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Research article

Pseudo-Stieltjes calculus: $\alpha-$pseudo-differentiability, the pseudo-Stieltjes integrability and applications

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Abstract: In this paper, the concepts of the $\alpha-$pseudo-differentiability and the pseudo-Stieltjes integrability are proposed, and the corresponding transformation theorems and Newton–Leibniz formula are established. The obtained results provide a framework for analyzing nonlinear differential equations.

Keywords: pseudo-analysis; pseudo-differentiability; pseudo-Stieltjes integrability

1. Introduction

Pseudo-analysis, originated by Pap [1–4], has enjoyed wide application in distinct domains, including measure theory, integration, integral operators, convolution, Laplace transform, optimization, nonlinear differential and difference equations, economics, game theory, etc. In fact, in many uncertain issues, such as probabilistic metric spaces, fuzzy logics and fuzzy sets theory, and fuzzy measures, operations differ from the usual addition and multiplication defined for real numbers, such as triangular norms, triangular conorms, pseudo-additions, and pseudo-multiplications, which are more effective. In particular, the triangular conorm decomposable measure was initially presented by Dubois and Prade [5] as a special class of key fuzzy measures [6]. Further, by using Aczel's representation [1, 7, 8], these could be represented with corresponding results of reals [9], such as the addition operator, multiplication operator, differentiability, and integrability.

However, the definition of g-integrability is inconsistent with the definition of the pseudo-integrability regarding a decomposable measure in extant research [4, 9]. Specifically, based on the integrability and the limit of an elementary function, [1, 2, 4, 10, 11] defined null-additive set functions, decomposable measures, nonlinear equations, fuzzy integrals based on pseudo-additions and multiplications, pseudo-additive measures, and integrals. Besides, according to the usual Riemann, Stieltjes, or Lebesgue integral of reals by Aczel's representation, [2, 4, 9] defined decomposable measures, nonlinear equations, and the double g-integral.

Recently, under the combination of the pseudo-differentiability and the pseudo-integrability presented by Gong [12], Newton–Leibniz formula has been developed and applied directly to the nonlinear differential equations. Also, the Jensen's and reverse Jensen's inequalities for Choquet integrals and asymmetric Choquet integrals are obtained [13, 14]. In the current work, first, the $\alpha-$pseudo-differentiability and the $\alpha-$pseudo-integrability are defined. Further, the corresponding transformation theorems are explored, and the Newton–Leibniz formula is investigated. Finally, the obtained results are directly utilized to discuss differential equations.

The remainder of the work is organized as follows: in Section 1, some basic results of pseudo-additions are recalled. Section 2 investigates the $\alpha-$pseudo-differentiability and the pseudo-Stieltjes integrability, and further gives the transformation theorems for them. Also, the Newton–Leibniz formula is obtained. In Section 4, we utilize the obtained results as a framework to directly discuss nonlinear differential equations.

2. Notations and preliminaries

According to [1, 10], let $[a, b]$ be a closed (in some cases it can be considered semiclosed) subinterval of $[-\infty, +\infty]$. Let $\preceq$ be the full order on $[a, b]$. A binary operation $\oplus$ on $[a, b]$ is pseudo-addition, if it is commutative, nondecreasing (with respect to $\preceq$), associative, and with a zero element $\mathbf{0}$. Let $[a, b]_+ \subseteq [a, b]$ with $0 \preceq x$. A binary operation $\otimes$ on $[a, b]$ is pseudo-multiplication, if it is commutative, positively nondecreasing, i.e., $x \preceq y$ implies $x \otimes z \preceq y \otimes z$ for all $z \in [a, b]_+$, associative, and with unit element $\mathbf{1} \in [a, b]$. We adopt the convention $\mathbf{0} \otimes x = \mathbf{0}$ for each $x \in [a, b]$, and $\otimes$ is distributive over $\oplus$. Further more, the convention that the operation $\otimes$ has priority with respect to the operation $\oplus$ will also be adopted. It is easy to verify that the structure $([a, b], \oplus, \otimes)$ is a (real) semiring.

