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Research article

Well-posedness for heat conducting non-Newtonian micropolar fluid equations

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Abstract: In this paper, we consider the first boundary value problem for a class of steady non-Newtonian micropolar fluid equations with heat convection in the three-dimensional smooth and bounded domain Ω. By using the fixed-point theorem and introducing a family of penalized problems, under the condition that the external force term and the vortex viscosity coefficient are appropriately small, the existence and uniqueness of strong solutions of the problem are obtained.

Keywords: non-Newtonian fluid; micropolar fluid; heat convection; strong solutions; existence and uniqueness

1. Introduction

The motion of an incompressible micropolar fluid with heat conduction and a constant density is described by the following system of partial differential equations (see [1]):

$$\begin{cases} \boldsymbol{u}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} - \mathrm{div}\boldsymbol{\tau} + \nabla\pi = 2\upsilon_r \mathrm{rot}\,\boldsymbol{\omega} + \boldsymbol{f}(\theta) \\ \qquad\qquad \mathrm{div}\,\boldsymbol{u} = 0 \\ \boldsymbol{\omega}_t + (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} - (c_a + c_d)\triangle\boldsymbol{\omega} - (c_0 + c_d - c_a)\nabla\mathrm{div}\boldsymbol{\omega} = 2\upsilon_r(\mathrm{rot}\boldsymbol{u} - 2\boldsymbol{\omega}) + \boldsymbol{g}(\theta) \\ \theta_t + (\boldsymbol{u}\cdot\nabla)\theta - \mathrm{div}(\kappa\nabla\theta) = \boldsymbol{\tau} : \mathcal{D} + 4\upsilon_r(\frac{1}{2}\mathrm{rot}\boldsymbol{u} - \boldsymbol{\omega})^2 + c_0(\mathrm{div}\boldsymbol{\omega})^2 \\ \qquad\qquad +(c_a + c_d)\nabla\boldsymbol{\omega} : \nabla\boldsymbol{\omega} + (c_d - c_a)\nabla\boldsymbol{\omega} : (\nabla\boldsymbol{\omega})^T + h \end{cases} \tag{1.1}$$

Equation (1.1) comprises the conservation laws of linear momentum, mass, angular momentum, and energy, respectively. The unknown $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t)$ is the velocity vector, $\pi(\boldsymbol{x}, t)$ is the pressure, $\boldsymbol{\omega} = \boldsymbol{\omega}(\boldsymbol{x}, t)$ is the angular velocity of internal rotation of a particle, and $\theta = \theta(\boldsymbol{x}, t)$ is the temperature. The vector-valued functions $\boldsymbol{f}$, $\boldsymbol{g}$ are given external forces, and the scalar-valued function h denotes the heat source. The positive constant υ_r in (1.1) represents the dynamic micro-rotation viscosity, c_0, c_a, c_d are constants called the coefficients of angular viscosities, and κ is the heat conductivity. The viscous stress tensor $\boldsymbol{\tau} = \boldsymbol{\tau}(\mathcal{D})$, where

$$\mathcal{D} = \mathcal{D}(\boldsymbol{u}) = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$$

is the rate of deformation tensor, is also called the shear rate tensor. If the relation between the stress $\boldsymbol{\tau}$ and the strain rate $\mathcal{D}$ is linear, then the fluid is called Newtonian. If the relation is non-linear, the fluid is called non-Newtonian. For an introduction to the mechanics of non-Newtonian fluids we refer the reader to references [2, 3].

If $\boldsymbol{\omega}, \boldsymbol{g}$, and the viscosity coefficients $c_0, c_a, c_d, \upsilon_r$ are zero, system (1.1) reduces to the system of field equations of classical hydrodynamics. For the Newtonian case (i.e., $\boldsymbol{\tau} = \mu\mathcal{D}$), several variants of system (1.1) have been studied by several authors in the literature. One well-known simplified model is the Oberbeck–Boussinesq approximation, which was obtained by ignoring the dissipation term $\boldsymbol{\tau} : \mathcal{D}$. The neglect of this term considerably simplifies the analysis, and it has been widely studied by several authors from a theoretical perspective; we could refer the reader to [4–9] (and the references cited therein) for related results. If the term $\boldsymbol{\tau} : \mathcal{D}$ is not neglected, the mathematical analysis for (1.1) becomes significantly more difficult. One of the main challenges stems from the fact that this viscous dissipation term belongs, a priori, only to $L^1(Q_T)$, which makes the application of compactness arguments problematic. Related results, such as the existence, uniqueness, regularity, and large time behavior of solutions, have been investigated in previous studies; see, e.g., [10–16] and the references therein. In the non-Newtonian case, a popular technique is to assume that the tensor $\boldsymbol{\tau}$ has a p-structure. Consiglieri [17] proved the existence of weak solutions to the coupled system of stationary equations given by (1.1) with the Dirichlet boundary conditions under more general assumptions on $\boldsymbol{\tau}$ with temperature-dependent coefficients. Consiglieri and Shilkin [18] proved the existence of a weak solution, where $\boldsymbol{u}$ possesses locally integrable second-order derivatives. Under the weak assumptions on the data of the problem, Consiglieri [19] proved the existence of weak solutions for a class of non-Newtonian heat-conducting fluids with a generalized nonlinear law of heat conduction. Roubíček [20] has shown the existence of the distributional solution to the steady-state system of equations for non-Newtonian fluids of the p-power type, coupled with the heat equation with heat sources to have L^1-structure and even to be measures. Beneš [21] considered the steady flow model with dissipative and adiabatic heating and temperature-dependent material coefficients in a plane bounded domain. The existence of a strong solution is proven by a fixed-point technique based on the Schauder theorem for sufficiently small external forces. For more results, we refer the reader to [22–24] and the references cited therein.

Under the condition that the angular momentum balance equation is considered (i.e., including the $\boldsymbol{\omega}$ equation in (1.1)), Kagei and Skowron [25] established the existence and uniqueness of solutions of problem by using the Banach fixed-point argument. Amorim, Loayza and Rojas-Medar [26] analyzed the existence, uniqueness, and regularity of the solutions in a bounded domain $\Omega \subset \mathbb{R}^3$ by using an iterative method; the convergence rates in several norms were also considered. Łukaszewicz, Waluś and Piskorek [27] studied the stationary problem associated with (1.1), and they showed that the boundary value problem has solutions in appropriate Sobolev spaces, provided the viscosities υ_r and $c_a + c_d$ are sufficiently large. The proof is based on a fixed-point argument. The above-mentioned results are all regarding the Newtonian case and, to the best of our knowledge, related results for such a problem of non-Newtonian type have not been considered yet.

In this paper, we study a stationary non-Newtonian version of the full system (1.1) in a smooth bounded domain $\Omega \subset \mathbb{R}^3$. More precisely, by neglecting the dissipation term $\boldsymbol{\tau} : \mathcal{D}$ and assuming that $\boldsymbol{\tau}$ has the p-structure

$$\boldsymbol{\tau}(\mathcal{D}\boldsymbol{u}) = 2\mu(1 + |\mathcal{D}\boldsymbol{u}|)^{(p-2)}\mathcal{D}\boldsymbol{u}, \qquad \mu > 0 \ \ const \tag{1.2}$$

after taking the viscosity coefficients properly, we consider the following non-Newtonian micropolar fluid equation with heat convection:

$$\begin{cases} \mathrm{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \mathrm{div}\boldsymbol{\tau}(\mathcal{D}\boldsymbol{u}) + \nabla\pi = 2\upsilon_r \mathrm{rot}\omega + \theta \boldsymbol{f}, & in \ \ \Omega \\ \qquad \mathrm{div}\boldsymbol{u} = 0, & in \ \ \Omega \\ (\boldsymbol{u} \cdot \nabla)\omega - 2\triangle\omega - 2\nabla\mathrm{div}\omega = 2\upsilon_r(\mathrm{rot}\boldsymbol{u} - 2\omega) + \theta \boldsymbol{g}, & in \ \ \Omega \\ -\mathrm{div}(\kappa(\cdot, \theta)\nabla\theta) + (\boldsymbol{u} \cdot \nabla)\theta = \Phi(\boldsymbol{u}, \omega) + h, & in \ \ \Omega \end{cases} \tag{1.3}$$

supplemented with the following first boundary value conditions:

$$\boldsymbol{u}|_{\partial\Omega} = 0, \qquad \omega|_{\partial\Omega} = 0, \qquad \theta|_{\partial\Omega} = 0 \tag{1.4}$$

where, in (1.3), $\Phi(\boldsymbol{u}, \omega) = \sum_{i=1}^{4} \Phi_i$ and

$$\Phi_1(\boldsymbol{u}, \omega) = (\frac{1}{2}\mathrm{rot}\boldsymbol{u} - \omega)^2, \quad \Phi_2(\omega) = (\mathrm{div}\omega)^2, \quad \Phi_3(\omega) = 2\sum_{i,j=1}^{3}(\omega_{i,j})^2, \quad \Phi_4(\omega) = \sum_{i,j=1}^{3} \omega_{i,j}\omega_{j,i}$$

here, for a vector-valued function $\boldsymbol{v}(x)$, we denote $\boldsymbol{v}_{i,j} = \partial_j \boldsymbol{v}_i(x)$. We assume that the heat conductivity $\kappa : \Omega \times \mathbb{R} \to \mathbb{R}$ is a C^1 function such that $0 < \kappa_1 \leq \kappa(x, \theta) \leq \kappa_2$ almost everywhere, with $x \in \Omega$ and, for all $\theta \in \mathbb{R}$, it satisfies that $|\kappa'(\cdot, a) - \kappa'(\cdot, b)| \leq \lambda'|a - b|$ for all $a, b \in \mathbb{R}$ and $\kappa'(\cdot, 0) = 0$, where κ_1, κ_2, and λ' are positive constants.

The goal of the present paper is to prove the existence and uniqueness of a strong solution to the system given by system (1.2)–(1.4) under a smallness condition on the external force term and the vortex viscosity coefficient. The procedure employs similar ideas to the ones presented in [28]. The main idea is to use the fixed-point theorem in combination with the regularized technique.

Let us briefly sketch the proof. First, after regularizing the term $|\mathcal{D}(\boldsymbol{u})|$ in the stress tensor with a parameter ε, we consider a penalized problem and rewrite it in a new form. Next, by the known results about the linear equation, we define the mapping by linearizing the above systems. Noticing that the first equation of the linearized systems is in a form of the Stokes type, by using the well-known regularity resuls (see [29]), we could obtain a pair $(\boldsymbol{u}_\varepsilon, \pi_\varepsilon) \in W^{2,q}(\Omega) \times W^{1,q}(\Omega)$. What needs to be pointed out is that, if we do not regularize the stress tensor, the right-hand side of this equation does not belong to $L^q(\Omega)$; this makes it impossible to apply the above theorem to get $u_\varepsilon \in W^{2,q}(\Omega)$. Then, by using the fixed-point theorem, we could prove the existence of an approximate solution $(\boldsymbol{u}_\varepsilon, \omega_\varepsilon, \theta_\varepsilon)$, and, finally, by taking $\varepsilon \to 0$, we prove the main result (Theorem 2.1).

