t) = -{}^H I_{1^+}^{\gamma-1} y(t) + C_1(\gamma - 1)(\ln t)^{\gamma-1}\frac{1}{t} + C_2(\gamma - 2)(\ln t)^{\gamma-3}\frac{1}{t} + \cdots + \ldots$$

By $u'(1) = 0$, we have $C_{n-1} = 0$, and taking the derivative step by step and combining $u^{(i)}(1) = 0, (i = 2, \cdots, n-2)$, we have $C_i = 0 (i = 2, 3, \cdots, n-2)$. Consequently, we get

$$u(t) = C_1(\ln t)^{\gamma-1} -{}^H I_{1^+}^{\gamma} y(t)$$

By some properties of the fractional integrals and fractional derivatives, we have

$$\begin{aligned} {}^H D_{1^+}^{r_1} u(t) &= C_1 \frac{\Gamma(\gamma)}{\Gamma(\gamma - r_1)} (\ln t)^{\gamma - r_1 - 1} -{}^H I_{1^+}^{\gamma - r_1} y(t), \\ {}^H D_{1^+}^{r_2} u(t) &= C_1 \frac{\Gamma(\gamma)}{\Gamma(\gamma - r_2)} (\ln t)^{\gamma - r_2 - 1} -{}^H I_{1^+}^{\gamma - r_2} y(t) \end{aligned} \tag{2.5}$$

On the other hand, ${}^H D_{1^+}^{r_1} u(1) = \sum_{j=1}^{\infty} \eta_j^H D_{1^+}^{r_2} u(\xi_j)$, and combining with (2.5), we get

$$\begin{aligned} C_1 &= \int_1^e \frac{(\ln e - \ln s)^{\gamma - r_1 - 1}}{\Gamma(\gamma - r_1)\Delta} y(s) \frac{ds}{s} - \sum_{j=1}^{\infty} \eta_j \int_1^{\xi_j} \frac{(\ln \xi_j - \ln s)^{\gamma - r_2 - 1}}{\Gamma(\beta - r_2)\Delta} y(s) \frac{ds}{s} \\ &= \int_1^e \frac{(\ln e - \ln s)^{\gamma - r_1 - 1} \Xi(s)}{\Delta} y(s) \frac{ds}{s} \end{aligned}$$

where $\Xi(s), \Delta$ are as (2.3), then,

$$\begin{aligned} u(t) &= C_1(\ln t)^{\gamma-1} -{}^H I_{1^+}^{\gamma} y(t) \\ &= -\int_1^t \frac{\Delta(\ln t - \ln s)^{\gamma-1}}{\Gamma(\gamma)\Delta} y(s) \frac{ds}{s} + \int_1^e \frac{(\ln e - \ln s)^{\gamma - r_1 - 1}(\ln t)^{\gamma-1}\Xi(s)}{\Delta} y(s) \frac{ds}{s} \\ &= \int_1^e \Psi(t, s) y(s) \frac{ds}{s} \end{aligned}$$

Therefore, $\Psi(t, s)$ is as (2.3). □

Lemma 2.5. *The Green functions* (2.3) *have the following properties:*

$(i)\Psi(t, s) > 0, \frac{\partial}{\partial t}\Psi(t, s) > 0, 1 < t, s < e$;

$(ii) \max_{t\in[1,e]} \Psi(t, s) = \Psi(e, s) = \frac{1}{\Delta\Gamma(\gamma)}\left[\Gamma(\gamma)\Xi(s)(\ln e - \ln s)^{\gamma - r_1 - 1} - \Delta(\ln e - \ln s)^{\gamma-1}\right]$;

$(iii)\Psi(t,s)\geq(\ln t)^{\alpha-1}\Psi(e,s), 1\leq t,s\leq e$

Proof. (i) By simple calculation, we have $\Xi'(s)>0$. For $s\in[1,e]$, we have $\Xi(s)\geq\Xi(1)$, then by the expression of $\Xi(s)$ and Δ, we have

$$\Gamma(\gamma)\Xi(s)\geq\Delta=\Gamma(\gamma)\Xi(1)$$

For $1<s\leq t<e$ by the preceding formula, we have

$$\begin{aligned}\Psi(t,s)=&\frac{1}{\Delta\Gamma(\gamma)}[\Gamma(\gamma)(\ln t)^{\gamma-1}\Xi(s)(\ln e-\ln s)^{\gamma-r_1-1}-\Delta(\ln t-\ln s)^{\gamma-1}]\\ =&\frac{1}{\Delta\Gamma(\gamma)}(\ln t)^{\gamma-1}[\Gamma(\gamma)\Xi(s)-(\ln e-\ln s)^{\gamma-r_1-1}-\Delta(1-\frac{\ln s}{\ln t})^{\gamma-1}]\\ \geq&\frac{1}{\Gamma(\gamma)}(\ln t)^{\gamma-1}[(\ln e-\ln s)^{\gamma-1}-(1-\frac{\ln s}{\ln t})^{\gamma-1}]\geq 0\end{aligned}$$

