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Research article

Prescribed-time trajectory tracking control for a class of nonlinear system

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Abstract: Previous works have analyzed finite/fixed-time tracking control for nonlinear systems. In these works, achieving the accurate time convergence of errors must be under the premise of known initial values and careful design of control parameters. Then, how to break through the constraints of initial values and design parameters for this issue is an unsolved problem. Motivated by this, we successfully studied prescribed-time tracking control for single-input single-output nonlinear systems with uncertainties. Specifically, we designed a state feedback controller on $[0, T_p)$, based on the backstepping method, to make the tracking error (TE) tend to zero at T_p, in which T_p is the arbitrarily selected prescribed-time. Furthermore, on $[T_p, \infty)$, another controller, similarly to that on $[0, T_p)$, was designed to keep TE within a precision after T_p, while TE may not stay at zero. Therefore, on $[T_p, \infty)$, another new controller, based on sliding mode control, was built to ensure that TE stays at zero after T_p.

Keywords: prescribed-time control; backstepping method; nonlinear system; sliding mode control; trajectory tracking

1. Introduction

The stability of automatic systems, being a significant property, has been extensively studied for several decades. Compared with asymptotic stability [1–7], finite-time stability is more practical, having been first introduced in [8]. Sliding mode control (SMC), a classical method, can achieve finite-time stability effectively due to its simplicity and high robustness. Furthermore, high-order sliding modes and terminal sliding modes were developed in subsequent research [9–11]. However, due to the discontinuous switching function, high-frequency chattering occurs in sliding mode control, restricting its practical application. To overcome this flaw, a method based on Lyapunov differential inequality was proposed to realize finite-time control [12,13]. For finite-time stability, although the system can converge to the equilibrium solution in a finite time, the estimation of the settling time always requires knowledge about the initial conditions. Nevertheless, in many real cases, the initial system conditions are unavailable, such as in state monitoring exceptions for unmanned aerial vehicles caused by sensor failure [14]. As a result, fixed-time stability (FxT stability) was introduced, in which the upper bound of settling time can be presented irrespective of initial values. The FxT control can be traced back to [15], which used polynomial feedback and modifications of the second-order SMC algorithm to achieve the stabilization of the linear system. Later, the FxT control, implemented by odd-order plus fractional-order feedback, was applied to higher-order nonlinear systems and multi-agent systems [16,17].

Although the FxT scheme can provide the upper bound of settling time by adjusting control parameters, choosing proper parameters is a complex problem. Furthermore, achieving convergence at a desired moment by FxT control is almost impossible. To overcome this, the prescribed-time (PT) stability was introduced [18] by converting the original system to a new one via a time-varying transformation; the desired convergent time was independent of arbitrarily designed parameters and initial conditions. In recent years, this interesting feature has attracted more and more attention [19–33], and various methods of prescribed-time control have been proposed, such as nonlinear feedback [21], extracting the characteristics of systems to design the controller [22], parametric Lyapunov equation [23,24], SMC [25], time transformation function [26], and Lyapunov differential inequality [29].

For special strict-feedback systems, our research focus, some results on PT control have been reported [27,28,30]. With the development of PT control, the PT tracking issue has emerged. By importing a new time-varying function, the PT tracking issue of nonlinear systems was achieved [31]. Based on the Barrier Lyapunov function, PT tracking control with pre-set properties for known nonlinear systems was achieved [32]. In [33], authors investigated PT tracking for completely certain systems in strict feedback form using the backstepping method. However, it should be noted that systems are not always fully observed in many cases. For systems with uncertainties, such as unknown functions and disturbance, the methods in [32,33] seem to be limited. Hence, a natural question arises: For nonlinear systems with uncertainties, can the prescribed-time tracking control be realized by the single-control approach, such as the backstepping approach, at the prescribed time and afterward? If not, another question emerges: can hybrid control approaches, such as the backstepping method and sliding mode control, be combined to achieve it? Answering these two questions is the main motivation of this paper.

Based on the above discussions, we will explore the single backstepping approach and the combined approach of backstepping and SMC to implement the PT tracking control for the single-input single-output (SISO) nonlinear system accompanied by uncertainties. The detailed contributions are listed as follows:

1) For the SISO system accompanied by unknown functions and disturbance, a state feedback controller on $[0, T_p)$, by backstepping method, is designed to make the tracking error (TE) tend to zero

at the prescribed time T_p (not dependent on initial values and designed parameters). Additionally, on $[0, T_p)$, the controller can always be kept bounded.

2) A further controller on $[T_p, \infty)$, by backstepping method similarly to 1), is imported to keep the TE within a precision after T_p. It should be pointed out that the TE may not stay at zero after T_p, which is the motivation of 3).

3) Another further controller on $[T_p, \infty)$, by SMC, is introduced to guarantee that the TE stays at zero after T_p, which compensates the deficiency of 2).

2. Preliminaries

Consider the SISO nonlinear system as follows:

$$\begin{cases} \dot{x}_j = x_{j+1} + f_j(\bar{x}_j), j = 1, \ldots, n-1, \\ \dot{x}_n = g(x)u + f_n(x) + d(t), \\ y = x_1, \end{cases} \tag{1}$$

in which $x_j \in R^n$ is the state, $g(x) \neq 0$ is a known continuous function, $x = [x_1, \ldots, x_n]^T \in R^n$ and $f_j(\bar{x}_j)$, $\bar{x}_j = [x_1, \ldots, x_j]^T \in R^j, j = 1, \ldots, n-1$ are known nonlinear and continuous functions, $f_n(x)$ is an unknown nonlinear continuous function, $u \in R$ denotes the control input, $d(t)$ denotes a bounded continuous disturbance, and $y \in R$ denotes the output.

The goal is to design a controller to allow $y(t)$ to track the pre-set trajectory $x_{1d}(t)$. To implement our control scheme, the following lemma and assumptions are listed.