Remark 1.1. *In our case, in the process of proof, we use an elementary inequality: for every $a, b \in R^+$, we have*

$$|(1 + a)^{p-2} - (1 + b)^{p-2}| \leq (|p - 2|, 1)^+|a - b|(1 + (a, b)^+)^{(p-2)^+-1}$$

If we allow the stress tensor to have singularity (i.e., $\boldsymbol{\tau}(\mathcal{D}\boldsymbol{u}) = 2\mu|\mathcal{D}\boldsymbol{u}|^{(p-2)}\mathcal{D}\boldsymbol{u}$), one similar estimate for $|a^{p-2} - b^{p-2}|$ is needed and this is not known. Therefore, our method is not suitable for the singular case. (See [28] for more details.)

The paper is organized as follows. In Section 2, we introduce basic notations and some preliminary results that will be used later; we then state the main results of this work. We prove the existence and uniqueness of strong solutions of an approximate problem described by (1.2)–(1.4) in Section 3 by employing a fixed-point argument. Finally, in Section 4, we prove the main result by letting the parameter $\varepsilon \to 0$.

2. Preliminaries and main result

Throughout the paper, we shall use the following functional spaces: $L^q(\Omega)$, $W^{m,q}(\Omega)$, and $W_0^{1,q}(\Omega)$ are the usual Lebesgue and Sobolev spaces; the norms in $L^q(\Omega)$ and $W^{m,q}(\Omega)$ we respectively denote by $\|\cdot\|_q$ and $\|\cdot\|_{m,q}$; $W^{-1,q}(\Omega)$ denotes the dual space of $W_0^{1,q}(\Omega)$, and its norm is represented by $\|\cdot\|_{-1,q;\Omega}$.

We also introduce the space

$$\begin{aligned} \mathcal{V} &:= \{\boldsymbol{u}|\ \boldsymbol{u} \in C_0^\infty(\Omega),\ \ \mathrm{div}\ \boldsymbol{u} = 0\} \\ V_p &:= \{\boldsymbol{u} \in W_0^{1,p}(\Omega) : \mathrm{div}\boldsymbol{u} = 0\} \\ V_{m,p} &:= \{\boldsymbol{u} \in W_0^{1,p}(\Omega) \cap W^{m,p}(\Omega) : \mathrm{div}\ \boldsymbol{u} = 0,\ \ \text{in } \Omega\} \end{aligned}$$

For $x, y \in \mathbb{R}$, $(x,\ y)^+ = \max\{x,\ y\}$, $x^+ = \max\{x,\ 0\}$, and $S_p = (|p-2|,\ 2)^+$. We introduce the following constants:

$$2r_p = 1 + (p-3)^+ - (p-4)^+, \quad \gamma_p = \frac{[(p,3)^+ - 2]^{(p,3)^+-2}}{[(p,3)^+ - 1]^{(p,3)^+-1}} \tag{2.1}$$

we also denote $C_p = C_p(n, s, \Omega)$ as the Poincaré constant of the Poincaré inequality.

For $q > r > s > 3$, let us consider the convex set B_ρ defined by

$$B_\rho = \left\{(\boldsymbol{\xi}, \boldsymbol{\eta}, \zeta) \in V_{2,q} \times W^{2,r}(\Omega) \times W^{2,s}(\Omega) : C_E\|\nabla\boldsymbol{\xi}\|_{1,q} \le \rho,\ C_{\widetilde{E}}\|\nabla\boldsymbol{\eta}\|_{1,r} \le \rho,\ C_{\overline{E}}\|\nabla\zeta\|_{1,s} \le \rho\right\} \tag{2.2}$$

where ρ is a constant to be determined; C_E, $C_{\widetilde{E}}$, and $C_{\overline{E}}$ are the embedding constants from $W^{1,q}(\Omega)$ into $L^\infty(\Omega)$, $W^{1,r}(\Omega)$ into $L^\infty(\Omega)$, and $W^{1,s}(\Omega)$ into $L^\infty(\Omega)$, respectively. Also, we consider the space $V_{2,q} \times (W^{2,r}(\Omega) \bigcap W_0^{1,r}(\Omega)) \times (W^{2,s}(\Omega) \bigcap W_0^{1,s}(\Omega))$, endowed with the norm

$$\|(\boldsymbol{\xi}, \boldsymbol{\eta}, \zeta)\|_{1,q,r,s} := \max\left\{\|\nabla\boldsymbol{\xi}\|_{1,q}, \|\nabla\boldsymbol{\eta}\|_{1,r}, \|\nabla\zeta\|_{1,s}\right\}$$

For later use, we state some useful lemmas.

Lemma 2.1. *[29] Let $m \ge -1$ be an integer, and let Ω be bounded in $\mathbb{R}^n (n = 2, 3)$ with a boundary $\partial\Omega$ of class C^k with $k = (m+2,\ 2)^+$. Then, for any $\boldsymbol{f} \in W^{m,q}(\Omega)$, the system given by*

$$\begin{cases} -\triangle\boldsymbol{u} + \nabla\pi = \boldsymbol{f}, & \text{in } \Omega \\ \quad \mathrm{div}\ \boldsymbol{u} = 0, & \text{in } \Omega \\ \boldsymbol{u}|_{\partial\Omega} = 0 \end{cases}$$

admits a unique $(\boldsymbol{u}, \pi) \in W^{m+2,q}(\Omega) \times W^{m+1,q}(\Omega)$. Moreover, the following estimate holds:

$$\|\nabla\boldsymbol{u}\|_{m+1,q} + \|\pi\|_{m+1,q/\mathbb{R}} \le C_m\|\boldsymbol{f}\|_{m,q}$$

where $C_m = C_m(n, q, \Omega)$ is a positive constant.

Lemma 2.2. *[28] Let γ_p be defined as (2.1), and let $L : \mathbb{R}^+ \to \mathbb{R}$ be defined by*

$$L(\delta) = A\delta^2 - \delta + E\delta\ell(\delta) + D$$

where A, D, E are positive constants and $\ell(x) = x(1 + x)^{(p-3)^+}$. Thus, if the following assertion holds:

$$AD + ED(1 + D)^{(p-3)^+} \leq \gamma_p$$

then L possesses at least one root δ_1. Moreover, $\delta_1 > D$, and, for every $\beta \in [1, 2]$, the following estimate holds:

$$\frac{\beta - 1}{\beta}\delta_1 + \frac{2 - \beta}{\beta}A\delta_1^2 + \frac{2 - \beta}{\beta}E\delta_1\ell(\delta_1) + \frac{E(p - 3)^+}{\beta}\delta_1^3(1 + \delta_1)^{(p-3)^+ - 1} \leq D \tag{2.3}$$

Lemma 2.3. *[30] Let X and Y be Banach spaces such that X is reflexive and $X \hookrightarrow Y$. Let B be a non-empty, closed, convex, and bounded subset of X, and let $T : B \to B$ be a mapping such that*

$$\|T(u) - T(v)\|_Y \leq K\|u - v\|_Y \quad \textit{for each } \; u, v \in B, \; 0 < K < 1$$

then, T has a unique fixed point in B.

The main result of our paper is as follows:

Theorem 2.1. *Let $\boldsymbol{f} \in L^q(\Omega)$, $\boldsymbol{g} \in L^r(\Omega)$, and $h \in L^s(\Omega)$, where $q > r > s > 3$, $p > 1$, and $\mu > 0$. There exist positive constants $\overline{\lambda} = \overline{\lambda}(C_0, C_{-1}, C_3, \lambda', \kappa_1, C_E, C_{\widetilde{E}}, C_{\overline{E}}, C_p)$, $m_1 = m_1(C_1, C_p, C_E, C_{\widetilde{E}}, \nu_r)$, and $m_2 = m_2(C, \lambda', C_p, C_E, C_{\widetilde{E}}, C_{\overline{E}})$ such that, if $\|\boldsymbol{g}\|_r < m_1$, $\|h\|_s < m_2k_1^2$, $\upsilon_r > 0$ small enough, and*

$$\begin{aligned}(\frac{1}{\mu} + 2)\frac{2\overline{\lambda}^2(\|\boldsymbol{f}\|_q^2 + \nu_r)}{\mu} + (\frac{1}{\mu} + 1)\overline{\lambda}\nu_r + \overline{\lambda}\frac{\|\boldsymbol{f}\|_q}{\mu} + \overline{\lambda}\|\boldsymbol{g}\|_r \\ + \overline{S}_p\overline{\lambda}^2\frac{\|\boldsymbol{f}\|_q^2 + \nu_r}{\mu} \cdot \left(1 + \frac{\overline{\lambda}(\|\boldsymbol{f}\|_q^2 + \nu_r)}{\mu}\right)^{(p-3)^+} < \frac{1}{4^{(p-2,1)^+}}\end{aligned} \tag{2.4}$$

then the problem given by (1.2)–(1.4) has a unique strong solution $(\boldsymbol{u}, \boldsymbol{\omega}, \theta) \in V_{2,q} \times (W^{2,r}(\Omega) \bigcap W_0^{1,r}(\Omega)) \times (W^{2,s}(\Omega) \bigcap W_0^{1,s}(\Omega))$.

Remark 2.1. *As usual, the pressure π has disappeared from the notion of the solution. Actually, the pressure may be recovered by the de Rham theorem, at least in $L^2(\Omega)$, such that $(\boldsymbol{u}, \pi, \boldsymbol{\omega}, \theta)$ satisfies (1.2)–(1.4) almost everywhere (see, e.g., [31]).*

3. Existence of the approximate solution

For $0 < \varepsilon < 1$, we consider the following family of penalized problems

$$\begin{cases} -\mathrm{div}(2\mu(1 + \sqrt{\varepsilon^2 + |\mathcal{D}\boldsymbol{u}|^2})^{p-2}\mathcal{D}\boldsymbol{u}) + \nabla\pi + \mathrm{div}(\boldsymbol{u} \otimes \boldsymbol{u}) = 2\upsilon_r \mathrm{rot}\, \boldsymbol{\omega} + \theta\boldsymbol{f}, & \textit{in } \Omega \\ \quad \mathrm{div}\, \boldsymbol{u} = 0, & \textit{in } \Omega \\ -2\triangle\boldsymbol{\omega} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - 2\nabla\mathrm{div}\boldsymbol{\omega} = 2\upsilon_r(\mathrm{rot}\boldsymbol{u} - 2\boldsymbol{\omega}) + \theta\boldsymbol{g}, & \textit{in } \Omega \\ -\mathrm{div}(\kappa(\cdot, \theta)\nabla\theta) + (\boldsymbol{u} \cdot \nabla)\theta = \Phi(\boldsymbol{u}, \boldsymbol{\omega}) + h, & \textit{in } \Omega \\ \boldsymbol{u}|_{\partial\Omega} = 0, \quad \boldsymbol{\omega}|_{\partial\Omega} = 0, \quad \theta|_{\partial\Omega} = 0 \end{cases} \tag{3.1}$$

The following result holds true.

