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Research article

On absolutely invertibles

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Abstract: In this manuscript, the notion of absolutely invertible was extended consistently from semi-normed rings to the class of general topological rings. Then, the closure of the absolutely invertibles multiplied by a certain element was proved to be contained in the set of topological divisors of the element. Also, a sufficient condition for the closed unit ball of a complete unital normed ring to become a closed unit neighborhood of zero was found. Finally, two applications to classical operator theory were provided, i.e., every Banach space of dimension of at least 2 could be equivalently re-normed in such a way that the group of surjective linear isometries was not a normal subgroup of the group of isomorphisms, and every infinite-dimensional Banach space, containing a proper complemented subspace isomorphic to it, could be equivalently re-normed so that the set of surjective linear operators was not dense in the Banach algebra of bounded linear operators.

Keywords: $*$-ring; absolutely invertible; topological ring; normal subgroup

1. Introduction

The exploration of topological rings and modules is gaining increasing prominence and importance in functional analysis [1]. For instance, classical operator theory on complex normed spaces [2–6] has been extended into the framework of normed modules over absolutely valued rings [7, 8], resulting in the development of a representation theory on the group of topological isomorphisms on a topological module. Later, in [9], the foundational concepts of operator theory were further expanded to encompass semi-normed modules over semi-normed rings and, in some instances, topological modules over topological rings, by utilizing convergence linear topologies to establish dual module topologies. Key notions from the classical geometry of Banach spaces, such as internal structure and extremal properties, have similarly been generalized to the broader setting of topological rings and modules [10].

Another significant area where topological rings and modules are notably present is spectral theory and quantum mechanics [11]. The examination of the algebraic characteristics of quantum systems led to the boost of C^*-algebras and effect algebras, as well as additional related algebraic structures, including $*$-rings and Hilbert C^*-modules [12–14].

In general, a key distinction between Banach spaces and general topological modules lies in the fact that finite-dimensional subspaces of a Banach space are always closed, whereas finitely spanned submodules of a topological module (including Hilbert C^*-modules) do not necessarily possess this property. This difference is one of the reasons why certain concepts and results from operator theory on Banach spaces cannot be directly applied to operators on topological modules. One illustrative example is the Fredholm and semi-Fredholm theory on the standard Hilbert module over a unital C^*-algebra, where the methods and proofs significantly diverge from those used in the classical Fredholm and semi-Fredholm theory on Banach spaces. Although Hilbert C^*-modules are complex Banach spaces as well, the semi-Fredholm theory described in [15–18] differs greatly from the classical theory precisely because finitely spanned sub-modules can exhibit different behavior from finite-dimensional subspaces. Another illustrative example comes from the Gelfand theory. One of its strongest results establishes that the boundary of the invertibles in a C^*-algebra is contained in the set of topological divisors. Diverse generalizations of this result have been recently provided in [19], some of them involving the novel notion of absolutely invertible.

The main objective of this manuscript is to extend consistently the notion of absolutely invertible to the class of general topological rings. The paper is organized as follows: Section 2 deals with the main properties of absolutely invertibles in (semi-normed) $*$-rings obtaining, in particular, that every Banach space of dimension of at least 2 can be equivalently re-normed in such a way that the group of surjective linear isometries is not a normal subgroup of the group of isomorphisms (Theorem 2), and that every infinite-dimensional Banach space, containing a proper complemented subspace isomorphic to it, can be equivalently re-normed so that the set of surjective linear operators is not dense in the Banach algebra of bounded linear operators (Theorem 3 and Corollary 1). In Section 3, the novel concepts of unit ball and absolutely invertible in the scope of topological rings are introduced in the literature (Definitions 1 and 2). Section 4 is devoted to the analysis of absolutely invertibles in semi-normed rings and modules satisfying topological properties such as compactness and completeness. More concretely, we obtain a new sufficient condition for the closed unit ball of a complete unital normed ring to be a closed unit neighborhood of zero (Theorem 5). Finally, in Theorem 7 of Section 5, we generalize, to the scope of Hausdorff topological rings via absolutely invertibles, the previously mentioned result in Gelfand theory, i.e., the boundary of the invertibles in a C^*-algebra is contained in the set of topological divisors.

2. Absolutely invertibles in semi-normed $*$-rings

If R is an absolutely semi-valued ring, then every invertible $u \in \mathcal{U}(R)$ satisfies that $\left|u^{-1}\right| = |u|^{-1}$ in virtue of the multiplicative character of the absolute semivalue. This motivates the definition of absolutely invertible in semi-normed rings. Let R be a semi-normed ring. We say that $u \in \mathcal{U}(R)$ is absolutely invertible (see [19]) provided that $\left\|u^{-1}\right\| = \|u\|^{-1}$. In the Banach algebra of operators on a Hilbert space, absolutely invertibles were fully characterized in [19] as those isomorphisms which are positive multiples of a surjective isometry. The subset of absolutely invertibles of a semi-normed ring R is denoted by $\mathcal{U}_1(R)$ and it is a subgroup of $\mathcal{U}(R)$ if R is unital ($\|1\| = 1$). In fact, the existence of absolutely invertibles is directly linked to the unital character of R in other words, $\|1\| = 1$ if, and only if, $\mathcal{U}_1(R) \neq \emptyset$, meaning in such a situation that $1 \in \mathcal{U}_1(R)$. Special attention will be paid to the normalized absolutely invertibles, that is, $\mathcal{U}_1(R) \cap \mathsf{S}_R$, with $\mathsf{S}_R = \{r \in R : \|r\| = 1\}$, which is itself a subgroup of $\mathcal{U}_1(R)$. Note that both $\mathcal{U}_1(R) \cap \mathsf{S}_R$ and $\mathcal{U}_1(R)$ are additively symmetric (a subset A of a ring is said to be additively symmetric provided that $A = -A$). Another interesting property satisfied by the absolutely invertibles is the fact that $\|ur\| = \|ru\| = \|r\|\|u\|$ for all $u \in \mathcal{U}_1(R)$ and all $r \in R$. Actually, if M is a semi-normed R-module, then $\|um\| = \|u\|\|m\|$ for all $m \in M$ and all $u \in \mathcal{U}_1(R)$.

