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Research article

Attractors for the nonclassical diffusion equations with the driving delay term in time-dependent spaces

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Abstract: In this study, we primarily investigate the asymptotic behavior of solutions associated with a nonclassical diffusion process by memory effects and a perturbed parameter that varies over time. A significant innovation is the consideration of a delay term governed by a function with minimal assumptions: merely measurability and a phase-space that is a time-dependent space of continuously-time-varying functions. By employing a novel analytical approach, we demonstrate the existence and regularity of time-varying pullback $\mathscr{D}$-attractors. Notably, the nonlinearity f is unrestricted by any upper limit on its growth rate.

Keywords: nonclassical diffusion equation; pullback attractors; nonlinear delay; polynomial growth

1. Introduction

This paper is concerned with the following nonclassical diffusion equation with memory and delay

$$\partial_t u - \varepsilon(t)\Delta\partial_t u - \Delta u + f(u) = g(t, u_t), \text{ in } \Omega \times \mathbb{R}_\tau, \tag{1.1}$$

and the problem is complemented by the boundary condition

$$u(x,t)|_{\partial\Omega\times(\tau,\infty)} = 0, \quad \forall t \geq \tau, \tau \in \mathbb{R}, \ \theta \in [-h, 0], \tag{1.2}$$

and initial condition

$$u_\tau = u(\tau + \theta) = \phi(\theta), \qquad x \in \Omega, \ \theta \in [-h, 0]. \tag{1.3}$$

Here, $g(\cdot,\cdot)$ represents an operator that incorporates hereditary characteristics, for each $t \geq \tau$, $u_t = u_t(\theta) = u(t + \theta)$ for $\theta \in [-h, 0]$ (where $h > 0$ denotes the duration of the delay effects), $\mathbb{R}_\tau = [\tau, +\infty)$, and $\Omega \subset \mathbb{R}^n$ $(n \geq 3)$. The initial value $\phi \in C([-h, 0]; H_0^1(\Omega))$. For simplicity, let C_X denote the Banach space $C([-h, 0]; X)$ endowed with the supremum norm. For any element $u \in C_X = C([-h, 0]; X)$, the norm is defined as

$$\|u\|_{C_X} = \max_{t\in[-h,0]} \|u(t)\|_X .$$

To analyze problems (1.1)–(1.3), we introduce several considerations regarding the time-varying disturbance factor $\varepsilon(t)$, the nonlinearity f and the driving delay term g, respectively:

(H_1) The time-dependent perturbed parameter $\varepsilon(t) \in C^1(\mathbb{R})$ is characterized by a monotonic decrease and boundedness that satisfies

$$\lim_{t\to+\infty} \varepsilon(t) = 0, \tag{1.4}$$

and there exists a positive constant L, such that

$$\sup_{t\in\mathbb{R}} \left(|\varepsilon(t)| + |\varepsilon'(t)|\right) \leq L. \tag{1.5}$$

(H_2) The function $f \in C^1(\Omega)$ satisfies:

$$f(s)s \geq -\alpha|s|^2 - \beta, \quad \forall s \in \mathbb{R}, \tag{1.6}$$

and

$$f'(s) \geq -l, \quad \forall s \in \mathbb{R}, \tag{1.7}$$

where $f(0) = 0$. The constants α,β, and l are positive, and $\alpha < \lambda_1$ (λ_1 is the primary eigenvalue of the Laplacian operator $-\Delta$ in $H_0^1(\Omega)$ under Dirichlet boundary conditions).

(H_3) The delay operator $g(\cdot,\cdot)$: $\mathbb{R} \times C_X \to L^2(\Omega)$ is subject to the following conditions:

(i) For any $\xi \in C_X$ and $t \in \mathbb{R}$, the function $t \mapsto g(t, \xi)$ is measurable when considered as a mapping from $\mathbb{R}$ to $L^2(\Omega)$;

(ii) $g(t, 0) = g_0(t) \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$, such that there exists a $\eta^0 \geq 0$ for which, for any $\eta \in [0, \eta^0]$,

$$\int_{-\infty}^{t} e^{\eta s} \|g_0(s)\|^2_{L^2(\Omega)} ds < +\infty \tag{1.8}$$

holds for all $t \in \mathbb{R}$;

(iii) For each $t \in \mathbb{R}$, one can find a constant $L_g > 0$ satisfying the following inequality

$$\|g(t, \varphi_2) - g(t, \varphi_1)\|_{L^2(\Omega)} \leq L_g \|\varphi_2 - \varphi_1\|_{C_{L^2(\Omega)}}, \ \forall\ \varphi_1, \varphi_2 \in C_{L^2(\Omega)}. \tag{1.9}$$

We denote F as the function

$$F(s) = \int_0^s f(\rho) d\rho.$$

Subsequently, there exists a positive constant β_1 such that the inequalities

$$F(s) \geq -\frac{1}{2}\alpha s^2 - \beta_1, \quad \forall s \in \mathbb{R}, \tag{1.10}$$

and

$$f(s)s \geq F(s) - \frac{1}{2}\alpha s^2 - \beta_1, \quad \forall s \in \mathbb{R} \tag{1.11}$$

hold true.

Remark 1.1. *Note that the assumption regarding the nonlinearity is akin to that in reference [1], but with some constraints being relaxed. Specifically, we no longer demand that* $l < \lambda_1$*. The class of nonlinearities examined in [2–5] and others is characterized by an upper growth limitation, which precludes the inclusion of exponential nonlinearities (e.g.,* $f(u) = e^u$*). Furthermore, the time-delayed driving component* $g(t, u_t)$ *complies with* $g(t, 0) = g_0(t) \in L^2_{loc}(\mathbb{R}, L^2(\Omega))$ *, rather than* $g(t, 0) = k(x)$ *(see e.g., [6–9]). We define*

$$\mathscr{G}_\sigma(t) = \int_{-\infty}^{t} e^{-\sigma(t-s)} \|g_0(s)\|^2_{L^2(\Omega)} ds.$$

Consequently, for every specified $t \in \mathbb{R}$*, the behavior of* $\mathscr{G}_\sigma(t)$ *is characterized by its boundedness and a monotonic decrease as the parameter* σ *varies, e.g.,*

$$\mathscr{G}_\sigma(t) < +\infty, \qquad \text{for any } \sigma > 0;$$
$$\mathscr{G}_{\sigma_2}(t) \le \mathscr{G}_{\sigma_1}(t), \qquad \text{for any } \sigma_2 \ge \sigma_1.$$

The diffusion-driven reaction model frequently serves as a mathematical framework for elucidating the nuances of heat transfer across the domains of fluid dynamics and solid mechanics. Moreover, it extends its applicability to the analysis of epidemiological systems, cellular neural networks, and stochastic environments. However, the influence of viscosity is significant in many such problems, necessitating an extension of the classical heat conduction equation. This extension is typically expressed in the following form (see e.g., [10–12]):

$$c\dot{u} - ca\Delta\dot{u} - k\Delta u = 0.$$

Furthermore, when examining polymers and highly viscous liquids, it is crucial to incorporate key elements like the historical impact of u and the disturbance coefficient of viscosity (see e.g., [13]), leading to the ensuing evolution formula:

$$u_t - \varepsilon\Delta u_t + f(u) = g(t, u_t). \tag{1.12}$$

The delay term $g(t, u_t)$ exemplifies the impact of an external force characterized by various types of time lags, memory effects, or hereditary attributes, and is capable of emulating some feedback controls. Recent research has identified a variety of delay terms within equations, with two being particularly typical. The first type features a distributed delay formulated as $\int_{-h}^{0} \mu(t-s)g(u(x, s))ds$ incorporating the kernel function μ. Moreover, memory effects also come in various forms, including general hereditary memory, which can be represented as $\int_0^\infty k(s)\Delta u(t-s)ds$. Numerous researchers have delved into the long-term behavior of solutions to Eq (1.12) with this type of memory (for example, see [9, 14–19] and the references therein). The second type involves a variable delay $g(t, u(x, t+\theta))$, where θ denotes a variable potentially related to t. Significant progress has also been made in understanding the long-term behavior of solutions to nonclassical reaction-diffusion equations that incorporate variable delays, as documented in several studies (refer to [8, 20, 21] and the references therein).

However, early research primarily concentrated on the nonclassical diffusion equation characterized by a fixed coefficient $\varepsilon = 1$. In [13], the incorporation of memory elements into diffusion equations played a crucial role in advancing the understanding of thermal conduction and the viscous relaxation dynamics of high-viscosity liquids. The convolution term encapsulates the influence of past states on future behavior, offering a more nuanced depiction of diffusion phenomena in specific materials. Examples include liquids with high viscosity at reduced temperatures and polymeric compounds. Hence, it is both necessary and scientifically significant to study the nonclassical diffusion equation that incorporates a coefficient varying with time, or a variable coefficient, along with memory effects. This exploration is encapsulated in the equation:

$$u_t - \varepsilon(t)\Delta u_t - \nu\Delta u + f(u) = g(t, u_t) + k. \tag{1.13}$$

Regarding Eq (1.13), the current research concentrates on analyzing the nonclassical diffusion equation, which is characterized by the presence of a variable delay and a time-varying perturbation parameter $\varepsilon(t)$. For example, the authors in [6–9, 22, 23] demonstrated the presence and regularity of the temporal global attractor within time-varying spaces, provided that the nonlinearity exhibits either critical exponential growth or polynomial growth of any order, with $g(t, 0) = 0$ and $k = k(x) \in L^2(\Omega)$ or $H^{-1}(\Omega)$.

Our objective in this paper is to introduce improvements to the conditions governing the nonclassical diffusion problem (1.1) with variable delay and time-dependent perturbed parameter $\varepsilon(t)$. Specifically, we consider a nonclassical diffusion problem under the sole requirement of measurability for the propelling delay components within the equation. Our investigation extends to examining the persistent behavior of the solutions over extended periods. Nonetheless, the existing literature offers limited insights into the asymptotic properties of solutions for Eq (1.1) within time-varying spaces, under the hypotheses (H_1) to (H_3). This can be attributed to two primary challenges in obtaining the presence of time-varying pullback $\mathscr{D}$-attractors within the realm of time-dependent continuous function spaces. First, due to the lack of restrictions on upper growth for the nonlinearity, achieving higher asymptotic regularity of the solutions to Eq (1.1) is not feasible using the methods employed in [24, 25]; Second, the effect of the time-varying perturbation parameter $\varepsilon(t)$ and the absence of the compact embedding theorem, renders it infeasible to directly formulate a contractive function to demonstrate the asymptotic compactness of the associated process $\{S(t, \tau)\}_{t\geq\tau}$ for Eq (1.1), as discussed in [26–28]. To address these issues, we employ a novel analysis technique combined with the operator decomposition technique to derive a contractive function. This enables us to demonstrate the pullback $\mathscr{D}$-asymptotic compactness for the process $\{S(t, \tau)\}_{t\geq\tau}$ associated with Eqs (1.1)–(1.3). Furthermore, utilizing this operator decomposition technique, we also demonstrate the long-term stability and pattern of the solutions across Eqs (1.1)–(1.3). As a result, this establishes the consistency of the pullback $\mathscr{D}$-attractors that are contingent on time for these equations.