Lemma 1 ([34])**:** Consider a scalar differential equation:

$$\dot{H}(s) = F(H(s), s), H(s_0) = H(0),$$

in which $F(H(s), s)$ is the continuity on s and local Lipschitz continuity on $H(s)$, for $\forall s \geq 0$, $H(s) \in M \subset R$. Denote $[s_0, T)$ by the maximal interval for solution $H(s)$, and suppose $H(s) \in M \subset R$ for $\forall s \in [s_0, T)$. Assume $Q(s)$ is continuous with Dini upper right-hand $D^+Q(s)$,

$$D^+Q(s) \leq F(H(s), s), Q(s_0) \leq H(s_0),$$

with $Q(s) \in M \subset R$ for $\forall s \in [s_0, T)$. Then, $Q(s) \leq H(s), \forall s \in [s_0, T)$.

Assumption 1: A smooth and bounded function $\bar{f}(x)$ and number λ exist for

$$|f_n(x)| \leq \bar{f}(x), |d(t)| \leq \lambda.$$

Assumption 2: For $\forall j = 1, \ldots, n$, $f_j(\bar{x}_j)$ and $x_{1d}(t)$ is smooth enough to be differentiable of order n.

This assumption is a general consideration of the backstepping design, see [32,33,35,36].

Before designing the controller, let us define the following n-dimensional error variables:

$$\begin{aligned} e_1 &= x_1 - \eta_1, \\ e_j &= x_j - \eta_j,\ j = 2, \ldots, n, \end{aligned} \tag{2}$$

where $\eta_1 = x_{1d}$ is the desired trajectory we will track, and $\eta_j, j = 2, \ldots, n$ are the virtual controllers that will be designed later.

3. Main results

3.1. Controller design based on a backstepping approach

In this section, we will design a controller to implement the PT tracking control for system (1). The controller is divided into two phases–prescribed-time tracking controller u_p when $0 \le t < T_p$ and infinite-time tracking controller $\tilde{u}$ when $t \ge T_p$.

Case 1 ($0 \le t < T_p$): We use the backstepping method to design the controller at this stage. It contains n steps.

Step 1: The derivative of e_1 is

$$\dot{e}_1 = e_2 + \eta_2 + f_1(\bar{x}_1) - \dot{\eta}_1, \tag{3}$$

According to Eq (3), we design the virtual controller η_2 as

$$\eta_2 = -\frac{ke_1}{T_p - t} + \dot{\eta}_1 - f_1(\bar{x}_1), \tag{4}$$

where $k > n > 0$, then

$$\dot{e}_1 = e_2 - \frac{ke_1}{T_p - t}. \tag{5}$$

Choose

$$V_1 = 0.5e_1^2, \tag{6}$$

then, along Eq (3),

$$\dot{V}_1 = -\frac{ke_1^2}{T_p - t} + e_1 e_2. \tag{7}$$

Step 2: The derivative of e_2 is

$$\dot{e}_2 = e_3 + \eta_3 - \dot{\eta}_2 + f_2(\bar{x}_2), \tag{8}$$

then the virtual controller η_3 is designed as

$$\eta_3 = -\frac{ke_2}{T_p - t} - e_1 + \dot{\eta}_2 - f_2(\bar{x}_2). \tag{9}$$

Substituting Eq (9) into Eq (8), we can get

$$\dot{e}_2 = e_3 - e_1 - \frac{ke_2}{T_p - t}. \tag{10}$$

Introduce Lyapunov function $V_2 = V_1 + 0.5e_2^2$, then the derivative of V_2 satisfies

$$\dot{V}_2 = \dot{V}_1 + e_2\dot{e}_2 = -\frac{ke_1^2}{T_p - t} - \frac{ke_2^2}{T_p - t} + e_2 e_3. \tag{11}$$

Step $j (j = 3, \dots n - 1)$: From Eq (7) to Eq (11), we can get the derivative of e_j:

$$\dot{e}_j = e_{j+1} + \eta_{j+1} - \dot{\eta}_j + f_j(\bar{x}_j). \tag{12}$$

The η_{j+1} can be set as

$$\eta_{j+1} = -\frac{ke_j}{T_p-t} - e_{j-1} + \dot{\eta}_j - f_j(\bar{x}_j). \tag{13}$$

Then

$$\dot{e}_j = e_{j+1} - e_{j-1} - \frac{ke_j}{T_p-t}. \tag{14}$$

Select $V_j = V_{j-1} + 0.5{e_j}^2$ and its derivative satisfies

$$\dot{V}_j = -\frac{k}{T_p-t}\sum_{i=1}^{j} e_i^2 + e_j e_{j+1}. \tag{15}$$

Step n: From Eq (2), the derivative of e_n is

$$\dot{e}_n = g(x)u + f_n(x) + d(t) - \dot{\eta}_n. \tag{16}$$

Choosing Lyapunov function as $V = V_{n-1} + 0.5{e_n}^2$, the controller u_p is designed

$$u_p = g(x)^{-1}\left(-\frac{ke_n}{T_p-t} - e_{n-1} - \frac{e_n(\bar{f}(x)+\lambda)^2}{2\delta} + \dot{\eta}_n\right), \tag{17}$$

where δ is a positive constant.