Theorem 3.1. *Let* $f \in L^q(\Omega)$, $\boldsymbol{g} \in L^r(\Omega)$, *and* $h \in L^s(\Omega)$, *where* $q > r > s > 3$, $p > 1$, $\mu > 0$, *and* $0 < \varepsilon < 1$. *There exist positive constants* $\overline{\lambda} = \overline{\lambda}(C_0, C_{-1}, C_3, \lambda', \kappa_1, C_E, C_{\widetilde{E}}, C_{\overline{E}}, C_p)$, $m_1 = m_1(C_1, C_p, C_E, C_{\widetilde{E}}, \nu_r)$, *and* $m_2 = m_2(C, \lambda', C_p, C_E, C_{\widetilde{E}}, C_{\overline{E}})$ *such that, if* $\|\boldsymbol{g}\|_r < m_1$, $\|h\|_s < m_2 k_1^2$, $\upsilon_r > 0$ *is small enough, and*

$$\Big(\frac{1}{\mu} + 2\Big)\frac{2\overline{\lambda}^{-2}(\|\boldsymbol{f}\|_q^2 + \nu_r)}{\mu} + (\frac{1}{\mu} + 1)\overline{\lambda}\nu_r + \overline{\lambda}\frac{\|\boldsymbol{f}\|_q}{\mu} + \overline{\lambda}\|\boldsymbol{g}\|_r + \overline{S}_p\overline{\lambda}^{-2}\frac{\|\boldsymbol{f}\|_q^2 + \nu_r}{\mu} \cdot \left(1 + \frac{\overline{\lambda}(\|\boldsymbol{f}\|_q^2 + \nu_r)}{\mu}\right)^{(p-3)^+} < \frac{1}{4^{(p-2,1)^+}} \tag{3.2}$$

then problem (3.1) has a unique strong solution:

$$(\boldsymbol{u}_\varepsilon, \boldsymbol{\omega}_\varepsilon, \theta_\varepsilon) \in V_{2,q} \times (W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)) \times (W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega))$$

Proof. We use a fixed-point argument to prove Theorem 3.1, and the proof will be divided into four steps.

Step 1: Linearization of the problem and construction of the mapping.

Reformulate the problem (3.1) as follows:

$$\begin{cases} -\mu(1+\varepsilon)^{(p-2)}\triangle \boldsymbol{u} + \nabla\pi = 2\upsilon_r \text{rot}\, \boldsymbol{\omega} + \theta \boldsymbol{f} - \text{div}(\boldsymbol{u} \otimes \boldsymbol{u}) + \text{div}(2\mu\sigma_\varepsilon(|\mathcal{D}\boldsymbol{u}|^2)\mathcal{D}\boldsymbol{u}) \\ \quad \text{div}\, \boldsymbol{u} = 0 \\ -2\triangle\boldsymbol{\omega} - 2\nabla \text{div}\boldsymbol{\omega} = 2\upsilon_r \text{rot}\, \boldsymbol{u} + \theta \boldsymbol{g} - 4\upsilon_r \boldsymbol{\omega} - (\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega} \\ -\kappa(\cdot,\theta)\triangle\theta = \kappa'(\cdot,\theta)|\nabla\theta|^2 - (\boldsymbol{u}\cdot\nabla)\theta + \Phi(\boldsymbol{u},\boldsymbol{\omega}) + h \\ \boldsymbol{u}|_{\partial\Omega} = 0, \quad \omega|_{\partial\Omega} = 0, \quad \theta|_{\partial\Omega} = 0 \end{cases} \tag{3.3}$$

where $\sigma_\varepsilon(x^2) = \left(1 + \sqrt{\varepsilon^2 + |x|^2}\,\right)^{(p-2)} - (1+\varepsilon)^{(p-2)}$.

We define the operator

$$T_\varepsilon : V_{2,q} \times (W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)) \times (W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega)) \mapsto V_{2,q} \times (W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)) \times (W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega))$$

given by $T_\varepsilon(\boldsymbol{\xi}, \boldsymbol{\eta}, \zeta) = (\boldsymbol{u}_\varepsilon, \boldsymbol{\omega}_\varepsilon, \theta_\varepsilon)$, where $(\boldsymbol{u}_\varepsilon, \boldsymbol{\omega}_\varepsilon, \theta_\varepsilon)$ is the solution of the following problem:

$$\begin{cases} -\mu(1+\varepsilon)^{(p-2)}\triangle \boldsymbol{u}_\varepsilon + \nabla\pi_\varepsilon = 2\upsilon_r \text{rot}\, \boldsymbol{\eta} + \zeta \boldsymbol{f} - \text{div}(\boldsymbol{\xi} \otimes \boldsymbol{\xi}) + \text{div}(2\mu\sigma_\varepsilon(|\mathcal{D}\boldsymbol{\xi}|^2)\mathcal{D}\boldsymbol{\xi}) \\ \quad \text{div}\, \boldsymbol{u}_\varepsilon = 0 \\ -2\triangle\boldsymbol{\omega}_\varepsilon - 2\nabla \text{div}\boldsymbol{\omega}_\varepsilon = 2\upsilon_r \text{rot}\, \boldsymbol{\xi} + \zeta \boldsymbol{g} - 4\upsilon_r \boldsymbol{\eta} - (\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\eta} \\ -\kappa(\cdot,\theta_\varepsilon)\triangle\theta_\varepsilon = \kappa'(\cdot,\zeta)|\nabla\zeta|^2 - (\boldsymbol{\xi}\cdot\nabla)\zeta + \Phi(\boldsymbol{\xi},\boldsymbol{\eta}) + h \\ \boldsymbol{u}_\varepsilon|_{\partial\Omega} = 0, \quad \boldsymbol{\omega}_\varepsilon|_{\partial\Omega} = 0, \quad \theta_\varepsilon|_{\partial\Omega} = 0 \end{cases} \tag{3.4}$$

Step 2: Proving T_ε maps B_ρ onto itself.

In this part, we will prove that there exists a constant $\rho > 0$ such that T_ε maps B_ρ onto B_ρ. We formulate the result as follows.

Proposition 3.1. *Let $\boldsymbol{f} \in L^q(\Omega)$, $\boldsymbol{g} \in L^r(\Omega)$, and $h \in L^s(\Omega)$, where $q > r > s > 3$, $p > 1$, and $\mu > 0$. There exist positive constants $\overline{\lambda}_1 = \overline{\lambda}_1(C_0, C_pC_E, C_{\widetilde{E}})$, $m_1 = m_1(C_1, C_E, C_{\widetilde{E}}, C_p, \nu_r)$, and $m_2 = m_2(C, \lambda', C_E, C_{\widetilde{E}}, C_{\overline{E}}, C_p)$ such that, if $\|\boldsymbol{g}\|_r < m_1$, $\|h\|_s < m_2k_1^2$, $\upsilon_r > 0$ is small enough, and*

$$\frac{\overline{\lambda}_1^2\left(\|\boldsymbol{f}\|_q^2+\nu_r\right)}{\mu^2}+\frac{\overline{\lambda}_1^2\overline{S}_p}{\mu}\left(\|\boldsymbol{f}\|_q^2+\nu_r\right)\left(1+\frac{\overline{\lambda}_1(\|\boldsymbol{f}\|_q^2+\nu_r)}{\mu}\right)^{(p-3)^+}\leq\gamma_p \tag{3.5}$$

then $T_\varepsilon(B_\rho) \subseteq B_\rho$ for some $\rho > 0$.

Proof. Let $(\boldsymbol{\xi}, \boldsymbol{\eta}, \zeta)$ be in B_ρ (see 2.2). Using Lemma 2.1, we obtain that $\boldsymbol{u}_\varepsilon \in V_{2,q}$ and

$$\|\nabla\boldsymbol{u}_\varepsilon\|_{1,q}\leq\frac{C_0}{(1+\varepsilon)^{(p-2)}\mu}\left(\|\zeta\boldsymbol{f}\|_q+\|\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}\|_q+\|2\nu_r\mathrm{rot}\boldsymbol{\eta}\|_q+\|\mathrm{div}[2\mu\sigma_\varepsilon(|\mathcal{D}\boldsymbol{\xi}|^2)\mathcal{D}\boldsymbol{\xi}]\|_q\right) \tag{3.6}$$

First, we have

$$\|2\nu_r\mathrm{rot}\boldsymbol{\eta}\|_q\leq 2\nu_rC\|\nabla\boldsymbol{\eta}\|_q\leq C\nu_r\|\nabla\boldsymbol{\eta}\|_{1,r}\leq\frac{C\nu_r}{C_{\widetilde{E}}}\rho \tag{3.7}$$

$$\|\zeta\boldsymbol{f}\|_q\leq\|\zeta\|_\infty\|\boldsymbol{f}\|_q\leq C_{\overline{E}}(C_p+1)\|\nabla\zeta\|_s\|\boldsymbol{f}\|_q\leq\rho(C_p+1)\|\boldsymbol{f}\|_q\leq\frac{\rho^2(C_p+1)^2}{2}+\frac{\|\boldsymbol{f}\|_q^2}{2} \tag{3.8}$$

Reasoning as in [28], we obtain

$$\|\boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}\|_q+\|\mathrm{div}[2\mu\sigma_\varepsilon(|\mathcal{D}\boldsymbol{\xi}|^2)\mathcal{D}\boldsymbol{\xi}]\|_q\leq\frac{C_p}{C_E}\rho^2+\frac{8\mu\overline{S_p}}{C_E}\rho\ell(\rho) \tag{3.9}$$

Combining (3.6)–(3.9), we conclude that

$$\begin{aligned}\|\nabla\boldsymbol{u}_\varepsilon\|_{1,q}&\leq\frac{C_0}{\mu}\left(\frac{\|\boldsymbol{f}\|_q^2}{2}+\frac{C\nu_r}{C_{\widetilde{E}}}\rho+\Big(\frac{(C_p+1)^2}{2}+\frac{C_p}{C_E}\Big)\rho^2+\frac{8\mu\overline{S_p}}{C_E}\rho\ell(\rho)\right)\\&\leq\frac{\overline{\lambda_1}}{\mu}\left(\|\boldsymbol{f}\|_q^2+\nu_r\rho+\rho^2+\mu\overline{S_p}\rho\ell(\rho)\right)\end{aligned} \tag{3.10}$$

where $\overline{\lambda_1} = C_0\max\left\{\frac{1}{2}, \frac{C}{C_{\widetilde{E}}}, \frac{(C_p+1)^2}{2}+\frac{C_p}{C_E}, \frac{8}{C_E}\right\}$, $\ell(x) = x\,(1+x)^{(p-3)^+}$, $\overline{S_p} = (|p-2|, 1)^+\,2^{(p-3)^+}$.

