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Research article

Well-posedness of the MHD boundary layer equations with small initial data in Sobolev space

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Abstract: The purpose of this paper is to prove the well-posedness of the 2D magnetohydrodynamic (MHD) boundary layer equations for small initial data in Sobolev space of polynomial weight and low regularity. Our proofs are based on the paralinearization method and an abstract bootstrap argument. We first obtain the systems (3.3)–(3.6) by paralinearizing and symmetrizing the system (1.2). Then, we establish the estimates of the solution in horizontal direction and vertical direction, respectively. Finally, we prove the well-posedness of the 2D MHD boundary layer equations by an abstract bootstrap argument.

Keywords: 2D MHD boundary layer equations; polynomial weight; well-posedness; bootstrap argument; paralinearization method

1. Introduction and main result

In this paper, we investigate the 2D magnetohydrodynamic (MHD) boundary layer equations on the upper half plane $\mathbb{R}^2_+ = \{(x, y) : x \in \mathbb{R},\ y \in \mathbb{R}_+\}$, which reads as:

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u = h\partial_x h + g\partial_y h + \partial_y^2 u - \partial_x p, \\ \partial_t h + \partial_y(vh - ug) = \partial_y^2 h, \\ \partial_x u + \partial_y v = 0,\ \partial_x h + \partial_y g = 0, \\ (u, v, \partial_y h, g)|_{y=0} = 0,\ \lim\limits_{y\to+\infty} (u, h) = (U(t, x), H(t, x)), \\ (u, h)|_{t=0} = (u_0, h_0)(x, y), \end{cases} \tag{1.1}$$

where (u, v) represents the velocity field of the boundary layer flow, (h, g) stands for the magnetic field and functions $U(t, x)$, $H(t, x)$, and $p(t, x)$ denote the trace of the tangential fluid and magnetic, the pressure of the outflow, respectively, which satisfy Bernoulli's law

$$\begin{cases} \partial_t U + U\partial_x U - H\partial_x H + \partial_x p = 0, \\ \partial_t H + U\partial_x H - H\partial_x U = 0. \end{cases}$$

System (1.1) is a boundary layer model, which is derived from the 2D incompressible MHD system with a non-slip boundary condition on the velocity and a perfectly conducting condition on the magnetic field [1, 2].

Before exhibiting the main result in this paper, let us recall some known results to system (1.1). Especially, when the magnetic field (h, g) are some constants in system (1.1), it reduces to the classical Prandtl equations, which were first introduced formally by Prandtl [3] in 1904. The Prandtl equations are the foundation of the boundary layer theory. It describes that the fluid near the boundary of a solid body can be divided into two regions: a very thin layer in the neighborhood of the body (the boundary layer) where viscous friction plays an essential part, and the remaining region outside this layer where friction may be neglected (the outer flow). The well-posedness theory of the Prandtl equations was well understood in [4–6] and the references therein for the recent progress.

When the velocity field equation is coupled with the magnetic field equation, the phenomenon of the boundary layer is different since the boundary layers of the magnetic field may exist [1] and they are more complicated than the classical Prandtl equations. It is worth pointing out that some results have been obtained about the well-posedness of the MHD boundary layer equations in weighed Sobolev space. Liu et al. [2] proved the local existence and uniqueness of solutions for the 2D nonlinear MHD boundary layer equations without monotonicity in weighted Sobolev space by using energy methods. Liu et al. [7] investigated the local well-posedness of the 2D MHD boundary layer equations without resistivity in Sobolev spaces. Finally, they also got the linear instability of the 2D MHD boundary layer when the tangential magnetic field is degenerate at one point. Besides, there are some well-posedness results for the MHD equations [8, 9].

There are some results in the analytic framework for the 2D MHD boundary layer equations, Xie and Yang [10] considered the global existence of solutions to the 2D MHD boundary layer equations in the mixed Prandtl and Hartmann regime when the initial datum is a small perturbation of the Hartmann profile and obtained the solution in analytic norm as an exponential decay in time. Recently, by using the cancellation mechanism, Xie and Yang [11] investigated the existence and uniqueness of solutions to the 2D MHD boundary layer system in an analytic space.

Besides, there are some known results about the well-posedness of boundary layer equations by using the paralinearization method. Chen et al. [12] obtained the local well-posedness to the classical Prandtl equations when the weighted function μ is an exponential function in $H^{3,1}_{\mu}(\mathbb{R}^2_+) \cap H^{1,2}_{\mu}(\mathbb{R}^2_+)$. Wang and Zhang [13] proved the local well-posedness of the classical Prandtl equation for monotonic data in a polynomial weighted Sobolev space. Chen et al. [14] studied the long-time well-posedness of the MHD equations for small initial data in an exponentially weighted Sobolev space $H^{3,0}_{\mu}(\mathbb{R}^2_+) \cap H^{1,2}_{\mu}(\mathbb{R}^2_+)$, and obtained the lifespan of solutions to depend on the initial data. Wang and Wang [15] investigated the global well-posedness of the 2D MHD equations in striped domain with small data, and proved the solutions of the anisotropic MHD equations convergence to the solutions of the hydrostatic MHD equations in L^∞. Chen and Li [16] obtained the long-time well-posedness of the 2D MHD boundary layer equations with small initial data in an exponentially weighted Sobolev space $H^{3,0}_{\mu}(\mathbb{R}^2_+) \cap H^{1,2}_{\mu}(\mathbb{R}^2_+) \cap H^{2,1}_{\mu}(\mathbb{R}^2_+)$, and proved the lifespan of solutions depends on the initial data. Inspired by the ideas in [13, 14], the aim of this paper is to investigate the well-posedness of the problem (1.1) by using the paralinearization method and an abstract bootstrap argument. Similar to the Prandtl equation, the difficulty of solving the problem (1.2) in the Sobolev framework is the loss of x-derivative in the terms like $v\partial_y u$, $v\partial_y h$, $g\partial_y u$ and $g\partial_y h$. To overcome this essential difficulty, inspired by recent results in [14], we will first paralinearize system (1.2) and introduce two new good functions to symmetrize the system, then establish the estimates of solutions to system (3.3).

Finally, the rest of the paper is arranged as follows. In Section 2, we introduce the Littlewood-Paley decomposition and paraproduct and some lemmas which that be used frequently. In Section 3, we paralinearize the system (1.2) and introduce the good unknown functions to symmetrize the system. In Section 4, we prove the Sobolev estimate in the horizontal direction. In Section 5, we get the high order energy estimate in the y variable and give the proof of Theorem 1.1.

Hereafter, let letter C be a general positive constant independent of ε, which may vary from line to line at each step.

For simplicity's sake, we consider a uniform outflow $(U, H) = (0, 1)$. Let $h(t, x, y) = 1 + \tilde{h}(t, x, y)$. Then $(u, \tilde{h})$ satisfies the following system:

$$\begin{cases} \partial_t u + u\partial_x u + v\partial_y u - h\partial_x \tilde{h} - g\partial_y \tilde{h} - \partial_y^2 u = 0, \\ \partial_t \tilde{h} + u\partial_x \tilde{h} + v\partial_y \tilde{h} - h\partial_x u - g\partial_y u - \partial_y^2 \tilde{h} = 0, \\ \partial_x u + \partial_y v = 0,\ \partial_x \tilde{h} + \partial_y g = 0, \\ (u, v, \partial_y \tilde{h}, g)|_{y=0} = 0,\ \lim\limits_{y\to+\infty} (u, \tilde{h}) = (0, 0), \\ (u, \tilde{h})|_{t=0} = (u_0, \tilde{h}_0)(x, y). \end{cases} \tag{1.2}$$

As in [14], we first introduce the following weighted Sobolev space. For $m, n, \alpha, \beta \in \mathbb{N}$, $\ell \geq \frac{3}{2}$, the space $H_\ell^{m,n}(\mathbb{R}_+^2)$ consists of all functions $f \in L_\ell^2$ satisfying

$$\|f\|_{H_\ell^{m,n}(\mathbb{R}_+^2)} = \sum_{\alpha=0}^{m} \sum_{\beta=0}^{n} \|\partial_x^\alpha \partial_y^\beta f\|_{L_\ell^2}^2 < +\infty,$$

where $\|f\|_{L_\ell^2} = \|\langle y\rangle^\ell f(x, y)\|_{L^2}$ with $\langle y\rangle = (1 + y)$.

We are now in a position to state the main result as follows:

Theorem 1.1. *Let $\ell \geq \frac{3}{2}$, $m, \beta \in \mathbb{N}$. For small enough $\varepsilon \in (0, \frac{1}{2\sqrt{10C}}]$, assume that initial data $(u_0, \tilde{h}_0)$ satisfy*

$$\sum_{m=0}^{2} \|\partial_y^m (u_0, \tilde{h}_0)\|_{H_{\ell-1+m}^{1,0}}^2 + \|(u_0, \tilde{h}_0)\|_{H_\ell^{3,0}}^2 \leq \varepsilon^2, \tag{1.3}$$

then for any given time T independent of ε such that the problem (1.2) *has a unique solution $(u, \tilde{h})$ satisfies*

$$\begin{aligned} &\Big(\sum_{m=0}^{2} \|\partial_y^m (u, \tilde{h})\|_{H_{\ell-1+m}^{1,0}}^2 + \|(u_\beta, \tilde{h}_\beta)\|_{H_\ell^{3,0}}^2\Big) \\ &+ \int_0^t \Big(\sum_{m=0}^{2} \|\partial_y^{m+1} (u, \tilde{h})\|_{H_{\ell-1+m}^{1,0}}^2 + \|(\partial_y u_\beta, \partial_y \tilde{h}_\beta)\|_{H_\ell^{3,0}}^2\Big)(s) ds \leq 5C\varepsilon^2, \end{aligned}$$

for any $t \in [0, T]$. Here C is a positive constant independent of ε.

Remark 1.1. *Theorem 1.1 obtains the local well-posedness of the solution, while the global well-posedness of the solution is still an open problem.*

2. Littlewood-Paley decomposition and paraproduct

As in [14], we first introduce the Littlewood-Paley decomposition in the horizontal direction $x \in \mathbf{R}$. Choose two smooth functions $\varphi(\tau)$ and $\chi(\tau)$ that satisfy

$$\text{Supp } \varphi \subset \Big\{\tau \in \mathbb{R} / \ \frac{3}{4} \leqq |\tau| \leqq \frac{8}{3}\Big\},$$
$$\text{Supp } \chi \subset \Big\{\tau \in \mathbb{R} / \ \ |\tau| \leqq \frac{4}{3}\Big\} \qquad \text{and} \qquad \tau \in \mathbb{R}, \ \chi(\tau) + \sum_{k\geqq 0} \varphi(2^{-k}\tau) = 1.$$

Then we define

$$\Delta_j f = \mathcal{F}^{-1}(\varphi(2^{-j}\xi)\hat{f}), \ S_j f = \mathcal{F}^{-1}(\chi(2^{-j}\xi)\hat{f}) \text{ for } j \geq 0,$$
$$\Delta_{-1} = S_0 f, \ \ S_j f = S_0 f \text{ for } j < 0.$$

Bony's paraproduct $T_f g$ is defined by

$$T_f g = \sum_{j\geq -1} S_{j-1}\Delta_j g.$$

Then we have the following Bony's paraproduct

$$fg = T_f g + T_g f + R(f, g), \tag{2.1}$$

where the remainder term $R(f, g)$ is defined by

$$R(f, g) = \sum_{|k-k'|\leq 1; k,k'\geq -1} \Delta_k f \Delta_{k'} g.$$

Next, let us introduce some classical paraproduct estimates and paraproduct calculus in Sobolev space [17], Chapter 2.

Lemma 2.1. *Let $s \in \mathbb{R}$, it holds that*

$$\|T_f g\|_{H^s} \leq C\|f\|_{L^\infty}\|g\|_{H^s}.$$

If $s > 0$, then we have

$$\|R(f, g)\|_{H^s} \leq C \min(\|f\|_{L^\infty}\|g\|_{H^s}, \|f\|_{H^s}\|g\|_{L^\infty}).$$

Lemma 2.2. *Let $s \in \mathbb{R}$ and $\sigma \in (0, 1]$, it holds that*

$$\|(T_a T_b - T_{ab})f\|_{H^s} \leq C(\|a\|_{W^{\sigma,\infty}}\|b\|_{L^\infty} + \|a\|_{L^\infty}\|b\|_{W^{\sigma,\infty}})\|f\|_{H^{s-\sigma}}.$$

Especially, we have

$$\|[T_a, T_b]f\|_{H^s} \leq C(\|a\|_{W^{\sigma,\infty}}\|b\|_{L^\infty} + \|a\|_{L^\infty}\|b\|_{W^{\sigma,\infty}})\|f\|_{H^{s-\sigma}},$$
$$\|(T_a - T_a^*)f\|_{H^s} \leq C\|a\|_{W^{\sigma,\infty}}\|f\|_{H^{s-\sigma}},$$

here T_a^* is the adjoint of T_a.