From Eqs (15)–(17), the derivatives of e_n and V_n satisfy

$$\dot{e}_n = -\frac{ke_n}{T_p-t} - e_{n-1} - \frac{e_n(\bar{f}(x)+\lambda)^2}{2\delta} + f_n(x) + d(t),$$

$$\begin{aligned} \dot{V}_n &= \dot{V}_{n-1} - \frac{ke_n^2}{T_p-t} + f_n(x)e_n + d(t)e_n - \frac{e_n^2(\bar{f}(x)+\lambda)^2}{2\delta} + e_n e_{n-1} \\ &= -\frac{k}{T_p-t}\sum_{j=1}^{n} e_j^2 + f_n(x)e_n + d(t)e_n - \frac{e_n^2(\bar{f}(x)+\lambda)^2}{2\delta} \\ &\le -\frac{k}{T_p-t}\sum_{j=1}^{n} e_j^2 + \bar{f}(x)|e_n| + \lambda|e_n| - \frac{e_n^2(\bar{f}(x)+\lambda)^2}{2\delta}. \end{aligned} \tag{18}$$

By Young's inequality, based on Eq (18), we can get

$$\dot{V}_n \le -\frac{k}{T_p-t}\sum_{j=1}^{n} e_j^2 + \frac{\delta}{2} = -2k\frac{V_n}{T_P-t} + \frac{\delta}{2}. \tag{19}$$

Case 2 ($t \ge T_p$): For this case, we will design an infinite-time controller to ensure the tracking effect when $t \ge T_p$. First, define a new serious of error variables

$$\begin{aligned} z_1 &= e_1 = x_1 - \eta_1, \\ z_j &= x_j - \tilde{\eta}_j, j = 2,\ldots,n. \end{aligned} \tag{20}$$

Step 1: Easily get

$$\dot{z}_1 = z_2 + \tilde{\eta}_2 - \dot{\eta}_1 + f_1(\bar{x}_1). \tag{21}$$

According to Eq (20), we design the new virtual controller $\tilde{\eta}_2$ as

$$\tilde{\eta}_2 = -\sigma z_1 + \dot{\eta}_1 - f_1(\bar{x}_1), \tag{22}$$

where $\sigma > 0$ is a positive constant, then $\dot{z}_1$ becomes

$$\dot{z}_1 = z_2 - \sigma z_1. \tag{23}$$

Choose

$$\tilde{V}_1 = 0.5{z_1}^2, \tag{24}$$

then along Eq (20),

$$\dot{\tilde{V}}_1 = z_1 \dot{z}_1 = -\sigma {z_1}^2 + z_1 z_2. \tag{25}$$

Similarly to Case 1, we can design the subsequent backstepping as follows:
Step j $(j = 2, \ldots, n)$: one can get

$$\dot{z}_j = z_{j+1} + \tilde{\eta}_{j+1} - \dot{\tilde{\eta}}_j + f_j(\bar{x}_j). \tag{26}$$

The $\tilde{\eta}_{j+1}$ can be set as

$$\tilde{\eta}_{j+1} = -\sigma z_j - z_{j-1} + \dot{\tilde{\eta}}_j - f_j(\bar{x}_j). \tag{27}$$

Then

$$\dot{z}_j = z_{j+1} - z_{j-1} - \sigma z_j. \tag{28}$$

Select $\tilde{V}_j = \tilde{V}_{j-1} + 0.5{z_j}^2$, and its derivative satisfies

$$\dot{\tilde{V}}_j = -\sigma \sum_{i=1}^{j} {z_i}^2 + z_j z_{j+1}. \tag{29}$$

Step n: From Eq (20),

$$\dot{z}_n = g(x)u + f_n(x) + d(t) - \dot{\tilde{\eta}}_n. \tag{30}$$

Choose $\tilde{V}_n = \tilde{V}_{n-1} + 0.5{z_n}^2$, the controller $\tilde{u}$ is designed as

$$\tilde{u} = g(x)^{-1}\left(-\sigma z_n - z_{n-1} - \frac{z_n(\bar{f}(x)+\lambda)^2}{2\delta} + \dot{\tilde{\eta}}_n\right). \tag{31}$$

From Eqs (29)–(31),

$$\dot{\tilde{V}}_n \leq -\sigma \sum_{j=1}^{n} {z_j}^2 + \frac{\delta}{2} = -2\sigma \tilde{V}_n + \frac{\delta}{2}. \tag{32}$$

Remark 1: We adopt different control gains in Cases 1 and 2 according to different control targets. The prescribed-time controller aims to make the error converge to zero at any desired prescribed-time T_p, hence a time-varying infinite gain function $k(T_p - t)^{-1}$ is imported into the controller design of Case 1. On the other hand, in Case 2, our goal is to keep the convergence of the tracking error at T_p, so a constant σ is adopted as control gain.

Based on the above analysis, we can get the following assertion.

Theorem 1: If Assumptions 1 and 2 hold, one can design the controllers (17) and (31) such that the system (1) can track the pre-set trajectory $x_{1d}(t)$ within the prescribed time T_p and keep tracking within a range (i.e., the tracking error is bounded). Controller (17) is bounded when $t \to {T_p}^-$.

Proof: The proof is divided into two parts due to complexity.

Part 1: Based on Eq (19), we provide the proof of prescribed-time convergence for $\dot{V}_n = -2k\frac{V_n}{T_P-t} + \frac{\delta}{2}$, and then apply Lemma 1 to the case of $\dot{V}_n < -2k\frac{V_n}{T_P-t} + \frac{\delta}{2}$.

It can be obtained by $\dot{V}_n = -2k\frac{V_n}{T_P-t} + \frac{\delta}{2}$ that

$$V_n(t) = C(1-\frac{t}{T_p})^{2k} + \frac{\delta}{2(2k-1)}\left(T_p - t\right), \tag{33}$$

where $C = \frac{V_n(e_j(0))}{{T_p}^{2k}} - \frac{\delta}{2(2k-1){T_p}^{2k-1}}$ is a constant. It can easily be found that $\lim_{t\to {T_p}^-} V_n(t) = 0$. For $V_n = \frac{1}{2}\sum_{j=1}^{n} e_j^2$, it means $\lim_{t\to {T_p}^-} e_j = 0, j = 1,\ldots,n.$

Then, we will verify the boundness of the virtual controllers η_j and u.