To keep our discussion concise, we shall employ the notation $|\cdot|_p$ instead of the norm of $L^p(\Omega)$ $(p \geq 1)$ throughout the rest of this text. Let $(\cdot,\cdot)$, $(\nabla\cdot, \nabla\cdot) = \langle\cdot,\cdot\rangle_{H_0^1(\Omega)}$ and $(\Delta\cdot, \Delta\cdot) = \langle\cdot,\cdot\rangle_{D(A)}$ denote the inner product of $L^2(\Omega)$, $H_0^1(\Omega)$, and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$ respectively.

Define $\|\cdot\|_{r-1} = |A^{\frac{r}{2}}\cdot|_2$ as the norm of $D(A^{\frac{r}{2}})$ $(1 \leq r \leq 2)$, and the time-dependent space $\mathcal{H}_t^r$ is equipped with the norms:

$$\|\cdot\|^2_{\mathcal{H}_t^r} = |A^{\frac{r-1}{2}}\cdot|_2^2 + \varepsilon(t)|A^{\frac{r}{2}}\cdot|_2^2.$$

Furthermore, the norm of time-dependent continuous function space $C_{\mathcal{H}_t^r}$ is given by

$$\|u_t\|^2_{C_{\mathcal{H}_t^r}} = \max_{\theta\in[-h,0]} \|u_t(\theta)\|^2_{\mathcal{H}_t^r}.$$

It is necessary to compare the relationship between $\|\cdot\|^2_{C_{\mathcal{H}^r_t}}$, $\|\cdot\|^2_{\mathcal{H}^r_t}$ and $\|\cdot\|^2_{C_{D(A^{(r-1)/2})}} + \varepsilon(t)\|\cdot\|^2_{C_{D(A^{r/2})}}$. Notably, the subsequent inequality is evident:

$$\begin{aligned}\|u_t\|^2_{C_{\mathcal{H}^r_t}} &= \max_{\theta\in[-h,0]}\left\{|A^{\frac{r-1}{2}}u(t+\theta)|^2_2 + \varepsilon(t+\theta)|A^{\frac{r}{2}}u(t+\theta)|^2_2\right\}\\ &\geq |A^{\frac{r-1}{2}}u(t)|^2_2 + \varepsilon(t)|A^{\frac{r}{2}}u(t)|^2_2 = \|u\|^2_{\mathcal{H}^r_t}.\end{aligned}$$

In addition, it is straightforward to drive the ensuing approximation:

$$\begin{aligned}&\max_{\theta\in[-h,0]}|A^{\frac{r-1}{2}}u(t+\theta)|^2_2 + \varepsilon(t)\max_{\theta\in[-h,0]}|A^{\frac{r}{2}}u(t+\theta)|^2_2\\ &\leq \max_{\theta\in[-h,0]}\left(|A^{\frac{r-1}{2}}u(t+\theta)|^2_2 + \varepsilon(t+\theta)|A^{\frac{r}{2}}u(t+\theta)|^2_2\right) + \max_{\theta\in[-h,0]}\varepsilon(t)|A^{\frac{r}{2}}u(t+\theta)|^2_2\\ &\leq \max_{\theta\in[-h,0]}\left(|A^{\frac{r-1}{2}}u(t+\theta)|^2_2 + \varepsilon(t+\theta)|A^{\frac{r}{2}}u(t+\theta)|^2_2\right) + \max_{\theta\in[-h,0]}\varepsilon(t+\theta)|A^{\frac{r}{2}}u(t+\theta)|^2_2\\ &\leq 2\max_{\theta\in[-h,0]}\left(|A^{\frac{r-1}{2}}u(t+\theta)|^2_2 + \varepsilon(t+\theta)|A^{\frac{r}{2}}u(t+\theta)|^2_2\right)\\ &= 2\|u_t\|^2_{C_{\mathcal{H}^r_t}}.\end{aligned}$$

As a result, we find that

$$\|\cdot\|^2_{\mathcal{H}^r_t} \leq \|\cdot\|^2_{C_{D(A^{\frac{r-1}{2}})}} + \varepsilon(t)\|\cdot\|^2_{C_{D(A^{\frac{r}{2}})}} \leq 2\|\cdot\|^2_{C_{\mathcal{H}^r_t}}. \tag{1.14}$$

Remark 1.2. *In general, the solution generated by problems* (1.1)–(1.3) *will be defined as* $(u, u_t) \in \mathcal{H}^1_t \times C_{\mathcal{H}^1_t}$*; and the norm of the time-dependent product space* $\mathcal{H}^1_t \times C_{\mathcal{H}^1_t}$ *is as follows:*

$$\|\cdot\|^2_{\mathcal{H}^1_t\times C_{\mathcal{H}^1_t}} = \|\cdot\|^2_{\mathcal{H}^1_t} + \|\cdot\|^2_{C_{\mathcal{H}^1_t}}.$$

Obviously, $\|\cdot\|^2_{\mathcal{H}^1_t\times C_{\mathcal{H}^1_t}}$ *and* $\|\cdot\|^2_{C_{\mathcal{H}^1_t}}$ *are equivalent. Consequently, the dynamic behavior of the process series* $\{S(t,\tau)\}_{t\in\mathbb{R}}$*, propelled by* u_t*, can fully encapsulate the dynamic behavior of* (u, u_t)*.*

The outline of this manuscript is as follows. Section 2 provides an overview of foundational ideas, including time-varying pullback attractors, along with pivotal results that will be utilized in subsequent discussions. Section 3 begins with the demonstration of asymptotic compactness in the pullback sense related to the process governed by problems (1.1)–(1.3) through the construction of a contractive function. Following this, we establish the existence and regularity of time-varying pullback $\mathscr{D}$- attractors for problem (1.1) with (1.2) and (1.3) in $C_{\mathcal{H}^1_t}$.

2. Foundations

In this chapter, we shall delve into the foundational principles of time-varying pullback $\mathscr{D}-$attractors and theories related to their existence (see e.g., [27, 29–31]).

Definition 2.1. *Denote* $\{X_t\}_{t\in\mathbb{R}}$ *as a collection of spaces equipped with norms. A pair of indexed operators* $\{S(t,\tau)\}_{t\geq\tau}$*, where* $S(t,\tau) : X_\tau \to X_t$*, is referred to as a dynamical process if the following conditions are met:*
(i) $S(\tau,\tau) = Id, \tau \in \mathbb{R}$ *(Identity operator on* X_τ*);*
(ii) $S(t,s)S(s,\tau) = S(t,\tau), \forall t \geq s \geq \tau \in \mathbb{R}$.

Definition 2.2. *Consider a collection of Banach spaces denoted by* $\{X_t\}_{t\in\mathbb{R}}$*, and let* $\{S(t,\tau)\}_{t\geq\tau}$ *represent a continuous evolution of operators on* $\{X_t\}_{t\in\mathbb{R}}$*. This evolution is characterized by the continuity of the mapping*

$$S(t,\tau) : X_\tau \to X_t$$

for all $\tau \leq t$*. Furthermore, the evolution* $\{S(t,\tau)\}_{t\geq\tau}$ *is designated as closed if, for any* $\tau \leq t$*, and for any sequence* $\{x_n\} \subset X_\tau$ *that converges to* $x \in X_\tau$ *with the corresponding sequence* $\{S(t,\tau)\}_{x_n}$ *converging to* $y \in X_t$*, it follows that* $S(t,\tau)x = y$*.*

If a dynamical process exhibits continuity, it can also be considered closed. Therefore, it is more general to develop a theoretical framework based on the notion of a closed process.

Define $\mathscr{D}$ as the assembly of all sequences $\hat{\mathcal{D}} = \{\mathcal{D}(t) : t \in \mathbb{R}\}$ where $\mathcal{D}(t) \in \mathscr{P}(X_t)$ for every $t \geq \tau \in \mathbb{R}$, with $\mathscr{P}(X_t)$ signifying the entirety of non-vacuous subsets within X_t.

Definition 2.3. *Consider a sequence of Banach spaces* $\{X_t\}_{t\in\mathbb{R}}$*. A process* $\{S(t,\tau)\}_{t\geq\tau}$ *is termed as a pullback* $\mathscr{D}$*-attracting process if there is a pullback* $\mathscr{D}$*-absorbing set* $\hat{\mathcal{D}}_0 = \{\mathcal{D}_0(t) : t \in \mathbb{R}\}$*, where each* $\mathcal{D}_0(t) \in \mathscr{P}(X_t)$*. That is, for each* $t \in \mathbb{R}$ *and for any* $\hat{\mathcal{D}} \in \mathscr{D}$*, there is a* $\tau_0 = \tau_0(t,\hat{\mathcal{D}}) \leq t$ *such that* $S(t,\tau)\mathcal{D}(\tau) \subset \mathcal{D}_0(t)$ *for all* $\tau \leq \tau_0(t,\hat{\mathcal{D}})$*.*

Definition 2.4. *Consider a collection of Banach spaces* $\{X_t\}_{t\in\mathbb{R}}$*. We characterize a two-parameter operator sequence* $\{S(t,\tau)\}_{t\geq\tau}$ *as being pullback* $\mathscr{D}$*-asymptotically compact under the following condition: for every* $t \in \mathbb{R}$*, any* $\hat{\mathcal{D}} \in \mathscr{D}$*, any decreasing sequence* $\{\tau_n\} \leq t$ *that tends to* $-\infty$ *as* n *approaches infinity, and* $\{x_n\} \in X_{\tau_n}$ *with* $x_n \in \mathcal{D}(\tau_n)$ *for all* $n \in \mathbb{N}$*, the set* $\{S(t,\tau_n)x_n\}_{n=1}^{\infty}$ *includes a subsequence that converges.*