Recall that a $*$-ring is a ring R endowed with an additive and anti-multiplicative involution $* : R \to R$. Note that involutions preserve the unity and the invertibles, that is, $1^* = 1$ and $\left(u^{-1}\right)^* = (u^*)^{-1}$.

Let R be a semi-normed $*$-ring. Observe that if $\|r\| \leq \|r^*\|$ for all $r \in R$, then $*$ is an isometry. Indeed, notice that $\|r^*\| \leq \|(r^*)^*\| = \|r\|$ for all $r \in R$. Therefore, $\|r^*\| = \|r\|$ for all $r \in R$, meaning that $*$ is an isometry.

We will say that the semi-norm of R is strongly $*$-multiplicative provided that the condition $\|rr^*\| = \|r\|^2$ holds for all $r \in R$; and is $*$-multiplicative if $\|rr^*\| = \|r\|\,\|r^*\|$ for all $r \in R$. Notice that the semi-norm of R is strongly $*$-multiplicative if and only if $*$ is an isometry and the semi-norm is $*$-multiplicative. Indeed, if $*$ is an isometry and the semi-norm is $*$-multiplicative, then it is clear that the semi-norm is strongly $*$-multiplicative. Conversely, if the semi-norm is strongly $*$-multiplicative, then $\|r\|^2 = \|rr^*\| \leq \|r\|\,\|r^*\|$ for all $r \in R$. Therefore, if $\|r\| > 0$, then $\|r\| \leq \|r^*\|$, and if $\|r\| = 0$, then we clearly have that $\|r\| \leq \|r^*\|$, reaching the conclusion that $\|r\| \leq \|r^*\|$ for all $r \in R$ and $*$ is an isometry, hence the semi-norm is $*$-multiplicative.

In the context of Banach $*$-algebras, it can be proved (see [20]) that the $*$-multiplicative condition ($\|xx^*\| = \|x\|\,\|x^*\|$) implies the strongly $*$-multiplicative condition ($\|xx^*\| = \|x\|^2$) without previously asking for the involution to be an isometry.

An invertible element u of a $*$-ring R is called unitary provided that $u^* = u^{-1}$. The subset of unitary elements is denoted by $\mathcal{U}_{\mathrm{u}}(R)$ and it is clearly an $*$-invariant additively symmetric subgroup of $\mathcal{U}(R)$. This fact motivates our first theorem.

Theorem 1. *Let R be a unital semi-normed $*$-ring. Then:*

1) *If the involution is an isometry, then $\mathcal{U}_1(R)^* = \mathcal{U}_1(R)$ and $\mathcal{U}_1(R) \cap \mathsf{S}_R$ is an $*$-invariant additively symmetric subgroup of $\mathcal{U}_1(R)$.*
2) *If the semi-norm is $*$-multiplicative, then $\mathcal{U}_{\mathrm{u}}(R) \subseteq \mathcal{U}_1(R)$.*
3) *If the involution is an isometry and the semi-norm is $*$-multiplicative, then $\mathcal{U}_{\mathrm{u}}(R) \subseteq \mathcal{U}_1(R) \cap \mathsf{S}_R$.*

Proof. The proof will follow as itemized as the statement of the theorem.

1) Let us prove first that $\mathcal{U}_1(R)^* = \mathcal{U}_1(R)$. Fix any arbitrary $r \in \mathcal{U}_1(R)$. Observe that $(r^*)^{-1} = \left(r^{-1}\right)^*$, so
$$\left\|(r^*)^{-1}\right\| = \left\|\left(r^{-1}\right)^*\right\| = \left\|r^{-1}\right\| = \|r\|^{-1} = \|r^*\|^{-1},$$
hence $r^* \in \mathcal{U}_1(R)$. This shows that $\mathcal{U}_1(R)^* \subseteq \mathcal{U}_1(R)$. Since $*$ is involutive, we obtain that $\mathcal{U}_1(R)^* = \mathcal{U}_1(R)$. As a consequence, $\mathcal{U}_1(R) \cap \mathsf{S}_R$ is an $*$-invariant additively symmetric subgroup of $\mathcal{U}_1(R)$ due to the assumption that $*$ is an isometry.

2) Indeed, if $u \in \mathcal{U}_u(R)$, since R is unital by hypothesis, then $1 = \|1\| = \left\|uu^{-1}\right\| = \|uu^*\| = \|u\|\,\|u^*\|$, meaning that $\|u\| = \|u^*\|^{-1} = \left\|u^{-1}\right\|^{-1}$, and hence, $u \in \mathcal{U}_1(R)$.
3) From the previous item, we already know that $\mathcal{U}_u(R) \subseteq \mathcal{U}_1(R)$, so it only remains to show that $\mathcal{U}_u(R) \subseteq \mathsf{S}_R$. Indeed, if $u \in \mathcal{U}_u(R)$, then $\|u\| = \|u^*\| = \left\|u^{-1}\right\| = \|u\|^{-1}$, meaning that $\|u\| = \left\|u^{-1}\right\| = 1$. Therefore, $u \in \mathsf{S}_R$.

□

Given a complex Hilbert space H, it is well-known that a bounded operator $T \in \mathcal{B}(H)$ is a unitary element of the C^*-algebra $\mathcal{B}(H)$ if, and only if, T is a surjective linear isometry [2]. In view of Theorem 1, $\mathcal{U}_1(\mathcal{B}(H)) \cap \mathsf{S}_{\mathcal{B}(H)} \supseteq \mathcal{U}_u(\mathcal{B}(H))$. However, in the case of the C^*-algebra $\mathcal{B}(H)$, it actually holds that $\mathcal{U}_1(\mathcal{B}(H)) \cap \mathsf{S}_{\mathcal{B}(H)} = \mathcal{U}_u(\mathcal{B}(H))$. This happens in accordance with [19] because $\mathcal{U}_1(\mathcal{B}(H))$ consists of the positive multiples of the surjective linear isometries. According to [19], if $\dim(H) \geq 2$, then neither $\mathcal{U}_1(\mathcal{B}(H))$ nor $\mathcal{U}_u(\mathcal{B}(H))$ is a normal subgroup of $\mathcal{U}(\mathcal{B}(H))$. However, it is obvious that $\mathcal{U}_u(\mathcal{B}(H))$ is a normal subgroup of $\mathcal{U}_1(\mathcal{B}(H))$ in virtue of the fact that $\mathcal{U}_1(\mathcal{B}(H)) \cap \mathsf{S}_{\mathcal{B}(H)} = \mathcal{U}_u(\mathcal{B}(H))$. In fact, the latest assertion works for general unital semi-normed rings.