Based on Eq (5), we can get

$$e_1(t) = e_1(0)(1-\frac{t}{T_p})^k + (T_p - t)^k \int_0^t \frac{e_2(s)}{(T_p-s)^k} ds, \tag{34}$$

then we have

$$\frac{e_1(t)}{T_p-t} = e_1(0){T_p}^{-1}(1-\frac{t}{T_p})^{k-1} + (T_p - t)^{k-1} \int_0^t \frac{e_2(s)}{(T_p-s)^k} ds. \tag{35}$$

By L'Hôpital's rule for $(T_p - t)^{k-1} \int_0^t \frac{e_2(s)}{(T_p-s)^k} ds$, we can obtain

$$\lim_{t\to {T_p}^-} \frac{e_1(t)}{T_p-t} = \lim_{t\to {T_p}^-} \frac{e_2(t)}{k-1} = 0, \tag{36}$$

which implies η_2 is bounded when $t \to {T_p}^-$from Eq (4). According to Eqs (5) and (36), we know $\lim_{t\to {T_P}^-} \dot{e}_1(t) = 0.$

By the same method, we can get

$$e_j(t) = e_j(0)(1-\frac{t}{T_p})^k + (T_p - t)^k \int_0^t \frac{e_{j+1}(s)-e_{j-1}(s)}{(T_p-s)^k} ds, j = 2,\ldots,n-1,$$

$$e_n(t) = e_n(0){T_p}^{-k}(T_p - t)^k \omega(t)^{-1} + (T_p - t)^k \omega(t)^{-1} \int_0^t \frac{(f_n(x(s)) + d(s) - e_{n-1}(s))\omega(s)}{(T_p-s)^k} ds,$$

$$\lim_{t\to {T_p}^-} \frac{e_j(t)}{T_p-t} = \lim_{t\to {T_p}^-} \frac{e_{j+1}(t)-e_{j-1}(t)}{k-1} = 0, j = 2, \ldots, n-1,$$

$$\lim_{t\to {T_p}^-} \frac{e_n(t)}{T_p-t} = \lim_{t\to {T_P}^-} \frac{(f_n(x(t)) + d(t) - e_{n-1}(t))\omega(t)}{k-1} \le \frac{(\bar{f}(x(T_p)) + \lambda)\omega(T_p)}{k-1}, \tag{37}$$

where $\omega(t) = e^{\int_0^t \frac{(\bar{f}(x(s))+\lambda)^2}{2\delta} ds}$, which means that all virtual controllers η_j and u_p are bounded when time $t \to {T_p}^-$ and $\lim_{t \to {T_P}^-} \dot{e}_j(t) = 0, j = 2, \ldots n$.

Part 2: From Eqs (34) and (37), using L'Hôpital's rule, we can get

$$\lim_{t \to {T_p}^-} \frac{e_1(t)}{(T_p - t)^m} = \lim_{t \to {T_p}^-} \frac{1}{k-m} \frac{e_2(t)}{(T_p - t)^{m-1}},$$

$$\lim_{t \to {T_p}^-} \frac{e_j(t)}{(T_p - t)^m} = \lim_{t \to {T_p}^-} \frac{1}{k-m} \left[\frac{e_{j+1}(t)}{(T_p - t)^{m-1}} - \frac{e_{j-1}(t)}{(T_p - t)^{m-1}}\right], j = 2, \ldots, n-1, \tag{38}$$

for any integer $m \in (0, k)$.

From Eq (38), we have

$$\lim_{t \to {T_p}^-} \frac{e_1(t)}{T_p - t} = \lim_{t \to {T_p}^-} \frac{e_2(t)}{k-1} = 0,$$

$$\lim_{t \to {T_p}^-} \frac{e_1(t)}{(T_p - t)^2} = \lim_{t \to {T_p}^-} \frac{1}{k-2} \frac{e_2(t)}{(T_p - t)} = 0,$$

$$\lim_{t \to {T_p}^-} \frac{e_1(t)}{(T_p - t)^3} = \lim_{t \to {T_p}^-} \frac{1}{k-3} \frac{e_2(t)}{(T_p - t)^2} = \lim_{t \to {T_p}^-} \frac{1}{k-3} \frac{1}{k-2} \left[\frac{e_3(t)}{T_p - t} - \frac{e_1(t)}{T_p - t}\right] = \lim_{t \to {T_p}^-} \frac{1}{k-3} \frac{1}{k-2} \frac{e_3(t)}{(T_p - t)} = 0. \tag{39}$$

Then, we can easily obtain

$$\lim_{t \to {T_p}^-} \frac{e_1(t)}{(T_p - t)^n} = \lim_{t \to {T_p}^-} \frac{(k-m)!}{(k-1)!} \frac{e_n(t)}{(T_p - t)}. \tag{40}$$

Due to the fact that $\lim_{t \to {T_p}^-} \frac{e_n(t)}{T_p - t}$ is bounded, from Assumption 2, using L'Hôpital's rule to Eq (40), we can get

$$\lim_{t \to {T_p}^-} \frac{e_1(t)^{(j)}(n-j)!}{(-1)^j n! (T_p - t)^{n-j}} = 0, j = 1, \ldots, n-1, \tag{41}$$

which means $\lim_{t \to {T_p}^-} e_1(t)^{(j)} = \lim_{t \to {T_p}^-} z_1(t)^{(j)} = 0, j = 1, \ldots, n-1$. From Eq (23), we can get

$$\dot{z}_1(T_p) = z_2(T_p) - \sigma z_1(T_p) = 0, \tag{42}$$

then $z_2(T_p) = 0$. From Eq (28), we can get

$$\ddot{z}_1(T_p) = z_3(T_p) - z_1(T_p) - \sigma z_2(T_p) - \sigma \dot{z}_1(T_p) = 0, \tag{43}$$

then $z_3(T_p) = 0$. Similarly, we can obtain $z_j(T_p) = 0, j = 4, \ldots, n$, which implies $\tilde{V}_n(T_p) = 0$.

According to Lemma 1, from Eq (32),

$$\tilde{V}_n(t) \leq (\tilde{V}_n(T_p) - \frac{\delta}{4\sigma}) e^{-2\sigma(t - T_p)} + \frac{\delta}{4\sigma} \leq \frac{\delta}{4\sigma}, \tag{44}$$

which implies $z_1 \leq \sqrt{\frac{\delta}{2\sigma}}$ when $t \geq T_p$. The proof is completed.