Definition 2.5. *Define a series of Banach spaces* $\{X_t\}_{t\in\mathbb{R}}$ *and let* $\mathscr{D}$ *be the collcetion of all families* $\hat{\mathcal{D}} = \{\mathcal{D}(t) : t \in \mathbb{R}\}$ *where each* $\mathcal{D}(t) \in \mathscr{P}(X_t)$*. An indexed collection of compact subsets* $\hat{\mathscr{A}} = \{\mathscr{A}(t) : t \in \mathbb{R}\}$ *with* $\mathscr{A}(t) \in \mathscr{P}(X_t)$ *is referred to as the time-varying pullback* $\mathscr{D}$*-attractor for the dynamical process* $\{S(t,\tau)\}_{t\geq\tau}$ *if the following conditions are met:*

(i) $\hat{\mathscr{A}}$ *remains unchanged under the action of the process* $S(t,\tau)$*, i.e.,*

$$S(t,\tau)\mathscr{A}(\tau) = \mathscr{A}(t) \text{ for } t \geq \tau;$$

(ii) $\hat{\mathscr{A}}$ *asymptotically attracts every* $\hat{\mathcal{D}} \in \mathscr{D}$*, i.e.,*

$$\lim_{\tau\to-\infty} dist_X(S(t,\tau)\mathcal{D}(\tau), \mathscr{A}(t)) = 0,$$

for all $\mathcal{D}(\tau) \in \hat{\mathcal{D}}$ *and all* $t \in \mathbb{R}$*; and*

(iii) $\hat{\mathscr{A}}$ *holds the property of minimality: If there exists another collection of closed sets* $\hat{C} = \{C(t) : t \in \mathbb{R}\}$ *fulfilling condition (ii), then it follows that* $\mathscr{A}(t) \subset C(t)$ *to every* $t \in \mathbb{R}$*.*

To avoid any ambiguity, we still refer to the $\mathscr{D}$-attractor with time-varying properties as the pullback $\mathscr{D}$-attractor.

In what follows, we will adhere to the presumption that the configuration of the pullback $\mathscr{D}$-attractors is as described in Theorem 2.6, and $\mathscr{D}$ is assumed to be a non-empty collection of parameterized families $\hat{\mathcal{D}} = \{\mathcal{D}(t) \in \mathscr{P}(X_t) : t \in \mathbb{R}\}$.

Theorem 2.6. *Consider a sequence of Banach spaces $\{X_t\}_{t\in\mathbb{R}}$ and a continuous processes $\{S(t,\tau)\}_{t\geq\tau}$. Suppose the subsequent assumptions are satisfied:*

(i) $\{S(t,\tau)\}_{t\geq\tau}$ possesses a pullback $\mathscr{D}$-ingesting family $\hat{\mathcal{D}}_0 = \{\mathcal{D}_0(t) : t \in \mathbb{R}\}$ with $\mathcal{D}_0(t) \subset X_t$;

(ii) $\{S(t,\tau)\}_{t\geq\tau}$ demonstrates retrogressive $\mathscr{D}$-asymptotic compactness within the set $\hat{\mathcal{D}}_0$.

Subsequently, the set $\hat{\mathscr{A}} = \{\mathscr{A}(t) : t \in \mathbb{R}\}$, where $\mathscr{A}(t) = \Lambda(\hat{\mathcal{D}}_0, t)$, is identified as a retrogressive $\mathscr{D}$-attractor for the dynamical process $\{S(t,\tau)\}_{t\geq\tau}$, where

$$\Lambda(\hat{\mathcal{D}}, t) = \bigcap_{s\leq t}\overline{\bigcup_{\tau\leq s} S(t,\tau)\mathcal{D}(\tau)}^{X_s},$$

for every $t \in \mathbb{R}$ and for any $\hat{\mathcal{D}} \in \mathscr{D}$. Moreover, $\hat{\mathscr{A}}$ fulfills $\mathscr{A}(t) = \overline{\bigcup_{\hat{\mathcal{D}}\in\mathscr{D}} \Lambda(\hat{\mathcal{D}}, t)}$, for every $t \in \mathbb{R}$. Furthermore, $\hat{\mathscr{A}}$ is minimal, meaning that if $\hat{C} = \{C(t) : t \in \mathbb{R}\}$ represents a family of nonempty closed subsets of X such that

$$\lim_{\tau\to-\infty} dist_{X_t}(S(t,\tau)\mathcal{D}(\tau), C(t)) = 0,$$

for all $t \in \mathbb{R}$, then $\mathscr{A}(t) \subset C(t)$ for any $t \in \mathbb{R}$.

Subsequently, we shall define the concept of a restrictive function and a restrictive process, instrumental in substantiating the asymptotic compactness for the sequence of operators $\{S(t,\tau)\}_{t\geq\tau}$ (as referenced in [19,27,32–35]).

Definition 2.7. *Suppose $\{X_t\}_{t\in\mathbb{R}}$ represents a series of Banach spaces, and $\hat{\mathcal{D}} = \{\mathcal{D}(t) : t \in \mathbb{R}\}$ with $\mathcal{D}(t) \in \mathscr{P}(X_t)$. A function $\psi(\cdot,\cdot)$ is termed as a contraction mapping on $\hat{\mathcal{D}} \times \hat{\mathcal{D}}$ if, given any infinite sequence $\{x_n\}_{n=1}^{\infty} \subset \mathcal{D}(t) \in \hat{\mathcal{D}}$, there exists a subsequence $\{x_{n_k}\}_{k=1}^{\infty} \subset \{x_n\}_{n=1}^{\infty}$ such that*

$$\lim_{k\to\infty}\lim_{l\to\infty}\psi(x_{n_k}, x_{n_l}) = 0.$$

For simplicity, we define the collection of all restrictive operations on $\hat{\mathcal{D}}$ by $Contr(\hat{\mathcal{D}})$.

Definition 2.8. *Consider a family of operators $\{S(t,\tau)\}_{t\geq\tau}$ that act on a set $\{X_t\}_{t\in\mathbb{R}}$, which is equipped with a retracting collection $\hat{\mathcal{D}} = \{\mathcal{D}(t) : t \in \mathbb{R}\}$. This system is characterized as being $\hat{\mathcal{D}}$-gradually convergent if, for any $\varepsilon > 0$, there exist a time $T = T(t, \hat{\mathcal{D}}, \varepsilon)$ and a mapping $\psi_T^t = \psi_T^t(\cdot,\cdot) \in Contr(\hat{\mathcal{D}})$ such that the inequality*

$$\|S(t,T)z_1 - S(t,T)z_2\|_{X_t} \leq \varepsilon + \psi_T^t(z_1, z_2), \quad \forall\, z_i \in \mathcal{D}(T)\ (i = 1, 2).$$

The function ψ_T^t is dependent on the choice of T.

In the upcoming theorem, we introduce an innovative approach (or technique) for establishing the existence of pullback $\mathscr{D}$-attractors for the processes $\{S(t,\tau)\}_{t\geq\tau}$ derived from evolutionary equations. This method will be instrumental in our forthcoming analysis.

Theorem 2.9. *Suppose $\{X_t\}_{t\in\mathbb{R}}$ represents a family of Banach spaces, and $\{S(t,\tau)\}_{t\geq\tau}$ with $S(t,\tau) : X_\tau \to X_t$ is a continuous manner. Then $\{S(t,\tau)\}_{t\geq\tau}$ possesses a pullback $\mathscr{D}$-attractor under the fulfillment of the following criteria:*

1) The family $\{S(t,\tau)\}_{t\geq\tau}$ contains a pullback $\mathscr{D}$-ingesting family $\hat{\mathcal{D}}_0 = \{\mathcal{D}_0(t) : t \in \mathbb{R}\}$;

2) The family $\{S(t,\tau)\}_{t\geq\tau}$ qualifies as a $\hat{\mathcal{D}}_0$-contracting process.

Proof. We only need to prove that the process $\{S(t,\tau)\}_{t\geq\tau}$ is pullback $\mathscr{D}$-asymptotically compact. According to Definition 2.4, we have to verify that, for any $\{x_n\}_{n=1}^{\infty}$ with $x_n \in \mathcal{D}_0(\tau_n)$ and $\tau_n \to -\infty$ as $n \to \infty$, the sequence $\{S(t,\tau_n)x_n\}_{n=1}^{\infty} \subset X_t$ has a convergent subsequence. For this purpose, we will employ diagonalization methods to demonstrate that $\{S(t,\tau_n)x_n\}_{n=1}^{\infty}$ contains a Cauchy subsequence in X_t.

Select the sequence $\{\varepsilon_m\}_{m=1}^{\infty} \subset \mathbb{R}^+$ with $\varepsilon_m \to 0$ as $m \to \infty$.

For $m = 1$ and ε_1, based on the given assumptions, there exist $T_0 = T_0(t,\hat{\mathcal{D}}_0;\varepsilon_1)$ and $\psi_{T_0}^t \in Contr(\hat{\mathcal{D}}_0)$ such that, for any $t \in \mathbb{R}$, we have

$$\|S(t,T_0)y_1 - S(t,T_0)y_2\|_{X_t} \leq \varepsilon_1 + \psi_{T_0}^t(y_1,y_2) \quad \forall y_i \in \mathcal{D}_0(T_0)\ (i=1,2). \tag{2.1}$$

Here, $\psi_{T_0}^t$ depends on T_0.