Proposition 1. *If R is a unital semi-normed ring, then $\mathcal{U}_1(R) \cap \mathsf{S}_R$ is a normal subgroup of $\mathcal{U}_1(R)$.*

Proof. Take arbitrary elements $u \in \mathcal{U}_1(R) \cap \mathsf{S}_R$ and $r \in \mathcal{U}_1(R)$ and we will show that $r^{-1}ur \in \mathcal{U}_1(R) \cap \mathsf{S}_R$. Indeed, it is clear that $r^{-1}ur \in \mathcal{U}_1(R)$, so it only remains to show that $r^{-1}ur \in \mathsf{S}_R$, which is a direct consequence of the properties satisfied by the absolutely invertibles because $\left\|r^{-1}ur\right\| = \left\|r^{-1}\right\| \|u\|\|r\| = \|r\|^{-1}\|r\| = 1$. □

A direct consequence of Proposition 1 is the fact that $\mathcal{U}_u(R)$ is a normal subgroup of $\mathcal{U}_1(R)$ provided that $\mathcal{U}_u(R) = \mathcal{U}_1(R) \cap \mathsf{S}_R$. However, the next example shows this assertion is not always true.

Example 1. *Consider the product ring $R := \mathbb{R} \times \mathbb{C}$ endowed with the 1-norm $\|(x,z)\|_1 := |x| + |z|$ and the involution given by right-conjugation $(x,z)^* := (x,\overline{z})$. It is not hard to check that this involution is an isometry but it is not $*$-multiplicative. Also, $\mathcal{U}_1(R) = \varnothing$ and $\mathcal{U}_u(R) = \{\pm 1\} \times \mathrm{bd}(\mathbb{D})$, where $\mathrm{bd}(\mathbb{D})$ stands for the boundary of the unit disk, also denoted by $\mathbb{T}$.*

A slight modification of Example 1 provides the existence of absolutely invertible elements.

Example 2. *Consider the unital semi-normed ring $R := \mathbb{C}$ endowed with the 1-norm $\|z\|_1 := \left|\Re z\right| + \left|\Im z\right|$ and the involution given by conjugation $z^* := \overline{z}$. It is not hard to check that this involution is an isometry but not $*$-multiplicative. Also, $\mathcal{U}_1(R) = \{z \in \mathbb{C} \setminus \{0\} : \Re z = 0 \text{ or } \Im z = 0\}$ and $\mathcal{U}_u(R) = \mathrm{bd}(\mathbb{D})$.*

Observe that if R is $\mathbb{C}$ endowed with its absolute value and involution given by conjugation, then the involution is clearly an isometry, the absolute value is $*$-multiplicative, $\mathcal{U}_u(R) = \mathsf{S}_{\mathbb{C}}$, and $\mathcal{U}_1(R) = \mathcal{U}(R) = \mathbb{C} \setminus \{0\}$, meaning that $\mathcal{U}_u(R) = \mathcal{U}_1(R) \cap \mathsf{S}_R$.

According to [19], if H is a complex Hilbert space, then $\mathcal{U}_1(\mathcal{B}(H))$ is precisely the positive multiples of the surjective linear isometries. On the other hand, $\mathcal{U}_1(\mathcal{B}(H)) \cap \mathsf{S}_{\mathcal{B}(H)} = \mathcal{U}_u(\mathcal{B}(H))$ consists of the surjective linear isometries on H. As previously mentioned, by [19], if $\dim(H) \geq 2$, then $\mathcal{U}_u(\mathcal{B}(H))$

is not a normal subgroup of $\mathcal{U}(\mathcal{B}(H))$. This result can be extended to general Banach spaces in an isomorphic way.

Theorem 2. *Let X be a Banach space of dimension of at least 2. There exists an equivalent norm on X in such a way that the group of surjective linear isometries is not a normal subgroup of the group of isomorphisms.*

Proof. Every Banach space of dimension of at least 2 is isomorphic to a Banach space of the form $H \oplus_2 X$, where H is a finite-dimensional Hilbert space of dimension of at least 2 and X is another Banach space. In view of [19], there exist a surjective linear isometry $T : H \to H$ and an isomorphism $S : H \to H$ in such a way that $S^{-1} \circ T \circ S$ is not an isometry on H. Consider the operators $S \oplus I_X$ and $T \oplus I_X$, which are an isomorphism and a surjective linear isometry on $H \oplus_2 X$, respectively. Observe that for every $h \in H$ and every $x \in X$,

$$\left[(S \oplus I_X)^{-1} \circ (T \oplus I_X) \circ (S \oplus I_X)\right](h + x) = S^{-1}(T(S(h))) + x,$$

in other words, $(S \oplus I_X)^{-1} \circ (T \oplus I_X) \circ (S \oplus I_X) = \left(S^{-1} \circ T \circ S\right) \oplus I_X$. Also, notice that

$$\begin{aligned} \left\|\left[(S \oplus I_X)^{-1} \circ (T \oplus I_X) \circ (S \oplus I_X)\right](h + x)\right\|_2^2 &= \left\|S^{-1}(T(S(h))) + x\right\|_2^2 \\ &= \left\|S^{-1}(T(S(h)))\right\|^2 + \|x\|^2, \end{aligned}$$

meaning that if $(S \oplus I_X)^{-1} \circ (T \oplus I_X) \circ (S \oplus I_X)$ is an isometry on $H \oplus_2 X$, then so is $S^{-1} \circ T \circ S$ on H because of the following chain of equalities:

$$\begin{aligned} \|h\|^2 + \|x\|^2 &= \|h + x\|_2^2 \\ &= \left\|\left[(S \oplus I_X)^{-1} \circ (T \oplus I_X) \circ (S \oplus I_X)\right](h + x)\right\|^2 \\ &= \left\|S^{-1}(T(S(h)))\right\|^2 + \|x\|^2. \end{aligned}$$

□

Our final efforts in this section are devoted to find examples of algebras of operators for which the group of isomorphisms is not dense. For this, we will strongly rely on the notion of E-projection, see [21]. A projection $P : X \to X$ on a Banach space X is said to be an E-projection provided that there exists a 2-dimensional real Banach space $E := \left(\mathbb{R}^2, \|\cdot\|_E\right)$ in such a way that $\{(1, 0), (0, 1)\}$ is a normalized 1-unconditional basis and $\|x\| = \|(\|P(x)\|, \|x - P(x)\|)\|_E$ for all $x \in X$. All ℓ_p-projections are E-projections for $1 \leq p \leq \infty$, but the converse is not true.