Lemma 2.3. *Let $s \in \mathbb{N}$, it holds that*

$$\|[\partial_x^s, T_a]f\|_{L^2} \leq C\|\partial_x a\|_{L^\infty}\|f\|_{H^{s-1}}.$$

In this position, we will show the Agmon inequality, whose proof is given in [18].

Lemma 2.4. *Let $f \in H^1(\mathbb{R}_+^2)$, then*

$$\|f\|_{L_x^\infty L_y^2} \leq C\|f\|_{L^2(\mathbb{R}_+^2)}^{1/2}\|\partial_x f\|_{L^2(\mathbb{R}_+^2)}^{1/2}.$$

Before proving the main Theorem 1.1 by using Lemmas 2.1–2.4, we first paralinearize and symmetrize the system (1.2) according to the method in [14].

3. Paralinearization and symmetrization

Similar to the Prandtl equations, the difficulty of solving problem (1.2) in the Sobolev framework is the loss of x-derivative in the terms $v\partial_y u - g\partial_y \tilde{h}$ and $v\partial_y \tilde{h} - g\partial_y u$ in the first and second equations of (1.2), respectively. In other words, $v = -\partial_y^{-1}\partial_x u$ and $g = -\partial_y^{-1}\partial_x \tilde{h}$ by the divergence-free conditions and boundary conditions. Thus, it creates a loss of the x-derivative and a y-integration to the y-variable. Then the standard energy estimates do not work. To overcome this essential difficulty, inspired by recent results in [14], we will first paralinearize the system (1.2) and then introduce the good unknown functions to symmetrize the system following the idea in [14].

Applying Bony's decomposition (2.1), we derive

$$\begin{cases} \partial_t u + T_u\partial_x u + T_{\partial_y u}v - T_h\partial_x \tilde{h} - T_{\partial_y \tilde{h}}g - \partial_y^2 u = f_1, \\ \partial_t \tilde{h} + T_u\partial_x \tilde{h} + T_{\partial_y \tilde{h}}v - T_h\partial_x u - T_{\partial_y u}g - \partial_y^2 \tilde{h} = f_2, \end{cases} \tag{3.1}$$

where

$$\begin{aligned} f_1 &= -R_{\partial_x u}u - R_v\partial_y u + R_{\partial_x \tilde{h}}h + R_g\partial_y \tilde{h}, \\ f_2 &= -R_v\partial_y \tilde{h} - R_{\partial_x h}u + R_{\partial_x u}\tilde{h} + R_g\partial_y u. \end{aligned}$$

We now define

$$h_1(t,x,y) = \int_0^y \tilde{h}(t,x,\tilde{y})d\tilde{y}.$$

From system $(3.1)_2$, we deduce that

$$\partial_t h_1 + T_h v - T_u g - \partial_y^2 h_1 = \int_0^y f_2(\tilde{y})d\tilde{y}.$$

Motivated by [14], we introduce the two good unknown functions

$$\begin{cases} u_\beta = u - T_{\frac{\partial_y u}{h}}h_1, \\ \tilde{h}_\beta = \tilde{h} - T_{\frac{\partial_y u}{h}}h_1. \end{cases} \tag{3.2}$$

Consequently, we can rewrite the system (3.1) as

$$\begin{cases} \partial_t u_\beta + T_u \partial_x u_\beta - T_h \partial_x \tilde{h}_\beta - \partial_y^2 u_\beta = G_1, \\ \partial_t \tilde{h}_\beta - T_h \partial_x u_\beta + T_u \partial_x \tilde{h}_\beta - \partial_y^2 \tilde{h}_\beta = G_2, \end{cases} \tag{3.3}$$

where

$$\begin{aligned} G_1 &= [T_{\frac{\partial_y u}{h}} T_h - T_{\partial_y u}]v - [T_{\frac{\partial_y u}{h}}, T_u]g - [T_h T_{\frac{\partial_y h}{h}} - T_{\partial_y h}]g - T_{(\partial_t - \partial_y^2)(\frac{\partial_y u}{h})} h_1 \\ &\quad + 2T_{\partial_y(\frac{\partial_y u}{h})} \tilde{h} - T_u T_{\partial_x(\frac{\partial_y u}{h})} h_1 + T_h T_{\partial_x \frac{\partial_y u}{h}} h_1 - T_{\frac{\partial_y u}{h}} \int_0^y f_2 d\tilde{y} + f_1 \\ &= G_{11} + \cdots + G_{19} \end{aligned} \tag{3.4}$$

and

$$\begin{aligned} G_2 &= [T_{\frac{\partial_y h}{h}} T_h - T_{\partial_y h}]v - [T_h T_{\frac{\partial_y u}{h}} - T_{\partial_y u}]g - [T_{\frac{\partial_y h}{h}}, T_u]g - T_{(\partial_t - \partial_y^2)(\frac{\partial_y h}{h})} h_1 \\ &\quad - 2T_{\partial_y(\frac{\partial_y h}{h})} \tilde{h} - T_u T_{\partial_x(\frac{\partial_y h}{h})} h_1 + T_h T_{\partial_x \frac{\partial_y h}{h}} h_1 - T_{\frac{\partial_y h}{h}} \int_0^y f_2 d\tilde{y} + f_2 \\ &= G_{21} + \cdots + G_{29}. \end{aligned} \tag{3.5}$$

Moreover, it is easy to check $(u_\beta, \tilde{h}_\beta)$ satisfies the following boundary conditions:

$$(u_\beta, \partial_y \tilde{h}_\beta)|_{y=0} = 0, \ \lim_{y \to +\infty} (u_\beta, \tilde{h}_\beta) = (0, 0). \tag{3.6}$$

The investigation of the local well-posedness of the solution to system (1.2) is equivalent to studying the local well-posedness of systems (3.3)–(3.6). We will establish a priori estimates of the solution to systems (3.3)–(3.6) in Sections 4 and 5.

4. Sobolev estimate in horizontal direction

In this section, we will establish the estimates of solutions. Let us first define the following energy functionals

$$E(t) = \sum_{m=0}^{2} \|\partial_y^m (u, \tilde{h})\|^2_{H^{1,0}_{\ell-1+m}} + \|(u_\beta, \tilde{h}_\beta)\|^2_{H^{3,0}_\ell}$$

and

$$D(t) = \sum_{m=0}^{2} \|\partial_y^{m+1} (u, \tilde{h})\|^2_{H^{1,0}_{\ell-1+m}} + \|(\partial_y u_\beta, \partial_y \tilde{h}_\beta)\|^2_{H^{3,0}_\ell}.$$

We assume that $(u, \tilde{h})$ is a smooth solution of (1.2) on $[0, T^*]$ and

$$\sup_{t \le T^*} E(t) \le (c_1 \varepsilon)^2, \tag{4.1}$$

where the positive constant $c_1 = \frac{1}{\sqrt{10C}}$.

4.1. Some technical lemmas

We first give the proof of the lower bound of $h(t,x,y)$.

Lemma 4.1. *Let $\ell \geq \frac{3}{2}$. For small enough $\varepsilon \in (0, \frac{1}{2\sqrt{10C}}]$ and any small $\delta \in (0,1)$, it holds that*

$$h(t,x,y) > \frac{1}{2} \quad \forall\ (t,x,y) \in [0,T] \times \mathbb{R}^2_+.$$

Proof. As $h = \tilde{h} + 1$, we can derive the following inequality from Lemma 2.4 and the condition $\lim\limits_{y\to+\infty} \tilde{h} = 0$

$$\begin{aligned}
\|\tilde{h}(t,x,y)\|_{L^\infty(\mathbb{R}^2_+)} &\leq \Big| \int_y^\infty \partial_y \tilde{h}(t,x,\tilde{y}) d\tilde{y} \Big|_{L^\infty(\mathbb{R}^2_+)} \\
&\leq \|\partial_y \tilde{h}(t,x,y)\|_{L^\infty(\mathbb{R}_x; L^2_{\frac{1}{2}+\delta}(\mathbb{R}))} \\
&\leq \|\partial_y \tilde{h}(t,x,y)\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)} + \|\partial_x \partial_y \tilde{h}(t,x,y)\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)} \\
&\leq E(t)^{\frac{1}{2}} \leq c_1 \varepsilon \leq \frac{1}{2}.
\end{aligned} \tag{4.2}$$

The proof is thus complete. □

The following lemma can be obtained by the Hölder inequality ([19], Theorem 1.4.3).

Lemma 4.2. *Let $\ell \geq \frac{3}{2}$, it holds that*

$$\Big\| \int_0^y f d\tilde{y} \Big\|_{L^\infty_y} \leq C \|f\|_{L^2_{y,\ell}}.$$

Especially, thanks to $\partial_x u + \partial_y v = 0$, $\partial_x \tilde{h} + \partial_y g = 0$, it holds that for $n \in \mathbb{N}$,

$$\|v\|_{H^n_x L^\infty_y} \leq C\|u\|_{H^{n+1,0}_\ell},\ \|g\|_{H^n_x L^\infty_y} \leq C\|\tilde{h}\|_{H^{n+1,0}_\ell}.$$

Lemma 4.3. *Let $\ell \geq \frac{3}{2}$. For $\varepsilon \in (0, \frac{1}{2\sqrt{10C}}]$ is small enough and any small $\delta \in (0,1)$, it holds that*

$$\begin{cases}
\|u(t,x,y)\|_{L^\infty} \leq E(t)^{\frac{1}{2}}, \quad \|\partial_x u(t,x,y)\|_{L^\infty} \leq D(t)^{\frac{1}{2}}, \\
\|\partial_y u(t,x,y)\|_{L^\infty} \leq D(t)^{\frac{1}{2}}, \quad \|\partial_y^2 u(t,x,y)\|_{L^\infty} \leq D(t)^{\frac{1}{2}}, \\
\|\partial_y \tilde{h}(t,x,y)\|_{L^\infty} \leq E(t)^{\frac{1}{2}}, \quad \|\partial_y \tilde{h}(t,x,y)\|_{L^\infty} \leq D(t)^{\frac{1}{2}}, \\
\|\partial_y^2 \tilde{h}(t,x,y)\|_{L^\infty(\mathbb{R}^2_+)} \leq D(t)^{\frac{1}{2}}.
\end{cases} \tag{4.3}$$

Proof. Integrating it over $[0,y]$ and using the boundary condition $u(t,x,0) = 0$ and Lemma 2.4, we have

$$\|u(t,x,y)\|_{L^\infty(\mathbb{R}^2_+)} \leq \Big| \int_0^y \partial_y u(t,x,\tilde{y}) d\tilde{y} \Big|_{L^\infty(\mathbb{R}^2_+)}$$

$$
\begin{aligned}
&\leq \|\partial_y u(t,x,y)\|_{L^\infty(\mathbb{R}_x;L^2_{\frac{1}{2}+\delta}(\mathbb{R}))}\\
&\leq \|\partial_y u(t,x,y)\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)} + \|\partial_x\partial_y u(t,x,y)\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}\\
&\leq \|\partial_y u(t,x,y)\|_{H^{1,0}_\ell(\mathbb{R}^2_+)}\\
&\leq E(t)^{\frac{1}{2}}
\end{aligned}
$$

and

$$
\begin{aligned}
\|\partial_x u(t,x,y)\|_{L^\infty(\mathbb{R}^2_+)} &\leq \Big|\int_0^y \partial_x\partial_y u(t,x,\tilde{y})d\tilde{y}\Big|_{L^\infty(\mathbb{R}^2_+)}\\
&\leq \|\partial_x\partial_y u(t,x,y)\|_{L^\infty(\mathbb{R}_x;L^2_{\frac{1}{2}+\delta}(\mathbb{R})}\\
&\leq \|\partial_x\partial_y u(t,x,y)\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)} + \|\partial_x^2\partial_y u(t,x,y)\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}\\
&\leq \|\partial_y u(t,x,y)\|_{H^{2,0}_\ell(\mathbb{R}^2_+)}\\
&\leq D(t)^{\frac{1}{2}}.
\end{aligned}
$$