On the other hand, by the theory of elliptic equations, there is a positive constant C_1 such that

$$\begin{aligned}\|\nabla\omega_\varepsilon\|_{1,r}&\leq C_1[\|\zeta\boldsymbol{g}\|_r+\|2\nu_r rot\boldsymbol{\xi}\|_r+\|4\nu_r\boldsymbol{\eta}\|_r+\|\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta}\|_r]\\&\leq C_1[\|\zeta\|_\infty\|\boldsymbol{g}\|_r+2\nu_rC\|\nabla\boldsymbol{\xi}\|_r+4\nu_rC_p\|\nabla\boldsymbol{\eta}\|_r+\|\boldsymbol{\xi}\|_\infty\|\nabla\boldsymbol{\eta}\|_r]\\&\leq C_1[C_{\overline{E}}(C_p+1)\|\nabla\zeta\|_s\|\boldsymbol{g}\|_r+C\nu_r\|\nabla\boldsymbol{\xi}\|_{1,q}+4\nu_rC_p\|\nabla\boldsymbol{\eta}\|_{1,r}+C_E\|\boldsymbol{\xi}\|_{1,q}\|\nabla\boldsymbol{\eta}\|_{1,r}]\\&\leq C_1[\rho(C_p+1)\|\boldsymbol{g}\|_r+C\nu_r\|\nabla\boldsymbol{\xi}\|_{1,q}+4\nu_rC_p\|\nabla\boldsymbol{\eta}\|_{1,r}+C_E(C_p+1)\|\nabla\boldsymbol{\xi}\|_q\|\nabla\boldsymbol{\eta}\|_{1,r}]\\&\leq C_1\left[\frac{\|\boldsymbol{g}\|_r^2}{2}+\frac{\rho^2(C_p+1)^2}{2}+C\nu_r\|\nabla\boldsymbol{\xi}\|_{1,q}+4\nu_rC_p\|\nabla\boldsymbol{\eta}\|_{1,r}+C_E(C_p+1)\|\nabla\boldsymbol{\xi}\|_{1,q}\frac{\rho}{C_{\widetilde{E}}}\right]\\&\leq C_1\left[\frac{\|\boldsymbol{g}\|_r^2}{2}+\frac{\rho^2(C_p+1)^2}{2}+\frac{C\nu_r}{C_E}\rho+\frac{4\nu_rC_p}{C_{\widetilde{E}}}\rho+\frac{(C_p+1)}{C_{\widetilde{E}}}\rho^2\right]\\&\leq\frac{C_1(C_p+1)[(C_p+1)C_{\widetilde{E}}+2]}{2C_{\widetilde{E}}}\rho^2+2\overline{\lambda_2}\nu_r\rho+\frac{C_1}{2}\|\boldsymbol{g}\|_r^2\end{aligned} \tag{3.11}$$

where $\overline{\lambda_2} = C_1 \max\left\{\frac{C}{C_E}, \frac{4C_p}{C_{\overline{E}}}\right\}$.

Also, from the elliptic equations of (3.4), there exists a positive constant C_2 such that

$$\begin{aligned} \|\nabla\theta_\varepsilon\|_{1,s} &\leq \frac{C_2}{\kappa_1}\|\kappa^{'}(\cdot,\zeta)|\mathcal{D}\zeta|^2\|_s + \frac{C_2}{\kappa_1}\|\Phi(\boldsymbol{\xi},\boldsymbol{\eta})\|_s + \frac{C_2}{\kappa_1}\|\boldsymbol{\xi}\cdot\nabla\zeta\|_s + \frac{C_2}{\kappa_1}\|h\|_s \\ &= \frac{C_2}{\kappa_1}\|\kappa^{'}(\cdot,\zeta)|\mathcal{D}\zeta|^2\|_s + \frac{C_2}{\kappa_1}\|\sum_{i=1}^{4}\Phi_i(\boldsymbol{\xi},\boldsymbol{\eta})\|_s + \frac{C_2}{\kappa_1}\|\boldsymbol{\xi}\cdot\nabla\zeta\|_s + \frac{C_2}{\kappa_1}\|h\|_s \end{aligned} \tag{3.12}$$

By the assumptions of $\kappa(\cdot,\theta)$, it follows that

$$\begin{aligned} \|\kappa^{'}(\cdot,\zeta)|\mathcal{D}\zeta|^2\|_s &= \|(\kappa^{'}(\cdot,\zeta)-\kappa^{'}(\cdot,0))|\mathcal{D}\zeta|^2\|_s \leq \lambda^{'}\|\zeta\|_\infty\|\nabla\zeta\|_{2s}^2 \\ &\leq \lambda^{'}C_{\overline{E}}(C_p+1)\|\nabla\zeta\|_s\|\nabla\zeta\|_{2s}^2 \leq \frac{\lambda^{'}C(C_p+1)}{C_{\overline{E}}^2}\rho^3 \end{aligned} \tag{3.13}$$

Since

$$\|\Phi_1(\boldsymbol{\xi},\boldsymbol{\eta})\|_s = \|(\frac{1}{2}\mathrm{rot}\boldsymbol{\xi}-\boldsymbol{\eta})^2\|_s \leq C\left(\int_\Omega((\nabla\boldsymbol{\xi})^2+\boldsymbol{\eta}^2)^s dx\right)^{1/s} \leq C(\|\nabla\boldsymbol{\xi}\|_{2s}^2 + \|\boldsymbol{\eta}\|_{2s}^2)$$

and

$$\begin{aligned} \|\Phi_2(\boldsymbol{\eta})\|_s + \|\Phi_3(\boldsymbol{\eta})\|_s + \|\Phi_4(\boldsymbol{\eta})\|_s &= \|(\mathrm{div}\boldsymbol{\eta})^2\|_s + \|\sum_{i=1}^{3}(\boldsymbol{\eta}_{i,j})^2\|_s + \|\sum_{i=1}^{3}\boldsymbol{\eta}_{i,j}\boldsymbol{\eta}_{j,i}\|_s \\ &\leq C\left(\int_\Omega((\nabla\boldsymbol{\eta})^2)^s dx\right)^{1/s} + C\left(\int_\Omega((\nabla\boldsymbol{\eta})^2)^s dx\right)^{1/s} + C\left(\int_\Omega((\nabla\boldsymbol{\eta})^2)^s dx\right)^{1/s} \\ &\leq C\|\nabla\boldsymbol{\eta}\|_{2s}^2 \end{aligned}$$

it follows that

$$\|\Phi(\boldsymbol{\xi},\boldsymbol{\eta})\|_s \leq C(\|\nabla\boldsymbol{\xi}\|_{2s}^2 + \|\nabla\boldsymbol{\eta}\|_{2s}^2) \leq C(\|\nabla\boldsymbol{\xi}\|_{1,q}^2 + \|\nabla\boldsymbol{\eta}\|_{1,r}^2) \leq \frac{C\rho^2}{C_E^2} + \frac{C\rho^2}{C_{\overline{E}}^2} \tag{3.14}$$

Finally,

$$\begin{aligned} \|\boldsymbol{\xi}\cdot\nabla\zeta\|_s &\leq \|\boldsymbol{\xi}\|_\infty\|\nabla\zeta\|_s \leq C_E\|\boldsymbol{\xi}\|_{1,q}\|\nabla\zeta\|_s \leq (C_p+1)C_E\|\nabla\boldsymbol{\xi}\|_q\|\nabla\zeta\|_s \\ &\leq (C_p+1)C_E\|\nabla\boldsymbol{\xi}\|_{1,q}\|\nabla\zeta\|_{1,s} \leq \frac{(C_p+1)}{C_{\overline{E}}}\rho^2 \end{aligned} \tag{3.15}$$

Combining (3.12)–(3.15), we obtain

$$\begin{aligned} \|\nabla\theta_\varepsilon\|_{1,s} &\leq \frac{C_2\lambda^{'}C(C_p+1)}{\kappa_1 C_{\overline{E}}^2}\rho^3 + \frac{C_2(C_p+1)}{\kappa_1 C_{\overline{E}}}\rho^2 + \frac{C_2}{\kappa_1}\left(\frac{C}{C_E^2} + \frac{C}{C_{\overline{E}}^2}\right)\rho^2 + \frac{C_2}{\kappa_1}\|h\|_s \\ &\leq \frac{C_2\lambda^{'}C(C_p+1)}{\kappa_1 C_{\overline{E}}^2}\rho^3 + 2\frac{\overline{\lambda_3}}{\lambda_1}\rho^2 + \frac{C_2}{\kappa_1}\|h\|_s \end{aligned} \tag{3.16}$$

where $\overline{\lambda_3} = C_2 \max\left\{\frac{(C_p+1)}{C_{\widetilde{E}}},\ \frac{C}{C_E^2} + \frac{C}{C_{\widetilde{E}}^2}\right\}$.

Without loss of generality, it can be assumed that $\rho \leq 1$. To ensure that $T_\varepsilon(B_\rho) \subseteq B_\rho$, it is sufficient to require that

$$\|\nabla \boldsymbol{u}_\varepsilon\|_{1,q} \leq \frac{\overline{\lambda_1}}{\mu}\left[\|\boldsymbol{f}\|_q^2 + \nu_r\rho + \rho^2 + \mu\overline{S}_p\rho\ell(\rho)\right] \leq \frac{\overline{\lambda_1}}{\mu}\left[\|\boldsymbol{f}\|_q^2 + \nu_r + \rho^2 + \mu\overline{S}_p\rho\ell(\rho)\right] \leq \rho \tag{3.17}$$

$$\|\nabla \boldsymbol{\omega}_\varepsilon\|_{1,r} \leq \frac{C_1(C_p+1)[(C_p+1)C_{\widetilde{E}}+2]}{2C_{\widetilde{E}}}\rho^2 + 2\overline{\lambda_2}\nu_r\rho + \frac{C_1}{2}\|\boldsymbol{g}\|_r^2 \leq \rho \tag{3.18}$$

$$\|\nabla \theta_\varepsilon\|_{1,s} \leq \frac{C_2\lambda' C(C_p+1)}{\kappa_1 C_{\widetilde{E}}^2}\rho^3 + 2\frac{\overline{\lambda_3}}{\kappa_1}\rho^2 + \frac{C_2}{\kappa_1}\|h\|_s \leq \left(\frac{C_2\lambda' C(C_p+1)}{\kappa_1 C_{\widetilde{E}}^2} + 2\frac{\overline{\lambda_3}}{\kappa_1}\right)\rho^2 + \frac{C_2}{\kappa_1}\|h\|_s \leq \rho \tag{3.19}$$