For $1<t\leq s<e$, obviously, $\Psi(t,s)>0$. Furthermore, by direct calculation, we get

$$\frac{\partial}{\partial t}\Psi(t,s)=\frac{1}{\Delta\Gamma(\gamma)}\begin{cases}\dfrac{(\gamma-1)\Gamma(\gamma)(\ln t)^{\gamma-2}\Xi(s)(\ln e-\ln s)^{\gamma-r_1-1}-\Delta(\ln t-\ln s)^{\gamma-1}}{t},\\ 1\leq s\leq t\leq e,\\ \dfrac{(\gamma-1)\Gamma(\gamma)(\ln t)^{\gamma-2}\Xi(s)(\ln e-\ln s)^{\gamma-r_1-1}}{t},\ 1\leq t\leq s\leq e\end{cases}\tag{2.6}$$

Clearly, $\frac{\partial}{\partial t}\Psi(t,s)$ is continuous on $[1,e]\times[1,e]$. By the similar method, for $1<s\leq t<e$, we get

$$\begin{aligned}\frac{\partial}{\partial t}\Psi(t,s)=&\frac{1}{\Delta\Gamma(\gamma)t}[(\gamma-1)\Gamma(\gamma)(\ln t)^{\gamma-2}\Xi(s)(\ln e-\ln s)^{\gamma-r_1-1}-\Delta(\gamma-1)(\ln t-\ln s)^{\gamma-2}]\\ =&\frac{\gamma-1}{\Delta\Gamma(\gamma)}(\ln t)^{\gamma-2}[\Gamma(\gamma)\Xi(s)(\ln e-\ln s)^{\gamma-r_1-1}-\Delta(1-\frac{\ln s}{\ln t})^{\gamma-2}]\\ \geq&\frac{\gamma-1}{\Gamma(\gamma)t}(\ln t)^{\gamma-2}[(\ln e-\ln s)^{\gamma-r_1-1}-(1-\frac{\ln s}{\ln t})^{\gamma-2}]\\ \geq&\frac{\gamma-1}{\Gamma(\gamma)t}(\ln t)^{\gamma-2}[(\ln e-\ln s)^{\gamma-2}-(1-\frac{\ln s}{\ln t})^{\gamma-2}]\geq 0\end{aligned}$$

(ii) By (i), we can easily get $\Psi(t,s)$ is increasing on t; hence, we get

$$\max_{t\in[1,e]}\Psi(t,s)=\Psi(e,s)=\frac{1}{\Delta\Gamma(\gamma)}[\Gamma(\gamma)\Xi(s)(\ln e-\ln s)^{\gamma-r_1-1}-\Delta(\ln e-\ln s)^{\gamma-1}], 1\leq s\leq e$$

(iii) For $1\leq s\leq t\leq e$, we have

$$\begin{aligned}\Psi(t,s)=&\frac{1}{\Delta\Gamma(\gamma)}[\Gamma(\gamma)(\ln t)^{\gamma-1}\Xi(s)(\ln e-\ln s)^{\gamma-r_1-1}-\Delta(\ln t-\ln s)^{\gamma-1}]\\ =&\frac{1}{\Delta\Gamma(\gamma)}(\ln t)^{\gamma-1}[\Gamma(\gamma)\Xi(s)-(\ln e-\ln s)^{\gamma-r_1-1}-\Delta(1-\frac{\ln s}{\ln t})^{\gamma-1}]\\ \geq&\frac{1}{\Delta\Gamma(\gamma)}(\ln t)^{\gamma-1}[\Gamma(\gamma)\Xi(s)(\ln e-\ln s)^{\gamma-r_1-1}-\Delta(1-\ln s)^{\gamma-1}]\\ =&(\ln t)^{\gamma-1}\Psi(e,s)\end{aligned}$$

For $1 \le t \le s \le e$, we have

$$\begin{aligned}\Psi(t,s) =& \frac{1}{\Delta}(\ln t)^{\gamma-1}\Xi(s)(\ln e - \ln s)^{\gamma-r_1-1}\\ =&(\ln t)^{\gamma-1}\cdot\frac{1}{\Delta}\Xi(s)(\ln e-\ln s)^{\gamma-r_1-1}\\ \ge&(\ln t)^{\gamma-1}\Psi(e,s)\end{aligned}$$