Applying the Gagliardo–Nirenberg inequality ([19], Theorem 1.1.18), the Young inequality ([19], Corollary 1.4.1) and Lemma 2.4, we deduce that

$$
\begin{aligned}
\|\partial_y u(t,x,y)\|_{L^\infty(\mathbb{R}^2_+)} &\leq \|\partial_y u(t,x,y)\|^{\frac{1}{2}}_{L^\infty_x L^2_y}\|\partial_y^2 u(t,x,y)\|^{\frac{1}{2}}_{L^\infty_x L^2_y}\\
&\leq \|\partial_y u(t,x,y)\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2_+)}\|\partial_x\partial_y u(t,x,y)\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2_+)}\\
&\quad\times\|\partial_y^2 u(t,x,y)\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2_+)}\|\partial_x\partial_y^2 u(t,x,y)\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2_+)}\\
&\leq \frac{1}{4}\Big(\|\partial_y u(t,x,y)\|_{L^2(\mathbb{R}^2_+)} + \|\partial_x\partial_y u(t,x,y)\|_{L^2(\mathbb{R}^2_+)}\\
&\qquad+\|\partial_y^2 u(t,x,y)\|_{L^2(\mathbb{R}^2_+)} + \|\partial_x\partial_y^2 u(t,x,y)\|_{L^2(\mathbb{R}^2_+)}\Big)\\
&\leq D(t)^{\frac{1}{2}}.
\end{aligned}
$$

Using the boundary condition $\partial_y^2 u(t,x,0) = 0$, Young's inequality ([19], Corollary 1.4.1) and Lemma 2.4, we conclude

$$
\begin{aligned}
\|\partial_y^2 u(t,x,y)\|_{L^\infty(\mathbb{R}^2_+)} &\leq \Big|\int_0^y \partial_y^3 u(t,x,y)d\tilde{y}\Big|\\
&\leq \|\partial_y^3 u(t,x,y)\|_{L^\infty_x L^2_{y,\frac{1}{2}+\delta}}\\
&\leq \|\partial_y^3 u(t,x,y)\|^{\frac{1}{2}}_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}\|\partial_x\partial_y^2 u(t,x,y)\|^{\frac{1}{2}}_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}\\
&\leq \frac{1}{2}\|\partial_y^3 u(t,x,y)\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)} + \frac{1}{2}\|\partial_x\partial_y^2 u(t,x,y)\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}\\
&\leq D(t)^{\frac{1}{2}}.
\end{aligned}
$$

In the same way, using the condition $\partial_y\tilde{h}|_{y=0}=0$, we derive

$$\begin{aligned}
\|\partial_y\tilde{h}(t,x,y)\|_{L^\infty(\mathbb{R}^2_+)} &\le \Big|\int_0^y \partial_y^2\tilde{h}(t,x,\tilde{y})d\tilde{y}\Big|_{L^\infty(\mathbb{R}^2_+)}\\
&\le \|\partial_y^2\tilde{h}(t,x,y)\|_{L^\infty_x L^2_{y,\frac{1}{2}+\delta}}\\
&\le \|\partial_y^2\tilde{h}(t,x,y)\|^{\frac{1}{2}}_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}\|\partial_x\partial_y^2\tilde{h}(t,x,y)\|^{\frac{1}{2}}_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}\\
&\le D(t)^{\frac{1}{2}}
\end{aligned}$$

or

$$\begin{aligned}
\|\partial_y\tilde{h}(t,x,y)\|_{L^\infty(\mathbb{R}^2_+)} &\le \|\partial_y^2\tilde{h}(t,x,y)\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}+\|\partial_x\partial_y^2\tilde{h}(t,x,y)\|_{L^2_{\frac{1}{2}+\delta}(\mathbb{R}^2_+)}\\
&\le E(t)^{\frac{1}{2}}.
\end{aligned}$$

By virtue of the Gagliardo-Nirenberg inequality ([19], Theorem 1.1.18), Lemmas 2.4 and 4.3 again yield

$$\begin{aligned}
\|\partial_y^2\tilde{h}(t,x,y)\|_{L^\infty(\mathbb{R}^2_+)} &\le \|\partial_y^2\tilde{h}(t,x,y)\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2_+)}\|\partial_x\partial_y^2\tilde{h}(t,x,y)\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2_+)}\\
&\quad\times\|\partial_y^3\tilde{h}(t,x,y)\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2_+)}\|\partial_x\partial_y^3\tilde{h}(t,x,y)\|^{\frac{1}{4}}_{L^2(\mathbb{R}^2_+)}\\
&\le D(t)^{\frac{1}{2}}.
\end{aligned}$$

The proof is now complete. □

The following lemma introduces the relationship of norms between good functions $(u_\beta,\tilde{h}_\beta)$ and $(u,\tilde{h})$.

Lemma 4.4. *Let $\ell \ge \frac{3}{2}$. For sufficient small $\varepsilon \in (0, \frac{1}{2\sqrt{10C}}]$, then for any $t\in[0,T^*]$,*

$$\|u\|_{H^{3,0}_\ell}+\|\tilde{h}\|_{H^{3,0}_\ell}\le 2E(t)^{\frac{1}{2}},\ \|\partial_y u\|_{H^{3,0}_\ell}+\|\partial_y\tilde{h}\|_{H^{3,0}_\ell}\le 4D(t)^{\frac{1}{2}}.$$

Proof. Using Lemmas 2.1, 2.4, 4.1 and 4.2, we can conclude that

$$\begin{aligned}
\|u\|_{H^{3,0}_\ell} &\le \|u_\beta\|_{H^{3,0}_\ell}+\|T_{\frac{\partial_y u}{h}}h_1\|_{H^{3,0}_\ell}\\
&\le \|u_\beta\|_{H^{3,0}_\ell}+C\|\partial_y u\|_{L^\infty_x L^2_{y,1}}\|\tilde{h}\|_{H^{3,0}_\ell}\\
&\le \|u_\beta\|_{H^{3,0}_\ell}+C\|\partial_y u\|^{\frac{1}{2}}_{L^2_1}\|\partial_x\partial_y u\|^{\frac{1}{2}}_{L^2_1}\|\tilde{h}\|_{H^{3,0}_\ell}\\
&\le \|u_\beta\|_{H^{3,0}_\ell}+CE(t)^{\frac{1}{2}}\|\tilde{h}\|_{H^{3,0}_\ell}\\
&\le \|u_\beta\|_{H^{3,0}_\ell}+Cc_1\varepsilon\|\tilde{h}\|_{H^{3,0}_\ell}.
\end{aligned}$$

Analogously, we have

$$\|\tilde{h}\|_{H^{3,0}_\ell}\le\|\tilde{h}_\beta\|_{H^{3,0}_\ell}+Cc_1\varepsilon\|\tilde{h}\|_{H^{3,0}_\ell}.$$

Therefore, by taking a small enough $\varepsilon \in (0, \frac{1}{2\sqrt{10C}}]$, we obtain

$$\|(u,\tilde{h})\|_{H_\ell^{3,0}} \leq 2\|(u_\beta,\tilde{h}_\beta)\|_{H_\ell^{3,0}} \leq 2E(t)^{\frac{1}{2}}.$$

Besides, we also obtain

$$\|(u_\beta,\tilde{h}_\beta)\|_{H_\ell^{3,0}} \leq 2\|(u,\tilde{h})\|_{H_\ell^{3,0}} \leq 2E(t)^{\frac{1}{2}}.$$

Applying Lemmas 2.1, 2.4, and 4.1–4.3, we deduce

$$\begin{aligned}
&\|\partial_y u\|_{H_\ell^{3,0}}\\
\leq\ & \|\partial_y u_\beta\|_{H_\ell^{3,0}} + \|\partial_y (T_{\frac{\partial_y u}{h}} h_1)\|_{H_\ell^{3,0}}\\
\leq\ & \|\partial_y u_\beta\|_{H_\ell^{3,0}} + C\Big(\|\partial_y^2 u\|_{L_x^\infty L_{y,\ell}^2} + \|\partial_y u\|_{L_x^\infty L_{y,\ell}^2}\|\partial_y \tilde{h}\|_{L^\infty} + \|\partial_y u\|_{L^\infty}\Big)\|\tilde{h}\|_{H_\ell^{3,0}}\\
\leq\ & \|\partial_y u_\beta\|_{H_\ell^{3,0}} + C\Big(\|\partial_y^2 u\|_{H_\ell^{1,0}} + \|\partial_y u\|_{H_\ell^{1,0}}\|\partial_y \tilde{h}\|_{L^\infty} + \|\partial_y u\|_{L^\infty}\Big)\|\tilde{h}\|_{H_\ell^{3,0}}\\
\leq\ & \|\partial_y u_\beta\|_{H_\ell^{3,0}} + CD(t)^{\frac{1}{2}}E(t)^{\frac{1}{2}} + CD(t)^{\frac{1}{2}}E(t)\\
\leq\ & 2D(t)^{\frac{1}{2}}.
\end{aligned}$$

Similarly, we also obtain

$$\|\partial_y \tilde{h}\|_{H_\ell^{3,0}} \leq 2D(t)^{\frac{1}{2}}.$$

The proof is therefore complete. □

Lemma 4.5. *Let $\ell \geq \frac{3}{2}$. For any $\varepsilon \in (0, \frac{1}{2\sqrt{10C}}]$, it holds that,*

$$\|\partial_x \tilde{h}\|_{L^\infty} \leq 2E(t)^{\frac{1}{4}} D(t)^{\frac{1}{4}}.$$

Proof. By virtue of the Gagliardo–Nirenberg inequality ([19], Theorem 1.1.18), Lemmas 2.4 and 4.4 yielding

$$\begin{aligned}
\|\partial_x \tilde{h}(t,x,y)\|_{L^\infty(\mathbb{R}_+^2)} \leq\ & \|\partial_x \tilde{h}(t,x,y)\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{4}} \|\partial_x^2 \tilde{h}(t,x,y)\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{4}}\\
& \times \|\partial_x\partial_y \tilde{h}(t,x,y)\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{4}} \|\partial_x^2\partial_y \tilde{h}(t,x,y)\|_{L^2(\mathbb{R}_+^2)}^{\frac{1}{4}}\\
\leq\ & 2E(t)^{\frac{1}{4}} D(t)^{\frac{1}{4}}.
\end{aligned}$$

The proof is hence complete. □

4.2. The estimates of the nonlinear terms G_1 and G_2

In this subsection, we establish the estimate of the nonlinear terms G_1 and G_2.

Lemma 4.6. *Let $\ell \geq \frac{3}{2}$, it holds that*

$$\begin{aligned}
&\|G_1\|_{H_\ell^{3,0}} + \|G_2\|_{H_\ell^{3,0}}\\
&\leq CD(t)^{\frac{1}{2}}E(t)^{\frac{1}{2}} + CD(t)^{\frac{1}{2}}E(t) + CD(t)^{\frac{1}{2}}E(t)^{\frac{3}{2}} + CD(t)^{\frac{1}{4}}E(t)^{\frac{3}{4}} + CD(t)^{\frac{1}{4}}E(t)^{\frac{5}{4}}.
\end{aligned}$$

Proof. We only establish the estimate of G_1. The estimate of G_2 could be derived in a similar way. By Lemmas 2.2, 2.4, and 4.1–4.3, we obtain

$$\begin{aligned}
\|G_{12}\|_{H^{3,0}_\ell} &\triangleq \|[T_{\frac{\partial_y u}{h}}, T_u]g\|_{H^{3,0}_\ell} \\
&\leq C\Big(\|\partial_y u\|_{L^2_{y,\ell}(W^{1,\infty}_x)}\|u\|_{L^\infty} + \|\partial_y u\|_{L^\infty}\|u\|_{L^2_{y,\ell}(W^{1,\infty}_x)}\Big)\|g\|_{L^\infty_y H^2_x} \\
&\leq C\Big(\|\partial_y u\|_{H^{2,0}_\ell}\|u\|_{L^\infty} + \|\partial_y u\|_{L^\infty}\|u\|_{H^{2,0}_\ell}\Big)\|\tilde{h}\|_{H^{3,0}_\ell} \\
&\leq CD(t)^{\frac{1}{2}}E(t).
\end{aligned}$$