Lemma 2.1. (Aczel's theorem [7, 8]) If $\oplus$ is continuous and strictly increasing in $(a, b) \times (a, b)$, then there exists a monotone function $g : [a, b] \to [-\infty, +\infty]$ such that $g(\mathbf{0}) = 0$ and

$$x \oplus y = g^{-1}(g(x) + g(y)),$$

where g is called a generator of $\oplus$.

The structure $([a, b], \oplus, \otimes)$ is a general g-semiring [1] with a continuous and strictly monotone generator g: $[a, b] \to [-\infty, +\infty]$, i.e., $x \oplus y = g^{-1}(g(x) + g(y))$, and $x \otimes y = g^{-1}(g(x)g(y))$, $x, y \in [a, b]$. And $\mathbf{0} = g^{-1}(0)$ holds. and $\mathbf{1} = g^{-1}(1)$.

Referring to [1], the following statements hold:

(a) If g is a strictly increasing generator, then $\mathbf{0} = a$, the usual order induced by $\oplus$ is as follows: $x \preceq y \Leftrightarrow g(x) \leq g(y)$.

(b) If g is a strictly decreasing generator, then $\mathbf{0} = b$, the usual order induced by $\oplus$ as follows: $x \preceq y \Leftrightarrow g(x) \geq g(y)$.

A metric can be induced as follows: $d(x, y) = |g(x) - g(y)|$.

Aczel's representation theorem is designed to solve computational problems in real valued non additive measure fuzzy calculus, so our theory is a further development and application of real valued non additive measure fuzzy calculus.

3. Pseudo-substraction and the pseudo-division

Definition 3.1. Let $\oplus$ be continuous and strictly increasing. For $x,\ y \in [a, b]$, if there exists $z \in [a, b]$ such that $x = y \oplus z$, then z is said to be a pseudo-difference of x and y, denoted as $z = x \ominus y$.

Remark 3.1. For simplicity, the operator $\ominus$ is called the pseudo-substraction.

The following results are direct consequences of Definition 3.1.

Corollary 3.1. As a general g-semiring ([1]) with a continuous and strictly monotone generator g, the pseudo-substraction of the structure $([a, b], \oplus, \otimes)$ exists, and $x \ominus y = g^{-1}(g(x) - g(y))$, where g is a generator of $\oplus$.

Proof. For any elements $x, y \in [a, b]$, there is $z = g^{-1}(g(x) - g(y))$ such that $g^{-1}(g(y) + g(z)) = x$, i.e., $g(y) + g(z) = g(x)$. Then there exists a z such that $x = y \oplus z$. Thus, there exists a $z \in [a, b]$ such that $x = y \ominus z$.

Definition 3.2. Let $\mathbf{0}, \mathbf{1} \in [a, b]$ be the zero (neutral) element and unit element respectively. Then $\ominus\mathbf{1}$ is defined by $\ominus\mathbf{1} = \mathbf{0} \ominus \mathbf{1} = g^{-1}(-g(\mathbf{1}))$.

Corollary 3.2. If $x \ominus y$ exists, then $a \oplus (\ominus\mathbf{1}) \otimes b) = a \ominus b$..

Definition 3.3. Let $x,\ y \in [a, b]$ and $y \neq \mathbf{0}$. If there exists $z \in [a, b]$ such that $x = y \otimes z$, then z is said to be a pseudo-quotient of x and y, denoted by $x \oslash y$.

Remark 3.2. For simplicity, the operator $\oslash$ is called the pseudo-division.

Corollary 3.3. As a general g-semiring[12] with a continuous and strictly monotone generator g, for any non-zero element $y \in [a, b]$, the pseudo-division of the structure $([a, b], \oplus, \otimes)$ exists, and $x \oslash y = g^{-1}(\frac{g(x)}{g(y)})$, where g is a generator of $\oplus$.

Proof. For any non-zero element $y \in [a, b]$, there is $z = g^{-1}(g(x)/g(y))$ such that $g^{-1}(g(y) \cdot g(z)) = x$, i.e., $g(y) \cdot g(z) = g(x)$. Thus, there exists a $z \in [a, b]$ such that $x = y \otimes z$.