Lemma 2.6. *Let $f \in C([1,e]\times(0,+\infty),[0,+\infty))$, then the BVP* (1.1, 1.2) *has a solution*

$$u(t) = \int_1^e \Psi(t,s)\varphi_q\left(\int_1^e H(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \tag{2.7}$$

where

$$H(t,s) = \frac{1}{\overline{\Delta}\Gamma(\alpha)}\begin{cases}(\ln t)^{\alpha-1}\Gamma(\alpha)\overline{\Xi}(s)(\ln e-\ln s)^{\alpha-1} - \overline{\Delta}(\ln t-\ln s)^{\alpha-1},\\ 1\le s\le t\le e,\\ (\ln t)^{\alpha-1}\Gamma(\alpha)\overline{\Xi}(s)(\ln e-\ln s)^{\alpha-1},\ 1\le t\le s\le e,\end{cases} \tag{2.8}$$

in which

$$\overline{\Xi}(s) = \frac{1}{\Gamma(\alpha)} - \frac{1}{\Gamma(\alpha)}\sum_{s\le\xi_j}\zeta_j\left(\frac{\ln\xi_j-\ln s}{\ln e-\ln s}\right)^{\alpha-1},$$

$$\overline{\Delta} = 1-\sum_{j=1}^{\infty}\zeta_j\ln\xi_j^{\alpha-1}\neq 0$$

Proof. Let $v = \varphi_p({}^H D_{1^+}^{\gamma}u), \overline{h}(t)\in C[1,e]$, then the Eqs (1.1), (1.2) can be changed into the following equation

$${}^H D_{1^+}^{\alpha}v(t)+\overline{h}(t)=0, 1<t<e, v(1)=0, v(e)=\sum_{j=1}^{\infty}\zeta_j v(\xi_j)$$

then by the similar method with Lemma 2.4, we get $v(t) = \int_1^e H(t,s)\frac{ds}{s}$, and $H(t,s)$ is as (2.8). Let $u(t)$ be the solution of BVP (1.1, 1.2) and let $\kappa(t) ={}^H D_{1^+}^{\gamma}u(t)$. By Lemma 2.4, we have

$$u(t) = \int_1^e \Psi(t,s)\kappa(s)\frac{ds}{s} \tag{2.9}$$

Putting $v(t) = \varphi_p(\kappa(t))$, we have

$$v(t) = \int_1^e H(t,s)f(s,u(s))\frac{ds}{s} \tag{2.10}$$

Combining (2.9) and (2.10), we have (2.7). The proof of Lemma 2.6 is completed.

Lemma 2.7. *The Green functions* (2.8) *have the following properties:*

$(i) H(t,s)>0, \frac{\partial}{\partial t}H(t,s)>0, 1<t,s<e;$

$(ii)\ \max_{t\in[1,e]} H(t,s) = H(e,s) = \frac{1}{\Delta\Gamma(\alpha)}\left[\Gamma(\alpha)\overline{\Xi}(s)(\ln e - \ln s)^{\alpha-1} - \overline{\Delta}(\ln e - \ln s)^{\alpha-1}\right];$

$(iii) H(t,s) \geq (\ln t)^{\alpha-1} H(e,s),\ 1 \leq t, s \leq e$

Proof. The proof is similar to Lemma 2.5 of this paper and we omit it here.

Lemma 2.8 [20]. Let E Be a real Banach space, $P \subset E$ be a cone, and $\Omega_r = u \in P : \|u\| \leq r$. Let the operator $T : P \cap \Omega_r \to P$ be completely continuous and satisfy $Tx \neq x, \forall x \in \partial\Omega_r$, then

(i) If $\|Tx\| \leq \|x\|, \forall x \in \partial\Omega_r$, then $i(T, \Omega_r, P) = 1$;

(ii) If $\|Tx\| \geq \|x\|, \forall x \in \partial\Omega_r$, then $i(T, \Omega_r, P) = 0$

3. Main results

Now, we define the Banach space $E = C([1,e], \mathbb{R})$, which is assigned a maximum norm, that is, $\|u\| = \sup_{1\leq t\leq e} |u(t)|$. Let $P = \{u \in E | u(t) \geq 0\}$, then P is a cone in E. Define an operator $T : P \to P$ by

$$(Tu)(t) = \int_1^e \Psi(t,s)\varphi_q\left(\int_1^e H(s,\tau) f(\tau, u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s} \tag{3.1}$$

then equation (1.1, 1.2) has a solution if, and only if, the operator T has a fixed point.