It is easy to check that

$$[T_{\frac{\partial_y u}{h}}T_h - T_{\frac{\partial_y u}{h}h}] = [T_{\frac{\partial_y u}{h}}T_{\tilde{h}} - T_{\frac{\partial_y u}{h}\tilde{h}}],\ [T_{\frac{\partial_y h}{h}}T_h - T_{\frac{\partial_y h}{h}h}] = [T_{\frac{\partial_y h}{h}}T_{\tilde{h}} - T_{\frac{\partial_y h}{h}\tilde{h}}].$$

Using Lemmas 2.2, 2.4, and 4.1–4.3 again, we attain

$$\begin{aligned}
\|G_{11}\|_{H^{3,0}_\ell} &\triangleq \|[T_{\frac{\partial_y u}{h}}T_{\tilde{h}} - T_{\frac{\partial_y u}{h}\tilde{h}}]v\|_{H^{3,0}_\ell} \\
&\leq C\Big(\|\partial_y u\|_{L^2_{y,\ell}(W^{1,\infty}_x)}\|\tilde{h}\|_{L^\infty} + \|\partial_y u\|_{L^\infty}\|\tilde{h}\|_{L^2_{y,\ell}(W^{1,\infty}_x)}\Big)\|v\|_{L^\infty_y H^2_x} \\
&\leq C\Big(\|\partial_y u\|_{H^{2,0}_\ell}\|\tilde{h}\|_{L^\infty} + \|\partial_y u\|_{L^\infty}\|\tilde{h}\|_{H^{2,0}_\ell}\Big)\|u\|_{H^{3,0}_\ell} \\
&\leq CD(t)^{\frac{1}{2}}E(t).
\end{aligned}$$

We now apply Lemmas 2.2, 2.4, and 4.1–4.3 again, with G_{11} replaced by G_{13}, to obtain

$$\begin{aligned}
\|G_{13}\|_{H^{3,0}_\ell} &\triangleq \|[T_h T_{\frac{\partial_y \tilde{h}}{h}} - T_{h\frac{\partial_y u}{h}}]v\|_{H^{3,0}_\ell} \\
&\leq C\Big(\|\partial_y \tilde{h}\|_{H^{2,0}_\ell}\|\tilde{h}\|_{L^\infty} + \|\partial_y \tilde{h}\|_{L^\infty}\|\tilde{h}\|_{H^{2,0}_\ell}\Big)\|u\|_{H^{3,0}_\ell} \\
&\leq CD(t)^{\frac{1}{2}}E(t).
\end{aligned}$$

Recall Lemmas 2.1 and 4.1–4.4; we derive

$$\begin{aligned}
\|G_{15}\|_{H^{3,0}_\ell} &\triangleq 2\|T_{\partial_y(\frac{\partial_y u}{h})}\tilde{h}\|_{H^{3,0}_\ell} \\
&\leq 2\|T_{\frac{\partial_y^2 u}{h}}\tilde{h}\|_{H^{3,0}_\ell} + 2\|T_{\frac{\partial_y u \partial_y \tilde{h}}{h}}\tilde{h}\|_{H^{3,0}_\ell} \\
&\leq C(\|\partial_y^2 u\|_{L^\infty} + \|\partial_y u\|_{L^\infty}\|\partial_y \tilde{h}\|_{L^\infty})\|\tilde{h}\|_{H^{3,0}_\ell} \\
&\leq CD(t)^{\frac{1}{2}}E(t)^{\frac{1}{2}} + CD(t)^{\frac{1}{2}}E(t).
\end{aligned}$$

By Lemmas 2.1 and 4.1–4.4, we have

$$\begin{aligned}
\|G_{17}\|_{H^{3,0}_\ell} &\triangleq \|T_h T_{\partial_x\frac{\partial_y h}{h}}h_1\|_{H^{3,0}_\ell} \leq \|h\|_{L^\infty}\|(\partial_x\partial_y\tilde{h} + \partial_x\tilde{h}\partial_y\tilde{h})\|_{L^\infty_x L^2_{y,\ell}}\|h_1\|_{H^3_x L^\infty_y} \\
&\leq C(\|\partial_y\tilde{h}\|_{H^{2,0}_\ell} + \|\partial_y\tilde{h}\|_{L^\infty}\|\tilde{h}\|_{H^{2,0}_\ell})\|\tilde{h}\|_{H^{3,0}_\ell} \\
&\leq CD(t)^{\frac{1}{2}}E(t)^{\frac{1}{2}} + CD(t)^{\frac{1}{2}}E(t).
\end{aligned}$$

In the same manner, we can obtain

$$\|G_{16}\|_{H^{3,0}_\ell} \triangleq \|T_h T_{\partial_x\frac{\partial_y u}{h}}h_1\|_{H^{3,0}_\ell} \leq CD(t)^{\frac{1}{2}}E(t)^{\frac{1}{2}} + CD(t)^{\frac{1}{2}}E(t).$$

Next, we will deal with the term G_{14}. From Eq (1.2), we can conclude that

$$
\begin{aligned}
&(\partial_t-\partial_y^2)(\frac{\partial_y u}{h}) = \frac{(\partial_t-\partial_y^2)\partial_y u}{h} - \frac{\partial_y u(\partial_t-\partial_y^2)h}{h^2} + \frac{2h\partial_y h\partial_y^2 u - 2\partial_y u(\partial_y h)^2}{h^3}\\
&= \frac{h\partial_{xy}^2\tilde{h} + g\partial_y^2\tilde{h} - u\partial_{xy}^2 u - v\partial_y^2 u}{h} - \frac{\partial_y u\partial_y(ug-vh)}{h^2} + \frac{2h\partial_y\tilde{h}\partial_y^2 u - 2\partial_y u(\partial_y\tilde{h})^2}{h^3}\\
&=: A_1 + A_2 + A_3.
\end{aligned}
$$

Using Lemmas 2.1 and 4.1–4.4, we are led to

$$
\begin{aligned}
&\|T_{A_1}h_1\|_{H_\ell^{3,0}}\\
\le\ & \|T_{\partial_{xy}^2\tilde{h}}h_1\|_{H_\ell^{3,0}} + \|T_{\frac{g\partial_y^2\tilde{h}}{h}}h_1\|_{H_\ell^{3,0}} + \|T_{\frac{u\partial_{xy}^2 u}{h}}h_1\|_{H_\ell^{3,0}} + \|T_{\frac{v\partial_y^2 u}{h}}h_1\|_{H_\ell^{3,0}}\\
\le\ & C\Big(\|\partial_y\tilde{h}\|_{H_\ell^{2,0}} + \|\tilde{h}\|_{H_\ell^{2,0}}\|\partial_y^2\tilde{h}\|_{H_\ell^{1,0}} + \|u\|_{L^\infty}\|\partial_y u\|_{H_\ell^{2,0}} + \|u\|_{H_\ell^{2,0}}\|\partial_y^2 u\|_{H_\ell^{1,0}}\Big)\|\tilde{h}\|_{H_\ell^{3,0}}\\
\le\ & CD(t)^{\frac12}E(t)^{\frac12} + CD(t)^{\frac12}E(t).
\end{aligned}
$$

In the same way, we deduce

$$
\begin{aligned}
&\|T_{A_2}h_1\|_{H_\ell^{3,0}}\\
\le\ & \|T_{\frac{g(\partial_y u)^2}{h^2}}h_1\|_{H_\ell^{3,0}} + \|T_{\frac{u\partial_y u\partial_x u}{h^2}}h_1\|_{H_\ell^{3,0}} + \|T_{\frac{\partial_y u\partial_x u}{h}}h_1\|_{H_\ell^{3,0}} + \|T_{\frac{v\partial_y u\partial_y\tilde{h}}{h^2}}h_1\|_{H_\ell^{3,0}}\\
\le\ & C\Big(\|\tilde{h}\|_{H_\ell^{2,0}}\|\partial_y u\|_{H_\ell^{1,0}}^2 + \|u\|_{L^\infty}\|\partial_y u\|_{L^\infty}\|u\|_{H_\ell^{2,0}}\\
&\quad + \|u\|_{H_\ell^{2,0}}\|\partial_y u\|_{L^\infty} + \|u\|_{H_\ell^{2,0}}\|\partial_y u\|_{L^\infty}\|\partial_y\tilde{h}\|_{H_\ell^{1,0}}\Big)\|\tilde{h}\|_{H_\ell^{3,0}}\\
\le\ & CD(t)^{\frac12}E(t)^{\frac32} + CD(t)^{\frac12}E(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&\|T_{A_3}h_1\|_{H_\ell^{3,0}}\\
\le\ & C\|T_{\frac{\partial_y\tilde{h}\partial_y^2 u}{h^2}}h_1\|_{H_\ell^{3,0}} + \|T_{\frac{\partial_y u(\partial_y\tilde{h})^2}{h^3}}h_1\|_{H_\ell^{3,0}}\\
\le\ & C\Big(\|\partial_y\tilde{h}\|_{H_\ell^{1,0}}\|\partial_y^2 u\|_{L^\infty} + \|\partial_y\tilde{h}\|_{H_\ell^{1,0}}^2\|\partial_y u\|_{L^\infty}\Big)\|\tilde{h}\|_{H_\ell^{3,0}}\\
\le\ & CD(t)^{\frac12}E(t)^{\frac32} + CD(t)^{\frac12}E(t).
\end{aligned}
$$

Thus, we can derive that

$$
\|G_{14}\|_{H_\ell^{3,0}} \triangleq \|T_{(\partial_t-\partial_y^2)(\frac{\partial_y u}{h})}h_1\|_{H_\ell^{3,0}} \le CD(t)^{\frac12}E(t)^{\frac32} + CD(t)^{\frac12}E(t) + CD(t)^{\frac12}E(t)^{\frac12}.
$$

Applying Lemmas 2.1, 2.4, and 4.1–4.5, we are led to

$$
\begin{aligned}
\|f_2\|_{H_\ell^{3,0}} \le\ & \|R_v\partial_y\tilde{h}\|_{H_\ell^{3,0}} + \|R_{\partial_x\tilde{h}}u\|_{H_\ell^{3,0}} + \|R_{\partial_x u}\tilde{h}\|_{H_\ell^{3,0}} + \|R_g\partial_y u\|_{H_\ell^{3,0}}\\
\le\ & \|u\|_{H_\ell^{2,0}}\|\partial_y\tilde{h}\|_{H_\ell^{3,0}} + \|\partial_x\tilde{h}\|_{L^\infty}\|u\|_{H_\ell^{3,0}}\\
&\quad + \|\partial_x u\|_{L^\infty}\|\tilde{h}\|_{H_\ell^{3,0}} + \|\tilde{h}\|_{H_\ell^{2,0}}\|\partial_y u\|_{H_\ell^{3,0}}
\end{aligned}
$$

$$\leq \quad CD(t)^{\frac{1}{2}}E(t)^{\frac{1}{2}} + CD(t)^{\frac{1}{4}}E(t)^{\frac{3}{4}},$$

which implies

$$\begin{aligned}\|G_{18}\|_{H^{3,0}_\ell} &= \left\|T_{\frac{\partial_y u}{h}}\int_0^y f_2 d\tilde{y}\right\|_{H^{3,0}_\ell} \leq C\|\partial_y u\|_{H^{1,0}_\ell}\|f_2\|_{H^{3,0}_\ell}\\ &\leq CD(t)^{\frac{1}{2}}E(t) + CD(t)^{\frac{1}{4}}E(t)^{\frac{5}{4}}.\end{aligned}$$

Analogously, we obtain

$$\|G_{19}\|_{H^{3,0}_\ell} \leq CD(t)^{\frac{1}{2}}E(t)^{\frac{1}{2}} + CD(t)^{\frac{1}{4}}E(t)^{\frac{3}{4}}.$$

Adding all the above estimates together gives the desired result.

The proof is now complete. □

4.3. *Tangential energy estimate*

In this subsection, we show the high-order derivative estimates of the solutions in the horizontal variable x.