Theorem 3. *Let X be an infinite dimensional Banach space. If there exists a proper subspace Y of X isometric to X and E-complemented in X, then the set of surjective operators is not dense in $\mathcal{B}(X)$.*

Proof. Let $T : X \to Y$ be a surjective linear isometry and $P : X \to X$ a continuous linear projection of range Y with $\|x\| = \|(\|P(x)\|, \|x - P(x)\|)\|_E$. Since all norms are equivalent in $\mathbb{R}^2$, there exists $K > 0$ such that $\|(\alpha, \beta)\|_\infty \leq K\|(\alpha, \beta)\|_E$ for all $(\alpha, \beta) \in \mathbb{R}^2$. If the surjective operators are dense in $\mathcal{B}(X)$, then we can find $S \in \mathcal{B}(X)$ surjective such that $\|S - T\| < \frac{1}{K}$. Fix an arbitrary $z_0 \in \ker(P) \setminus \{0\}$.

The surjectivity of S allows the existence of $x_0 \in X \setminus \{0\}$ such that $S(x_0) = z_0$. Then we obtain the following contradiction:

$$\begin{aligned} \|x_0\| \le \|(\|z_0\|, \|x_0\|)\|_\infty &\le K\,\|(\|z_0\|, \|x_0\|)\|_E = K\,\|(\|z_0\|, \|T(x_0)\|)\|_E \\ &= K\|z_0 - T(x_0)\| = K\|S(x_0) - T(x_0)\| \\ &\le K\|S - T\|\|x_0\| < K\frac{1}{K}\|x_0\| = \|x_0\|. \end{aligned}$$

□

Theorem 3 provides plenty of examples of algebras of operators for which the group of invertibles is not dense.

Corollary 1. *Let X be an infinite dimensional Banach space. If there exists a proper subspace Y of X isomorphic to X which is complemented in X, then X can be equivalently re-normed so that the set of surjective operators is not dense in $\mathcal{B}(X)$.*

Proof. Let $T : Y \to X$ be an isomorphism and $P : X \to X$ a continuous linear projection whose range is Y. Consider on Y the equivalent norm given by $\|y\|_0 := \max\{\|P(T(y))\|, \|T(y) - P(T(y))\|\}$ for all $y \in Y$ and denote Y' to Y endowed with the new norm. If $W := \ker(P) \oplus_\infty Y'$, then W and X are isomorphic, W is linearly isometric to Y' by construction of $\| \bullet \|_0$, and Y' is E-complemented in W. Finally, Theorem 3 serves to assure the desired result. □

3. Absolutely invertibles in topological rings

In order to extend the notion of absolutely invertible to topological rings, we need to rely on the following concept in associative ring theory, which is novel from this work. Recall that a multiplicatively idempotent subset of a ring is a set A satisfying $AA = A$.

Definition 1 (Unit ball)**.** *Let R be a topological ring. A unit ball in R is an additively symmetric and multiplicatively idempotent closed neighborhood of* 0 *in R containing* 1.

The closed unit ball of a unital semi-normed ring is the most representative example of the above notion. Obviously, the entire ring is trivially a unit ball. In fact, if a ring R is endowed with the trivial topology, then R is the only unit ball. However, if R is endowed with the discrete topology, then $\{-1, 0, 1\}$ is the smallest unit ball.

A unit ball B in a topological ring R is said to be unital provided that $1 \in \mathrm{bd}(B)$. This fact implies that $B \neq R$ because R is trivially open and its boundary is empty. A nontrivial example of a non-unital unit ball is $\{-1, 0, 1\}$ in a discrete ring. For instance, $\mathbb{Z}$ endowed with its absolute value provides an example of a unital semi-normed ring whose closed unit ball is a non-unital unit ball.

Definition 2 (Absolutely invertible)**.** *Let R be a topological ring endowed with a unit ball B. An invertible element $u \in \mathcal{U}(R)$ is said to be absolutely invertible provided that $u, u^{-1} \in B$. The set of absolutely invertibles of R is denoted again by $\mathcal{U}_1(R)$.*

Notice that the notion of absolutely invertible depends on the unit ball. Like we mentioned above, in a unital semi-normed ring R its closed unit ball B_R is an additively symmetric and multiplicatively idempotent closed neighborhood of 0 containing 1. As a consequence, the definition of absolutely invertible in topological rings extends properly that of semi-normed rings. However, we will pay special attention to a particular class of unit balls: the closed unit neighborhoods of zero.

A subset U of a topological space X is called regular open provided that $U = \text{int}(\text{cl}(U))$. A subset B of X is called regular closed provided that $B = \text{cl}(\text{int}(B))$. If $U \subseteq X$ is open and $B \subseteq X$ is closed, then $\text{cl}(U)$ and $\text{int}(B)$ are regular closed and regular open, respectively. Unit neighborhoods of zero, also known as unit zero-neighborhoods, constitute a new concept in associative ring theory with plenty of applications in topological modules. We refer the reader to [7, 22] for a wider perspective on this new notion.

Let R be a topological ring. Let U, B be subsets of R. Then, U is called an open unit neighborhood of 0 provided that U is an additively symmetric and multiplicatively idempotent regular open neighborhood of 0 such that $1 \in \text{cl}(U)$; and B is called a closed unit neighborhood of 0 provided that B is regular closed and its interior is an open unit neighborhood of 0. Notice that B is additively symmetric and multiplicatively idempotent as well.