For a fixed T_0, owing to that τ_n tends to $-\infty$, we can, without loss of specificity, presume that $\tau_n \leq T_0$ ensures $S(T_0,\tau_n)x_n \in D_0(T_0)$ for every $n \in \mathbb{N}$. Define $\omega_n = S(T_0,\tau_n)x_n$. By Eq (2.1), we get

$$\begin{aligned}
&\|S(t,\tau_n)x_n - S(t,\tau_m)x_m\|_{X_t}\\
=\ &\|S(t,T_0)S(T_0,\tau_n)x_n - S(t,T_0)S(T_0,\tau_m)x_m\|_{X_t}\\
=\ &\|S(t,T_0)\omega_n - S(t,T_0)\omega_m\|_{X_t}\\
\leq\ &\varepsilon_1 + \psi_{T_0}^t(\omega_n,\omega_m).
\end{aligned} \tag{2.2}$$

According to the definition of $Contr(\hat{\mathcal{D}}_0)$ and given that $\psi_{T_0}^t \in Contr(\hat{\mathcal{D}}_0)$, it implies that there is a subsequence $\{x_{n_k}^{(1)}\}_{k=1}^{\infty}$ of $\{x_n\}_{n=1}^{\infty}$ that satisfies

$$\lim_{k\to\infty}\lim_{l\to\infty}\psi_{T_0}^t(\omega_{n_k}^{(1)},\omega_{n_l}^{(1)}) = 0. \tag{2.3}$$

Here, $\omega_j = S(T_0,\tau_j)x_j$ $(j = n,m)$, and thus we obtain

$$\begin{aligned}
&\lim_{k\to\infty}\sup_{p\in\mathbb{N}}\|S(t,\tau_{n_{k+p}}^{(1)})x_{n_{k+p}}^{(1)} - S(t,\tau_{n_k}^{(1)})x_{n_k}^{(1)}\|_{X_t}\\
\leq\ &\lim_{k\to\infty}\sup_{p\in\mathbb{N}}\limsup_{l\to\infty}\|S(t,\tau_{n_{k+p}}^{(1)})x_{n_{k+p}}^{(1)} - S(t,\tau_{n_l}^{(1)})x_{n_l}^{(1)}\|_{X_t}\\
&+\limsup_{k\to\infty}\limsup_{l\to\infty}\|S(t,\tau_{n_k}^{(1)})x_{n_k}^{(1)} - S(t,\tau_{n_l}^{(1)})x_{n_l}^{(1)}\|_{X_t}\\
\leq\ &2\varepsilon_1 + \lim_{k\to\infty}\sup_{p\in\mathbb{N}}\limsup_{l\to\infty}\psi_{T_0}^t(\omega_{n_{k+p}}^{(1)},\omega_{n_l}^{(1)})\\
&+\lim_{k\to\infty}\lim_{l\to\infty}\psi_{T_0}^t(\omega_{n_k}^{(1)},\omega_{n_l}^{(1)})\\
\leq\ &4\varepsilon_1 + \lim_{k\to\infty}\lim_{l\to\infty}\psi_{T_0}^t(\omega_{n_k}^{(1)},\omega_{n_l}^{(1)}).
\end{aligned} \tag{2.4}$$

Combining with Eq (2.3), we deduce that

$$\lim_{k\to\infty}\sup_{p\in\mathbb{N}}\|S(t,\tau_{n_{k+p}}^{(1)})x_{n_{k+p}}^{(1)} - S(t,\tau_{n_k}^{(1)})x_{n_k}^{(1)}\|_{X_t} \leq 4\varepsilon_1. \tag{2.5}$$

Hence, one can find an $N_1 \in \mathbb{N}$, ensuring that

$$\|S(t,\tau_{n_k}^{(1)})x_{n_k}^{(1)} - S(t,\tau_{n_l}^{(1)})x_{n_l}^{(1)}\|_{X_t} \leq 5\varepsilon_1 \quad \forall k,l \geq N_1. \tag{2.6}$$

Subsequently, we can identify a subsequence $\{S(t, \tau_{n_k}^{(m+1)})x_{n_k}^{(m+1)}\}_{k=1}^{\infty}$ of $\{S(t, \tau_{n_k}^{(m)})x_{n_k}^{(m)}\}_{k=1}^{\infty}$ for each $m > 1$ and a certain N_{m+1} so that

$$\|S(t, \tau_{n_k}^{(m+1)})x_{n_k}^{(m+1)} - S(t, \tau_{n_l}^{(m+1)})x_{n_l}^{(m+1)}\|_{X_t} \le 5\varepsilon_{m+1} \tag{2.7}$$

holds for all $k, l \ge N_{m+1}$.

Moving forward, we focus on the diagonal sequence of the subsequence $\{S(t, \tau_{n_k}^{(k)})x_{n_k}^{(k)}\}_{k=1}^{\infty}$. Thanks to the fact that for each $m \in \mathbb{N}$, $\{S(t, \tau_{n_k}^{(k)})x_{n_k}^{(k)}\}_{k=1}^{\infty}$ is nested within $\{S(t, \tau_{n_k}^{(m)})x_{n_k}^{(m)}\}_{k=1}^{\infty}$, thereby, for any $k, l \ge max\{m, N_m\}$, we deduce that

$$\|S(t, \tau_{n_k}^{(k)})x_{n_k}^{(k)} - S(t, \tau_{n_l}^{(l)})x_{n_l}^{(l)}\|_{X_t} \le 5\varepsilon_{m+1}. \tag{2.8}$$

Therefore, $\{S(t, \tau_{n_k}^{(k)})x_{n_k}^{(k)}\}_{k=1}^{\infty}$ is Cauchy sequence in X_t with $\varepsilon_m \to 0$ as $m \to \infty$, this entails that $\{S(t, \tau_n)x_n\}_{n=1}^{\infty}$ possesses a convergent subsequence. This proof is complete.

Lemma 2.10. *[23] Consider the Banach spaces X, H, and Y such that X is compactly embedded in H and H is continuously embedded in Y, with X being reflexive. Suppose the sequence* $\{u_n\}_{n=0}^{\infty}$ *adheres to a uniform limit in* $L^2(\tau, T; X)$*, and its time derivative* du_n/dt *has a uniform bound in* $L^p(\tau, T; Y)$*, for some* $p > 1$. *Under these conditions, there exists a subsequence of* $\{u_n\}_{n=0}^{\infty}$ *that converges strongly in* $L^2(\tau, T; H)$.

3. Pullback $\mathscr{D}$-attractors

In this segment, we are dedicated to investigating the existence of a time-varying pullback $\mathscr{D}$-attractor in $\{C_{\mathcal{H}_t^1}\}_{t\in\mathbb{R}}$. To this end, we must initially address the well-posedness of the Eq (1.1) in conjunction with (1.2) and (1.3).

3.1. The well-posed nature of equation

The well-posed nature of the Eq (1.1), along with (1.2) and (1.3), can be deduced through the Faedo-Galerkin approach (refer to [31, 36, 37] for instance). To begin with, we define what constitutes a weak solution.

Definition 3.1. *Given* $T > \tau$, *the function* $u \in C([\tau - h, T]; \mathcal{H}_t^1) \cap L^2([\tau, T]; H_0^1(\Omega))$ *is termed a weak solution to the systems* (1.1)–(1.3) *with the initial condition* $u(\tau) = \phi(\theta) \in C_{\mathcal{H}_\tau^1}$, *provided that the subsequent equation*

$$(\partial_t u, \omega) + \varepsilon(t)(\nabla\partial_t u, \nabla\omega) + (\nabla u, \nabla\omega) + \langle f(u), \omega\rangle = (g(t, u_t), \omega) \tag{3.1}$$

holds for all $\omega \in H_0^1(\Omega)$ *and a.e.,* $t \in [\tau, T]$.

Lemma 3.2. *Let* $\varepsilon(t)$ *satisfy* (H_1), *and f and g satisfy* (H_2) *and* (H_3), *respectively. Given any* $T > \tau$ *and initial condition* $u_\tau = \phi(\theta) \in C_{\mathcal{H}_\tau^1}$, *the systems* (1.1)–(1.3) *yields a sole weak solution* $u = u(t, \tau, \phi)$, *which fulfills the conditions for any* $T > \tau$ *as follows:*

$$u(t) \in C([\tau - h, T]; \mathcal{H}_t^1) \cap L^2([\tau, T]; H_0^1(\Omega)),$$
$$\partial_t u \in L^2([\tau, T]; H_0^1(\Omega)),$$

which maintains a continuous dependence on the initial state in $C_{\mathcal{H}_\tau^1}$, i.e., there is a constant $\kappa > 0$, independent of t, ensuring that the family of operators $\{S(t,\tau)\}_{t\geq\tau}$ exhibits Lipschitz continuity:

$$\|S(t,\tau)\phi_1(\theta) - S(t,\tau)\phi_2(\theta)\|_{C_{\mathcal{H}_t^1}} \leq Ce^{\kappa(T-\tau)}\|\phi_1 - \phi_2\|_{C_{\mathcal{H}_\tau^1}}, \quad \forall t \in [\tau, T]. \tag{3.2}$$

According to Lemma 3.2, we may establish the process of solutions on the time-dependent space $\{C_{\mathcal{H}_t^1}\}_{t\in\mathbb{R}}$:

$$S(t,\tau) : C_{\mathcal{H}_\tau^1} \to C_{\mathcal{H}_t^1},\ S(t,\tau)\phi = u(t,\tau,\phi), \quad \forall t \geq \tau,\ \theta \in [-h, 0]. \tag{3.3}$$

Furthermore, it is straightforward to deduce that the sequence of solutions $\{S(t,\tau)\}_{t\geq\tau}$ forms a process that is continuously evolving on the phase space that changes over time, $\{C_{\mathcal{H}_t^1}\}_{t\in\mathbb{R}}$.

3.2. *Pullback $\mathscr{D}$-ingesting family*

In the subsequent analysis, we always presume that: the assumptions (H_1)–(H_3) are true. Furthermore, we consider $u = u(t,\tau,\phi) = S(t,\tau)\phi(\theta)$ to be a solution of (1.1)–(1.3) that possesses ample regularity.

Lemma 3.3. *Let $\varepsilon(t)$ satisfy (H_1), and f and g satisfy (H_2) and (H_3), respectively. Consequently, there are positive constants σ and β_1, ensuring that*

$$\|u_t\|^2_{C_{\mathcal{H}_t^1}} \leq \beta_1\left(\|\phi\|^2_{C_{\mathcal{H}_\tau^1}} e^{-\sigma(t-\tau)} + 1 + \mathscr{G}_\sigma(t)\right)$$

holds for each $\tau \leq t$.

Proof. By multiplying Eq (1.1) with u in $L^2(\Omega)$, then it follows that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{\mathcal{H}_t^1} - \frac{1}{2}\varepsilon'(t)|\nabla u|_2^2 + |\nabla u|_2^2 + \int_\Omega f(u)u \leq L_g\|u_t\|_{C_{\mathcal{H}_t^1}}|u|_2 + |g_0|_2|u|_2. \tag{3.4}$$

Thanks to (1.6) and Young's inequality, (3.4) can be rewritten as

$$\frac{d}{dt}\|u\|^2_{\mathcal{H}_t^1} + 2\left(1 - \frac{\alpha}{\lambda_1}\right)|\nabla u|_2^2 \leq \frac{L_g^2}{\delta_1}\|u_t\|^2_{C_{\mathcal{H}_t^1}} + \frac{1}{\delta_2}|g_0|_2^2 + (\delta_1 + \delta_2)|u|_2^2 + 2\beta|\Omega|, \tag{3.5}$$

where α is from (1.6), $1 - \frac{\alpha}{\lambda_1} > 0$, δ_i $(i = 1, 2)$ are undetermined constants, and $\lambda = \delta_1 + \delta_2$.