Lemma 4.7. *Let $\ell \geq \frac{3}{2}$, it holds that*

$$\begin{aligned}&\frac{d}{dt}\|(u_\beta, \tilde{h}_\beta)\|^2_{H^{3,0}_\ell} + \|(\partial_y u_\beta, \partial_y \tilde{h}_\beta)\|^2_{H^{3,0}_\ell}\\ &\leq CD(t)^{\frac{1}{2}}E(t)^{\frac{1}{2}} + CD(t)^{\frac{1}{2}}E(t) + CD(t)^{\frac{1}{2}}E(t)^{\frac{3}{2}} + CD(t)^{\frac{1}{2}}E(t)^2\\ &\quad + CD(t)^{\frac{1}{4}}E(t)^{\frac{5}{4}} + CD(t)^{\frac{1}{4}}E(t)^{\frac{7}{4}}.\end{aligned}$$

Proof. Multiplying Eq (3.3) by $(u_\beta, \tilde{h}_\beta)$ in $H^{3,0}_\ell$, respectively, we derive

$$\begin{aligned}&(\partial_t u_\beta, u_\beta)_{H^{3,0}_\ell} + (\partial_t \tilde{h}_\beta, \tilde{h}_\beta)_{H^{3,0}_\ell} - (\partial_y^2 u_\beta, u_\beta)_{H^{3,0}_\ell} - (\partial_y^2 \tilde{h}_\beta, \tilde{h}_\beta)_{H^{3,0}_\ell}\\ &+(T_u\partial_x u_\beta, u_\beta)_{H^{3,0}_\ell} - (T_h\partial_x \tilde{h}_\beta, u_\beta)_{H^{3,0}_\ell} - (T_h\partial_x u_\beta, \tilde{h}_\beta)_{H^{3,0}_\ell} + (T_u\partial_x \tilde{h}_\beta, \tilde{h}_\beta)_{H^{3,0}_\ell}\\ &= (G_1, u_\beta)_{H^{3,0}_\ell} + (G_2, \tilde{h}_\beta)_{H^{3,0}_\ell}.\end{aligned}$$

First of all, using (3.6) and integrating it by parts, we obtain

$$\begin{aligned}&(\partial_t u_\beta, u_\beta)_{H^{3,0}_\ell} + (\partial_t \tilde{h}_\beta, \tilde{h}_\beta)_{H^{3,0}_\ell} - (\partial_y^2 u_\beta, u_\beta)_{H^{3,0}_\ell} - (\partial_y^2 \tilde{h}_\beta, \tilde{h}_\beta)_{H^{3,0}_\ell}\\ =\ &\frac{1}{2}\frac{d}{dt}\|(u_\beta, \tilde{h}_\beta)\|_{H^{3,0}_\ell} + \|(\partial_y u_\beta, \partial_y \tilde{h}_\beta)\|_{H^{3,0}_\ell}\\ &\quad +2\ell(\partial_y \tilde{h}_\beta, \tilde{h}_\beta)_{H^{3,0}_{\ell-\frac{1}{2}}} + 2\ell(\partial_y u_\beta, u_\beta)_{H^{3,0}_{\ell-\frac{1}{2}}}\\ =\ &\frac{1}{2}\frac{d}{dt}\|(u_\beta, \tilde{h}_\beta)\|_{H^{3,0}_\ell} + \|(\partial_y u_\beta, \partial_y \tilde{h}_\beta)\|_{H^{3,0}_\ell}\\ &\quad -\ell(2\ell-1)\|(u_\beta, \tilde{h}_\beta)\|^2_{H^{3,0}_{\ell-1}} - \ell(2\ell-1)\|\tilde{h}_\beta|_{y=0}\|^2_{H^{3,0}}\\ \geq\ &\frac{1}{2}\frac{d}{dt}\|(u_\beta, \tilde{h}_\beta)\|^2_{H^{3,0}_\ell} + \|(\partial_y u_\beta, \partial_y \tilde{h}_\beta)\|^2_{H^{3,0}_\ell}.\end{aligned}$$

An easy computation shows that

$$
\begin{aligned}
(T_u\partial_x u_\beta, u_\beta)_{H^{3,0}_\ell} &= \sum_{m=0}^{3}(\partial_x^k T_u\partial_x u_\beta, \partial_x^k u_\beta)_{L^2_\ell} \\
&= \sum_{m=0}^{3}(T_u\partial_x^k\partial_x u_\beta, \partial_x^k u_\beta)_{L^2_\ell} + \sum_{m=1}^{3}([\partial_x^k, T_u]\partial_x u_\beta, \partial_x^k u_\beta)_{L^2_\ell} \\
&=: D + \sum_{m=1}^{3}([\partial_x^k, T_u]\partial_x u_\beta, \partial_x^k u_\beta)_{L^2_\ell},
\end{aligned}
$$

where

$$
\begin{aligned}
D &= -\frac{1}{2}\sum_{m=0}^{3}(T_{\partial_x u}\partial_x^k u_\beta, \partial_x^k u_\beta)_{L^2_\ell} + \frac{1}{2}\sum_{m=0}^{3}\Big((T_u - T_u^*)\partial_x^k\partial_x u_\beta, \partial_x^k u_\beta\Big)_{L^2_\ell} \\
&=: D_1 + D_2.
\end{aligned}
$$

By Lemmas 2.1, 2.2, 4.3, and 4.5, we conclude

$$
\begin{aligned}
\left|D_1\right| &\leq \sum_{m=0}^{3}\|T_{\partial_x u}\partial_x^k u_\beta\|_{H^{3,0}_\ell}\|\partial_x^k u_\beta\|_{H^{3,0}_\ell} \leq C\|\partial_x u\|_{L^\infty}\|\partial_x^k u_\beta\|^2_{H^{3,0}_\ell} \\
&\leq CD(t)^{\frac{1}{2}}E(t)
\end{aligned}
$$

and

$$
\begin{aligned}
\left|D_2\right| &\leq \sum_{k=0}^{3}\|(T_u - T_u^*)\partial_x^k\partial_x u_\beta\|_{L^2_\ell}\|\partial_x^k u_\beta\|_{L^2_\ell} \\
&\leq \sum_{k=0}^{3}\|u\|_{W^{1,\infty}_x L^\infty_y}\|\partial_x^k u_\beta\|^2_{L^2_\ell} \\
&\leq CD(t)^{\frac{1}{2}}E(t)
\end{aligned}
$$

and by Lemmas 2.3 and 4.3 gives,

$$
\sum_{k=1}^{3}([\partial_x^k, T_u]\partial_x u_\beta, \partial_x^k u_\beta)_{L^2_\ell} \leq C\|\partial_x u\|_{L^\infty}\|u_\beta\|^2_{H^{3,0}_\ell} \leq CD(t)^{\frac{1}{2}}E(t).
$$

Thus, we have

$$
(T_u\partial_x u_\beta, u_\beta)_{H^{3,0}_\ell} \leq CD(t)^{\frac{1}{2}}E(t).
$$

Likewise, we can also obtain

$$
\begin{aligned}
(T_u\partial_x\tilde{h}_\beta, \tilde{h}_\beta)_{H^{3,0}_\ell} &\leq CD(t)^{\frac{1}{2}}E(t), \\
(T_h\partial_x u_\beta, u_\beta)_{H^{3,0}_\ell} &\leq CD(t)^{\frac{1}{4}}E(t)^{\frac{5}{4}},
\end{aligned}
$$

$$(T_h\partial_x\tilde{h}_\beta, \tilde{h}_\beta)_{H^{3,0}_\ell} \leq CD(t)^{\frac{1}{4}}E(t)^{\frac{5}{4}}.$$

A simple calculation yields

$$\begin{aligned}
&(T_h\partial_x\tilde{h}_\beta, u_\beta)_{H^{3,0}_\ell} + (T_h\partial_x u_\beta, \tilde{h}_\beta)_{H^{3,0}_\ell} \\
&= (T_h\partial_x(\tilde{h}_\beta + u_\beta), (\tilde{h}_\beta + u_\beta))_{H^{3,0}_\ell} - (T_h\partial_x u_\beta, u_\beta)_{H^{3,0}_\ell} - (T_h\partial_x\tilde{h}_\beta, \tilde{h}_\beta)_{H^{3,0}_\ell}.
\end{aligned}$$

Therefore, we also obtain

$$(T_h\partial_x\tilde{h}_\beta, u_\beta)_{H^{3,0}_\ell} + (T_h\partial_x u_\beta, \tilde{h}_\beta)_{H^{3,0}_\ell} \leq CD(t)^{\frac{1}{4}}E(t)^{\frac{5}{4}}.$$

It follows from Lemma 4.6 that

$$\begin{aligned}
&(G_1, u_\beta)_{H^{3,0}_\ell} + (G_2, \tilde{h}_\beta)_{H^{3,0}_\ell} \\
&\leq \Big(CD(t)^{\frac{1}{2}}E(t)^{\frac{1}{2}} + CD(t)^{\frac{1}{2}}E(t) + CD(t)^{\frac{1}{2}}E(t)^{\frac{3}{2}} \\
&\quad + CD(t)^{\frac{1}{4}}E(t)^{\frac{3}{4}} + CD(t)^{\frac{1}{4}}E(t)^{\frac{5}{4}}\Big)\|(u_\beta, \tilde{h}_\beta)\|_{H^{3,0}_\ell} \\
&\leq CD(t)^{\frac{1}{2}}E(t) + CD(t)^{\frac{1}{2}}E(t)^{\frac{3}{2}} + CD(t)^{\frac{1}{2}}E(t)^2 \\
&\quad + CD(t)^{\frac{1}{4}}E(t)^{\frac{5}{4}} + CD(t)^{\frac{1}{4}}E(t)^{\frac{7}{4}}.
\end{aligned}$$

Summing up all the above estimates, we deduce

$$\begin{aligned}
&\frac{d}{dt}\|(u_\beta, \tilde{h}_\beta)\|^2_{H^{3,0}_\ell} + \|(\partial_y u_\beta, \partial_y\tilde{h}_\beta)\|^2_{H^{3,0}_\ell} \\
&\leq CD(t)^{\frac{1}{2}}E(t) + CD(t)^{\frac{1}{2}}E(t)^{\frac{3}{2}} + CD(t)^{\frac{1}{2}}E(t)^2 \\
&\quad + CD(t)^{\frac{1}{4}}E(t)^{\frac{5}{4}} + CD(t)^{\frac{1}{4}}E(t)^{\frac{7}{4}}.
\end{aligned}$$

The proof is then complete. □

5. High order energy estimate in y variable

In Lemma 4.7, we just prove the high-order derivative estimates of the solutions in the horizontal variable x. To make the energy estimate more complete, we need to derive the high-order derivative estimates in variable y. We again assume that $(u, \tilde{h})$ is a smooth solution of (1.2) on $[0, T^*]$ satisfying (4.1).

Lemma 5.1. *Let $\ell \geq \frac{3}{2}$, it holds that for any $t \in [0, T^*]$,*

$$\begin{aligned}
&\frac{d}{dt}\Big(\sum_{m=0}^{2}\|(\partial_y^m u, \partial_y^m\tilde{h})\|^2_{H^{1,0}_{\ell-1+m}}\Big) + \sum_{m=0}^{2}\|(\partial_y^{m+1} u, \partial_y^{m+1}\tilde{h})\|^2_{H^{1,0}_{\ell-1+m}} \\
&\leq CD(t)^{\frac{1}{4}}E(t)^{\frac{5}{4}} + CE(t)^{\frac{3}{2}} + CD(t)^{\frac{1}{2}}E(t) + CD(t)^{\frac{1}{2}}E(t)^{\frac{1}{2}}.
\end{aligned}$$

Proof. The proof will be divided into the following three steps.
Step 1. ($H^{1,0}_{\ell-1}$ estimate) Taking $H^{1,0}_{\ell-1}$-inner product between (1.2) with $(\partial_y u, \partial_y \tilde{h})$, we obtain

$$\begin{aligned}
&(\partial_t u, u)_{H^{1,0}_{\ell-1}} + (\partial_t \tilde{h}, \tilde{h})_{H^{1,0}_{\ell-1}} - (\partial_y^2 u, u)_{H^{1,0}_{\ell-1}} - (\partial_y^2 \tilde{h}, \tilde{h})_{H^{1,0}_{\ell-1}} \\
&= -(u\partial_x u, u)_{H^{1,0}_{\ell-1}} - (u\partial_x \tilde{h}, \tilde{h})_{H^{1,0}_{\ell-1}} + (h\partial_x \tilde{h}, u)_{H^{1,0}_{\ell-1}} + (h\partial_x u, \tilde{h})_{H^{1,0}_{\ell-1}} \\
&\quad -(v\partial_y u, u)_{H^{1,0}_{\ell-1}} + (g\partial_y \tilde{h}, u)_{H^{1,0}_{\ell-1}} - (v\partial_y \tilde{h}, \tilde{h})_{H^{1,0}_{\ell-1}} + (g\partial_y u, \tilde{h})_{H^{1,0}_{\ell-1}} \\
&=: A_1 + B_1.
\end{aligned}$$