Observe that, if U is an open unit neighborhood of zero, then $\text{cl}(U)$ is a closed unit neighborhood of zero, and, if B is a closed unit neighborhood of zero, then $\text{int}(B)$ is an open unit neighborhood of zero. In [22], it is proved that the only proper closed unit neighborhood of 0 in $\mathbb{R}$ is $[-1, 1]$, and the only proper closed unit neighborhood of 0 in $\mathbb{C}$ containing the unit complex numbers $\mathbb{T}$ is the closed unit disk $\overline{\mathbb{D}}$. In [23], other proper closed unit neighborhoods of 0 in $\mathbb{C}$ are provided. As far as we know, it is still unknown whether every topological ring not endowed with the trivial topology has a proper closed unit neighborhood of zero.

Lemma 1. *Let R be a topological ring endowed with a unital unit ball B. Then, $\mathcal{U}_1(R)$ is an additively symmetric subgroup of $\mathcal{U}(R)$ entirely contained in* $\text{bd}(B)$.

Proof. Observe that $\mathcal{U}_1(R)$ is trivially an additively symmetric subgroup of $\mathcal{U}(R)$ in view of the fact that B is additively symmetric and multiplicatively idempotent. Let us show first that $\mathcal{U}_1(R) \subseteq \text{bd}(B)$. Indeed, fix an arbitrary $u \in \mathcal{U}_1(B)$. Suppose on the contrary that $u \in \text{int}(B)$. Note that $u^{-1}\text{int}(B)$ is an open neighborhood of 0 contained in $BB = B$, hence $u^{-1}\text{int}(B) \subseteq \text{int}(B)$, meaning that $1 = u^{-1}u \in \text{int}(B)$, which contradicts the fact that B is unital. As a consequence, $u \in \text{bd}(B)$. Therefore, $\mathcal{U}_1(R) \subseteq \text{bd}(B)$. □

Proposition 2. *Let R be a topological $*$-ring endowed with a unit ball B. Then $\mathcal{U}_\text{u}(R) \cap B \cap B^* \subseteq \mathcal{U}_1(R)$.*

Proof. Fix an arbitrary $u \in \mathcal{U}_\text{u}(R) \cap B \cap B^*$. All we need to show is that $u, u^{-1} \in B$. By assumption, $u \in B \cap B^*$, meaning that $u^* \in B \cap B^*$. Finally, since $u \in \mathcal{U}_\text{u}(R)$, we obtain that $u^{-1} = u^* \in B$. □

Under the assumptions of Proposition 2, if B is $*$-invariant ($B^* = B$), then $\mathcal{U}_\text{u}(R) \cap B \subseteq \mathcal{U}_1(R)$. The following example shows that, in general, we cannot achieve that $\mathcal{U}_\text{u}(R) \subseteq \mathcal{U}_1(R)$.

Example 3. *Consider the topological ring $R := \mathbb{C}$ endowed with the involution given by conjugation $z^* := \bar{z}$. According to [23], the closed unit ball of the 1-norm $B := \{z \in \mathbb{C} : \|z\|_1 \leq 1\}$ is in fact a closed unit neighborhood of 0 in R. It is not hard to check that B is $*$-invariant. Also, $\mathcal{U}_1(R) = \{1, -1, i, -i\}$ and $\mathcal{U}_u(R) = \mathrm{bd}(\mathbb{D})$.*

The condition of $*$-invariance for a closed unit zero-neighborhood is the topological version of the isometry condition in the case of semi-normed rings.

Recall that a subset A of a topological module M over a topological ring R is said to be bounded (see [24–26]), provided that for every neighborhood U of 0 there exists an invertible $v \in \mathcal{U}(R)$ such that $A \subseteq vU$. In the context of topological rings, left bounded refers to bounded when the ring is a left module over itself.

Lemma 2. *Let R be a topological ring, M, N topological R-modules, and $T : M \to N$ linear. If T is continuous, then $T(B)$ is bounded in N for every $B \subseteq M$ bounded. If there exists a 0-neighborhood $V \subseteq M$ such that $T(V)$ is bounded, then T is continuous.*

Proof. Suppose first that T is continuous. Let U be a 0-neighborhood in N. Then, $T^{-1}(U)$ is a 0-neighborhood in M by the continuity of T. Since $B \subseteq M$ is bounded, there exists $u \in \mathcal{U}(R)$ with $B \subseteq uT^{-1}(U)$. Thus, $T(B) \subseteq uU$. This implies that $T(B)$ is bounded. Conversely, let us assume now that T is linear and there exists a 0-neighborhood $V \subseteq M$ such that $T(V)$ is bounded. We will prove that T is continuous. For this, we will show the continuity of T at 0. Fix any arbitrary 0-neighborhood $W \subseteq N$. By hypothesis, there exists an invertible $u \in \mathcal{U}(R)$ in such a way that $T(V) \subseteq uW$, meaning that $u^{-1}V \subseteq T^{-1}(W)$, so $T^{-1}(W)$ is a neighborhood of 0 in M. This proves the continuity of T at 0. By additivity, T is everywhere continuous. □

Lemma 2 has strong consequences on involutions.

Corollary 2. *Let R be a topological $*$-ring. If there exists a left- or right-bounded and $*$-invariant neighborhood W of 0, then $*$ is a homeomorphism.*

Proof. Since $*^{-1} = *$, it only suffices to prove is that if $V \subseteq R$ is a neighborhood of 0, then V^* is also a neighborhood of 0. Indeed, there exists an invertible $u \in \mathcal{U}(R)$ such that $uW \subseteq V$ (here we are assuming that W is left-bounded, and a similar proof applies for the right-bounded case). Then $Wu^* \subseteq V^*$, meaning that V^* is a neighborhood of 0 in R because u^* is also invertible. □

Our next results deal with absolutely invertibles in a dense subring. We will strongly rely on the following technical remark, which can be found in almost any basic Topology textbook.

Remark 1. *Let X be a topological space, Y a subset of X, and $A \subseteq X$. Then, $\mathrm{int}(A) \cap Y \subseteq \mathrm{int}_Y(A \cap Y)$, $\mathrm{cl}_Y(A \cap Y) = \mathrm{cl}(A \cap Y) \cap Y$, and $\mathrm{bd}_Y(A \cap Y) \subseteq \mathrm{bd}(A \cap Y) \cap \mathrm{bd}(A)$. If, in addition, Y is dense in X and A is closed in X, then $\mathrm{int}(A) \cap Y = \mathrm{int}_Y(A \cap Y)$, $\mathrm{bd}(A) \cap Y = \mathrm{bd}_Y(A \cap Y)$, and $\mathrm{cl}(A) \cap Y = \mathrm{cl}_Y(A \cap Y)$. As a consequence, if Y is dense in X and $U, B \subseteq X$ are regular open and regular closed, respectively, then $U \cap Y$ is regular open in Y and $B \cap Y$ is regular closed in Y.*

Recall that topological rings in which multiplicative inversion is continuous are called topological inversion rings. Topological division rings are topological inversion rings by default, whereas division topological rings need not be necessary.