Corollary 3.4. Let $\mathbf{1} \in [a, b]$ be the unit element, respectively. For any $x \in [a, b]$ and $x \neq \mathbf{0}$. Then $x^{(-1)}$ is defined by $x^{(-1)} = g^{-1}(\frac{1}{g(x)})$. It is easily to prove that $x \oslash y = x \otimes y^{(-1)} = g^{-1}(\frac{g(x)}{g(y)})$.

4. The definitions of the pseudo-differentiability and its properties

Given $x \in [a, b]$, its pseudo-absolute value $|x|_\oplus$is defined as

$$|x|_\oplus = |g(x)|$$

where g is a generator of $\oplus$.

The metric on $[a, b]$ is given by

$$d(x, y) = |g(x) - g(y)|$$

for $x,\ y \in [a, b]$, wherein g is a generator of $\oplus$. Obviously, mapping d is a metric.

Furthermore, we have the following representation:

Remark 4.1. Let $\oplus$ be continuous and strictly increasing. If $x \ominus y$ exists, then the pseudo-metric d on $[a, b]$ can be represented by

$$d(x, y) = |x \ominus y|_\oplus,$$

where $|\cdot|_\oplus$ is a pseudo-absolute value.

Remark 4.2. Let d be the metric defined on $[a, b]$, and $x, y, \lambda_1,\ \lambda_2 \in [a, b]$. Then

$$d(\lambda_1 \otimes x, \lambda_2 \otimes y) = |g(\lambda_1)g(x) - g(\lambda_2)g(y)|$$

where g is a generator of $\oplus$.

Definition 4.1. Let $\oplus$ be strictly increasing and continuous, let α be a nondecreasing function, and let $f:[c,d]\to[a,b]$. Then f is said to be pseudo-differentiable with respect to α at the point $x\in[c,d]$, if there exists $\frac{d_\alpha^{\oplus}f(x)}{dx}\in[a,b]$ such that

$$\lim_{h\to 0}[(f(x+h)\ominus f(x))\oslash(\alpha(x+h)\ominus\alpha(x)]$$

exists and equals to $\frac{d_\alpha^{\oplus}f(x)}{dx}$. $\frac{d_\alpha^{\oplus}f(x)}{dx}$ (or wrote $f_\alpha'^{\oplus}(x_0)$) is called the $\alpha-$pseudo-derivative of $f(x)$ at x. For $x=c,\ x=d$, only consider the single $\alpha-$pseudo-derivative: $\lim_{h\to 0^+}[(f(c+h)\ominus f(c))\oslash((\alpha(c+h)\ominus\alpha(c)]$ or $\lim_{h\to 0^-}[(f(d)\ominus f(d-h))\oslash(\alpha(d)\ominus\alpha(d-h))]$.

Obviously, we have the following statements.

Remark 4.3. It is clear that if $\alpha(x)=x$, then Definition 4.1 degenerate to the definition of the pseudo-differentiability of f introduced in [12], and the $\alpha-$pseudo-derivative of $f(x)$ at the point x is written to be $\frac{d^{\oplus}f(x)}{dx}$ (or written $f'^{\oplus}(x_0)$).

Remark 4.4. Let $\oplus$ be strictly increasing and continuous, let α be a nondecreasing function, and let $f:[c,d]\to[a,b]$. Then f is $\alpha-$pseudo-differentiable at the point $x_0\in[c,d]$ ($f_\alpha'^{\oplus}(x_0)$ is the α-derivative at x_0), if and only if for every $\epsilon>0$, there exists a $\delta>0$ such that for any u,v satisfying $x_0\in[u,v]\subset(x_0-\delta,x_0+\delta)$,

$$d((f(v)\ominus f(u))\oslash((\alpha(v)\ominus\alpha(u)),f_\alpha'^{\oplus}(x_0))<\epsilon$$

holds.

Theorem 4.1. Let $\oplus$ be strictly increasing and continuous, and let α be a nondecreasing function with $\alpha\in C_g^1[a,b]$ (i.e., $g(\alpha)\in C^1[a,b]$), $f:[c,d]\to[a,b]$. Then f is $\alpha-$pseudo-differentiable at the point $x_0\in[c,d]$, and $f_\alpha'^{\oplus}(x_0)$ is the α-derivative at x_0, if and only if f is pseudo-differentiable at $x_0\in[c,d]$, and

$$f'^{\oplus}(x_0)=\alpha'(x_0)\otimes f_\alpha'^{\oplus}(x_0),$$

where $C^1[a,b]$ is the continuously differentiable function space.