Applying Lemma 2.2 with $A = \frac{\overline{\lambda_1}}{\mu}$, $E = \overline{\lambda_1 S_p}$, and $D = \frac{\overline{\lambda_1}(\|\boldsymbol{f}\|_q^2+\nu_r)}{\mu}$, there exists $\rho_1 > \frac{\overline{\lambda_1}(\|\boldsymbol{f}\|_q^2+\nu_r)}{\mu}$ such that

$$\frac{\overline{\lambda_1}}{\mu}\left[\|\boldsymbol{f}\|_q^2 + \nu_r + \rho_1^2 + \mu\overline{S}_p\rho_1\ell(\rho_1)\right] \leq \rho_1$$

moreover, by taking $\beta = 2$ in (2.3), we have

$$\rho_1 \leq \frac{2\overline{\lambda_1}(\|\boldsymbol{f}\|_q^2 + \nu_r)}{\mu}$$

Reformulate (3.18) as follows

$$\frac{C_1(C_p+1)[(C_p+1)C_{\widetilde{E}}+2]}{2C_{\widetilde{E}}}\rho^2 + (2\overline{\lambda_2}\nu_r - 1)\rho + \frac{C_1}{2}\|\boldsymbol{g}\|_r^2 \leq 0 \tag{3.20}$$

since the discriminant $\Delta = (2\overline{\lambda_2}\nu_r - 1)^2 - \frac{C_1^2(C_p+1)[(C_p+1)C_{\widetilde{E}}+2]}{C_{\widetilde{E}}}\|\boldsymbol{g}\|_r^2 > 0$, namely,

$$\|\boldsymbol{g}\|_r^2 < \frac{(2\overline{\lambda_2}\nu_r - 1)^2 C_{\widetilde{E}}}{C_1^2(C_p+1)[(C_p+1)C_{\widetilde{E}}+2]} \equiv m_1$$

we deduce that the inequality (3.20) is valid for some ρ .

Take a constant D satisfying that $\rho_2^- < D < \rho_2^+$, where

$$\begin{aligned}\rho_2^\pm = &\frac{C_{\widetilde{E}}}{C_1(C_p+1)[(C_p+1)C_{\widetilde{E}}+2]} \\ &\cdot\left[(1-2\overline{\lambda_2}\nu_r) \pm \sqrt{(2\overline{\lambda_2}\nu_r-1)^2 - \frac{C_1^2(C_p+1)[(C_p+1)C_{\widetilde{E}}+2]}{C_{\widetilde{E}}}\|\boldsymbol{g}\|_r^2}\ \right] \\ = &\frac{C_1 m_1}{1-2\overline{\lambda_2}\nu_r}\left[1 \pm \sqrt{1 - \frac{\|\boldsymbol{g}\|_r^2}{m_1}}\ \right]\end{aligned}$$

since, for every $\rho \in [\rho_2^-, \rho_2^+]$, (3.20) holds true, we could choose $\rho_2 \in (\rho_2^-, D)$ such that

$$\frac{C_1(C_p+1)[(C_p+1)C_{\widetilde{E}}+2]}{2C_{\widetilde{E}}}\rho_2^2 + 2\overline{\lambda_2}\nu_r\rho_2 + \frac{C_1}{2}\|\boldsymbol{g}\|_r^2 \leq \rho_2$$

On the other hand, we rewrite (3.19) as follows

$$\left(\frac{C_2\lambda' C(C_p+1)}{\kappa_1 C_{\overline{E}}^2}+2\frac{\overline{\lambda_3}}{\kappa_1}\right)\rho^2-\rho+\frac{C_2}{\kappa_1}\|\boldsymbol{h}\|_s\leq 0 \tag{3.21}$$

since $\Delta = 1-4\left(\frac{C_2\lambda' C(C_p+1)}{\kappa_1 C_{\overline{E}}^2}+2\frac{\overline{\lambda_3}}{\kappa_1}\right)\frac{C_2}{\kappa_1}\|\boldsymbol{h}\|_s > 0$, namely,

$$\frac{\|\boldsymbol{h}\|_s}{k_1^2}<\frac{C_{\overline{E}}^2}{4C_2[C_2\lambda' C(C_p+1)+2\overline{\lambda_3}C_{\overline{E}}^2]}\equiv m_2$$

it follows that (3.21) is valid for some ρ.

The above D could also be selected to satisfy that $\rho_3^- < 2D < \rho_3^+$, where

$$\begin{aligned}\rho_3^{\pm} &= \frac{\kappa_1 C_{\overline{E}}^2}{2[C_2\lambda' C(C_p+1)+2\overline{\lambda_3}C_{\overline{E}}^2]}\cdot\left[1\pm\sqrt{1-4(\frac{C_2\lambda' C(C_p+1)}{\kappa_1 C_{\overline{E}}^2}+2\frac{\overline{\lambda_3}}{\kappa_1})\frac{C_2}{\kappa_1}\|h\|_s}\right]\\ &= \frac{2m_2}{\kappa_1}\left[1\pm\sqrt{1-\frac{\|h\|_s}{k_1^2 m_2}}\right]\end{aligned}$$

since (3.21) is valid for every $\rho\in[\rho_3^-,\rho_3^+]$, we can choose $\rho_3\in(2D,\rho_3^+)$ such that

$$\left(\frac{C_2\lambda' C(C_p+1)}{\kappa_1 C_{\overline{E}}^2}+2\frac{\overline{\lambda_3}}{\kappa_1}\right)\rho_3^2+\frac{C_2}{\kappa_1}\|h\|_s\leq\rho_3$$

In conclusion, we have obtained

$$\rho_2<\frac{\overline{\lambda_1}(\|\boldsymbol{f}\|_q^2+\nu_r)}{\mu}<\rho_1\leq\frac{2\overline{\lambda_1}(\|\boldsymbol{f}\|_q^2+\nu_r)}{\mu}<\rho_3 \tag{3.22}$$

which completes the proof by taking $\rho=\rho_1$.

Step 3: Proving $T_\varepsilon : B_\rho \mapsto B_\rho$ is a contraction.

In this step, we concentrate on proving that the map $T_\varepsilon : B_\rho \mapsto B_\rho$ is a contraction. Our aim is to prove the following result.

Proposition 3.2. *There is a positive constant $\overline{\lambda}_0=\overline{\lambda}_0(C_{-1},C_3,\lambda',\kappa_1,C_p,C_E,C_{\widetilde{E}},C_{\overline{E}})$ such that, if*

$$\begin{aligned}\overline{\lambda}_0\Bigg[&\left(\frac{1}{\mu}+2\right)\frac{2\overline{\lambda}_1(\|\boldsymbol{f}\|_q^2+\nu_r)}{\mu}+(\frac{1}{\mu}+1)\nu_r+\frac{\|\boldsymbol{f}\|_q}{\mu}+\|\boldsymbol{g}\|_r\\ &+\overline{S}_p\frac{\overline{\lambda}_1(\|\boldsymbol{f}\|_q^2+\nu_r)}{\mu}\cdot\left(1+\frac{\overline{\lambda}_1(\|\boldsymbol{f}\|_q^2+\nu_r)}{\mu}\right)^{(p-3)^+}\Bigg]<\frac{1}{4^{(p-2,1)^+}}\end{aligned} \tag{3.23}$$

then $T_\varepsilon : B_\rho \mapsto B_\rho$ is a contraction in $W_0^{1,q}(\Omega)\times W_0^{1,r}(\Omega)\times W_0^{1,s}(\Omega)$.

Proof. Let $(\boldsymbol{\xi},\boldsymbol{\eta},\zeta),(\widehat{\boldsymbol{\xi}},\widehat{\boldsymbol{\eta}},\widehat{\zeta})\in B_\rho$, and let $[\boldsymbol{u}_\varepsilon,\omega_\varepsilon,\theta_\varepsilon],[\widehat{\boldsymbol{u}}_\varepsilon,\widehat{\omega}_\varepsilon,\widehat{\theta}_\varepsilon]$ be their respective images under T_ε. Then, from (3.4), we obtain

$$\begin{cases} -\mu(1+\varepsilon)^{(p-2)}\triangle(\boldsymbol{u}_\varepsilon-\widehat{\boldsymbol{u}}_\varepsilon)+\nabla(p_\varepsilon-\widehat{p}_\varepsilon)=\boldsymbol{F}_\varepsilon \\ \quad \operatorname{div}(\boldsymbol{u}_\varepsilon-\widehat{\boldsymbol{u}}_\varepsilon)=0 \\ -2\triangle(\omega_\varepsilon-\widehat{\omega}_\varepsilon)-2\nabla\operatorname{div}(\omega_\varepsilon-\widehat{\omega}_\varepsilon)=\boldsymbol{G} \\ -\kappa(\cdot,\theta_\varepsilon)\triangle\theta_\varepsilon+\kappa(\cdot,\widehat{\theta}_\varepsilon)\triangle\widehat{\theta}_\varepsilon=H+\kappa'(\cdot,\zeta)|\nabla\zeta|^2-\kappa'(\cdot,\widehat{\zeta})|\nabla\widehat{\zeta}|^2 \\ (\boldsymbol{u}_\varepsilon-\widehat{\boldsymbol{u}}_\varepsilon)|_{\partial\Omega}=0,\quad (\omega_\varepsilon-\widehat{\omega}_\varepsilon)|_{\partial\Omega}=0,\quad (\theta_\varepsilon-\widehat{\theta}_\varepsilon)|_{\partial\Omega}=0 \end{cases} \tag{3.24}$$

where

$$\boldsymbol{F}_\varepsilon:=\operatorname{div}(\widehat{\boldsymbol{\xi}}\otimes\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi}\otimes\boldsymbol{\xi})+2\nu_r\operatorname{rot}(\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}})+2\mu\operatorname{div}[\sigma_\varepsilon(|\mathcal{D}\boldsymbol{\xi}|^2)\mathcal{D}\xi-\sigma_\varepsilon(|\mathcal{D}\widehat{\boldsymbol{\xi}}|^2)\mathcal{D}\widehat{\boldsymbol{\xi}}]+(\zeta-\widehat{\zeta})\boldsymbol{f}$$
$$\boldsymbol{G}:=2\nu_r\operatorname{rot}(\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}})-4\nu_r(\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}})-(\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\eta}+(\widehat{\boldsymbol{\xi}}\cdot\nabla)\widehat{\boldsymbol{\eta}}+(\zeta-\widehat{\zeta})\boldsymbol{g}$$
$$H:=\Phi(\boldsymbol{\xi},\boldsymbol{\eta})-\Phi(\widehat{\boldsymbol{\xi}},\widehat{\boldsymbol{\eta}})-(\boldsymbol{\xi}\cdot\nabla)\zeta+(\widehat{\boldsymbol{\xi}}\cdot\nabla)\widehat{\zeta}$$