Remark 2: The designed controller based on the backstepping approach u_p can make the system error converge to zero at T_p, which is consistent with the effect achieved in [33]. That means we can achieve tracking control of nonlinear systems with uncertain function and disturbance by the backstepping approach. Nevertheless, we must point out that the effect of the unknown function $f_n(x)$ and disturbance $d(t)$ cannot be precisely eliminated, and that the tracking error cannot be maintained at zero after T_p, which can be later illustrated by our example. Of course, we can adjust the control parameters to keep the tracking error within a controllable range. This gives a direct answer to the first question mentioned above.

3.2. Controller design based on the backstepping approach and SMC

It should be noted that, from Theorem 1, the single backstepping approach can only ensure that the error is within a controllable range after the prescribed time rather than staying at zero. However, some practical applications need to ensure accurate tracking, such as the tracking control of spacecraft [25]. This means that, other than the backstepping approach, other approaches need to be imported to guarantee the tracking error remains at zero after the prescribed time. Considering that SMC has a great advantage in countering disturbances through the input channel, SMC method is imported to the design controller, so as to achieve zero error tracking when $t \geq T_p$. The controller design is as follows.

When $0 \leq t < T_p$, the controller design based on the backstepping method can be referred to Case 1 of Section 3.1. Then, we mainly design the controller using SMC to ensure that the tracking error is kept at zero.

First, we use $e_1(t)$ and its derivatives to build the following system for the SMC design:

$$\xi_1(t) = e_1(t) = x_1(t) - x_{1d}(t),$$

$$\xi_j(t) = \dot{\xi}_{j-1}(t) = e_1(t)^{(j-1)}, j = 2, \ldots, n. \tag{45}$$

The SMC variable is set as

$$s(t) = \sum_{j=1}^{n} \xi_j(t), j = 1, \ldots, n, \tag{46}$$

then

$$\dot{s} = \sum_{j=2}^{n} x_j + \sum_{j=1}^{n-1} f_j(\bar{x}_j) - \sum_{j=1}^{n} x_{1d}{}^{(j)} + \sum_{l=1}^{n-1} \sum_{j=1}^{n-l} f_j(\bar{x}_j)^{(l)} + g(x)u + f_n(x) + d(t). \tag{47}$$

The controller u is designed as

$$u_s = g(x)^{-1}[-(\bar{f}(x) + \lambda)\, sgn(s) - \varepsilon s - \sum_{j=2}^{n} x_j - \sum_{j=1}^{n-1} f_j(\bar{x}_j) + \sum_{j=1}^{n} x_{1d}{}^{(j)} - \sum_{l=1}^{n-1} \sum_{j=1}^{n-l} f_j(\bar{x}_j)^{(l)}], \tag{48}$$

where $\varepsilon > 0$ is a constant.

Theorem 2: If Assumptions 1 and 2 hold, one can design the controllers (17) and (48) such that the system (1) can track the pre-set trajectory $x_{1d}(t)$ within the prescribed time T_p and keep the tracking error stay at zero after T_p, and controller (17) is bounded when $t \to T_p{}^{-}$.

Proof: Substituting Eq (48) into Eq (47), we can obtain

$$\dot{s} = f_n(x) + d(t) - (\bar{f}(x) + \lambda)\, sgn(s) - \varepsilon s. \tag{49}$$

Choose $V_s = 0.5s^2$, then from Assumption 2, the derivative of V_s satisfies

$$\dot{V}_s = (f_n(x) + d(t))s - (\bar{f}(x) + \lambda)s\, sgn(s) - \varepsilon s^2 \leq 0. \tag{50}$$

From assertions Eq (38) to Eq (41), we have $\xi_1(T_p)^{(j-1)} = \xi_j(T_p) = \dot{\xi}_{j-1}(T_p) = 0, j = 2, \ldots, n,$ which means error system (45) converge to zero when $t = T_p$. Additionally, from Eq (50), we can obtain that error $\xi_1(t)$ stays at zero when $t \geq T_p$. The proof is completed.

Remark 3: It can be shown from Theorems 1 and 2 that, compared with the infinite time controller (31), the sliding mode controller (48) introduces a sign function sign(·) to forcibly cancel the effects of unknown functions and perturbation, and then zero tracking error is achieved. Of course, the sliding mode controller (48) requests greater control costs due to the introduction of a sign function. Therefore, in the application, we can choose the appropriate controller, whether infinite controller (31) or sliding mode controller (48), according to the practiced requirements.

Remark 4: On the FxT tracking control scheme, the tracking error can converge to zero without relying on the initial state. However, it is necessary to adjust the design parameters carefully to achieve arbitrary time convergence through the FxT method, see [35,36]. In our control scheme, the error convergence can be realized at any desired time without considering any initial conditions and designed parameters, which is simpler and more convenient. Contrary to [18,19], the prescribed-time control method here is feasible for $t \in [0, \infty)$ rather than only valid for $[0, T_p)$. Prescribed-time attitude tracking control of spacecraft was proposed by SMC [25], where the upper bound of the uncertainty is a constant. Compared to [25] and [33], our assumptions are broader, which makes our method applicable to more general systems.

Remark 5: Compared with predefined-time control giving an upper bound on settling time [37], prescribed-time control can give an exact settling time. The prescribed-time tracking control, as an application of prescribed-time control, also gives an exact settling time. For the difference between predefined-time control and prescribed-time control, readers are referred to [38]. Prescribed performance control, such as [39], always refers to a control method with system state and convergence speed as expected, with no explicit requirement for settling time. The advantage of our proposed method lies in that it can give an exact settling time that does not depend on control parameters and initial values.