First of all, integrating it by parts and using the Hölder inequality ([19], Theorem 1.4.3), we deduce

$$\begin{aligned}
&(\partial_t u, u)_{H^{1,0}_{\ell-1}} + (\partial_t \tilde{h}, \tilde{h})_{H^{1,0}_{\ell-1}} - (\partial_y^2 u, u)_{H^{1,0}_{\ell-1}} - (\partial_y^2 \tilde{h}, \tilde{h})_{H^{1,0}_{\ell-1}} \\
= \;& \frac{1}{2}\frac{d}{dt}\|(u, \tilde{h})\|_{H^{1,0}_{\ell-1}} + \|(\partial_y u, \partial_y \tilde{h})\|_{H^{1,0}_{\ell-1}} \\
&-2(\ell-1)(\partial_y u, u)_{H^{1,0}_{\ell-1}} - 2(\ell-1)(\partial_y \tilde{h}, \tilde{h})_{H^{1,0}_{\ell-1}} \\
&\geq \frac{1}{2}\frac{d}{dt}\|(u, \tilde{h})\|_{H^{1,0}_{\ell-1}} + \|(\partial_y u, \partial_y \tilde{h})\|_{H^{1,0}_{\ell-1}} - C\|u\|_{H^{1,0}_{\ell-1}}\|\partial_y u\|_{H^{1,0}_{\ell-1}} - C\|\tilde{h}\|_{H^{1,0}_{\ell-1}}\|\partial_y \tilde{h}\|_{H^{1,0}_{\ell-1}}.
\end{aligned}$$

By integration by parts and applying Lemma 4.3, we have

$$\begin{aligned}
&(u\partial_x u, u)_{H^{1,0}_{\ell-1}} = (u\partial_x u, u)_{L^2_{\ell-1}} + \frac{1}{2}(\partial_x u \partial_x u, \partial_x u)_{L^2_{\ell-1}} \\
&\leq C\|\partial_x u\|_{L^\infty}\|u\|^2_{H^{1,0}_{\ell-1}} \leq CD^{\frac{1}{2}}E(t).
\end{aligned}$$

It infers from Lemmas 4.3 and 4.5 that

$$\begin{aligned}
&(h\partial_x \tilde{h}, u)_{H^{1,0}_{\ell-1}} + (h\partial_x u, \tilde{h})_{H^{1,0}_{\ell-1}} \\
&= (h\partial_x \tilde{h}, u)_{L^2_{\ell-1}} + (h\partial_x u, \tilde{h})_{L^2_{\ell-1}} + (\partial_x \tilde{h}\partial_x \tilde{h}, \partial_x u)_{L^2_{\ell-1}} \\
&\leq CD^{\frac{1}{4}}E(t)^{\frac{5}{4}} + CD^{\frac{1}{2}}E(t).
\end{aligned}$$

In the same way, we can obtain

$$(u\partial_x \tilde{h}, \tilde{h})_{H^{1,0}_{\ell-1}} \leq CD^{\frac{1}{4}}E(t)^{\frac{5}{4}}.$$

This shows that

$$|A_1| \leq CD(t)^{\frac{1}{4}}E(t)^{\frac{5}{4}} + CD^{\frac{1}{2}}E(t).$$

In view of $v = \displaystyle\int_0^y \partial_x u d\tilde{y}$, from Lemmas 4.2 and 4.4, we can derive that

$$(v\partial_y u, u)_{H^{1,0}_{\ell-1}} \leq \|v\|_{W^{1,\infty}_x L^\infty_y}\|\partial_y u\|_{H^{1,0}_{\ell-1}} \leq C\|u\|_{H^{2,0}_{\ell-1}}\|\partial_y u\|_{H^{1,0}_{\ell-1}}\|u\|_{H^{1,0}_{\ell-1}} \leq CD(t)^{\frac{1}{2}}E(t).$$

The estimate of B_1 could be deduced in a similar way. Then we have

$$|B_1| \leq CD(t)^{\frac{1}{2}}E(t).$$

Therefore, we conclude that

$$\frac{d}{dt}\|(u,\tilde{h})\|^2_{H^{1,0}_{\ell-1}}+\|(\partial_y u,\partial_y\tilde{h})\|^2_{H^{1,0}_{\ell-1}}\le CD(t)^{\frac14}E(t)^{\frac54}+CD(t)^{\frac12}E(t)^{\frac12}+CD(t)^{\frac12}E(t). \tag{5.1}$$

Step 2.($H^{1,1}_\ell$ estimate) Taking ∂_y to (1.2) and then taking $H^{1,0}_\ell$-inner product with $(\partial_y u,\partial_y\tilde{h})$, we attain

$$\begin{aligned}
&\frac{d}{dt}\|(\partial_y u,\partial_y\tilde{h})\|^2_{H^{1,0}_\ell}+\|(\partial_y^2 u,\partial_y^2\tilde{h})\|^2_{H^{1,0}_\ell}\\
&\le -(u\partial_x\partial_y u,\partial_y u)_{H^{1,0}_\ell}-(u\partial_x\partial_y\tilde{h},\partial_y\tilde{h})_{H^{1,0}_\ell}+(h\partial_x\partial_y\tilde{h},\partial_y u)_{H^{1,0}_\ell}+(h\partial_x\partial_y u,\partial_y\tilde{h})_{H^{1,0}_\ell}\\
&\quad-(v\partial_y^2 u,\partial_y u)_{H^{1,0}_\ell}+(g\partial_y^2\tilde{h},\partial_y u)_{H^{1,0}_\ell}-(v\partial_y^2\tilde{h},\partial_y\tilde{h})_{H^{1,0}_\ell}+(g\partial_y^2 u,\partial_y\tilde{h})_{H^{1,0}_\ell}\\
&\quad-2(\partial_x\tilde{h}\partial_y u-\partial_x u\partial_y\tilde{h},\partial_y\tilde{h})_{H^{1,0}_\ell}\\
&\quad+C\|\partial_y^2 u\|_{H^{1,0}_\ell}\|\partial_y u\|_{H^{1,0}_\ell}+C\|\partial_y^2\tilde{h}\|_{H^{1,0}_\ell}\|\partial_y\tilde{h}\|_{H^{1,0}_\ell}\\
&=: A_2+B_2+C_2+D_2.
\end{aligned}$$

For the estimate of term A_2, by integration by parts and using Lemma 4.3, we have

$$\begin{aligned}
&(u\partial_x\partial_y u,\partial_y u)_{H^{1,0}_\ell}=(u\partial_x\partial_y u,\partial_y u)_{L^2_\ell}+\frac12(\partial_x u\partial_x\partial_y u,\partial_x\partial_y u)_{L^2_\ell}\\
&\le C\|\partial_x u\|_{L^\infty}\|\partial_y u\|^2_{H^{1,0}_\ell}\le CD(t)^{\frac12}E(t)
\end{aligned}$$

and

$$\begin{aligned}
&(h\partial_x\partial_y\tilde{h},\partial_y u)_{H^{1,0}_\ell}+(h\partial_x\partial_y u,\partial_y\tilde{h})_{H^{1,0}_\ell}\\
&=(h\partial_x\partial_y\tilde{h},\partial_y u)_{L^2_\ell}+(h\partial_x\partial_y u,\partial_y\tilde{h})_{L^2_\ell}+(\partial_x\tilde{h}\partial_x\partial_y\tilde{h},\partial_x\partial_y u)_{L^2_\ell}\\
&\le CD(t)^{\frac14}E(t)^{\frac54}+CD(t)^{\frac12}E(t).
\end{aligned}$$

On account of the above calculations, we can obtain

$$|A_2|\le CD(t)^{\frac14}E(t)^{\frac54}+CD(t)^{\frac12}E(t).$$

For the estimate of term B_2, using Lemma 4.2, we thus obtain

$$\begin{aligned}
(v\partial_y^2 u,\partial_y u)_{H^{1,0}_\ell}&=(v\partial_y^2 u,\partial_y u)_{L^2_\ell}+\left(\partial_x v\partial_y^2 u+v\partial_x\partial_y^2 u,\partial_x\partial_y u\right)_{L^2_\ell}\\
&\le C\|v\|_{H^1_xL^\infty_y}\|\partial_y^2 u\|_{H^{1,0}_\ell}\|\partial_y u\|_{H^{1,0}_\ell}\\
&\le C\|u\|_{H^{2,0}_\ell}\|\partial_y^2 u\|_{H^{1,0}_\ell}\|\partial_y u\|_{H^{1,0}_\ell}.
\end{aligned}$$

The remainder terms of B_2 can be handled in much the same way, which gives

$$\begin{aligned}
&(g\partial_y^2\tilde{h},\partial_y u)_{H^{1,0}_\ell}\le C\|\tilde{h}\|_{H^{2,0}_\ell}\|\partial_y^2\tilde{h}\|_{H^{1,0}_\ell}\|\partial_y u\|_{H^{1,0}_\ell},\\
&(v\partial_y^2\tilde{h},\partial_y\tilde{h})_{H^{1,0}_\ell}\le C\|u\|_{H^{2,0}_\ell}\|\partial_y^2\tilde{h}\|_{H^{1,0}_\ell}\|\partial_y\tilde{h}\|_{H^{1,0}_\ell},\\
&(g\partial_y^2 u,\partial_y\tilde{h})_{H^{1,0}_\ell}\le C\|\tilde{h}\|_{H^{2,0}_\ell}\|\partial_y^2 u\|_{H^{1,0}_\ell}\|\partial_y\tilde{h}\|_{H^{1,0}_\ell}.
\end{aligned}$$

Therefore, we obtain

$$|B_2| \le CD(t)^{\frac{1}{2}} E(t).$$

For the estimate of term C_2, we have

$$\begin{aligned}|C_2| &\le \|(\partial_x \tilde{h} \partial_y u - \partial_x u \partial_y \tilde{h})\|_{H^{1,0}_\ell} \|\partial_y \tilde{h}\|_{H^{1,0}_\ell} \\ &\le C\Big(\|\tilde{h}\|_{H^{1,0}_\ell} \|\partial_y u\|_{L^\infty} + \|u\|_{H^{1,0}_\ell} \|\partial_y \tilde{h}\|_{L^\infty}\Big) \|\partial_y \tilde{h}\|_{H^{1,0}_\ell}.\end{aligned}$$

Thanks to Lemmas 4.3 and 4.4, we conclude

$$|C_2| \le CD(t)^{\frac{1}{2}} E(t).$$

It is obvious that

$$|D_2| \le CD(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}}.$$

Consequently, we deduce

$$\frac{d}{dt}\|(\partial_y u, \partial_y \tilde{h})\|^2_{H^{1,0}_\ell} + \|(\partial_y^2 u, \partial_y^2 \tilde{h})\|^2_{H^{1,0}_\ell} \le CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}} + CD(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}} + CD(t)^{\frac{1}{2}} E(t). \tag{5.2}$$

Step 3. ($H^{1,2}_{\ell+1}$ estimate) Taking ∂_y^2 to (1.2) and then taking $H^{1,0}_{\ell+1}$-inner product with $(\partial_y^2 u, \partial_y^2 \tilde{h})$, we attain

$$\begin{aligned}&\frac{d}{dt}\|(\partial_y^2 u, \partial_y^2 \tilde{h})\|^2_{H^{1,0}_{\ell+1}} + \|(\partial_y^3 u, \partial_y^3 \tilde{h})\|^2_{H^{1,0}_{\ell+1}} \\ &\le -(u\partial_x \partial_y^2 u, \partial_y^2 u)_{H^{1,0}_{\ell+1}} - (u\partial_x \partial_y^2 \tilde{h}, \partial_y^2 \tilde{h})_{H^{1,0}_{\ell+1}} + (h\partial_x \partial_y^2 \tilde{h}, \partial_y^2 u)_{H^{1,0}_{\ell+1}} + (h\partial_x \partial_y^2 u, \partial_y^2 \tilde{h})_{H^{1,0}_{\ell+1}} \\ &\quad -(v\partial_y^3 u, \partial_y^2 u)_{H^{1,0}_{\ell+1}} + (g\partial_y^3 \tilde{h}, \partial_y^2 u)_{H^{1,0}_{\ell+1}} - (v\partial_y^3 \tilde{h}, \partial_y^2 \tilde{h})_{H^{1,0}_{\ell+1}} + (g\partial_y^3 u, \partial_y^2 \tilde{h})_{H^{1,0}_{\ell+1}} \\ &\quad -(\partial_y u \partial_x \partial_y u, \partial_y^2 u)_{H^{1,0}_{\ell+1}} - 3(\partial_y u \partial_x \partial_y \tilde{h}, \partial_y^2 \tilde{h})_{H^{1,0}_{\ell+1}} + (\partial_y \tilde{h} \partial_x \partial_y \tilde{h}, \partial_y^2 u)_{H^{1,0}_{\ell+1}} + 3(\partial_y \tilde{h} \partial_x \partial_y u, \partial_y^2 \tilde{h})_{H^{1,0}_{\ell+1}} \\ &\quad -(\partial_y v \partial_y^2 u, \partial_y^2 u)_{H^{1,0}_{\ell+1}} + (\partial_y g \partial_y^2 \tilde{h}, \partial_y^2 u)_{H^{1,0}_{\ell+1}} - 3(\partial_y v \partial_y^2 \tilde{h}, \partial_y^2 \tilde{h})_{H^{1,0}_{\ell+1}} + 3(\partial_y g \partial_y^2 u, \partial_y^2 \tilde{h})_{H^{1,0}_{\ell+1}} \\ &\quad +C\|\partial_y^3 u\|_{H^{1,0}_{\ell+1}} \|\partial_y^2 u\|_{H^{1,0}_{\ell+1}} + C\|\partial_y^3 \tilde{h}\|_{H^{1,0}_{\ell+1}} \|\partial_y^2 \tilde{h}\|_{H^{1,0}_{\ell+1}} \\ &=: A_3 + B_3 + C_3 + D_3 + E_3.\end{aligned}$$