Additionally, we derive the following inequality:

$$\frac{d}{dt}\|u\|^2_{\mathcal{H}_t^1} + \lambda\|u\|^2_{\mathcal{H}_t^1} \leq \frac{L_g^2}{\delta_1}\|u_t\|^2_{C_{\mathcal{H}_t^1}} + \frac{1}{\delta_2}|g_0|_2^2 + 2\beta|\Omega|, \tag{3.6}$$

where $\lambda = \left(1 - \frac{\alpha}{\lambda_1}\right)\min\left\{\lambda_1, \frac{1}{L}\right\}$. Applying *Gronwall's* Lemma, we find that for any $\theta \in [-h, 0]$,

$$e^{\lambda t}\|u\|^2_{\mathcal{H}_t^1} \leq e^{\lambda\tau}\|\phi\|^2_{C_{\mathcal{H}_\tau^1}} + \frac{L_g^2}{\delta_1}\int_\tau^t e^{\lambda s}\|u_s\|^2_{C_{\mathcal{H}_s^1}}ds + \frac{1}{\delta_2}e^{\lambda t}\mathscr{G}_\lambda(t) + \frac{2\beta|\Omega|}{\lambda}e^{\lambda t}.$$

Furthermore, $u_t(\theta) = u(t+\theta) \in C([-h, 0]; \mathcal{H}_t^1)$ from Lemma 3.2, and by substituting $t + \theta$ for t where $\theta \in [-h, 0]$, we deduce that

$$e^{\lambda t}\|u_t\|^2_{C_{\mathcal{H}_t^1}} - \frac{L_g^2 e^{\lambda h}}{\delta_1}\int_\tau^t e^{\lambda s}\|u_s\|^2_{C_{\mathcal{H}_s^1}}ds \leq e^{\lambda(\tau+h)}\|\phi\|^2_{C_{\mathcal{H}_\tau^1}} + \frac{1}{\delta_2}e^{\lambda t}\mathscr{G}_\lambda(t) + \frac{2\beta|\Omega|}{\lambda}e^{\lambda t}.$$

By *Gronwall′s Lemma*, it yields

$$e^{-\lambda t}\int_{\tau}^{t} e^{\lambda s}\|u_s\|^2_{C_{\mathcal{H}^1_s}}ds \le \frac{\delta_1}{L_g^2}e^{-\sigma(t-\tau)}\|\phi\|^2_{C_{\mathcal{H}^1_\tau}} + \frac{1}{\delta_2}\mathcal{G}_\lambda(t) + \frac{2\beta|\Omega|}{\lambda}, \tag{3.7}$$

and

$$\|u_t\|^2_{C_{\mathcal{H}^1_t}} \le 2e^{\lambda h}\|\phi\|^2_{C_{\mathcal{H}^1_\tau}}e^{-\sigma(t-\tau)} + \left(1+\frac{L_g^2e^{\lambda h}}{\delta_1}\right)\frac{1}{\delta_2}\mathcal{G}_\lambda(t) + \left(1+\frac{L_g^2e^{\lambda h}}{\delta_1}\right)\frac{2\beta|\Omega|}{\lambda}, \tag{3.8}$$

where $\sigma = \lambda - \frac{L_g^2e^{\lambda h}}{\delta_1}$.

Define $\delta_1 = L_g e^{\lambda h/2}$, $\delta_2 = \lambda - L_g e^{\lambda h/2}$, and assume that

$$\delta_2 = \lambda - L_g e^{\lambda h/2} > 0.$$

Then, it follows that

$$\lambda > \sigma = \delta_2 = \lambda - L_g e^{\lambda h/2} > 0.$$

Furthermore, let

$$\beta_1 = \left(1+\frac{L_g^2e^{\lambda h}}{\delta_1}\right)\max\left\{2e^{\lambda h}, \frac{1}{\delta_2}, \frac{2\beta|\Omega|}{\lambda}\right\}.$$

With this, the proof is complete.

Corollary 3.4. *The family of processes* $\{S(t,\tau)\}_{t\in\mathbb{R}}$ *associated with problems* (1.1)–(1.3) *possesses a pullback* $\mathscr{D}-$ *absorbing set*

$$\hat{D}_0 = \left\{D_0(t) = \left\{u \in C_{\mathcal{H}^1_t} : \|u\|^2_{C_{\mathcal{H}^1_t}} \le 2\beta_1\left(1+\mathcal{G}_\sigma(t)\right)\right\} : t \in \mathbb{R}\right\},$$

that is, for each $t \in \mathbb{R}$ *and* $\hat{\mathcal{D}} \in \mathscr{D} \subset \mathscr{P}(C_{\mathcal{H}^1_\tau})$*, there exists a* $\tau_0 = \tau_0(t,\hat{\mathcal{D}}) \le t$ *such that*

$$S(t,\tau)\mathcal{D}(\tau) \subset \mathcal{D}_0(t)$$

for all $\tau \le \tau_0(t,\hat{\mathcal{D}})$.

In fact, let

$$\|\hat{\mathcal{D}}\|^2_{C_{\mathcal{H}^1_\tau}} = \max_{\phi\in\mathcal{D}(\tau)\subset\hat{\mathcal{D}}}\|\phi\|^2_{C_{\mathcal{H}^1_\tau}}; \tag{3.9}$$

$$\tau_0 = \tau_0(t,\hat{\mathcal{D}}) = t - \frac{1}{\sigma}\ln\frac{\|\hat{\mathcal{D}}\|^2_{C_{\mathcal{H}^1_\tau}}}{1+\mathcal{G}_\sigma(t)} < t.$$

Then, the conclusion can be directly obtained from Lemma 3.3.

From Lemma 3.3, we have the following corollary.

Corollary 3.5. *For each* $t \in \mathbb{R}$ *and* $\hat{\mathcal{D}} \in \mathscr{D} \subset \mathscr{P}(C_{\mathcal{H}^1_\tau})$*, then there exists* $\mathcal{K}_0 = \mathcal{K}_0(t,\hat{\mathcal{D}})$ *such that*

$$\|u_t\|^2_{C_{\mathcal{H}^1_t}} = \max_{\theta\in[-h,0]}\left(|u(t+\theta)|_2^2 + \varepsilon(t+\theta)|\nabla u(t+\theta)|_2^2\right) \le \mathcal{K}_0$$

holds for all $t - \tau \ge 0$.

In fact, let

$$\mathcal{K}_0 = \beta_1 \left(\|\phi\|^2_{C_{\mathcal{H}^1_\tau}} + 1 + \mathcal{G}_\sigma(t) \right).$$

Then, the conclusion can be directly inferred from Lemma 3.3.

Lemma 3.6. *Let $\varepsilon(t)$ satisfy (H_1), and f and g satisfy (H_2) and (H_3), respectively. Then, there exists a positive constant β_2, such that the following estimate*

$$\int_{t-1}^{t} \left(|u(s)|_2^2 + |\nabla u(s)|_2^2 + \int_\Omega (f(u(s))u(s) + \alpha|u|^2 + \beta) \right) ds \le \beta_2 \left(\|\phi\|^2_{C_{\mathcal{H}^1_\tau}} e^{-\sigma(t-\tau)} + 1 + \mathcal{G}_\sigma(t) \right)$$

holds for any $\tau \le t-1$.

Proof. From (3.4), it is easy to obtain

$$\begin{aligned} &\frac{d}{dt}\left(|u|_2^2 + \varepsilon(t)|\nabla u|_2^2 \right) + (1 - \frac{\alpha}{\lambda_1})|\nabla u|_2^2 + 2\int_\Omega ((f(u) + \alpha u)u + \beta) \\ &\le \frac{L_g^2}{\delta\lambda_1}\|u_t\|^2_{C_{\mathcal{H}^1_t}} + \frac{1}{\lambda_1\delta}|g_0|_2^2 + 2\beta|\Omega|, \end{aligned} \tag{3.10}$$

where $\delta = \frac{1}{2}(1 - \frac{\alpha}{\lambda_1})$. Let $b_1 = \min\{2, \frac{\lambda_1-\alpha}{2}, \frac{\lambda_1-\alpha}{2\lambda_1}\}$, $f_1(u) = f(u) + \alpha u$, and by integrating inequality (3.10) on $[t-1, t]$, we obtain

$$\begin{aligned} &\int_{t-1}^{t} \left(|u(s)|_2^2 + |\nabla u(s)|_2^2 + \int_\Omega (f_1(u(s))u(s) + \beta) \right) ds \\ &\le \frac{1}{b_1}\left(\frac{L_g^2}{\delta\lambda_1}\int_{t-1}^{t} \|u_s\|^2_{C_{\mathcal{H}^1_s}} ds + \frac{1}{\lambda_1\delta}\int_{t-1}^{t} |g_0(s)|_2^2 ds + 2\beta|\Omega| + \|u\|^2_{\mathcal{H}^1_t} \right) \\ &\le \frac{1}{b_1}\left(\frac{L_g^2 e^\lambda}{\delta\lambda_1}\int_{\tau}^{t} e^{-\lambda(t-s)}\|u_s\|^2_{C_{\mathcal{H}^1_s}} ds + \frac{e^\lambda}{\lambda_1\delta}\int_{-\infty}^{t} e^{-\sigma(t-s)}|g_0(s)|_2^2 ds + 2\beta|\Omega| + \|u_t\|^2_{C_{\mathcal{H}^1_t}} \right). \end{aligned}$$

Combining with (3.7) and (3.8), it then follows that

$$\begin{aligned} &\int_{t-1}^{t} \left(|u(s)|_2^2 + |\nabla u(s)|_2^2 + \int_\Omega (f_1(u(s))u(s) + \beta) \right) ds \\ &\le \frac{1}{b_1}(e^{\lambda-\sigma} + 2e^{\lambda h})\|\phi\|^2_{C_{\mathcal{H}^1_\tau}} e^{-\sigma(t-\tau)} + \frac{\beta|\Omega|}{\lambda b_1}\left(2 + (1 + \frac{2L_g^2 e^{\lambda h}}{\delta\lambda_1}) \right) + \frac{1}{\delta\lambda_1 b_1}\left(\frac{2L_g^2 e^\lambda}{\delta} + e^\lambda \right)\int_{-\infty}^{t} e^{-\sigma(t-s)}|g_0(s)|_2^2 ds. \end{aligned}$$

Let

$$\beta_2 = \max\left\{ \frac{1}{b_1}(e^{\lambda-\sigma} + 2e^{\lambda h}),\ \frac{\beta|\Omega|}{\lambda b_1}\left(2 + (1 + \frac{2L_g^2 e^{\lambda h}}{\delta\lambda_1}) \right),\ \frac{1}{\delta\lambda_1 b_1}\left(\frac{2L_g^2 e^\lambda}{\delta} + e^\lambda \right) \right\},$$

and $\tau \le t$. The proof is complete.