Lemma 3.1 *If $f \in C([1,e] \times [0,+\infty), [0,+\infty))$, then the operator $T : P \to P$ is completely continuous.*

Proof. From the continuity and nonnegativeness of $\Psi(t,s)$ and $f(t, u(t))$, we know that $T : P \to P$ is continuous. Let $\Omega \subset P$ be bounded, then for all $t \in [1,e]$ and $u \in \Omega$, there exists a positive constant M such that $|f(t, u(t))| \leq M$. Thus,

$$\begin{aligned}
|(Tu)(t)| =& \left|\int_1^e \Psi(t,s)\varphi_q\left(\int_1^e H(s,\tau) f(\tau, u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\right| \\
\leq& \left|\int_1^e \Psi(e,s)\varphi_q\left(\int_1^e H(e,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s} \times M^{q-1}\right| \\
=& \omega M^{q-1} L
\end{aligned}$$

where

$$L = \int_1^e \Psi(e,s)\frac{ds}{s},\quad \omega = \varphi_q\left(\int_1^e H(e,\tau)\frac{d\tau}{\tau}\right)$$

which means that $T(\Omega)$ is uniformly bounded.

On the other hand, from the continuity of $\Psi(t,s)$ on $[1,e] \times [1,e]$, we have $\Psi(t,s)$ is uniformly continuous on $[1,e] \times [1,e]$. Hence, for fixed $s \in [1,e]$ and for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that $t_1, t_2 \in [1,e]$ and $|t_1 - t_2| < \delta$, $|\Psi(t_1,s) - \Psi(t_2,s)| < \frac{1}{\omega M^{q-1}}\varepsilon$, then for all $u \in \Omega$, we have

$$
\begin{aligned}
&|(Tu)(t_2) - (Tu)(t_1)| \\
&\le \int_1^e |\Psi(t_2, s) - \Psi(t_1, s)| \varphi_q \left(\int_1^e H(s,\tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \\
&\le \omega M^{q-1} \int_1^e |\Psi(t_2, s) - \Psi(t_1, s)| \frac{ds}{s} \\
&< \varepsilon
\end{aligned}
$$

which implies that $T(\Omega)$ is equicontinuous. By the Arzela-Ascoli theorem, we have that $T : P \to P$ is completely continuous. The proof is complete.

Theorem 3.1 *If $f \in C([1,e] \times [0,+\infty), [0,+\infty))$, $f(t,u)$ is nondecreasing in u, and $\lambda \in (0,+\infty)$, then BVP(1.1,1.2) has a minimal positive solution $\overline{v}$ in B_r and a maximal positive solution $\overline{\varrho}$ in B_r. Moreover, $v_m(t) \to \overline{v}(t)$, $\varrho_m(t) \to \overline{\varrho}(t)$ as $m \to \infty$ uniformly on $[1,e]$, where*

$$
v_m(t) = \int_1^e \Psi(t,s) \varphi_q \left(\int_1^e H(s,\tau) f(\tau, v_{m-1}(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \tag{3.2}
$$

and

$$
\varrho_m(t) = \int_1^e \Psi(t,s) \varphi_q \left(\int_1^e H(s,\tau) f(\tau, \varrho_{m-1}(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \tag{3.3}
$$

Proof. Let

$$
B_r = \{u \in P : \|u\| \le r\},
$$

$$
r \ge \omega M_1^{q-1} \int_1^e \Psi(e,s) \frac{ds}{s}
$$

For $u \in B_r$, there exists a positive constant M_1 such that $|f(t,u(t))| \le M_1$,

$$
\begin{aligned}
|(Tu)(t)| &= \left| \int_1^e \Psi(t,s) \varphi_q \left(\int_1^e H(s,\tau) f(\tau, u(\tau)) \frac{d\tau}{\tau} \right) \frac{ds}{s} \right| \\
&\le \left| \int_1^e \Psi(e,s) \varphi_q \left(\int_1^e H(e,\tau) \frac{d\tau}{\tau} \right) \frac{ds}{s} \times M_1^{q-1} \right| \\
&= \omega M_1^{q-1} \int_1^e G(e,s) \frac{ds}{s} \le r
\end{aligned}
$$

hence,

$$
T : B_r \to B_r
$$

By Lemma 3.1, we get that $T : B_r \to B_r$ is completely continuous. Therefore, by the Schauder fixed point theorem, the operator T has at least one fixed point, and so BVP(1.1,1.2) has at least one solution in B_r. Next, we show that BVP(1.1,1.2) has a positive solution in B_r, which is a minimal positive solution.

From (3.1) and (3.2), we have that

$$
v_m(t) = (Tv_{m-1})(t), t \in [1,e], m = 1, 2, 3, \ldots \tag{3.4}
$$

This, together with $f(t,u)$ being nondecreasing in u, yields that

$$0 = v_0(t) \le v_1(t) \le \ldots \le v_m(t) \le \ldots, t \in [1, e]$$

Since T is compact, we have that $\{v_m\}$ is a sequentially compact set. Hence, there exists $\overline{v} \in B_r$ such that $v_m \to \overline{v}(m \to \infty)$.