Lemma 3. *Let R be a topological inversion ring. Let U be an open subset of R. If $1 \in \mathrm{cl}(U \cap \mathcal{U}(R))$, then $U \subseteq UU$.*

Proof. Fix an arbitrary $u \in U$. Let $(u_i)_{i\in I} \subseteq U \cap \mathcal{U}(R)$ be a net converging to 1. By the continuity of the inversion, we deduce that $\left(u_i^{-1}\right)_{i\in I}$ also converges to 1 therefore, $\left(u_i^{-1}u\right)_{i\in I}$ converges to u, which means that we can find $j \in I$ with $u_j^{-1}u \in U$. Finally, $u = u_j\left(u_j^{-1}u\right) \in UU$. □

Proposition 3. *Let R be a topological inversion ring. Let S be a dense subring of R. If $U \subseteq R$ is an open unit neighborhood of 0 in R such that $1 \in \mathrm{cl}(U \cap \mathcal{U}(S))$, then $U \cap S$ is an open unit neighborhood of 0 in S.*

Proof. According to Remark 1, $U \cap S$ is a regular open neighborhood of 0 in S and $\mathrm{cl}_S(U \cap S) = \mathrm{cl}(U) \cap S$. Notice that $U \cap S$ is additively symmetric and $(U \cap S)(U \cap S) \subseteq U \cap S$. Finally, since $1 \in \mathrm{cl}(U \cap \mathcal{U}(S))$ by hypothesis, Lemma 3 guarantees that $(U \cap S)(U \cap S) = U \cap S$. □

4. Absolutely invertibles in semi-normed rings and modules

As mentioned earlier, semi-normed rings are topological rings and the closed unit ball of a unital semi-normed ring fits the definition of unit ball for the ring topology induced by the semi-norm. Therefore, every time that we consider absolutely invertibles in semi-normed rings, it will be in the original sense [19].

Our first lemma in this section improves a result on the closedness of the invertibles of norm 1 in an absolutely valued ring given in [7]. In what follows throughout this section and the rest of the manuscript, B_M, U_M, and S_M stand for the closed unit ball, the open unit ball, and the unit sphere of a semi-normed module M (and the same notation is valid for a semi-normed ring R).

Lemma 4. *Let R be a unital normed ring. Then, $\mathcal{U}_1(R) \cap \mathsf{S}_R$ is closed if either one of the following conditions hold:*

1) *R is complete.*
2) *S_R is compact.*

Proof. Let $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}_1(R) \cap \mathsf{S}_R$ be a sequence converging to an element $r \in \mathsf{S}_R$. We will distinguish between the two conditions above:

1) R is complete. In this case, the equality
$$u_p^{-1} - u_q^{-1} = u_q^{-1}(u_q - u_p)u_p^{-1}$$
for all $p, q \in \mathbb{N}$ shows that $\left(u_n^{-1}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence, hence it converges to an element $s \in \mathsf{S}_R$. Now it is clear that $rs = sr = 1$.

2) In this case, the compactness of S_R allows us to deduce the existence of a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ and an element $s \in \mathsf{S}_R$ such that $u_{n_k}^{-1} \to s$ as $k \to \infty$. Finally, $rs = 1 = sr$.

□

Notice that, given a semi-normed module M over a semi-normed ring R, in order for U_M to be dense in B_M, it only suffices that $1 \in \mathrm{cl}(\mathsf{U}_R)$. Indeed, if $(r_n)_{n\in\mathbb{N}} \subseteq \mathsf{U}_R$ converges to 1, then $(r_n m)_{n\in\mathbb{N}}$ converges to m for all $m \in \mathsf{S}_M$ and $\|r_n m\| \leq \|r_n\|\|m\| < 1$ for all $n \in \mathbb{N}$.

Lemma 5. *Let R be a unital semi-normed ring. Then:*

1) If $1 \in \mathrm{cl}\,(\mathcal{U}_1(R) \setminus \mathsf{S}_R)$, then $1 \in \mathrm{cl}(\mathsf{U}_R)$.
2) If B_R is compact and $1 \in \mathrm{cl}\,(\{\|u\| : u \in \mathcal{U}_1(R) \setminus \mathsf{S}_R\})$, then $1 \in \mathrm{cl}(\mathsf{U}_R)$.

Proof. Suppose first that $1 \in \mathrm{cl}\,(\mathcal{U}_1(R) \setminus \mathsf{S}_R)$. Take a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}_1(R) \setminus \mathsf{S}_R$ converging to 1. Notice that if $\|u_n\| > 1$ for some $n \in \mathbb{N}$, then $\left\|u_n^{-1}\right\| = \|u_n\|^{-1} < 1$ therefore, it only suffices to switch u_n^{-1} for u_n whenever $\|u_n\| > 1$ to obtain a sequence in U_R converging to 1. Next, assume that B_R is compact and $1 \in \mathrm{cl}\,(\{\|u\| : u \in \mathcal{U}_1(R) \setminus \mathsf{S}_R\})$. Consider a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}_1(R) \setminus \mathsf{S}_R$ such that $(\|u_n\|)_{n\in\mathbb{N}}$ converges to 1. By using a similar argument as before, we may assume without any loss of generality that $\|u_n\| > 1$ for all $n \in \mathbb{N}$. The compactness of B_R allows us to assume, also without any loss, that $(u_n)_{n\in\mathbb{N}}$ converges to some $r \in \mathsf{S}_R$. For every $n \in \mathbb{N}$, $\left\|1 - ru_n^{-1}\right\| = \left\|u_n u_n^{-1} - ru_n^{-1}\right\| \leq \|u_n - r\| \left\|u_n^{-1}\right\| = \|u_n - r\| \|u_n\|^{-1} \to 0$ as $n \to \infty$. Finally, simply observe that $\left\|ru_n^{-1}\right\| \leq \|r\| \left\|u_n^{-1}\right\| = \|u_n\|^{-1} < 1$ for every $n \in \mathbb{N}$. □

A direct consequence of the first observation of this section together with Lemma 5 is the following theorem, whose proof is omitted and which improves [7].