From Lemma 2.1, we obtain

$$\begin{aligned}\|\nabla(\boldsymbol{u}_\varepsilon-\widehat{\boldsymbol{u}}_\varepsilon)\|_q\le\frac{C_{-1}}{\mu}\Big[&\|\operatorname{div}(\widehat{\boldsymbol{\xi}}\otimes\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi}\otimes\boldsymbol{\xi})\|_{-1,q}+2\nu_r\|\operatorname{rot}(\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}})\|_{-1,\mathrm{q}}\\ &+2\mu\|\operatorname{div}[\sigma_\varepsilon(|\mathcal{D}\boldsymbol{\xi}|^2)\mathcal{D}\boldsymbol{\xi}-\sigma_\varepsilon(|\mathcal{D}\widehat{\boldsymbol{\xi}}|^2)\mathcal{D}\widehat{\boldsymbol{\xi}}]\|_{-1,q}+\|(\zeta+\widehat{\zeta})\boldsymbol{f}\|_{-1,q}\Big]\end{aligned} \tag{3.25}$$

We estimate each term on the right-hand side of (3.25) as follows:

$$\|\operatorname{div}(\widehat{\boldsymbol{\xi}}\otimes\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi}\otimes\boldsymbol{\xi})\|_{-1,q}\le C\|\widehat{\boldsymbol{\xi}}\otimes\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi}\otimes\boldsymbol{\xi}\|_q\le 2CC_p(C_p^q+1)^{\frac{1}{q}}\rho_1\|\nabla(\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}})\|_q \tag{3.26}$$

$$\begin{aligned}2\mu\|\operatorname{div}[\sigma_\varepsilon(|\mathcal{D}\boldsymbol{\xi}|^2)\mathcal{D}\boldsymbol{\xi}-\sigma_\varepsilon(|\mathcal{D}\widehat{\boldsymbol{\xi}}|^2)\mathcal{D}\widehat{\boldsymbol{\xi}}]\|_{-1,\mathrm{q}}&\le C\mu\|\sigma_\varepsilon(|\mathcal{D}\boldsymbol{\xi}|^2)\mathcal{D}\boldsymbol{\xi}-\sigma_\varepsilon(|\mathcal{D}\widehat{\boldsymbol{\xi}}|^2)\mathcal{D}\widehat{\boldsymbol{\xi}}\|_q\\ &\le C\mu\overline{S}_p\ell(2\rho_1)\|\nabla(\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}})\|_q\end{aligned} \tag{3.27}$$

$$2\nu_r\|\operatorname{rot}(\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}})\|_{-1,\mathrm{q}}\le \mathrm{C}\nu_\mathrm{r}\|\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}}\|_\mathrm{q}\le\mathrm{C}\nu_\mathrm{r}\|\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}}\|_{1,\mathrm{r}}\le\mathrm{C}(\mathrm{C_p}+1)\nu_\mathrm{r}\|\nabla(\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}})\|_\mathrm{r} \tag{3.28}$$

$$\|(\zeta-\widehat{\zeta})\boldsymbol{f}\|_{-1,q}\le C\|(\zeta-\widehat{\zeta})\boldsymbol{f}\|_q\le C\|\zeta-\widehat{\zeta}\|_\infty\|\boldsymbol{f}\|_q\le CC_{\overline{E}}(C_p+1)\|\boldsymbol{f}\|_q\|\nabla(\zeta-\widehat{\zeta})\|_q \tag{3.29}$$

Inserting (3.26)–(3.29) into (3.25), we obtain

$$\begin{aligned}\|\nabla(\boldsymbol{u}_\varepsilon-\widehat{\boldsymbol{u}}_\varepsilon)\|_q&\le\frac{CC_{-1}}{\mu}\Big[2C_p(C_p^q+1)^{\frac{1}{q}}\rho_1\|\nabla(\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}})\|_q+(C_p+1)\nu_r\|\nabla(\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}})\|_r\\ &\qquad+\mu\overline{S}_p\ell(2\rho_1)\|\nabla(\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}})\|_q+C_{\overline{E}}(C_p+1)\|\boldsymbol{f}\|_q\|\nabla(\zeta-\widehat{\zeta})\|_s\Big]\\ &\le\overline{\lambda}_4\Big[\frac{1}{\mu}\rho_1+\frac{1}{\mu}\nu_r+\overline{S}_p\ell(2\rho_1)\Big]\cdot\max\Big\{\|\nabla(\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}})\|_q,\|\nabla(\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}})\|_r,\|\nabla(\zeta-\widehat{\zeta})\|_s\Big\}\end{aligned} \tag{3.30}$$

where $\overline{\lambda}_4=CC_{-1}\max\Big\{2C_p(C_p^q+1)^{\frac{1}{q}},C_p+1,1,C_{\overline{E}}(C_p+1)\Big\}$.

On the other hand, by the theory of elliptic equations, there exists a positive constant C_3 such that

$$\begin{aligned}\|\nabla(\omega_\varepsilon-\widehat{\omega}_\varepsilon)\|_r&\le\|\nabla(\omega_\varepsilon-\widehat{\omega}_\varepsilon)\|_{1,r}\\ &\le C_3\Big[\|2\nu_r\operatorname{rot}(\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}})\|_r+\|4\nu_r(\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}})\|_r+\|(\widehat{\boldsymbol{\xi}}\cdot\nabla)\widehat{\boldsymbol{\eta}}-(\boldsymbol{\xi}\cdot\nabla)\boldsymbol{\eta}\|_r\|(\zeta-\widehat{\zeta})\boldsymbol{g}\|_r\Big]\end{aligned} \tag{3.31}$$

For each term on the right-hand side of (3.31), we have

$$\|2\nu_r \mathrm{rot}(\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}})\|_r \le C\nu_r\|\nabla(\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}})\|_r \le C\nu_r\|\nabla(\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}})\|_q \tag{3.32}$$

$$\|4\nu_r(\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}})\|_r \le 4\nu_r C_p\|\nabla(\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}})\|_r \tag{3.33}$$

$$\|(\zeta-\widehat{\zeta})\boldsymbol{g}\|_r \le \|\zeta-\widehat{\zeta}\|_\infty\|\boldsymbol{g}\|_r \le C_{\widetilde{E}}(C_p+1)\|\nabla(\zeta-\widehat{\zeta})\|_s\|\boldsymbol{g}\|_r \tag{3.34}$$

$$\begin{aligned}
\|\widehat{\boldsymbol{\xi}}\cdot\nabla\widehat{\boldsymbol{\eta}}-\boldsymbol{\xi}\cdot\nabla\boldsymbol{\eta}\|_r &= \|(\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi})\nabla\widehat{\boldsymbol{\eta}}+\boldsymbol{\xi}\nabla(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r\\
&\le \|(\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi})\nabla\widehat{\boldsymbol{\eta}}\|_r+\|\boldsymbol{\xi}\nabla(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r\\
&\le \|\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi}\|_\infty\|\nabla\widehat{\boldsymbol{\eta}}\|_r+\|\boldsymbol{\xi}\|_\infty\|\nabla(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r\\
&\le C_E\|\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi}\|_{1,q}\|\nabla\widehat{\boldsymbol{\eta}}\|_r+C_E\|\boldsymbol{\xi}\|_{1,q}\|\nabla(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r\\
&\le C_E(C_p+1)\|\nabla(\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi})\|_q\|\nabla\widehat{\boldsymbol{\eta}}\|_r+C_E(C_p+1)\|\nabla\boldsymbol{\xi}\|_q\|\nabla(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r\\
&\le C_E(C_p+1)\|\nabla(\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi})\|_q\|\nabla\widehat{\boldsymbol{\eta}}\|_{1,r}+C_E(C_p+1)\|\nabla\boldsymbol{\xi}\|_{1,q}\|\nabla(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r\\
&\le \frac{C_E(C_p+1)}{C_{\widetilde{E}}}\rho_1\|\nabla(\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi})\|_q+(C_p+1)\rho_1\|\nabla(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r
\end{aligned}\tag{3.35}$$

Combining (3.31)–(3.35), it follows that

$$\|\nabla(\omega_\varepsilon-\widehat{\omega}_\varepsilon)\|_r \le \overline{\lambda}_5(2\nu_r+2\rho_1+\|\boldsymbol{g}\|_r)\cdot\max\left\{\|\nabla(\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi})\|_q,\|\nabla(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r,\|\nabla(\zeta-\widehat{\zeta})\|_s\right\} \tag{3.36}$$

where $\overline{\lambda}_5 = C_3\max\left\{4C_p,\ C,\ \frac{C_E(C_p+1)}{C_{\widetilde{E}}},\ C_p+1,\ C_{\widetilde{E}}(C_p+1)\right\}$.

Noticing that

$$-\kappa(\cdot,\theta_\varepsilon)\triangle\theta_\varepsilon+\kappa(\cdot,\widehat{\theta}_\varepsilon)\triangle\widehat{\theta}_\varepsilon = \kappa(\cdot,\widehat{\theta}_\varepsilon)\triangle(\widehat{\theta}_\varepsilon-\theta_\varepsilon)+(\kappa(\cdot,\widehat{\theta}_\varepsilon)-\kappa(\cdot,\theta_\varepsilon))\triangle\theta_\varepsilon \tag{3.37}$$

it follows from $(3.24)_4$ that

$$\|\nabla(\theta_\varepsilon-\widehat{\theta}_\varepsilon)\|_s \le \frac{1}{\kappa_1}\|H\|_s+\frac{1}{\kappa_1}\|\kappa'(\cdot,\zeta)|\nabla\zeta|^2-\kappa'(\cdot,\widehat{\zeta})|\nabla\widehat{\zeta}|^2\|_s+\frac{1}{\kappa_1}\|(\kappa(\cdot,\widehat{\theta}_\varepsilon)-\kappa(\cdot,\theta_\varepsilon))\triangle\theta_\varepsilon\|_s \tag{3.38}$$

Recall that $H = \Phi(\boldsymbol{\xi},\boldsymbol{\eta})-\Phi(\widehat{\boldsymbol{\xi}},\widehat{\boldsymbol{\eta}})-(\boldsymbol{\xi}\cdot\nabla)\zeta+(\widehat{\boldsymbol{\xi}}\cdot\nabla)\widehat{\zeta}$ and $\Phi(\boldsymbol{u},\omega)=\sum_{i=1}^{4}\Phi_i$. In the sequel, we shall derive estimates for each term on the right-hand side of (3.38) one by one.