Let $u(t)$ be any positive solution of BVP(1.1,1.2) in B_r. Obviously, $0 = v_0(t) \le u(t) = (Tu)(t)$; thus,

$$v_m(t) \le u(t), m = 0, 1, 2, \ldots. \tag{3.5}$$

Taking limits as $m \to \infty$ in (3.5), we have $\overline{v}(t) \le u(t)$ for $t \in [1, e]$. Thus, $\overline{v}$ is a minimal positive solution, then, we show BVP(1.1,1.2)has a positive solution in B_r, which is a maximal positive solution.

Let $\varrho_0(t) = r, t \in [1, e]$ and $\varrho_1(t) = T\varrho_0(t)$. From $T : B_r \to B_r$, we get $\varrho_1 \in B_r$. Therefore,

$$0 \le \varrho_1(t) \le r = \varrho_0$$

This, together with $f(t,u)$ being nondecreasing in u, yields that

$$\ldots \le \varrho_m(t) \le \ldots \le \varrho_1(t) \le \varrho_0(t), t \in [1, e]$$

Using a proof similar to that of the above, we can show that

$$\varrho_m(t) \to \overline{\varrho}(t)(m \to \infty)$$

and

$$\overline{\varrho}(t) = \int_1^e \Psi(t,s)\varphi_q\left(\int_1^e H(s,\tau) f(\tau, \overline{\varrho}(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}$$

Let $u(t)$ be any positive solution of BVP(1.1,1.2) in B_r. It is obvious that

$$u(t) \le \varrho_0(t)$$

so

$$u(t) \le \varrho_m(t) \tag{3.6}$$

Taking limits as $m \to \infty$ in (3.6), we have $u(t) \le \overline{\varrho}(t)$ for $t \in [1, e]$. The proof is complete.

Define

$$f^0 = \lim_{u\to 0^+} \sup_{t\in[1,e]} \frac{f(t,u)}{\varphi_p(l_1\|u\|)}, f_0 = \lim_{u\to 0^+} \sup_{t\in[1,e]} \frac{f(t,u)}{\varphi_p(l_2\|u\|)},$$

$$f^\infty = \lim_{u\to +\infty} \sup_{t\in[1,e]} \frac{f(t,u)}{\varphi_p(l_3\|u\|)}, f_\infty = \lim_{u\to +\infty} \sup_{t\in[1,e]} \frac{f(t,u)}{\varphi_p(l_4\|u\|)}$$

Let

$$B_1 = \int_1^e G(e,s)(\ln s)^{(\alpha-1)(q-1)}\frac{ds}{s}$$

Theorem 3.2 *If* $f \in C([1, e] \times [0, +\infty), [0, +\infty))$ *and the following conditions hold:*

(H_1) $f_0 = f_\infty = +\infty$.

(H_2) *There exists constants* $\lambda, \hbar > 0$ *such that* $f(t,u) \leq \lambda^{q-1}\varphi_p(l_5\|u\|)$ *for* $t \in [1,e], u \in [0,\hbar]$, *then BVP(1.1,1.2)has at least two positive solutions* u_1 *and* u_2 *such that*

$$0 < \|u_1\| < \hbar < \|u_2\|$$

for

$$\lambda \in \left(\frac{1}{(l_2B_1)^{\frac{1}{q-1}}\kappa}, \frac{1}{(l_5L)^{\frac{1}{q-1}}\kappa}\right) \cap \left(\frac{1}{(l_4B_1)^{\frac{1}{q-1}}\kappa}, \frac{1}{(l_5l)^{\frac{1}{q-1}}\kappa}\right) \tag{3.7}$$

where

$$l_2B_1 > l_5L \;\; and \;\;\; l_4B_1 > l_5L$$

Proof. Since

$$f_0 = \lim_{u\to 0^+} \sup_{t\in[1,e]} \frac{f(t,u)}{\varphi_p(l_2\|u\|)} = +\infty$$

there is $\hbar_0 \in (0,\hbar)$ such that

$$f(t,u) \geq \varphi_p(l_2\|u\|) \text{ for } t \in [1,e], u \in [0,\hbar_0]$$

Let

$$\Omega_{\hbar_0} = \{u \in P : \|u\| \leq \hbar_0\}$$

then, for any $u \in \partial\Omega_{\hbar_0}$, it follows from Lemma 2.4 that

$$\begin{aligned}
(Tu)(t) = &\int_1^e \Psi(t,s)\varphi_q\left(\int_1^e H(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
\geq &\lambda^{q-1}\int_1^e (\ln t)^{\gamma-1}\Psi(e,s)\varphi_q\left(\int_1^e H(s,\tau)\varphi_p(l_2\|u\|)\frac{d\tau}{\tau}\right)\frac{ds}{s}\\
\geq &\lambda^{q-1}l_2\int_1^e (\ln t)^{\gamma-1}\Psi(e,s)\varphi_q\left(\int_1^e (\ln s)^{\alpha-1}H(e,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s}\|u\|\\
= &\lambda^{q-1}l_2\varphi_q(\kappa)\int_1^e (\ln t)^{\gamma-1}\Psi(e,s)(\ln s)^{(\alpha-1)(q-1)}\frac{ds}{s}\|u\|\\
= &\lambda^{q-1}l_2B_1\varphi_q(\kappa)\|u\|
\end{aligned} \tag{3.8}$$