We establish the estimates of the nonlinear terms as follows:

For the estimates of the term A_3, a direct calculation yields

$$(u\partial_x \partial_y^2 u, \partial_y^2 u)_{H^{1,0}_{\ell+1}} = (u\partial_x \partial_y^2 u, \partial_y^2 u)_{L^2_{\ell+1}} + \frac{1}{2}(\partial_x u \partial_x \partial_y^2 u, \partial_x \partial_y^2 u)_{L^2_{\ell+1}},$$

$$(u\partial_x \partial_y^2 \tilde{h}, \partial_y^2 \tilde{h})_{H^{1,0}_{\ell+1}} = (u\partial_x \partial_y^2 \tilde{h}, \partial_y^2 \tilde{h})_{L^2_{\ell+1}} + \frac{1}{2}(\partial_x u \partial_x \partial_y^2 \tilde{h}, \partial_x \partial_y^2 \tilde{h})_{L^2_{\ell+1}}$$

and

$$(h\partial_x \partial_y^2 \tilde{h}, \partial_y^2 u)_{H^{1,0}_{\ell+1}} + (h\partial_x \partial_y^2 u, \partial_y^2 \tilde{h})_{H^{1,0}_{\ell+1}}$$

$$
\begin{aligned}
&= (h\partial_x\partial_y^2(\tilde{h}+u), \partial_y^2(\tilde{h}+u))_{H^{1,0}_{\ell+1}} - (h\partial_x\partial_y^2\tilde{h}, \partial_y^2\tilde{h})_{H^{1,0}_{\ell+1}} - (h\partial_x\partial_y^2 u, \partial_y^2 u)_{H^{1,0}_{\ell+1}}\\
&= (h\partial_x\partial_y^2(\tilde{h}+u), \partial_y^2(\tilde{h}+u))_{L^2_{\ell+1}} + \frac{1}{2}(\partial_x h\partial_x\partial_y^2(\tilde{h}+u), \partial_x\partial_y^2(\tilde{h}+u))_{L^2_{\ell+1}}\\
&\quad -(\tilde{h}\partial_x\partial_y^2\tilde{h}, \partial_y^2\tilde{h})_{L^2_{\ell+1}} - \frac{1}{2}(\partial_x\tilde{h}\partial_x\partial_y^2\tilde{h}, \partial_x\partial_y^2\tilde{h})_{L^2_{\ell+1}}\\
&\quad -(\tilde{h}\partial_x\partial_y^2 u, \partial_y^2 u)_{L^2_{\ell+1}} - \frac{1}{2}(\partial_x\tilde{h}\partial_x\partial_y^2 u, \partial_x\partial_y^2 u)_{L^2_{\ell+1}}.
\end{aligned}
$$

Therefore, applying the Hölder inequality ([19], Theorem 1.4.3), Lemmas 4.1, 4.3 and 4.5 for the above equality, we have

$$
\begin{aligned}
\left|A_3\right| &\le C\Big(\|(u,\tilde{h})\|_{L^\infty} + \|(\partial_x u, \partial_x\tilde{h})\|_{L^\infty}\Big)\Big(\|\partial_y^2 u\|^2_{H^{1,0}_{\ell+1}} + \|\partial_y^2\tilde{h}\|^2_{H^{1,0}_{\ell+1}}\Big)\\
&\le CD(t)^{\frac{1}{4}}E(t)^{\frac{5}{4}} + CD(t)^{\frac{1}{2}}E(t) + CE(t)^{\frac{3}{2}}.
\end{aligned}
$$

For the estimates of the term B_3, applying the Hölder inequality ([19], Theorem 1.4.3), Lemmas 2.4, 4.2 and 4.4, we can conclude that

$$
\begin{aligned}
\left|B_3\right| &\le C\Big(\|v\|_{W^{1,\infty}_x L^\infty_y} + \|g\|_{W^{1,\infty}_x L^\infty_y}\Big)\Big(\|\partial_y^3 u\|_{H^{1,0}_{\ell+1}} + \|\partial_y^3\tilde{h}\|_{H^{1,0}_{\ell+1}}\Big)\Big(\|\partial_y^2 u\|_{H^{1,0}_{\ell+1}} + \|\partial_y^2\tilde{h}\|_{H^{1,0}_{\ell+1}}\Big)\\
&\le CD(t)^{\frac{1}{2}}E(t).
\end{aligned}
$$

Analogously, using the Hölder inequality ([19], Theorem 1.4.3), Lemmas 2.4, 4.3 and 4.4, for $\ell \ge \frac{3}{2}$, we can deduce that

$$
\begin{aligned}
C_3 &= -(\partial_y u\partial_x\partial_y u, \partial_y^2 u)_{H^{1,0}_{\ell+1}} - 3(\partial_y u\partial_x\partial_y\tilde{h}, \partial_y^2\tilde{h})_{H^{1,0}_{\ell+1}} + (\partial_y\tilde{h}\partial_x\partial_y\tilde{h}, \partial_y^2 u)_{H^{1,0}_{\ell+1}} + 3(\partial_y\tilde{h}\partial_x\partial_y u, \partial_y^2\tilde{h})_{H^{1,0}_{\ell+1}}\\
&= (u\partial_x\partial_y^2 u, \partial_y^2 u)_{H^{1,0}_{\ell+1}} + (u\partial_x\partial_y u, \partial_y^3 u)_{H^{1,0}_{\ell+1}} + 2(\ell+1)(u\partial_x\partial_y u, \partial_y^2 u)_{H^{1,0}_{\ell+\frac{1}{2}}}\\
&\quad +3(u\partial_x\partial_y^2\tilde{h}, \partial_y^2\tilde{h})_{H^{1,0}_{\ell+1}} + 3(u\partial_x\partial_y\tilde{h}, \partial_y^3\tilde{h})_{H^{1,0}_{\ell+1}} + 6(\ell+1)(u\partial_x\partial_y\tilde{h}, \partial_y^2\tilde{h})_{H^{1,0}_{\ell+\frac{1}{2}}}\\
&\quad +(\partial_y\tilde{h}\partial_x\partial_y\tilde{h}, \partial_y^2 u)_{H^{1,0}_{\ell+1}} + 3(\partial_y\tilde{h}\partial_x\partial_y u, \partial_y^2\tilde{h})_{H^{1,0}_{\ell+1}}\\
&= (u\partial_x\partial_y^2 u, \partial_y^2 u)_{L^2_{\ell+1}} + \frac{1}{2}(\partial_x u\partial_x\partial_y^2 u, \partial_x\partial_y^2 u)_{L^2_{\ell+1}} + (u\partial_x\partial_y u, \partial_y^3 u)_{H^{1,0}_{\ell+1}} + 2(\ell+1)(u\partial_x\partial_y u, \partial_y^2 u)_{H^{1,0}_{\ell+\frac{1}{2}}}\\
&\quad +3(u\partial_x\partial_y^2\tilde{h}, \partial_y^2\tilde{h})_{L^2_{\ell+1}} + \frac{3}{2}(\partial_x u\partial_x\partial_y^2\tilde{h}, \partial_x\partial_y^2\tilde{h})_{L^2_{\ell+1}} + 3(u\partial_x\partial_y\tilde{h}, \partial_y^3\tilde{h})_{H^{1,0}_{\ell+1}} + 6(\ell+1)(u\partial_x\partial_y\tilde{h}, \partial_y^2\tilde{h})_{H^{1,0}_{\ell+\frac{1}{2}}}\\
&\quad +(\partial_y\tilde{h}\partial_x\partial_y\tilde{h}, \partial_y^2 u)_{H^{1,0}_{\ell+1}} + 3(\partial_y\tilde{h}\partial_x\partial_y u, \partial_y^2\tilde{h})_{H^{1,0}_{\ell+1}}\\
&\le \|u\|_{L^\infty}\|\partial_x\partial_y^2 u\|_{L^2_{\ell+1}}\|\partial_y^2 u\|_{L^2_{\ell+1}} + \frac{1}{2}\|\partial_x u\|_{L^\infty}\|\partial_x\partial_y^2 u\|^2_{L^2_{\ell+1}}\\
&\quad +\|\langle y\rangle u\|_{W^{1,\infty}_x L^\infty_y}\|\partial_x\partial_y u\|_{L^2_\ell}\|\partial_y^3 u\|_{H^{1,0}_{\ell+1}} + \|\langle y\rangle u\|_{L^\infty}\|\partial_x\partial_y u\|_{H^{1,0}_\ell}\|\partial_y^3 u\|_{H^{1,0}_{\ell+1}}\\
&\quad +2(\ell+1)\|u\|_{W^{1,\infty}_x L^\infty_y}\|\partial_x\partial_y u\|_{H^{1,0}_\ell}\|\partial_y^2 u\|_{H^{1,0}_{\ell+1}}\\
&\quad +3\|u\|_{L^\infty}\|\partial_x\partial_y^2\tilde{h}\|_{L^2_{\ell+1}}\|\partial_y^2\tilde{h}\|_{L^2_{\ell+1}} + \frac{3}{2}\|\partial_x u\|_{L^\infty}\|\partial_x\partial_y^2\tilde{h}\|^2_{L^2_{\ell+1}}\\
&\quad +3\|\langle y\rangle u\|_{W^{1,\infty}_x L^\infty_y}\|\partial_x\partial_y\tilde{h}\|_{L^2_\ell}\|\partial_y^3\tilde{h}\|_{H^{1,0}_{\ell+1}} + 3\|\langle y\rangle u\|_{L^\infty(\mathbb{R}^2_+)}\|\partial_x\partial_y\tilde{h}\|_{H^{1,0}_\ell}\|\partial_y^3\tilde{h}\|_{H^{1,0}_{\ell+1}}\\
&\quad +6(\ell+1)\|u\|_{W^{1,\infty}_x L^\infty_y}\|\partial_x\partial_y\tilde{h}\|_{H^{1,0}_\ell}\|\partial_y^2\tilde{h}\|_{H^{1,0}_{\ell+1}}
\end{aligned}
$$

$$
\begin{aligned}
&+\|\langle y\rangle\partial_y\tilde h\|_{L^\infty(\mathbb{R}^2_+)}\|\partial_x\partial_y\tilde h\|_{H^{1,0}_\ell}\|\partial_y^2u\|_{H^{1,0}_{\ell+1}}+\|\langle y\rangle\partial_y\tilde h\|_{H^1_xL^\infty_y}\|\partial_x\partial_y\tilde h\|_{L^\infty_xL^2_{y,\ell}}\|\partial_y^2u\|_{H^{1,0}_{\ell+1}}\\
&+3\|\langle y\rangle\partial_y\tilde h\|_{L^\infty(\mathbb{R}^2_+)}\|\partial_x\partial_yu\|_{H^{1,0}_\ell}\|\partial_y^2\tilde h\|_{H^{1,0}_{\ell+1}}+3\|\langle y\rangle\partial_y\tilde h\|_{H^1_xL^\infty_y}\|\partial_x\partial_yu\|_{L^\infty_xL^2_{y,\ell}}\|\partial_y^2\tilde h\|_{H^{1,0}_{\ell+1}}\\
\leq\ & CE(t)^{\frac32}+CD(t)^{\frac12}E(t),
\end{aligned}
$$