4. Simulation results

Consider a nonlinear system as follows

$$\begin{cases} \dot{x}_1 = x_2 + {x_1}^{2/3} - 0.5x_1, \\ \dot{x}_2 = ({x_1}^2 + 1)u + ax_1 + b\, sin(x_1 x_2) + c\, cos\, t, \\ y = x_1, \end{cases} \tag{51}$$

where $ax_1 + b\, sin(x_1x_2)$ and $c\, cos(t)$ represent the uncertain function $f_n(x)$ and disturbance $d(t)$, respectively, $g(x) = {x_1}^2 + 1$. The target trajectory is selected as $x_{1d} = sin(t)$. Here, we select the parameters as $a = 0.1$, $b = 1$, $c = 0.2$. Then, we can obtain the upper bound of $f_n(x)$ and $d(t)$ as follows:

$$\bar{f}(x) = 0.1|x_1| + 1,$$

$$\lambda = 0.2.$$

Based on the proposed method, the prescribed-time controller u_p can be designed as

$$u_p = \frac{1}{{x_1}^2+1}\left[\frac{-ke_2}{T_p-t} - e_1 + \dot{\eta}_2 - \frac{e_2(1+0.1|x_1|+0.2)^2}{2\delta}\right], \tag{52}$$

where $\eta_2 = \frac{-k(x_1-x_{1d})}{T_p-t} + \dot{x}_{1d} - {x_1}^{2/3} + 0.5x_1$, $e_1 = x_1 - x_{1d}$, $e_2 = x_2 - \eta_2$.

The infinite time controller $\tilde{u}$ is

$$\tilde{u} = -\sigma z_2 + \dot{\tilde{\eta}}_2 - z_1 - \frac{z_2(1+0.1|x_1|+0.2)^2}{2\delta}, \tag{53}$$

where $\tilde{\eta}_2 = -\sigma z_2 + \dot{x}_{1d} - {x_1}^{2/3} + 0.5x_1$, $z_1 = x_1 - x_{1d}$, $z_2 = x_2 - \tilde{\eta}_2$.

The sliding mode controller u_s is designed as

$$u_s = \frac{1}{{x_1}^2+1}\left[-(1.2+0.1|x_1|)\,sgn(s) - \varepsilon s + \dot{x}_{1d} - x_2 + \ddot{x}_{1d} - {x_1}^{2/3} + 0.5x_1 - \left(\tfrac{2}{3}{x_1}^{-1/3} + 0.5\right)\dot{x}_1\right], \tag{54}$$

where $s = \xi_1 + \dot{\xi}_1$, $\xi_1 = x_1 - x_{1d}$. We design two examples with different initial states and control parameters for simulation.

Example 1: The controller parameters are selected as $k = 4$, $\delta = 0.09$, $\sigma = 0.5$, $T_p = 4$, $\varepsilon = 0.1$ and the initial state is $[2, 2]$; then, the track error under the action of $\tilde{u}$ satisfies $|e_1| \le 0.3$ when $t \ge 4$.

Example 2: The controller parameters are selected as $k = 3$, $\delta = 0.04$, $\sigma = 2$, $T_p = 4$, $\varepsilon = 0.2$ and the initial state is $[3, 2.5]$; then, the track error under the action of $\tilde{u}$ satisfies $|e_1| \le 0.1$ when $t \ge 4$.

The simulation results of the two examples under the action of the prescribed-time controller u_p and the infinite time controller $\tilde{u}$ are shown in Figures 1–4.

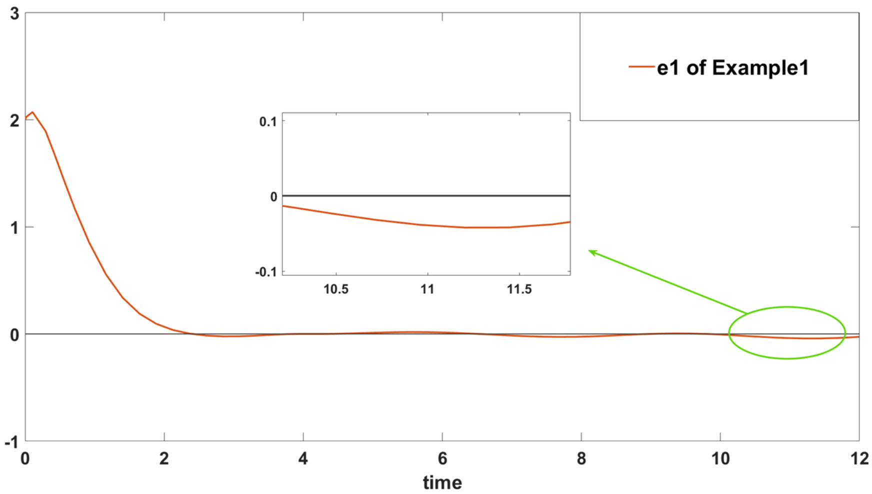

Figure 1. Simulation of e_1 under controllers u_p and $\tilde{u}$ for Example 1.

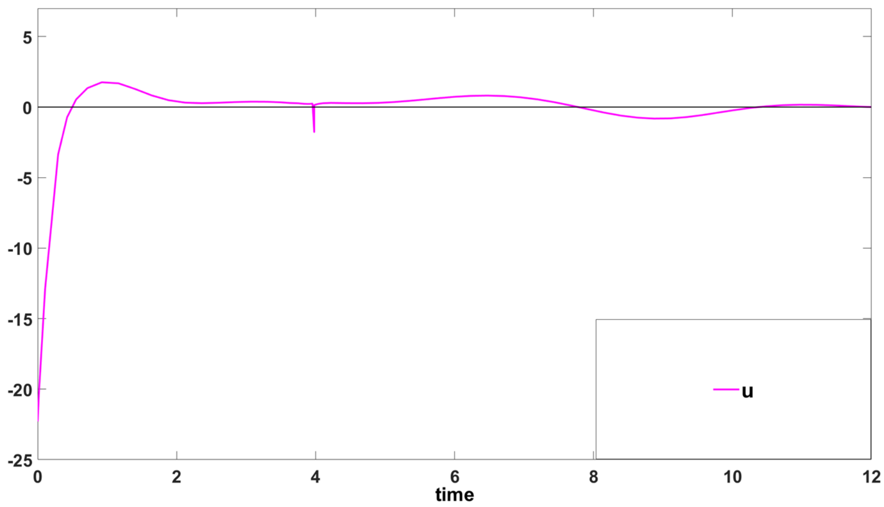

Figure 2. Simulations of u_p and $\tilde{u}$ for Example 1.