Theorem 4. *Let M be a semi-normed module over a unital semi-normed ring R. If $1 \in \mathrm{cl}\,(\mathcal{U}_1(R) \setminus \mathsf{S}_R)$, then $\mathrm{cl}(\mathsf{U}_M) = \mathsf{B}_M$.*

In [27], it was proved that the closed unit ball of a unital absolutely semi-valued real algebra is a closed unit neighborhood of zero. The proof of the previous result can be easily adapted to prove that the closed unit ball of a unital real Banach algebra is a closed unit neighborhood of zero. We will extend this result to normed rings with few extra hypotheses. Let M be a semi-normed module over a semi-normed ring R. An element $e \in \mathsf{B}_M$ is said to be an *extreme point* of M provided that the following condition holds: if $e + e = m + n$ with $m, n \in \mathsf{B}_M$, then $e = m = n$; see [22]. According to [28], 1 is an extreme point of the closed unit ball of every unital real or complex Banach algebra.

Theorem 5. *Let R be a complete unital normed ring containing a closed subring S such that $\mathcal{U}(S) = \mathcal{U}_1(S)$ and $0 \in \mathrm{cl}(\mathcal{U}(S))$. If B_R is regular closed, $\mathsf{U}_R = \mathrm{int}(\mathsf{B}_R)$, and 1 is an extreme point of B_R, then B_R is a closed unit neighborhood of 0.*

Proof. By hypothesis, B_R is regular closed and $\mathsf{U}_R = \mathrm{int}(\mathsf{B}_R)$. As a consequence, U_R is regular open. We will prove that U_R is an open unit neighborhood of 0, which suffices to assure that B_R is a closed unit neighborhood of 0 in view of the definition of such notion. First off, U_R is clearly additively symmetric because so is the norm of R. Next, by hypothesis, $1 \in \mathsf{B}_R = \mathrm{cl}(\mathrm{int}(\mathsf{B}_R)) = \mathrm{cl}(\mathsf{U}_R)$. Thus, it only remains to show that U_R is multiplicatively idempotent, that is, $\mathsf{U}_R\mathsf{U}_R = \mathsf{U}_R$. Observe that $\mathsf{U}_R\mathsf{U}_R \subseteq \mathsf{U}_R$ because of the submultiplicative character of the norm of R. Let us show that $\mathsf{U}_R \subseteq \mathsf{U}_R\mathsf{U}_R$. To accomplish this, we will show that $1 \in \mathrm{cl}(\mathsf{U}_R \cap \mathcal{U}(R))$ and apply Lemma 3. Indeed, by hypothesis, there exists a sequence $(s_k)_{k\in\mathbb{N}} \subseteq \mathcal{U}(S)$ convergent to 0. We may assume that $\|s_k\| < 1$ for all $k \in \mathbb{N}$. In view of [19], for every $k \in \mathbb{N}$ we have that $1 - s_k$ and $1 + s_k$ are invertible in R and their inverses are $\sum_{n=0}^{\infty} s_k^n$ and $\sum_{n=0}^{\infty}(-s_k)^n$, respectively. Since S is closed, we conclude that $(1 - s_k)^{-1}, (1 + s_k)^{-1} \in S$ for all $k \in \mathbb{N}$. Observe that by hypothesis, $\mathcal{U}(S) = \mathcal{U}_1(S)$, meaning that $(1 - s_k)^{-1}, (1 + s_k)^{-1} \in \mathcal{U}_1(S)$ for all $k \in \mathbb{N}$. Suppose that there exists $k_0 \in \mathbb{N}$ such that $\left\|1 - s_{k_0}\right\| = \left\|1 + s_{k_0}\right\| = 1$. Since $1 + 1 = (1 + s_{k_0}) + (1 - s_{k_0})$ and 1 is an extreme point of B_R, we deduce that $1 = 1 + s_{k_0} = 1 - s_{k_0}$, which implies the contradiction that $s_{k_0} = -s_{k_0} = 0$. As a consequence, for every $k \in \mathbb{N}$, $1 - s_k$ and $1 + s_k$ cannot have both norm 1 at the same time. Now, we construct the sequence $(u_k)_{k\in\mathbb{N}}$ defined by

$$u_k := \begin{cases} 1 - s_k & \text{if } \|1 - s_k\| < 1, \\ (1 - s_k)^{-1} & \text{if } \|1 - s_k\| > 1, \\ 1 + s_k & \text{if } \|1 - s_k\| = 1 \ \& \ \|1 + s_k\| < 1, \\ (1 + s_k)^{-1} & \text{if } \|1 - s_k\| = 1 \ \& \ \|1 + s_k\| > 1. \end{cases}$$

In virtue of [19], the inverse map $u \mapsto u^{-1}$ in $\mathcal{U}(R)$ is continuous, so both sequences $\left((1 - s_k)^{-1}\right)_{k\in\mathbb{N}}$ and $\left((1 + s_k)^{-1}\right)_{k\in\mathbb{N}}$ converge to 1. As a consequence, $(u_k)_{k\in\mathbb{N}}$ is a sequence in $\mathsf{U}_R \cap \mathcal{U}(R)$ converging to 1. □

Theorem 5 improves considerably [7]. There are plenty of examples of complete unital normed rings containing a closed subring satisfying the hypothesis of Theorem 5. For example, any real or complex Banach algebra in particular, or more generally, any complete unital normed algebra, over a non-discrete absolutely valued field, whose unity is an extreme point.