where e is as (3.8), $\kappa = \int_1^e H(e,\tau)\frac{d\tau}{\tau}$, $B_1 = \int_1^e \Psi(e,s)(\ln s)^{(\alpha-1)(q-1)}\frac{ds}{s}$. Thus,

$$\|Tu\| \geq \lambda^{q-1}l_2B_1\varphi_q(\kappa)\|u\|$$

This, together with (3.7), yields that

$$\|Tu\| \geq \|u\|, \forall u \in \partial\Omega_{\hbar_0}$$

By Lemma 2.6, we get

$$i(T,\Omega_{\hbar_0},P) = 0 \tag{3.9}$$

In view of

$$f_\infty = \lim_{u\to+\infty} \sup_{t\in[1,e]} \frac{f(t,u)}{\varphi_p(l_4\|u\|)} = +\infty$$

there is $\hbar_0^\star, \hbar_0^\star > \hbar$, such that

$$f(t,u) \geq \lambda^{q-1}\varphi_p(l_4\|u\|), \text{for } \ t \in [1,e], u \in [\hbar_0^\star, +\infty)$$

Let

$$\Omega_{\hbar_0^\star} = \{u \in P : \|u\| \leq \hbar_0^\star\}$$

then, for any $u \in \partial\Omega_{\hbar_0^\star}$, it follows from Lemma 2.4 that

$$\begin{aligned}(Tu)(t) &= \int_1^e \Psi(t,s)\varphi_q\left(\int_1^e H(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\ &\geq \lambda^{q-1}\int_1^e (\ln t)^{\gamma-1}\Psi(e,s)\varphi_q\left(\int_1^e H(s,\tau)\varphi_p(l_4\|u\|)\frac{d\tau}{\tau}\right)\frac{ds}{s}\\ &\geq \lambda^{q-1}l_4\int_1^e (\ln t)^{\gamma-1}\Psi(e,s)\varphi_q\left(\int_1^e (\ln s)^{\alpha-1}H(e,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s}\\ &= \lambda^{q-1}l_4\varphi_q(\kappa)\int_1^e (\ln t)^{\gamma-1}\Psi(e,s)(\ln s)^{(\alpha-1)(q-1)}\frac{ds}{s}\|u\|\\ &= \lambda^{q-1}l_4B_1\varphi_q(\kappa)\|u\|\end{aligned}$$

Thus, by (3.7), we have

$$\|Tu\| \geq \lambda^{q-1}L_4B_1\varphi_q(\kappa)\|u\|$$

This, together with (3.7), yields that

$$\|Tu\| \geq \|u\|, \forall u \in \partial\Omega_{\hbar_0^\star}$$

By Lemma 2.6, we get

$$i(T, \Omega_{\hbar_0^\star}, P) = 0 \tag{3.10}$$

Finally, let $\Omega_\hbar = \{u \in P : \|u\| \leq \hbar\}$. For any $u \in \partial\Omega_\hbar$, it follows from Lemma 2.3 and (H_2) that

$$\begin{aligned}(Tu)(t) &= \int_1^e \Psi(t,s)\varphi_q\left(\int_1^e H(s,\tau)f(\tau,u(\tau))\frac{d\tau}{\tau}\right)\frac{ds}{s}\\ &\leq \lambda^{q-1}\int_1^e \Psi(e,s)\varphi_q\left(\int_1^e H(e,\tau)\varphi_p(l_5\|u\|)\frac{d\tau}{\tau}\right)\frac{ds}{s}\\ &\leq \lambda^{q-1}l_5\int_1^e \Psi(e,s)\varphi_q\left(\int_1^e H(e,\tau)\frac{d\tau}{\tau}\right)\frac{ds}{s}\cdot\|u\|\\ &= \lambda^{q-1}l_5\varphi_q(\kappa)\int_1^e \Psi(e,s)\frac{ds}{s}\|u\|\\ &= \lambda^{q-1}l_5L\varphi_q(\kappa)\|u\|\end{aligned}$$

where e is as (3.8), $L = \int_1^e \Psi(e,s)\frac{ds}{s}$. Thus,

$$\|Tu\| \leq \lambda^{q-1}l_5L\varphi_q(\kappa)\|u\|$$

This, together with (3.7), yields that

$$\|Tu\| \leq \|u\|, \forall u \in \partial\Omega_\hbar$$

By Lemma 2.6, we get

$$i(T,\Omega_{\hbar},P)=1 \tag{3.11}$$

From (3.9)–(3.11) and $\hbar_0<\hbar<\hbar_0^{\star}$, we get

$$i(T,\Omega_{\hbar_0}^{\star}\backslash\overline{\Omega_{\hbar}},P)=-1, i(T,\Omega_{\hbar}^{\star}\backslash\overline{\Omega}_{\hbar_0},P)=1$$