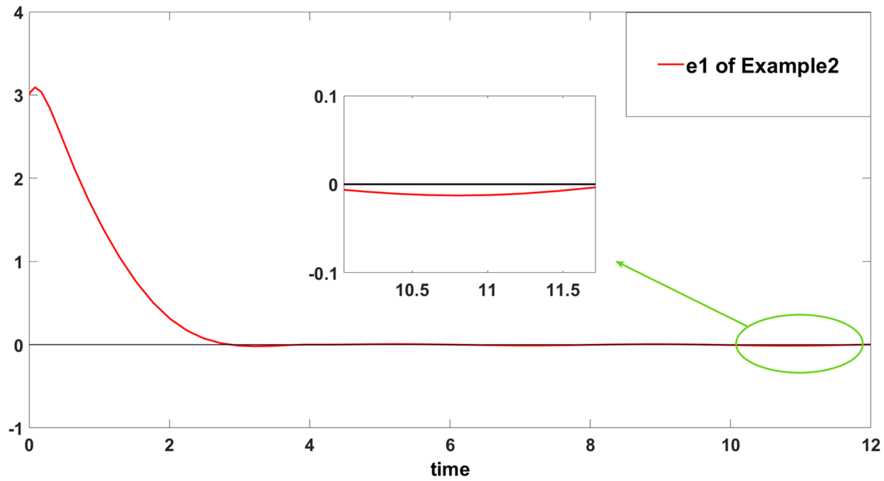

Figure 3. Simulation of e_1 under controllers u_p and $\tilde{u}$ for Example 2.

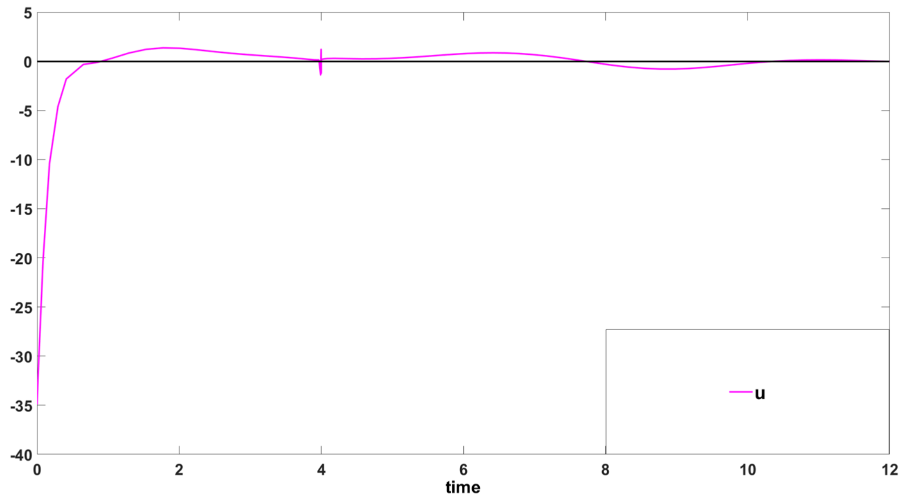

Figure 4. Simulations of u_p and $\tilde{u}$ for Example 2.

As we can see from Figures 1 and 3, although the initial conditions and designed parameters of two examples are different, all tracking errors converge to zero at $T_p = 4$, and the output y keeps tracking $x_{1d}(t)$ after T_p with the precision being less than $\sqrt{\delta / 2\sigma}$ under the single action of backstepping control, which is consistent with Remark 2. Additionally, Figures 2 and 4 illustrate that all control inputs u are bounded for $t \to T_p$.

In order to verify the effectiveness of the sliding mode controller u_s, Figures 5–8 provide the simulation results of two examples under the action of the prescribed-time controller u_p and the sliding mode controller u_s.

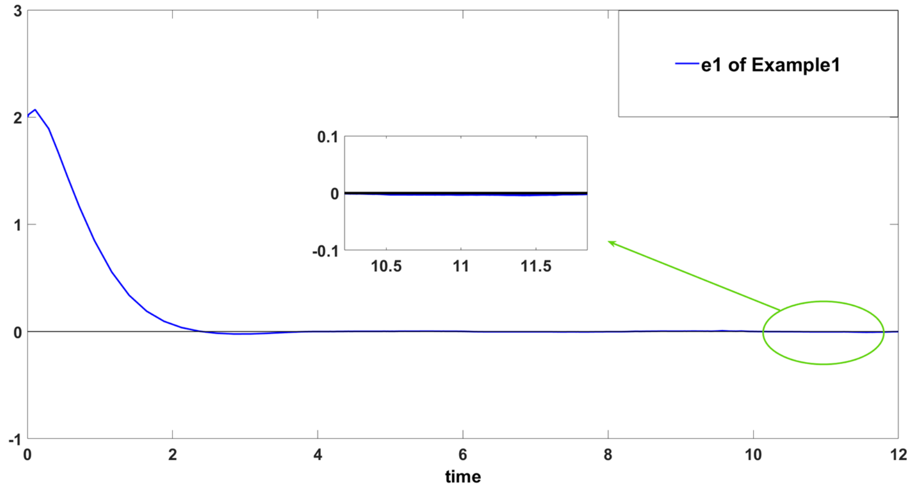

Figure 5. Simulation of $e_1$under controllers u_p and u_s for Example 1.