5. Absolutely invertibles and topological divisors

Given a ring R and an element $s \in R$, the set of left divisors of s is defined as $\mathtt{ld}(s) := \{r \in R : \exists t \in R \setminus \{0\} \quad s = rt\}$. In a similar way, the set of right divisors $\mathtt{rd}(s)$ of s can be defined. In [19, 29–31], topological zero-divisors were deeply studied. Let R be a topological ring and consider $s \in R$. An element $r \in R$ is said to be a topological left divisor of s provided that there exists a subset $T \subseteq R$ such that $0 \notin \mathrm{cl}(T)$ and $s \in \mathrm{cl}(rT)$. The set of topological left divisors of s is denoted by $\mathtt{tld}(s)$. In a similar way, topological right divisors of s can be defined and they are denoted by $\mathtt{trd}(s)$. The following two novel lemmas establish that the condition $0 \notin \mathrm{cl}(T)$ in the previous definition is only necessary when $s = 0$.

Lemma 6. *Let X be a Hausdorff topological space. Let $T \subseteq X$. If $x, y \in \mathrm{cl}(T)$ with $x \neq y$, then there exists a subset $S \subseteq T$ such that $x \in \mathrm{cl}(S)$ and $y \notin \mathrm{cl}(S)$.*

Proof. Since X is Hausdorff, we can take U, V neighborhoods of x, y, respectively, such that $U \cap V = \varnothing$. Notice that $I := \{W \subseteq U : W \text{ is a neighborhood of } x\}$ is a directed set when endowed with the partial order $W_1 \leq W_2 \iff W_2 \subseteq W_1$. Since $x \in \mathrm{cl}(T)$, for every $W \in I$ there exists $t_w \in T \cap W$. It is clear that the net $(t_W)_{W\in I}$ converges to x. However, by construction, $y \notin \mathrm{cl}(\{t_W : W \in I\})$. Finally, simply take $S := \{t_W : W \in I\}$. □

Lemma 6 can be adapted to prove that the condition $0 \notin \mathrm{cl}(T)$ in the definition of topological divisor is only necessary when $s = 0$.

Lemma 7. *Let R be a Hausdorff topological ring. Let $s \in R \setminus \{0\}$. Then,* $\mathtt{tld}(s) = \{r \in R : \exists T \subseteq R \;\; s \in \mathrm{cl}(rT)\}$.

Proof. By definition, it is clear that $\mathtt{tld}(s) \subseteq \{r \in R : \exists T \subseteq R \;\; s \in \mathrm{cl}(rT)\}$. Suppose that $r \in R$ is so that there exists $T \subseteq R$ with $s \in \mathrm{cl}(rT)$. Since R is Hausdorff, we can take U, V neighborhoods of $s, 0$, respectively, such that $U \cap V = \varnothing$. Notice that $I := \{W \subseteq U : W \text{ is a neighborhood of } s\}$ is a directed set when endowed with the partial order $W_1 \leq W_2 \iff W_2 \subseteq W_1$. Since $s \in \mathrm{cl}(rT)$, for every $W \in I$ there exists $t_w \in T$ such that $rt_w \in W$. It is clear that the net $(rt_W)_{W \in I}$ converges to s. However, by construction, $0 \notin \mathrm{cl}(\{t_W : W \in I\})$. Finally, simply take $S := \{t_W : W \in I\}$, and we have that $0 \notin \mathrm{cl}(S)$ but $s \in \mathrm{cl}(rS)$, meaning that $r \in \mathtt{tld}(s)$. □

We will first show the relationships between absolutely invertibles and topological divisors in normed rings.

Theorem 6. *Let R be a unital normed ring. Let $s \in R \setminus \{0\}$. Then,* $\mathrm{cl}(s\mathcal{U}_1(R)) \setminus \{0\} \subseteq \mathtt{tld}(s)$.

Proof. Fix an arbitrary $r \in \mathrm{cl}(s\mathcal{U}_1(R)) \setminus \{0\}$. Take a sequence $(u_n)_{n\in\mathbb{N}} \subseteq \mathcal{U}_1(R)$ such that $(su_n)_{n\in\mathbb{N}}$ converges to r. Since $r \neq 0$, no subsequence of $(u_n)_{n\in\mathbb{N}}$ converges to 0. Hence, there exist n_0 and $K > 0$ such that $\|u_n\| \geq K$ for all $n \geq n_0$. Then, $\left\|u_n^{-1}\right\| = \|u_n\|^{-1} \leq \frac{1}{K}$ for all $n \geq n_0$. Finally, for every $n \geq n_0$,

$$\left\|ru_n^{-1} - s\right\| \leq \|r - su_n\| \left\|u_n^{-1}\right\| \leq \frac{\|r - su_n\|}{K},$$

meaning that $\left(ru_n^{-1}\right)_{n\in\mathbb{N}}$ converges to s. In view of Lemma 7, $r \in \mathtt{tld}(s)$. □

The previous theorem can be reformulated for general topological rings by assuming they are endowed with a left-bounded closed unit neighborhood B of zero.

Theorem 7. *Let R be a Hausdorff topological ring endowed with a left-bounded unit ball B. Let $s \in R \setminus \{0\}$. Then,* $\mathrm{cl}(s\mathcal{U}_1(R)) \setminus \{0\} \subseteq \mathtt{tld}(s)$.

Proof. Let V_0 be a fixed neighborhood of 0. By hypothesis, there exist a 0-neighborhood W such that $WW \subseteq V_0$ and $u_0 \in \mathcal{U}(R)$ such that $u_0B \subseteq W$. Let $x \in \mathrm{cl}(s\mathcal{U}_1(R)) \setminus \{0\}$ and denote $T := \mathcal{U}_1(R)$. It follows that $(x + Wu_0) \cap sT \neq \emptyset$. As a consequence, there exists $t \in \mathcal{U}_1(R)$ such that $st - x \in Wu_0$, or, equivalently, $s - xt^{-1} \in W(u_0t^{-1}) \subseteq W(u_0B) \subseteq WW \subseteq V_0$. This implies $(s + V_0) \cap xT \neq \emptyset$ and then $x \in \mathtt{tld}(s)$ by Lemma 7. □

If the Hausdorff hypothesis is dropped from Theorem 6, then we need to ensure that $0 \notin \mathrm{cl}(\mathcal{U}_1(R))$. However, many algebras, such as $\mathcal{B}(H)$, trivially satisfy that $0 \in \mathrm{cl}(\mathcal{U}_1(\mathcal{B}(H)))$, since $\mathcal{U}_1(\mathcal{B}(H))$ is precisely the positive multiples of the surjective linear isometries on H (see [19]).

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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