Hence, T has a fixed point $u_1\in\Omega_{\hbar}^{\star}\backslash\overline{\Omega}_{\hbar_0}$ and a fixed point $u_2\in\Omega_{\hbar_0}^{\star}\backslash\overline{\Omega_{\hbar}}$. Obviously, u_1,u_2 are both positive solutions of BVP (1.1,1.2) and $0<\|u_1\|<\hbar<\|u_2\|$. The proof of Theorem 3.2 is completed. In a similar way, we get the following result.

Corollary 3.1 *If* $f\in C([1,e]\times[0,+\infty),[0,+\infty))$, *and the following conditions hold:*

(H_3) $f^0=f^\infty=0$.

(H_4) *There exists constants* $\lambda,\hbar_2>0$ *such that* $f(t,u)\geq\lambda^{q-1}\varphi_p(l_6\|u\|)$ *for* $t\in[1,e],u\in[0,\hbar_2]$, *then BVP(1.1,1.2)has at least two positive solutions* u_1 *and* u_2 *such that*

$$0<\|u_1\|<\hbar_2<\|u_2\|$$

for

$$\lambda\in\left(\frac{1}{(l_6B_1)^{\frac{1}{q-1}}\kappa},\frac{1}{(l_3L)^{\frac{1}{q-1}}\kappa}\right)\cap\left(\frac{1}{(l_6B_1)^{\frac{1}{q-1}}\kappa},\frac{1}{(l_1L)^{\frac{1}{q-1}}\kappa}\right) \tag{3.12}$$

where

$$l_6B_1>l_3L \text{ and } l_6B_1>l_1L$$

4. An example

Example. Consider the following infinite-point $p-$Laplacian fractional differential equations

$$\begin{cases} {}^HD_{1^+}^{\frac{3}{2}}\left(\varphi_3\left({}^HD_{1^+}^{\frac{5}{2}}u\right)\right)(t)+f(t,u(t))=0,\ 1<t<e,\\ u(1)=u'(1)=0;{}^HD_{1^+}^{r_1}u(1)=\sum\limits_{j=1}^{\infty}\eta_j^H D_{1^+}^{r_2}u(\xi_j),\\ {}^HD_{1^+}^{\alpha}u(1)=0;\varphi_p\left({}^HD_{1^+}^{\frac{5}{2}}u(e)\right)=\sum\limits_{j=1}^{\infty}\zeta_i\varphi_p\left({}^HD_{1^+}^{\frac{5}{2}}u(\eta_i)\right)\end{cases} \tag{4.1}$$

where $\gamma=\frac{5}{2},\alpha=\frac{3}{2},r_1=\frac{3}{2},r_2=\frac{1}{2},p=3,q=\frac{3}{2},\eta_j=\frac{1}{2j^2},\xi_j=e^{\frac{1}{j^4}},\zeta_j=\frac{1}{2j^2}$,

$$f(t,u)=(\ln t+2)\frac{|u(t)|}{3+|u(t)|}$$

Clearly, for $f\in C([1,e]\times[0,\infty),[0,\infty))$, one can have

$$|f(t,u)|=|(\ln t+2)\frac{|u(t)|}{3+|u(t)|}|\leq 3$$

By simple calculation, we have

$$\begin{aligned}\Delta &= \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_1)} - \frac{\Gamma(\gamma)}{\Gamma(\gamma-r_2)}\sum_{j=1}^{\infty}\eta_j \ln \xi_j^{\gamma-r_2-1}\\ &= \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2}-\frac{3}{2})} - \frac{\Gamma(\frac{5}{2})}{\Gamma(\frac{5}{2}-\frac{1}{2})}\sum_{j=1}^{\infty}\frac{1}{2j^2}\left(\frac{1}{j^4}\right)^2\\ &\approx 38.18,\end{aligned}$$

$$\overline{\Delta} = 1-\sum_{i=1}^{\infty}\zeta_i \ln \xi_i^{\alpha-1} = 1-\sum_{i=1}^{\infty}\zeta_i \ln \xi_i^{\frac{1}{2}} = 1-\frac{1}{2}\sum_{i=1}^{\infty}\frac{1}{j^4} \approx 1-0.5411 = 0.4589,$$