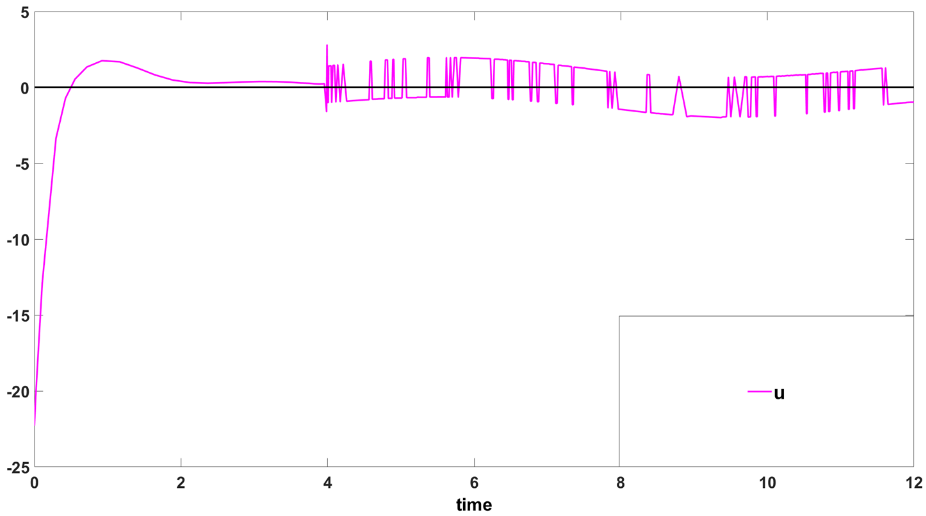

Figure 6. Simulations of u_p and u_s for Example 1.

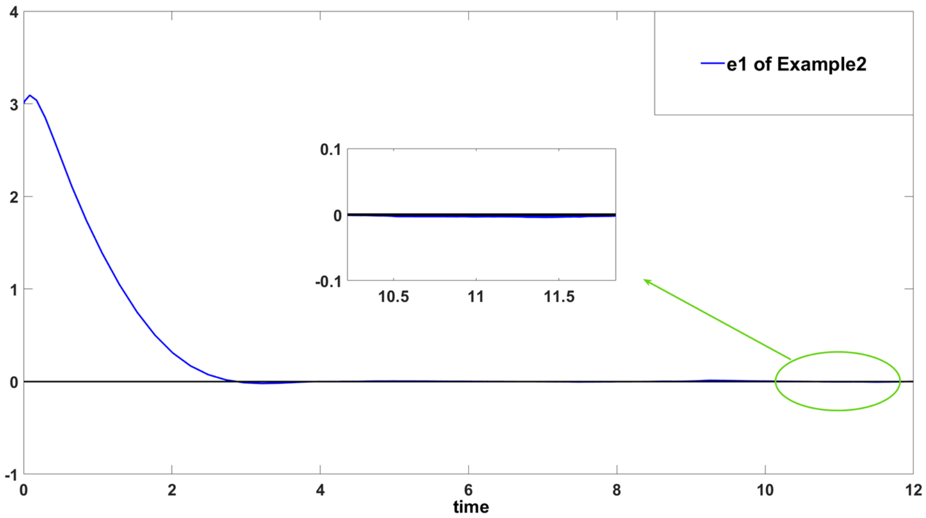

Figure 7. Simulation of e_1 under controllers u_p and u_s for Example 2.

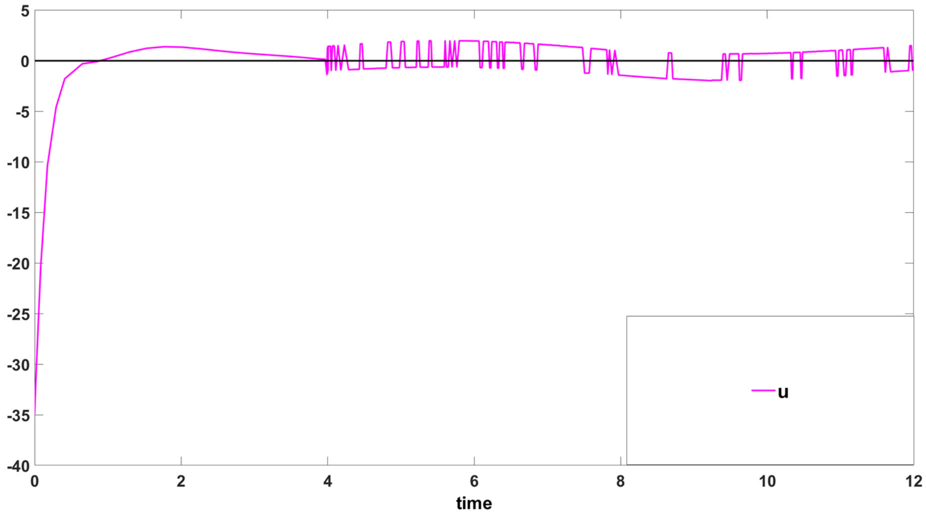

Figure 8. Simulations of u_p and u_s for Example 2.

Figures 5 and 7 show that all tracking errors converge to zero at T_p and stay at zero after T_p under the hybrid action of backstepping controller and sliding mode controller. At the same time, from Figures 6 and 8, the control input remains bounded and has discontinuity after $t \geq 4$ due to the introduction of the sign function, which is consistent with Remark 3. Numerical simulations are demonstrated to verify the proposed theory.

5. Conclusions

In our manuscript, a PT tracking control for the SISO nonlinear system is proposed. Under the prescribed-time controller and infinite controller designed by the backstepping method in Theorem 1, the system output can track the expected trajectory at any desired time, and the tracking error can be limited to a range, which gives an answer to the first question in the Introduction. Additionally, we design a controller in Theorem 2 combining the backstepping method and sliding mode method to make the tracking error stay at zero rather than in a range, which gives an answer to the second question.

Although an infinite gain function is introduced, the control behavior exhibits boundedness over the entire time domain. It is worth mentioning that the convergence time does not depend on initial values and designed parameters. Future work may focus on prescribed-time control for other nonlinear systems, such as network control systems [40].

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

Jinde Cao is one of the special issue editors for Electronic Research Archive, and was not involved in the editorial review or the decision to publish this article. All authors declare that there are no competing interests.

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