where we used the following facts:

$$
\begin{aligned}
\|\langle y\rangle\partial_x^mu(t,x,\tilde y)\|_{L^\infty(\mathbb{R}^2_+)} &\leq \left|\int_0^y(\partial_x^mu(t,x,\tilde y)+\langle y\rangle\partial_x^m\partial_yu(t,x,\tilde y)d\tilde y\right|\\
&\leq \|\partial_x^mu(t,x,y)\|_{L^\infty_xL^2_{y,\frac12}}+\|\partial_x^m\partial_yu(t,x,y)\|_{L^\infty_xL^2_{y,\frac32}}\\
&\leq \|\partial_x^mu(t,x,y)\|^{\frac12}_{L^2_{\frac12}(\mathbb{R}^2_+)}\|\partial_x^{m+1}u(t,x,y)\|^{\frac12}_{L^2_{\frac12}(\mathbb{R}^2_+)}\\
&\quad+\|\partial_x^m\partial_yu(t,x,y)\|^{\frac12}_{L^2_{\frac32}(\mathbb{R}^2_+)}\|\partial_x^{m+1}\partial_yu(t,x,y)\|^{\frac12}_{L^2_{\frac32}(\mathbb{R}^2_+)}\\
&\leq \|\partial_yu(t,x,y)\|_{H^{m+1,0}_{\frac32}(\mathbb{R}^2_+)},\ \text{for some } m\in\mathbb{N},
\end{aligned}
$$

$$
\begin{aligned}
\|\langle y\rangle\partial_y\tilde h(t,x,\tilde y)\|_{L^\infty(\mathbb{R}^2_+)} &\leq \left|\int_0^y(\partial_y\tilde h(t,x,\tilde y)+\langle y\rangle\partial_y^2\tilde h(t,x,\tilde y)d\tilde y\right|\\
&\leq \|\partial_y\tilde h(t,x,y)\|_{L^\infty_xL^2_{y,\frac12}}+\|\partial_y^2\tilde h(t,x,y)\|_{L^\infty_xL^2_{y,\frac32}}\\
&\leq \|\partial_y\tilde h(t,x,y)\|^{\frac12}_{L^2_{\frac12}(\mathbb{R}^2_+)}\|\partial_x\partial_y\tilde h(t,x,y)\|^{\frac12}_{L^2_{\frac12}(\mathbb{R}^2_+)}\\
&\quad+\|\partial_y^2\tilde h(t,x,y)\|^{\frac12}_{L^2_{\frac32}(\mathbb{R}^2_+)}\|\partial_x\partial_y^2\tilde h(t,x,y)\|^{\frac12}_{L^2_{\frac32}(\mathbb{R}^2_+)}\\
&\leq \|\partial_y^2\tilde h(t,x,y)\|_{H^{1,0}_{\frac32}(\mathbb{R}^2_+)}
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\langle y\rangle\partial_x^m\partial_y\tilde h(t,x,\tilde y)\|_{L^\infty(\mathbb{R}_+)}\\
\leq\ & \left|\int_0^y(\partial_x^m\partial_y\tilde h(t,x,\tilde y)+\langle y\rangle\partial_x^m\partial_y^2\tilde h(t,x,\tilde y)d\tilde y\right|\\
\leq\ & \|\partial_x^m\partial_y\tilde h(t,x,\tilde y)\|_{L^2_{y,\frac12}}+\|\partial_x^m\partial_y^2\tilde h(t,x,\tilde y)\|_{L^2_{y,\frac32}}\\
\leq\ & \|\partial_y^2\tilde h(t,x,\tilde y)\|_{H^{m,0}_{\frac32}(\mathbb{R}^2_+)},\ \text{for some } m\in\mathbb{N}.
\end{aligned}
$$

For the estimates of the term D_3, applying $(1.2)_3$, the Hölder inequality ([19], Theorem 1.4.3), Lemmas 2.4 and 4.3–4.5 again, we can also derive that

$$
\begin{aligned}
\left|D_3\right| &\leq C\|(\partial_xu,\partial_x\tilde h)\|_{L^\infty(\mathbb{R}^2_+)}\Big(\|\partial_y^2u\|^2_{H^{1,0}_{\ell+1}}+\|\partial_y^2\tilde h\|^2_{H^{1,0}_{\ell+1}}\Big)\\
&\leq CD(t)^{\frac12}E(t)+CD(t)^{\frac14}E(t)^{\frac54}.
\end{aligned}
$$

The estimate of the term E_3 is obvious that

$$
\left|E_3\right|\leq CD(t)^{\frac12}E(t)^{\frac12}.
$$

Thus, we obtain

$$\begin{aligned}&\frac{d}{dt}\|(\partial_y^2 u, \partial_y^2 \tilde{h})\|^2_{H^{1,0}_{\ell+1}} + \|(\partial_y^3 u, \partial_y^3 \tilde{h})\|^2_{H^{1,0}_{\ell+1}} \\ \leq\ & CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}} + CE(t)^{\frac{3}{2}} + CD(t)^{\frac{1}{2}} E(t) + CD(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}}. \qquad (5.3)\end{aligned}$$

Summarizing all the above estimates (5.1)–(5.3), the proof is thus complete.

The proof of Theorem 1.1

According to the initial data condition (1.3) and Lemma 4.3, we can get

$$\begin{aligned}&\sum_{m=0}^{2} \|\partial_y^m (u_0, \tilde{h}_0)\|^2_{H^{1,0}_{\ell-1+m}} \leq \varepsilon^2, \\ &\|(u_\beta(0), \tilde{h}_\beta(0))\|^2_{H^{3,0}_{\ell}} \leq 4\|(u(0), \tilde{h}(0))\|^2_{H^{3,0}_{\ell}} \leq 4\varepsilon^2.\end{aligned}$$

Therefore, $E(0) \leq 5\varepsilon^2$.

Based on the classical bootstrap argument [20], we can obtain the uniform estimates of solutions to problem (1.2). First, we assume that $[0, T^*]$ is the maximal time interval such that

$$E(t) \leq (c_1\varepsilon)^2, \ t \in [0, T^*], \qquad (5.4)$$

where the positive constant c_1 is determined later.

It follows from Lemmas 4.7 and 5.1 that

$$\begin{aligned}\frac{d}{dt}E(t) + D(t) \leq\ & CD(t)^{\frac{1}{4}} E(t)^{\frac{5}{4}} + CD(t)^{\frac{1}{4}} E(t)^{\frac{7}{4}} + CE(t)^{\frac{3}{2}} \\ & + CD(t)^{\frac{1}{2}} E(t)^{\frac{1}{2}} + CD(t)^{\frac{1}{2}} E(t) + CD(t)^{\frac{1}{2}} E(t)^{\frac{3}{2}} + CD(t)^{\frac{1}{2}} E(t)^2. \qquad (5.5)\end{aligned}$$

Using the assumption condition $E(t) \leq (c_1\varepsilon)^2$ and the smallness property of ε, (5.5) implies

$$\frac{d}{dt}E(t) + D(t) \leq CE(t).$$

Then using the Gronwall's inequality, we can conclude that for any $t \in [0, T^*]$,

$$E(t) + \int_0^t D(s)ds \leq E(0)\exp\{Ct\} \leq 5C\varepsilon^2,$$

if we take $c_1 = \sqrt{10C}$, the theorem 1.1 follows by a bootstrap argument. □

6. Conclusions

This paper mainly investigates the well-posedness of the 2D MHD boundary layer equations for small initial data in Sobolev space of polynomial weight and low regularity. The main steps include the following two parts: (i) We first obtain the systems (3.3)–(3.6) by paralinearizing and symmetrizing the system (1.2). (ii) We establish the estimates of the solution in horizontal direction and vertical direction, respectively. In addition, the method in this article can also be used to investigate the well-posedness of the other boundary layer equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declare there is no conflicts of interest.

References

1. D. Gérard-Varet, M. Prestipino, Formal derivation and stability analysis of boundary layer models in MHD, *Z. Angew. Math. Phys.*, **68** (2017), 76. https://doi.org/10.1007/s00033-017-0820-x
2. C. Liu, F. Xie, T. Yang, MHD Boundary layers theory in Sobolev spaces without monotonicity I: Well-Posedness Theory, *Commun. Pure Appl. Math.*, **72** (2019), 63–121. https://doi.org/10.1002/cpa.21763
3. L. Prandtl, Über Flüssigkeitsbewegung bei sehr kleiner Reibung, in *Verhandlungen des III. Internationalen Mathematiker-Kongresses Heidelberg*, (1904), 484–491.
4. R. Alexandre, Y. Wang, C. Xu, T. Yang, Well-posedness of the Prandtl equation in sobolev spaces, *J. Am. Math. Soc.*, **28** (2015), 745–784. http://doi.org/10.1090/S0894-0347-2014-00813-4
5. N. Masmoudi, T. K. Wong, Local-in-time existence and uniqueness of solutions to the Prandtl equations by energy methods, *Commun. Pure Appl. Math.*, **68** (2015), 1683–1741. https://doi.org/10.1002/cpa.21595
6. O. A. Oleinik, V. N. Samokhin, *Mathematical Models in Boundary Layer Theory*, Routledge, New York, 1999. https://doi.org/10.1201/9780203749364
7. C. Liu, D. Wang, F. Xie, T. Yang, Magnetic effects on the solvability of 2D MHD boundary layer equations without resistivity in Sobolev spaces, *J. Funct. Anal.*, **279** (2020), 108637. https://doi.org/10.1016/j.jfa.2020.108637
8. N. Aarach, Hydrostatic approximation of the 2D MHD system in a thin strip with a small analytic data, *J. Math. Anal. Appl.*, **509** (2022), 125949. https://doi.org/10.1016/j.jmaa.2021.125949
9. X. Yang, P. Huang, Y. He, A Voigt-regularization of the thermally coupled inviscid, resistive magnetohydrodynamic, *Int. J. Numer. Anal. Model.*, **21** (2024), 476–503 https://doi.org/10.4208/ijnam2024-1019
10. F. Xie, T. Yang, Global-in-time stability of 2D MHD boundary layer in the Prandtl-Hartmann regime, *SIAM. J. Math. Anal.*, **50** (2018), 5749–5760. https://doi.org/10.1137/18M1174969

11. F. Xie, T. Yang, Lifespan of solutions to MHD Boundary layer equations with analytic perturbation of general shear flow, *Acta Math. Appl. Sin. Engl. Ser.*, **35** (2019), 209–229. https://doi.org/10.1007/s10255-019-0805-y
12. D. Chen, Y. Wang, Z. Zhang, Well-posedness of the Prandtl equation with monotonicity in Sobolev spaces, *J. Differ. Equations*, **264** (2018), 5870–5893. https://doi.org/10.1016/j.jde.2018.01.024
13. Y. Wang, Z. Zhang, Paralinearization method about the well-posedness of the Prandtl equation, *Sci. Sin. Math.*, **51** (2021), 1037–1056. https://doi.org/10.1360/SSM-2020-0231
14. D. Chen, S. Ren, Y. Wang, Z. Zhang, Long time well-posedness of the MHD boundary layer equation in Sobolev space, *Anal. Theory Appl.*, **36** (2020), 1–18. https://doi.org/10.4208/ata.OA-0015
15. C. Wang, Y. Wang, On the hydrostatic approximation of the MHD equations in a thin strip, *SIAM J. Math. Anal.*, **54** (2022), 1241–1269. https://doi.org/10.1137/21M1425360
16. D. Chen, X. Li, Long time well-posedness of two dimensional Magnetohydrodynamic boundary layer equation without resistivity, *Math. Meth. Appl. Sci.*, **46** (2023), 10186–10202. https://doi.org/10.1002/mma.9110
17. H. Bahouri, J. Chemin, R. Danchin, *Fourier Analysis and Nonlinear Partial Differential Equations*, Springer Berlin, Heidelberg, 2011. https://doi.org/10.1007/978-3-642-16830-7
18. M. Ignatova, V. Vicol, Almost global existence for the Prandtl boundary layer equations, *Arch. Ration. Mech. Anal.*, **220** (2016), 809–848. https://doi.org/10.1007/s00205-015-0942-2
19. Y. Qin, *Nonlinear Parabolic-Hyperbolic Coupled Systems and Their Attractors*, Birkhäuser, Basel, 2008. https://doi.org/10.1007/978-3-7643-8814-0
20. T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, American Mathematical Society, Provindence, 2006. https://doi.org/10.1090/cbms/106

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