Lemma 3.7. *Let $\varepsilon(t)$ satisfy (H_1), and f and g satisfy (H_2) and (H_3), respectively. Then, there exists a positive constant β_3, such that*

$$|u(t)|_2^2 + |\nabla u(t)|_2^2 + \int_\Omega F(u(t)) \le \beta_3 \left(\|\phi\|^2_{C_{\mathcal{H}^1_\tau}} e^{-\sigma(t-\tau)} + 1 + \mathcal{G}_\sigma(t) \right),$$

is valid for any $\tau \le t-1$.

Proof. Multiplying Eq (1.1) by $\partial_t u$ in $L^2(\Omega)$, then we get

$$\frac{d}{dt}\left(\frac{1}{2}|\nabla u|_2^2+\int_\Omega F(u)\right)+\frac{1}{2}\left(|\partial_t u|_2^2+\varepsilon(t)|\nabla\partial_t u|_2^2\right)\le L_g^2\|u_t\|_{C_{\mathcal{H}_t^1}}^2+|g_0(t)|_2^2. \tag{3.11}$$

Thus, the inequality (3.11) can be rewritten as follows

$$\frac{d}{dt}\left(\frac{1}{2}|\nabla u|_2^2+\int_\Omega F(u)\right)\le L_g^2\|u_t\|_{C_{\mathcal{H}_t^1}}^2+|g_0(t)|_2^2.$$

Then, for any $t>s>t-1\ge\tau$, it follows that

$$\begin{aligned}\frac{1}{2}|\nabla u(t)|_2^2+\int_\Omega F(u(t))\le{}&\frac{1}{2}|\nabla u(s)|_2^2+\int_\Omega F(u(s))\\&+L_g^2\int_s^t\|u_s\|_{C_{\mathcal{H}_s^1}}^2ds+\int_s^t|g_0(s)|_2^2ds.\end{aligned}$$

From Lemma 3.3, Lemma 3.6, and (3.7), we can get

$$\begin{aligned}&\frac{1}{2}|\nabla u(t)|_2^2+\int_\Omega F(u(t))\le\int_{t-1}^t\left(\frac{1}{2}|\nabla u(s)|_2^2+\int_\Omega F(u(s))\right)ds\\&+L_g^2\int_{t-1}^t\|u_s\|_{C_{\mathcal{H}_s^1}}^2ds+\int_{t-1}^t|g_0(s)|_2^2ds\\&\le\frac{1}{2}\int_{t-1}^t|\nabla u(s)|_2^2ds+\int_{t-1}^t\int_\Omega\left(f(u(s))u(s)+\frac{1}{2}\alpha|u(s)|_2^2+\beta_1\right)ds\\&+L_g^2\int_{t-1}^t\|u_s\|_{C_{\mathcal{H}_s^1}}^2ds+\int_{t-1}^t|g_0(s)|_2^2ds\\&\le\int_{t-1}^t\left(|u(s)|_2^2+|\nabla u(s)|_2^2+\int_\Omega\left(f(u(s))u(s)+\alpha|u(s)|_2^2+\beta\right)\right)ds\\&+L_g^2e^\lambda\int_\tau^t e^{-\lambda(t-s)}\|u_s\|_{C_{\mathcal{H}_s^1}}^2ds+e^\sigma\int_{-\infty}^t e^{-\sigma(t-s)}|g_0(s)|_2^2ds+|\beta-\beta_1||\Omega|\\&\le\beta_2\left(\|\phi\|_{C_{\mathcal{H}_\tau^1}}^2e^{-\sigma(t-\tau)}+1+\int_{-\infty}^t e^{-\sigma(t-s)}|g_0(s)|_2^2ds\right)\\&+\delta_1\lambda_1e^\lambda\|\phi\|_{C_{\mathcal{H}_\tau^1}}^2e^{-\sigma(t-\tau)}+\frac{L_g^2}{\delta_2\lambda_1}\int_{-\infty}^t e^{-\sigma(t-s)}|g_0(s)|_2^2ds+\frac{L_g^2\beta|\Omega|}{\lambda}\\&=b_2\left(\|\phi\|_{C_{\mathcal{H}_\tau^1}}^2e^{-\sigma(t-\tau)}+1+\int_{-\infty}^t e^{-\sigma(t-s)}|g_0(s)|_2^2ds\right),\end{aligned}\tag{3.12}$$

where $b_2=\beta_2+\max\{\delta_1\lambda_1e^\lambda,\frac{L_g^2}{\delta_2\lambda_1},\frac{L_g^2\beta|\Omega|}{\lambda}\}$.

Furthermore, by associating with (1.10), it is straightforward to obtain the followings inequality:

$$
\begin{aligned}
&\frac{1}{2}|\nabla u(t)|_2^2+\int_\Omega F(u(t))\\
&=\frac{1}{2}|\nabla u(t)|_2^2+\int_\Omega\left(F(u(t))+\frac{1}{2}\alpha|u(t)|^2+\beta_1\right)-\frac{1}{2}\alpha|u(t)|_2^2-\beta_1|\Omega|\\
&\geq\frac{\lambda_1}{4}|u(t)|_2^2+\frac{1}{4}|\nabla u(t)|_2^2+\int_\Omega\left(F(u(t))+\frac{1}{2}\alpha|u(t)|^2+\beta_1\right)-\frac{1}{2}\alpha|u(t)|_2^2-\beta_1|\Omega|\\
&\geq\frac{1}{4}\min\{1,\lambda_1\}\left(|u(t)|_2^2+|\nabla u(t)|_2^2+\int_\Omega F(u(t))\right)-\frac{1}{2}\alpha|u(t)|_2^2-\beta_1|\Omega|. \qquad (3.13)
\end{aligned}
$$

Let

$$b_3=\frac{4}{\min\{1,\lambda_1\}}.$$

Then, we can obtain that

$$
\begin{aligned}
&|u(t)|_2^2+|\nabla u(t)|_2^2+\int_\Omega F(u(t))\\
&\leq b_3\left(\frac{1}{2}|\nabla u(t)|_2^2+\int_\Omega F(u(t))+\frac{1}{2}\alpha\|u_t\|^2_{C_{\mathcal{H}_t^1}}+\beta_1|\Omega|\right)\\
&\leq(b_2b_3+\frac{1}{2}\alpha b_3\beta_1)\left(\|\phi\|^2_{C_{\mathcal{H}_\tau^1}}e^{-\sigma(t-\tau)}+1+\int_{-\infty}^t e^{-\sigma(t-s)}|g_0(s)|_2^2ds\right)+b_3\beta_1|\Omega|\\
&\leq\beta_3\left(\|\phi\|^2_{C_{\mathcal{H}_\tau^1}}e^{-\sigma(t-\tau)}+1+\int_{-\infty}^t e^{-\sigma(t-s)}|g_0(s)|_2^2ds\right) \qquad (3.14)
\end{aligned}
$$

holds for all $\tau\leq t$, and where $\beta_3=b_2b_3+\frac{1}{2}\alpha b_3\beta_1+b_3\beta_1|\Omega|$. With this, the proof is complete.

3.3. Time-varying pullback-$\mathscr{D}$ attractors

In this segment, we aim to establish the presence of temporally varying pullback-$\mathscr{D}$ attractors within the space $\{C_{\mathcal{H}_t^1}\}_{t\in\mathbb{R}}$ via the process $S(t,\tau)$ outlined in by Eq (3.3). To substantiate Theorem 3.12, we introduce several preliminary lemmas.

Lemma 3.8. *For each $t\in\mathbb{R}$ and $\hat{\mathcal{D}}\in\mathscr{D}\subset\mathscr{P}(C_{\mathcal{H}_\tau^1})$, there exists a constant $\mathcal{K}_1=\mathcal{K}_1(t,\hat{\mathcal{D}})$ such that*

$$\int_t^{t+1}(|\partial_t u(t)|_2^2+\varepsilon(t)|\nabla\partial_t u(t)|_2^2)dt\leq\mathcal{K}_1$$

holds for any $t\geq\tau$.

Proof. From the inequality (3.11), we can derive that

$$
\begin{aligned}
&\frac{1}{2}|\nabla u(t+1)|_2^2+\int_\Omega F(u(t+1))+\frac{1}{2}\int_t^{t+1}\left(|\partial_t u(s)|_2^2+\varepsilon(s)|\nabla\partial_t u(s)|_2^2\right)ds\\
&\leq\frac{1}{2}|\nabla u(t)|_2^2+\int_\Omega F(u(t))+L_g^2\int_t^{t+1}\|u_s\|^2_{C_{\mathcal{H}_t^1}}ds+\int_t^{t+1}|g_0(s)|_2^2ds. \qquad (3.15)
\end{aligned}
$$

By combining this with the inequalities (3.7) and (3.12), and after organizing, we obtain

$$
\begin{aligned}
&\int_t^{t+1} \left(|\partial_t u(t)|_2^2 + \varepsilon(t)|\nabla \partial_t u(t)|_2^2\right) ds \\
&\leq b_2 \left(\|\phi\|^2_{C_{\mathcal{H}^1_\tau}} e^{-\sigma(t-\tau)} + 1 + \int_{-\infty}^{t} e^{-\sigma(t-s)} |g_0(s)|_2^2 ds \right) \\
&+ L_g^2 e^{\lambda} \int_\tau^{t+1} e^{-\lambda(t+1-s} \|u_s\|^2_{C_{\mathcal{H}^1_s}} ds + e^{\sigma} \int_{-\infty}^{t+1} e^{-\sigma(t+1-s)} |g_0(s)|_2^2 ds \\
&\leq b_2 e^{\sigma} \left(\|\phi\|^2_{C_{\mathcal{H}^1_\tau}} e^{-\sigma(t+1-\tau)} + 1 + \int_{-\infty}^{t+1} e^{-\sigma(t+1-s)} |g_0(s)|_2^2 ds \right) \\
&+ \delta\lambda_1 e^{\lambda} \|\phi\|^2_{C_{\mathcal{H}^1_\tau}} e^{-\sigma(t+1-\tau)} + \left(\frac{e^{\lambda} L_g^2}{\delta\lambda_1} + e^{\sigma} \right) \int_{-\infty}^{t+1} e^{-\sigma(t+1-s)} |g_0(s)|_2^2 ds + \frac{e^{\lambda} L_g^2 \beta}{\lambda} |\Omega|,
\end{aligned}
$$

which holds for any $\tau < t$. Then, let

$$
\begin{aligned}
b_4 &= b_2 e^{\sigma} + \max \left\{ \delta\lambda_1 e^{\lambda}, \frac{e^{\lambda} L_g^2}{\delta\lambda_1} + e^{\sigma}, \frac{e^{\lambda} L_g^2 \beta}{\lambda} |\Omega| \right\}, \\
\mathcal{K}_1 &= b_4 \left(\|\phi\|^2_{C_{\mathcal{H}^1_\tau}} e^{-\sigma(t+1-\tau)} + 1 + \mathcal{G}_\sigma(t) \right).
\end{aligned}
$$

The proof is complete.