$$\begin{aligned}P(s) &= \frac{1}{\Gamma(\gamma-r_1)} - \frac{1}{\Gamma(\gamma-r_2)}\sum_{s\le\xi_j}\eta_j\left(\frac{\ln\xi_j-\ln s}{\ln e-\ln s}\right)^{\gamma-r_2-1}(\ln e-\ln s)^{r_1-r_2}\\ &= \frac{1}{\Gamma(\frac{5}{2}-\frac{3}{2})} - \frac{1}{\Gamma(\frac{5}{2}-\frac{1}{2})}\sum_{s\le\frac{1}{j^4}}\frac{1}{2j^2}\left(\frac{\frac{1}{j^4}-\ln s}{\ln e-\ln s}\right)^{\frac{5}{2}-\frac{1}{2}-1}(\ln e-\ln s)^{\frac{3}{2}-\frac{1}{2}}\\ &= 1-\frac{1}{2}\sum_{s\le\xi_j}\left(\frac{\frac{1}{j^4}-\ln s}{\ln e-\ln s}\right)(1-s),\end{aligned}$$

$$\begin{aligned}L = \int_1^e \Psi(e,s)\frac{ds}{s} =& \frac{1}{\Delta}\int_1^e P(s)(\ln e-\ln s)^{\gamma-r_1-1}ds - \frac{1}{\Gamma(\gamma)}\int_1^e(\ln e-\ln s)^{\gamma-1}\frac{ds}{s}\\ \approx& \frac{1}{38.18}\int_1^e\left(1-\frac{1}{2}\sum_{s\le\xi_j}\left(\frac{\frac{1}{j^4}-\ln s}{\ln e-\ln s}\right)(\ln e-\ln s)\right)(\ln e-\ln s)\frac{ds}{s}\\ &-\frac{1}{\Gamma(\frac{5}{2})}\int_1^e(\ln e-\ln s)^{\frac{3}{2}}\frac{ds}{s}\\ \approx& 0.5600,\end{aligned}$$

$$\begin{aligned}\kappa = \int_1^e H(e,s)\frac{ds}{s} =& \frac{1}{\overline{\Delta}}\int_1^e \overline{P}(s)(\ln e-\ln s)^{\alpha-1}\frac{ds}{s} - \frac{1}{\Gamma(\alpha)}\int_1^e(\ln e-\ln s)^{\alpha-1}\frac{ds}{s}\\ \approx& \frac{1}{0.25}\int_1^e\left(\frac{1}{\Gamma(\frac{3}{2})}-\frac{1}{\Gamma(\frac{3}{2})}\sum_{s\le\xi_j}\eta_j\left(\frac{\ln\xi_j-\ln s}{\ln e-\ln s}\right)^{\frac{1}{2}}\right)(\ln e-\ln s)^{\frac{1}{2}}\frac{ds}{s}\\ &-\frac{1}{\Gamma(\frac{3}{2})}\int_1^e(\ln e-\ln s)^{\frac{1}{2}}\frac{ds}{s}\\ \approx& 5.08\end{aligned}$$

and

$$
\begin{aligned}
B_1 &= \int_1^e \Psi(e,s)(\ln s)^{(\alpha-1)(q-1)}\frac{ds}{s}\\
&= \int_1^e \Psi(e,s)(\ln s)^{\frac{1}{4}}\frac{ds}{s}\\
&\approx \frac{1}{38.18}\int_1^e \left(1-\frac{1}{2}\sum_{s\le\xi_j}\left(\frac{\frac{1}{j^4}-\ln s}{\ln e-\ln s}\right)(\ln e-\ln s)\right)(\ln e-\ln s)(\ln s)^{\frac{1}{4}}\frac{ds}{s}\\
&\quad -\frac{1}{\Gamma(\frac{5}{2})}\int_1^e (\ln e-\ln s)^{\frac{3}{2}}(\ln s)^{\frac{1}{4}}\frac{ds}{s}\\
&\approx 3.0235
\end{aligned}
$$

Hence, take $l_2 = 10, l_5 = 8, L = 0.56, l_4 = 6$, and we have

$$
l_2B_1 = 10\times 3.0235 > l_5L = 8\times 0.56, l_4B_1 = 6\times 3.0235 > l_5L = 8\times 0.56,
$$

then there exists $\lambda = 0.005$ such that

$$
\begin{aligned}
\lambda \in &\left(\frac{1}{(l_2B_1)^{\frac{1}{q-1}}\kappa}, \frac{1}{(l_5L)^{\frac{1}{q-1}}\kappa}\right)\cap\left(\frac{1}{(l_4B_1)^{\frac{1}{q-1}}\kappa}, \frac{1}{(l_5l)^{\frac{1}{q-1}}\kappa}\right)\\
&= (0.00022, 0.01)\cap(0.0006, 0.01),
\end{aligned}
$$

Hence, all the conditions of Theorem 3.2 hold and boundary value problem (4.1) has two positive solutions u_1, u_2.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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