The first term can be estimated as follows.

$$\begin{aligned}
\|\Phi_1(\boldsymbol{\xi},\boldsymbol{\eta})-\Phi_1(\widehat{\boldsymbol{\xi}},\widehat{\boldsymbol{\eta}})\|_s &\le C\left(\int_\Omega |[\nabla(\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}})+(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta})][\nabla(\boldsymbol{\xi}+\widehat{\boldsymbol{\xi}})-(\widehat{\boldsymbol{\eta}}+\boldsymbol{\eta})]|^s dx\right)^{1/s}\\
&\le \sup_{x\in\Omega}[|\nabla(\boldsymbol{\xi}+\widehat{\boldsymbol{\xi}})|+|(\widehat{\boldsymbol{\eta}}+\boldsymbol{\eta})|]\left(\int_\Omega |[\nabla(\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}})+(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta})]|^s dx\right)^{1/s}\\
&\le (C\rho_1+C\rho_1)\left(\|\nabla(\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}})\|_s+\|\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}}\|_s\right)\\
&\le C\rho_1\left(\|\nabla(\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}})\|_q+\|\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}}\|_\infty\right)\\
&\le C\rho_1\|\nabla(\boldsymbol{\xi}-\widehat{\boldsymbol{\xi}})\|_q+C\rho_1 C_{\widetilde{E}}(C_p+1)\|\nabla(\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}})\|_r
\end{aligned}$$

$$\sum_{i=2}^{4}\|\Phi_i(\boldsymbol{\eta})-\Phi_i(\widehat{\boldsymbol{\eta}})\|_s \le C\rho_1\|\nabla(\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}})\|_s \le C\rho_1\|\nabla(\boldsymbol{\eta}-\widehat{\boldsymbol{\eta}})\|_r$$

$$
\begin{aligned}
\|(\widehat{\boldsymbol{\xi}}\cdot\nabla)\widehat{\zeta}-(\boldsymbol{\xi}\cdot\nabla)\zeta\|_s &= \|(\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi})\nabla\widehat{\zeta}+\boldsymbol{\xi}\nabla(\widehat{\zeta}-\zeta)\|_s \\
&\le \|(\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi})\nabla\widehat{\zeta}\|_s+\|\boldsymbol{\xi}\nabla(\widehat{\zeta}-\zeta)\|_s \\
&\le \|\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi}\|_\infty\|\nabla\widehat{\zeta}\|_s+\|\boldsymbol{\xi}\|_\infty\|\nabla(\widehat{\zeta}-\zeta)\|_s \\
&\le C_E\|\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi}\|_{1,q}\|\nabla\widehat{\zeta}\|_s+C_E\|\boldsymbol{\xi}\|_{1,q}\|\nabla(\widehat{\zeta}-\zeta)\|_s \\
&\le \frac{C_E(C_p+1)}{C_{\overline{E}}}\rho_1\|\nabla(\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi})\|_q+(C_p+1)\rho_1\|\nabla(\widehat{\zeta}-\zeta)\|_s
\end{aligned}
\tag{3.39}
$$

it follows that

$$
\|H\|_s \le \overline{\lambda}_6(4\rho_1)\cdot\max\left\{\|\nabla(\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi})\|_q,\|\nabla(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r,\|\nabla(\widehat{\zeta}-\zeta)\|_s\right\} \tag{3.40}
$$

where $\overline{\lambda}_6=\max\left\{C,C[1+C_{\overline{E}}(C_p+1)],\frac{C_E(C_p+1)}{C_{\overline{E}}},C_p+1\right\}$.

For the second term

$$
\begin{aligned}
\|\kappa'(\cdot,\zeta)|\nabla\zeta|^2-\kappa'(\cdot,\widehat{\zeta})|\nabla\widehat{\zeta}|^2\|_s &= \|(\kappa'(\cdot,\zeta)-\kappa'(\cdot,\widehat{\zeta}))|\nabla\zeta|^2+\kappa'(\cdot,\widehat{\zeta})(|\nabla\zeta|^2-|\nabla\widehat{\zeta}|^2)\|_s \\
&\le \lambda'\|\zeta-\widehat{\zeta}\|_\infty\||\nabla\zeta|^2\|_s+\|(\kappa'(\cdot,\widehat{\zeta})-\kappa'(\cdot,0))(|\nabla\zeta|^2-|\nabla\widehat{\zeta}|^2)\|_s \\
&\le \lambda'\|\zeta-\widehat{\zeta}\|_\infty\||\nabla\zeta|^2\|_s+\lambda'\|\widehat{\zeta}\|_\infty\|(|\nabla\zeta|^2-|\nabla\widehat{\zeta}|^2)\|_s \\
&\le \lambda' C_{\overline{E}}(C_p+1)C\|\nabla\zeta\|_{1,s}\|\nabla(\zeta-\widehat{\zeta})\|_s+\lambda' C_{\overline{E}}\|\widehat{\zeta}\|_{1,s}\|\nabla(\zeta-\widehat{\zeta})\cdot\nabla(\zeta+\widehat{\zeta})\|_s \\
&\le \lambda'(C_p+1)C\rho_1\|\nabla(\zeta-\widehat{\zeta})\|_s+2\lambda'(C_p+1)\rho_1^2\|\nabla(\zeta-\widehat{\zeta})\|_s
\end{aligned}
\tag{3.41}
$$

Finally, because $|\kappa(\cdot,a)-\kappa(\cdot,b)|\le\lambda'(|a|+|b|),\forall a,b\in\mathbb{R}$ and $\|\triangle\theta_\varepsilon\|_s\le\|\nabla\theta_\varepsilon\|_{1,s}$, we have

$$
\|(\kappa(\cdot,\widehat{\theta_\varepsilon})-\kappa(\cdot,\theta_\varepsilon))\triangle\theta_\varepsilon\|_s\le 2\lambda'\rho_1^2(C_p+1)^2\|\nabla(\theta_\varepsilon-\widehat{\theta_\varepsilon})\|_s \tag{3.42}
$$

Combining (3.38) and (3.40)–(3.42), we obtain

$$
\begin{aligned}
(1-\frac{2}{\kappa_1}\lambda'\rho_1^2(C_p+1)^2)\|\nabla(\theta_\varepsilon-\widehat{\theta_\varepsilon})\|_s &\le \frac{4\overline{\lambda}_6}{\kappa_1}\rho_1\max\left\{\|\nabla(\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi})\|_q,\|\nabla(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r,\|\nabla(\widehat{\zeta}-\zeta)\|_s\right\} \\
&\quad+\frac{\lambda'}{\kappa_1}C(C_p+1)\rho_1\|\nabla(\zeta-\widehat{\zeta})\|_s+\frac{2\lambda'}{\kappa_1}(C_p+1)\rho_1^2\|\nabla(\zeta-\widehat{\zeta})\|_s \\
&\le \rho_1\overline{\lambda}_7\max\left\{\|\nabla(\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi})\|_q,\|\nabla(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r,\|\nabla(\widehat{\zeta}-\zeta)\|_s\right\}
\end{aligned}
$$

where $\overline{\lambda}_7=\frac{3}{\kappa_1}\max\left\{4\overline{\lambda}_6,\lambda' C(C_p+1),2\lambda'(C_p+1)\right\}$.

Combining the above estimates, taking ρ_1 such that $\frac{2}{\kappa_1}\lambda'\rho_1^2(C_p+1)^2\le\frac{1}{2}$, we conclude that

$$
\begin{aligned}
&\max\left\{\|\nabla(\boldsymbol{u}_\varepsilon-\widehat{\boldsymbol{u}}_\varepsilon)\|_q,\ \|\nabla(\boldsymbol{\omega}_\varepsilon-\widehat{\boldsymbol{\omega}}_\varepsilon)\|_r,\ \|\nabla(\theta_\varepsilon-\widehat{\theta_\varepsilon})\|_s\right\} \\
&\quad\le\left[\frac{\overline{\lambda}_4\rho_1}{\mu}+\frac{\overline{\lambda}_4\nu_r}{\mu}+\overline{\lambda}_4\overline{S}_p\ell(2\rho_1)+\frac{\overline{\lambda}_4}{\mu}\|\boldsymbol{f}\|_q+2\overline{\lambda}_5\nu_r+2\overline{\lambda}_5\rho_1+\overline{\lambda}_5\|\boldsymbol{g}\|_r+2\overline{\lambda}_7\rho_1\right] \\
&\qquad\cdot\max\left\{\|\nabla(\widehat{\boldsymbol{\xi}}-\boldsymbol{\xi})\|_q,\|\nabla(\widehat{\boldsymbol{\eta}}-\boldsymbol{\eta})\|_r,\|\nabla(\widehat{\zeta}-\zeta)\|_s\right\}
\end{aligned}
$$

Choosing $\overline{\lambda}_0=\max\left\{\overline{\lambda}_4,2\overline{\lambda}_5,\overline{\lambda}_7\right\}$, noticing that $\rho_1\le\frac{2\overline{\lambda}_1(\|\boldsymbol{f}\|_q^2+\nu_r)}{\mu}$, and taking into account that the function ℓ is nondecreasing, and $\ell(4y) \le 4^{(p-2,1)^+}\ell(y)$, we finally obtain

$$\begin{aligned}
&\max\Big\{\|\nabla(\boldsymbol{u}_\varepsilon - \widehat{\boldsymbol{u}}_\varepsilon)\|_q, \|\nabla(\boldsymbol{\omega}_\varepsilon - \widehat{\boldsymbol{\omega}}_\varepsilon)\|_r, \|\nabla(\theta_\varepsilon - \widehat{\theta}_\varepsilon)\|_s\Big\}\\
&\le \overline{\lambda}_0\Big[\frac{\rho_1}{\mu} + \frac{\nu_r}{\mu} + \overline{S}_p\ell(2\rho_1) + \frac{\|\boldsymbol{f}\|_q}{\mu} + \|\boldsymbol{g}\|_r + \nu_r + 2\rho_1\Big]\cdot\max\Big\{\|\nabla(\widehat{\boldsymbol{\xi}} - \boldsymbol{\xi})\|_q, \|\nabla(\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta})\|_r, \|\nabla(\widehat{\zeta} - \zeta)\|_s\Big\}\\
&\le \overline{\lambda}_0\Big[(\frac{1}{\mu} + 2)\frac{2\overline{\lambda}_1(\|\boldsymbol{f}\|_q^2 + \nu_r)}{\mu} + (\frac{1}{\mu} + 1)\nu_r + \frac{\|\boldsymbol{f}\|_q}{\mu} + \|\boldsymbol{g}\|_r + \overline{S}_p 4^{(p-2,1)^+}\Big(\frac{\overline{\lambda}_1(\|\boldsymbol{f}\|_q^2 + \nu_r)}{\mu}\Big)\\
&\qquad\cdot\Big(1 + \frac{\overline{\lambda}_1(\|\boldsymbol{f}\|_q^2 + \nu_r)}{\mu}\Big)^{(p-3)^+}\Big]\cdot\max\Big\{\|\nabla(\widehat{\boldsymbol{\xi}} - \boldsymbol{\xi})\|_q, \|\nabla(\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta})\|_r, \|\nabla(\widehat{\zeta} - \zeta)\|_s\Big\}\\
&\le 4^{(p-2,1)^+}\overline{\lambda}_0\Big[(\frac{1}{\mu} + 2)\frac{2\overline{\lambda}_1(\|\boldsymbol{f}\|_q^2 + \nu_r)}{\mu} + (\frac{1}{\mu} + 1)\nu_r + \frac{\|\boldsymbol{f}\|_q}{\mu} + \|\boldsymbol{g}\|_r + \overline{S}_p\Big(\frac{\overline{\lambda}_1(\|\boldsymbol{f}\|_q^2 + \nu_r)}{\mu}\Big)\\
&\qquad\cdot\Big(1 + \frac{\overline{\lambda}_1(\|\boldsymbol{f}\|_q^2 + \nu_r)}{\mu}\Big)^{(p-3)^+}\Big]\cdot\max\Big\{\|\nabla(\widehat{\boldsymbol{\xi}} - \boldsymbol{\xi})\|_q, \|\nabla(\widehat{\boldsymbol{\eta}} - \boldsymbol{\eta})\|_r, \|\nabla(\widehat{\zeta} - \zeta)\|_s\Big\}
\end{aligned} \tag{3.43}$$