Next, we will verify the asymptotic regularity of a family of solution processes $\{S(t,\tau)\}_{t\geq\tau}$ corresponding to problems (1.1)–(1.3), and thus obtain the compactness of $\{S(t,\tau)\}_{t\geq\tau}$. Based on this purpose, we decompose the solution $S(t,\tau)u_\tau = u(t,\theta) = u$ into the following sum:

$$
S(t,\tau)u_\tau = U_1(t,\tau)u_\tau + K(t,\tau)u_\tau, \tag{3.16}
$$

where $U_1(t,\tau)u_\tau = v(t,\theta) = v$ and $K(t,\tau)u_\tau = \omega(t,\theta) = \omega$ solve the following equations respectively:

$$
\begin{cases}
\partial_t v - \varepsilon(t)\Delta\partial_t v - \Delta v + f(u) - f(\omega) + lv = 0, & (x,t) \in \Omega \times (\tau, \infty), \\
v(x,\ t)|_{\partial\Omega} = 0, & \forall\, t \in (\tau, \infty), \\
v(x,\tau) = u_\tau = u(\tau+\theta) = \phi(\theta), & x \in \Omega, \tau \in \mathbb{R},\ \ \theta \in [-h, 0],
\end{cases} \tag{3.17}
$$

and

$$
\begin{cases}
\partial_t \omega - \varepsilon(t)\Delta\partial_t \omega - \Delta\omega + f(\omega) - lv = g(t, u_t), & (x,t) \in \Omega \times (\tau, \infty), \\
\omega(x,\ t)|_{\partial\Omega} = 0, & \forall\, t \in (\tau, \infty), \\
\omega(x,\tau) = 0, & x \in \Omega, \tau \in \mathbb{R},\ \theta \in [-h, 0].
\end{cases} \tag{3.18}
$$

Lemma 3.9. *Let $\varepsilon(t)$ satisfy (H_1), and f and g satisfy (H_2) and (H_3), respectively. Furthermore, assume that $U_1(t,\tau)u_\tau = v(t,\theta) = v$ is the solution of the initial-boundary value system (3.17). Then,*

$$
\lim_{\tau\to-\infty} \|U_1(t,\tau)u_\tau\|_{C_{\mathcal{H}^1_t}} = 0
$$

holds for every $t \geq \tau \in \mathbb{R}$ fixed.

Proof. Multiplying the first equation of (3.17) by $v(t)$ and integrating over $L^2(\Omega)$, we obtain

$$\frac{d}{dt}\left(|v|_2^2 + \varepsilon(t)|\nabla v|_2^2\right) + 2|\nabla v|_2^2 \leq 0. \tag{3.19}$$

Furthermore, since $0 < \sigma < \lambda < \alpha_1 = \min\{\lambda_1, 1/L\}$, then

$$\frac{d}{dt}\left(|v|_2^2 + \varepsilon(t)|\nabla v|_2^2\right) + \alpha_1\left(|v|_2^2 + \varepsilon(t)|\nabla v|_2^2\right) \leq 0.$$

By *Gronwall's* Lemma, we deduce that

$$|v(t)|_2^2 + \varepsilon(t)|\nabla v(t)|_2^2 \leq \|u_\tau\|^2_{C_{\mathcal{H}^1_\tau}} e^{-\alpha_1(t-\tau)} \leq \|u_\tau\|^2_{C_{\mathcal{H}^1_\tau}} e^{-\sigma(t-\tau)}.$$

Setting $t+\theta$ instead of t with $\theta \in [-h, 0]$, we infer that

$$|v(t+\theta)|_2^2 + \varepsilon(t+\theta)|\nabla v(t+\theta)|_2^2 \leq e^{\sigma h}\|u_\tau\|^2_{C_{\mathcal{H}^1_\tau}} e^{-\sigma(t-\tau)},$$

for any $t \geq \tau$ and $\theta \in [-h, 0]$. Then, it follows that

$$\|v_t\|^2_{C_{\mathcal{H}^1_t}} \leq e^{\sigma h}\|\phi\|^2_{C_{\mathcal{H}^1_\tau}} e^{-\sigma(t-\tau)}. \tag{3.20}$$

Combining with the initial value of system (3.17), we then conclude that:

$$\lim_{\tau\to-\infty}\|U_1(t,\tau)u_\tau\|^2_{C_{\mathcal{H}^1_t}} = 0.$$

Lemma 3.10. *Assume that $K(t,\tau)u_\tau = \omega(t)$ is the solution of the Eq* (3.18). *Then, there exist positive constants k_i $(i = 3, 4)$, such that*

$$|\omega(t)|_2^2 + |\nabla\omega(t)|_0^2 + \mathcal{F}(\omega(t)) \leq k_3\left(\|\phi\|^2_{C_{\mathcal{H}^1_\tau}} e^{-\sigma(t-\tau)} + 1 + \mathcal{G}_\sigma(t)\right),$$

and

$$\int_{t-1}^{t}\left(|\partial_t\omega(t)|_2^2 + \varepsilon(t)|\nabla\partial_t\omega(t)|_2^2\right)dt \leq k_4\left(\|\phi\|^2_{C_{\mathcal{H}^1_\tau}} e^{-\sigma(t-\tau)} + 1 + \mathcal{G}_\sigma(t)\right),$$

hold for any $\tau \leq t-1$.

Proof. The proof of this lemma can be obtained by imitating the proof of Lemmas 3.3, 3.7, and 3.8. It should be noted that Eqs (3.7) and (3.20), $\alpha_1 > \lambda > \sigma$, and $\|\omega_\tau\|_{C_{\mathcal{H}^1_\tau}} = 0$ are crucial. The details are omitted for brevity.

Lemma 3.11. *There exists a positive constant k_4, such that*

$$|\nabla\omega(t)|_2^2 + \varepsilon(t)|\triangle\omega(t)|_2^2 \leq k_4\left(\|\phi\|^2_{C_{\mathcal{H}^1_\tau}} e^{-\sigma(t-\tau)} + \mathcal{G}_\sigma(t) + 1\right) \tag{3.21}$$

holds for any $\tau \leq t \in \mathbb{R}$.

Proof. By operating on the first equation of (3.18) with $-\Delta\omega(t)$ in $L^2(\Omega)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(|\nabla\omega|_2^2 + \varepsilon(t)|\Delta\omega|_2^2\right) + \frac{1}{2}|\Delta\omega|_2^2 \le l^2|u|_2^2 + L_g^2\|u_t\|_{C_{\mathcal{H}_t^1}}^2. \tag{3.22}$$

Then,

$$\begin{aligned}
&\frac{d}{dt}\left(|\nabla\omega|_2^2 + \varepsilon(t)|\Delta\omega|_2^2\right) + \alpha_1\left(|\nabla\omega|_2^2 + \varepsilon(t)|\Delta\omega|_2^2\right) \\
&\le 2l^2|u|_2^2 + 2L_g^2\|u_t\|_{C_{\mathcal{H}_t^1}}^2 \\
&\le 2(l^2 + L_g^2)\|u_t\|_{C_{\mathcal{H}_t^1}}^2,
\end{aligned} \tag{3.23}$$

where α_1 is from Lemma 3.9.

Therefore, by *Gronwall's* lemma, for any $\tau \le t \in \mathbb{R}$, we have

$$|\nabla\omega(t)|_2^2 + \varepsilon(t)|\Delta\omega(t)|_2^2 \le 2(l^2 + L_g^2)\int_\tau^t e^{-\alpha_1(t-s)}\|u_s\|_{C_{\mathcal{H}_s^1}}^2 ds.$$

Considering $\alpha_1 > \lambda > \sigma$ and Eq (3.7), then

$$\begin{aligned}
&|\nabla\omega(t)|_2^2 + \varepsilon(t)|\Delta\omega(t)|_2^2 \le 2(l^2 + L_g^2)\int_\tau^t e^{-\alpha_1(t-s)}\|u_s\|_{C_{\mathcal{H}_s^1}}^2 ds \\
&\le 2(l^2 + L_g^2)\int_\tau^t e^{-\lambda(t-s)}\|u_s\|_{C_{\mathcal{H}_s^1}}^2 ds \\
&\le 2(l^2 + L_g^2)\left(\frac{\delta\lambda_1}{L_g^2}e^{-\sigma(t-\tau)}\|\phi\|_{C_{\mathcal{H}_\tau^1}}^2 + \frac{1}{\delta\lambda_1}\int_{-\infty}^t e^{-\sigma(t-s)}|g_0(s)|_2^2 ds + \frac{\beta|\Omega|}{\lambda}\right).
\end{aligned}$$

Let

$$k_4 = 2(l^2 + L_g^2)\max\left\{\frac{\delta\lambda_1}{L_g^2}, \frac{1}{\delta\lambda_1}, \frac{\beta|\Omega|}{\lambda}\right\}.$$

Then, it follows that

$$|\nabla\omega(t)|_2^2 + \varepsilon(t)|\Delta\omega(t)|_2^2 \le k_4\left(\|\phi\|_{C_{\mathcal{H}_\tau^1}}^2 e^{-\sigma(t-\tau)} + \mathscr{G}_\sigma(t) + 1\right),$$

holds for any $\tau \le t \in \mathbb{R}$.

Next, we will demonstrate the existence and regularity of pullback $\mathscr{D}-$attractors $\hat{\mathscr{A}}$ for the equations defined in (1.1)–(1.3).