Considering the space $Y := W_0^{1,q}(\Omega) \times W_0^{1,r}(\Omega) \times W_0^{1,s}(\Omega)$ with the norm $\max\Big\{\|\nabla\cdot\|_q, |\nabla\cdot\|_r, |\nabla\cdot\|_s\Big\}$, (3.43) implies that

$$\begin{aligned}
\|T_\varepsilon(\widehat{\boldsymbol{\xi}}, \widehat{\boldsymbol{\eta}}, \widehat{\zeta}) - T_\varepsilon(\boldsymbol{\xi}, \boldsymbol{\eta}, \zeta)\|_Y \le 4^{(p-2,1)^+}\overline{\lambda}_0\Big[&(\frac{1}{\mu} + 2)\frac{2\overline{\lambda}_1(\|\boldsymbol{f}\|_q^2 + \nu_r)}{\mu} + (\frac{1}{\mu} + 1)\nu_r + \frac{\|\boldsymbol{f}\|_q}{\mu} + \|\boldsymbol{g}\|_r\\
&+ \overline{S}_p\Big(\frac{\overline{\lambda}_1(\|\boldsymbol{f}\|_q^2 + \nu_r)}{\mu}\Big)\cdot\Big(1 + \frac{\overline{\lambda}_1(\|\boldsymbol{f}\|_q^2 + \nu_r)}{\mu}\Big)^{(p-3)^+}\Big]\cdot\|(\widehat{\boldsymbol{\xi}}, \widehat{\boldsymbol{\eta}}, \widehat{\zeta}) - (\boldsymbol{\xi}, \boldsymbol{\eta}, \zeta)\|_Y
\end{aligned}$$

From this and hypothesis (3.23), we obtain that $T_\varepsilon : B_\rho \mapsto B_\rho$ is a contraction in $W_0^{1,q}(\Omega) \times W_0^{1,r}(\Omega) \times W_0^{1,s}(\Omega)$.

Step 4: Proof of Theorem 3.1.

We observe that, for $p \le 3$, $\gamma_p = \frac{1}{4} = \frac{1}{4^{(p-2,1)^+}}$, and, for $p > 3$, $\gamma_p > \frac{1}{4^{(p-2,1)^+}}$. Thus, by taking $\overline{\lambda} = (\overline{\lambda}_0, \overline{\lambda}_1)^+$, and because (3.2) implies (3.5) and (3.23), taking $X = V_{2,q} \times (W^{2,r}(\Omega) \cap W_0^{1,r}(\Omega)) \times (W^{2,s}(\Omega) \cap W_0^{1,s}(\Omega))$, $Y = W_0^{1,q}(\Omega) \times W_0^{1,r}(\Omega) \times W_0^{1,s}(\Omega)$, and $B = B_{\rho_1}$, according to Lemma 2.3, we know that T_ε has a unique fixed point on B_{ρ_1}. This completes the proof of Theorem 3.1.

4. Proof of Theorem 2.1

Notice that for each $\varepsilon > 0$, $(\boldsymbol{u}_\varepsilon, \boldsymbol{\omega}_\varepsilon, \theta_\varepsilon)$ satisfies the following weak formula:

$$\begin{aligned}
\int_\Omega 2\mu(1 + \sqrt{\varepsilon^2 + |\mathcal{D}\boldsymbol{u}_\varepsilon|^2})^{(p-2)}\mathcal{D}\boldsymbol{u}_\varepsilon : \mathcal{D}(\boldsymbol{\Psi})dx - \int_\Omega(\boldsymbol{u}_\varepsilon \otimes \boldsymbol{u}_\varepsilon) : \mathcal{D}(\boldsymbol{\Psi})dx&\\
= 2\nu_r\int_\Omega \mathrm{rot}\boldsymbol{\omega}_\varepsilon \cdot \boldsymbol{\Psi}dx + \int_\Omega \theta_\varepsilon\boldsymbol{f}\cdot\boldsymbol{\Psi}dx, \qquad \forall\boldsymbol{\Psi} \in \mathcal{V}&
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
\int_\Omega \nabla\boldsymbol{\omega}_\varepsilon \cdot \nabla\boldsymbol{\psi}dx + \int_\Omega(\boldsymbol{u}_\varepsilon\cdot\nabla)\boldsymbol{\omega}_\varepsilon\cdot\boldsymbol{\psi}dx + 4\nu_r\int_\Omega \boldsymbol{\omega}_\varepsilon\cdot\boldsymbol{\psi}dx - \int_\Omega \mathrm{div}\boldsymbol{\omega}_\varepsilon\mathrm{div}\boldsymbol{\psi}&\\
= 2\nu_r\int_\Omega \mathrm{rot}\boldsymbol{u}_\varepsilon\cdot\boldsymbol{\psi}dx + \int_\Omega \theta_\varepsilon\boldsymbol{g}\cdot\boldsymbol{\psi}dx, \qquad \forall\boldsymbol{\psi} \in C_0^\infty(\Omega)&
\end{aligned} \tag{4.2}$$

$$\int_\Omega \kappa(x,\theta_\varepsilon)\nabla\theta_\varepsilon\cdot\nabla\phi dx+\int_\Omega \phi\boldsymbol{u}_\varepsilon\cdot\nabla\theta_\varepsilon dx=\int_\Omega \Phi(\boldsymbol{u}_\varepsilon,\boldsymbol{\omega}_\varepsilon)\phi dx+\int_\Omega h\phi dx,\qquad \forall\phi\in C_0^\infty(\Omega) \tag{4.3}$$

From (3.10), (3.11), (3.16), and (3.22), we have that the sequence $\{(\boldsymbol{u}_\varepsilon,\boldsymbol{\omega}_\varepsilon,\theta_\varepsilon)\}_\varepsilon$ is uniformly bounded in $V_{2,q}\times(W^{2,r}(\Omega)\cap W_0^{1,r}(\Omega))\times(W^{2,s}(\Omega)\cap W_0^{1,s}(\Omega))$. Then, there exists a subsequence of $\{(\boldsymbol{u}_\varepsilon,\boldsymbol{\omega}_\varepsilon,\theta_\varepsilon)\}_\varepsilon$, still indexed by ε, and $(\boldsymbol{u},\boldsymbol{\omega},\theta)$ such that

$$\begin{aligned}
&((\boldsymbol{u}_\varepsilon,\boldsymbol{\omega}_\varepsilon,\theta_\varepsilon)\rightharpoonup(\boldsymbol{u},\boldsymbol{\omega},\theta)\ \text{ weakly in }\ V_{2,q}\times(W^{2,r}(\Omega)\cap W_0^{1,r}(\Omega))\times(W^{2,s}(\Omega)\cap W_0^{1,s}(\Omega))\\
&((\boldsymbol{u}_\varepsilon,\boldsymbol{\omega}_\varepsilon,\theta_\varepsilon)\to(\boldsymbol{u},\boldsymbol{\omega},\theta)\ \text{ strongly in }\ C^{1,\alpha_1}(\overline{\Omega})\times C^{1,\alpha_2}(\overline{\Omega})\times C^{1,\alpha_3}(\overline{\Omega})\\
&\qquad\alpha_1<1-\frac{3}{q},\quad \alpha_2<1-\frac{3}{r},\quad \alpha_3<1-\frac{3}{s}
\end{aligned}$$

Therefore, noticing that $\kappa:\Omega\times\mathbb{R}\to\mathbb{R}$ is a C^1-function and letting ε tend to 0 in (4.1)–(4.3), we have

$$\begin{aligned}
&\int_\Omega 2\mu(1+|\mathcal{D}\boldsymbol{u}|)^{(p-2)}\mathcal{D}\boldsymbol{u}:\mathcal{D}(\boldsymbol{\Psi})dx-\int_\Omega(\boldsymbol{u}\otimes\boldsymbol{u}):\mathcal{D}(\boldsymbol{\Psi})dx\\
&\qquad=2\nu_r\int_\Omega \text{rot}\boldsymbol{\omega}\cdot\boldsymbol{\Psi}dx+\int_\Omega\theta\boldsymbol{f}\cdot\boldsymbol{\Psi}dx,\quad \forall\boldsymbol{\Psi}\in\mathcal{V}
\end{aligned}\tag{4.4}$$

$$\begin{aligned}
&\int_\Omega\nabla\boldsymbol{\omega}\cdot\nabla\boldsymbol{\psi}dx+\int_\Omega(\boldsymbol{u}\cdot\nabla)\boldsymbol{\omega}\cdot\boldsymbol{\psi}dx+4\nu_r\int_\Omega\boldsymbol{\omega}\cdot\boldsymbol{\psi}dx-\int_\Omega\text{div}\boldsymbol{\omega}\text{div}\boldsymbol{\psi}\\
&\qquad=2\nu_r\int_\Omega\text{rot}\boldsymbol{u}\cdot\boldsymbol{\psi}dx+\int_\Omega\theta\boldsymbol{g}\cdot\boldsymbol{\psi}dx,\qquad\forall\boldsymbol{\psi}\in C_0^\infty(\Omega)
\end{aligned}\tag{4.5}$$

$$\int_\Omega\kappa(x,\theta)\nabla\theta\cdot\nabla\phi dx+\int_\Omega\phi\boldsymbol{u}\cdot\nabla\theta dx=\int_\Omega\Phi(\boldsymbol{u},\boldsymbol{\omega})\phi dx+\int_\Omega h\phi dx,\quad\forall\phi\in C_0^\infty(\Omega)\tag{4.6}$$

The regularity of $(\boldsymbol{u},\boldsymbol{\omega},\theta)$ follows from (3.10), (3.11), and (3.16). Theorem 2.1 is proved.

5. Conclusions

In this paper, we proved the existence and uniqueness of strong solutions for a class of steady non-Newtonian micropolar fluid equations with heat convection. As far as we can see, the known results are all regarding the Newtonian case, and related results for such a problem of non-Newtonian type have not been considered yet. The results in this paper are new and generalize many related problems in the literature.

Use of AI tools declaration

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there are no conflicts of interest.

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