Theorem 3.12. *The process* $\{S(t,\tau)\}_{t\ge\tau}$ *for Eq* (1.1) *with* (1.2) *and* (1.3) *is a* $C_{\mathcal{H}_t^1}-$*contractive process on* $\hat{\mathcal{D}}_0$ *(from Corollary 3.4).*

Proof. Let $u^i = u^i(t,\theta) = S(t,\tau)u_\tau^i$ $(i = 1,2)$ denote the solutions to Eq (1.1), characterized by the parameter $\varepsilon(t)$ and the initial data $u_\tau^i = \phi^i \in D(\tau) \in \hat{D}_0$ $(i = 1,2)$ ($\hat{D}_0$ is from *Corollary* 3.4).

By (3.16), we have

$$u^i = S(t,\tau)u_\tau^i = U_1(t,\tau)u_\tau^i + K(t,\tau)u_\tau^i = v^i + \omega^i.$$

This yields

$$\begin{aligned}&\|S(t,\tau)u_\tau^1 - S(t,\tau)u_\tau^2\|^2_{C_{\mathcal{H}_t^1}}\\&\le 2\|U_1(t,\tau)u_\tau^1 - U_1(t,\tau)u_\tau^2\|^2_{C_{\mathcal{H}_t^1}} + 2\|K(t,\tau)u_\tau^1 - K(t,\tau)u_\tau^2\|^2_{C_{\mathcal{H}_t^1}},\end{aligned}\tag{3.24}$$

and

$$\|U_1(t,\tau)u_\tau^1 - U_1(t,\tau)u_\tau^2\|^2_{C_{\mathcal{H}_t^1}} \le 2\left(\|U_1(t,\tau)u_\tau^1\|^2_{C_{\mathcal{H}_t^1}} + \|U_1(t,\tau)u_\tau^2\|^2_{C_{\mathcal{H}_t^1}}\right).$$

By (3.20), for any $\varepsilon > 0$, let

$$\|\hat{D}_0\|^2_{C_{\mathcal{H}_\tau^1}} = \max_{\phi\in\mathcal{D}(\tau)\in\hat{\mathcal{D}},} \|\phi\|^2_{C_{\mathcal{H}_\tau^1}},$$

$$\tau_1 = \tau_1(t,\varepsilon,\hat{\mathcal{D}},) \le t - \frac{1}{\sigma}\ln\frac{2e^{\sigma h}\|\hat{\mathcal{D}},\|^2_{C_{\mathcal{H}_\tau^1}}}{\varepsilon}.$$

Then,

$$2\|S(t,\tau_1)u_\tau^1 - S(t,\tau_1)u_\tau^2\|^2_{C_{\mathcal{H}_t^1}} < \varepsilon \tag{3.25}$$

holds for any $\tau \le \tau_1$.

Let $\varpi(t) = \omega^1 - \omega^2$ be the solution of the following system:

$$\partial_t\varpi - \varepsilon(t)\Delta\partial_t\varpi - \Delta\varpi + f(\omega^1) - f(\omega^2) + l\varpi = l(u^1 - u^2) + g(t,u_t^1) - g(t,u_t^2),$$

it is subject to initial and boundary value conditions

$$\begin{aligned}&\varpi(x,\tau) = 0, \quad x\in\Omega, \tau\in\mathbb{R},\\&\varpi(x,\ t)|_{\partial\Omega} = 0, \quad \forall\, t\in(\tau,\infty).\end{aligned}$$

This yields

$$\begin{aligned}&\frac{d}{dt}\left(|\varpi(t)|_2^2 + \varepsilon(t)|\nabla\varpi(t)|_2^2\right) + \alpha_2\left(|\varpi(t)|_2^2 + \varepsilon(t)|\nabla\varpi(t)|_2^2\right)\\&\le 2l|u^1(t) - u^2(t)|_2|\varpi(t)|_2 + 2Lg\|u_t^1 - u_t^2\|_{C_{\mathcal{H}_t^1}}|\varpi(t)|_2\\&\le 2(l + L_g)(\|u_t^1\|_{C_{\mathcal{H}_t^1}} + \|u_t^2\|_{C_{\mathcal{H}_t^1}})|\omega^1 - \omega^2|_2,\end{aligned}$$

where $\alpha_2 = 2\alpha_1$. Let $\tau = T \le \min\{\tau_0,\tau_1\}$ be fixed. We get

$$|\varpi(t)|_2^2 + \varepsilon(t)|\nabla\varpi(t)|_2^2 \le 2(l + L_g)\int_T^t e^{-\alpha_2(t-s)}(\|u_s^1\|_{C_{\mathcal{H}_s^1}} + \|u_s^2\|_{C_{\mathcal{H}_s^1}})|\varpi(s)|_2 ds.$$

Note that $\sigma < \lambda < \alpha_2$ and (3.7). Then,

$$
\begin{aligned}
& e^{-\alpha_2 t}\int_T^t e^{\alpha_2 s}(\|u_s^1\|_{C_{\mathcal{H}_s^1}}+\|u_s^2\|_{C_{\mathcal{H}_s^1}})|\varpi(s)|_2 ds \\
\leq & \left(e^{-\alpha_2 t}\int_T^t e^{\alpha_2 s}(\|u_s^1\|^2_{C_{\mathcal{H}_s^1}}+\|u_s^2\|^2_{C_{\mathcal{H}_s^1}})ds\right)^{\frac{1}{2}}\left(e^{-\alpha_2 t}\int_T^t e^{\alpha_2 s}|\omega^1(s)-\omega^2(s)|_2^2 ds\right)^{\frac{1}{2}} \\
\leq & \left(e^{-\lambda t}\int_T^t e^{\lambda s}(\|u_s^1\|^2_{C_{\mathcal{H}_s^1}}+\|u_s^2\|^2_{C_{\mathcal{H}_s^1}})ds\right)^{\frac{1}{2}}\left(\int_T^t |\omega^1(s)-\omega^2(s)|_2^2 ds\right)^{\frac{1}{2}} \\
\leq & \mathcal{K}_2\left(\int_T^t |\omega^1(s)-\omega^2(s)|_2^2 ds\right)^{\frac{1}{2}},
\end{aligned}
$$

where $\mathcal{K}_2=\mathcal{K}_2(t,\hat{D}_0)=4\left(\frac{\delta_1\lambda_1}{L_g^2}\|\hat{D}_0\|^2_{C_{\mathcal{H}_\tau^1}}+\frac{1}{\delta_2\lambda_1}e^{-\lambda t}\int_{-\infty}^t e^{-\lambda(t-s)}|g_0(s)|_2^2 ds+\frac{\beta|\Omega|}{\lambda}\right)^{\frac{1}{2}}$.

Then,

$$
\|\varpi_t\|^2_{C_{\mathcal{H}_t^1}}\leq 4(2l+L_g)\mathcal{K}_2\left(\int_T^t |\omega^1(s)-\omega^2(s)|_2 ds\right)^{\frac{1}{2}}.
$$

We set

$$
\psi_T^t(u^1,u^2)=4(2l+L_g)\mathcal{K}_2\left(\int_T^t\int_\Omega |\omega^1(s)-\omega^2(s)|^2 dxds\right)^{\frac{1}{2}}. \tag{3.26}
$$

By *Corollary* 3.5, Lemma 3.10, and using Lemma 2.10, we find that the sequence $\{\omega_n(s)\}_{n=1}^{\infty}$ is relatively compact in $L^2(T,t;L^2(\Omega))$. To put it differently, for any sequences $\{u_n(T)=\phi_n\}\subset D_0(T)\in\hat{D}_0$, $\{\omega_n(t)\}$ constitutes the solution of system (3.18) with the initial values $\{u_n(T)\}$ respectively. Then there exists a subsequence $\{\omega_{n_k}\}\subset\{\omega_n\}$ satisfying:

$$
\lim_{k\to\infty}\lim_{l\to\infty}\psi_T^t(u_{n_k},u_{n_l})=0.
$$

So, we have $\varphi_T^t\in Contr(\hat{\mathcal{D}}_0)$. Substituting (3.26) and (3.25) into (3.24), we get

$$
\|S(t,T)x-S(t,T)y\|^2_{C_{\mathcal{H}_t^1}}\leq\varepsilon+\psi_T^t(x,y).
$$

By Definitions 2.7 and 2.8, then $\psi_T^t\in Contr(\hat{\mathcal{D}}_0)$. Therefore, it is straightforward to conclude that the process $\{S(t,\tau)\}_{t\geq\tau}$ is a $C_{\mathcal{H}_t^1}$−contractive process on $\hat{\mathcal{D}}_0$.

As the concluding remark of this article, we will derive the main result presented in the following theorem.

Theorem 3.13. *The process $\{S(t,\tau)\}_{t\geq\tau}$ defined by Eq* (3.3) *possesses a pullback $\mathscr{D}$-attractor $\hat{\mathscr{A}}$ in $\left\{C_{\mathcal{H}_t^1}\right\}_{t\in\mathbb{R}}$, and $\hat{\mathscr{A}}$ is non-empty, compact, invariant in $\left\{C_{\mathcal{H}_t^1}\right\}_{t\in\mathbb{R}}$, and pullback attracting in $\left\{C_{\mathcal{H}_t^1}\right\}_{t\in\mathbb{R}}$. Furthermore,*

$$
\hat{\mathscr{A}}=\{\mathscr{A}(t)\subset C_{\mathcal{H}_t^r}: t\in\mathbb{R}\}\quad \textit{for all}\quad 1\leq r<2.
$$

Proof. Thanks to Theorem 2.9, Lemma 3.3, and Theorem 3.12, we can easily establish the existence of the pullback $\mathscr{D}$-attractor, denoted as $\hat{\mathscr{A}}$, for the process $\{S(t,\tau)\}_{t\geq\tau}$ defined by (3.3) in time-dependent spaces $\left\{C_{\mathcal{H}_t^1}\right\}_{t\in\mathbb{R}}$. Based on Lemmas 3.9–3.11, we can prove the asymptotic regularity of solutions to the problems (1.1)–(1.3). Furthermore, since $\mathcal{H}_t^r \hookrightarrow \mathcal{H}_t^2$ for $1 \leq r < 2$, it follows that $\omega_t = \omega(t+\theta) \in C_{\mathcal{H}_t^r}$. This leads us to conclude the regularity of the pullback $\mathscr{D}$-attractor $\hat{\mathscr{A}}$. By combining these findings with Theorem 2.9 and (3.2), we conclude that the pullback $\mathscr{D}$-attractor $\hat{\mathscr{A}}$ is invariant.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

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