Proof. If f is $\alpha-$pseudo-differentiable at the point $x_0\in[c,d]$, then for any $\epsilon>0$, there exists a $\delta>0$ such that for any interval $[u,v]$ satisfying $x_0\in[u,v]\subset(x_0-\delta,x_0+\delta)$, we have

$$d((f(v)-f(u))\oslash((\alpha(v)\ominus\alpha(u)),f_\alpha'^{\oplus}(x_0))<\epsilon.$$

Note that

$$\begin{aligned}
&d((f(v)-f(u))\oslash(v\ominus u),\alpha'(x_0)\otimes f_\alpha'^{\oplus}(x_0)\\
=&d([(f(v)-f(u))\oslash((\alpha(v)\ominus\alpha(u))]\otimes[(\alpha(v)\ominus\alpha(u))\oslash(v\ominus u),\alpha'(x_0)\otimes f_\alpha'^{\oplus}(x_0))\\
=&|g([(f(v)-f(u))\oslash((\alpha(v)\ominus\alpha(u))])g([(\alpha(v)\ominus\alpha(u))\oslash(v\ominus u)])-g(\alpha'(x_0))g(f_\alpha'^{\oplus}(x_0))|\\
=&|g([(f(v)-f(u))\oslash((\alpha(v)\ominus\alpha(u))])-g(f_\alpha'^{\oplus}(x_0))|\cdot|g([(\alpha(v)\ominus\alpha(u))\oslash(v\ominus u)])|\\
+&|g([(\alpha(v)\ominus\alpha(u))\oslash(v\ominus u)])-g(\alpha'(x_0))|\cdot|g(f_\alpha'^{\oplus}(x_0))|\\
\leq&\varepsilon\cdot|g([(\alpha(v)\ominus\alpha(u))\oslash(v\ominus u)])|+\varepsilon\cdot|g(f_\alpha'^{\oplus}(x_0))|\\
=&\varepsilon\cdot|g([(\alpha(v)\ominus\alpha(u))\oslash(v\ominus u)])-g(\alpha'(x_0))|+|g(\alpha'(x_0))|)+\varepsilon\cdot|g(f_\alpha'^{\oplus}(x_0))|\\
\leq&\varepsilon\cdot(\varepsilon+M)+\varepsilon\cdot|g(f_\alpha'^{\oplus}(x_0))|.
\end{aligned}$$

Remark 4.5. Let $\oplus$ be strictly increasing and continuous, and let α be a nondecreasing function with $\alpha \in C_g^1[a,b]$, $f : [c,d] \to [a,b]$. Further assume f be $\alpha-$pseudo-differentiable $x_0 \in [c,d]$. Then f is pseudo-continuous at x_0, i.e., $\lim_{h\to 0} f(x_0+h) = f(x_0)$ for any $x_0 \in [c,d]$.

Proof. Fixed $x_0 \in [c,d]$. Follows that Remark 4.4, f is pseudo-differentiable at x_0, and

$$\lim_{h\to 0}[(f(x_0+h) \ominus f(x)) \oslash (\alpha(x_0+h) \ominus \alpha(x))] = f_\alpha'^{\oplus}(x_0).$$

For the generator g of $\oplus$, we have

$$\lim_{h\to 0}|g[(f(x_0+h) \ominus f(x)) \oslash (\alpha(x_0+h) \ominus \alpha(x))] - g(f_\alpha'^{\oplus}(x_0))| = 0.$$

It follows that

$$\lim_{h\to 0}|\frac{g(f(x_0+h)) - g(f(x_0))}{g((\alpha(x_0+h)) - g((\alpha(x_0))} - g(f_\alpha'^{\oplus}(x_0))| = 0.$$

That is to say,

$$\lim_{h\to 0}|(g(f(x_0+h)) - g(f(x_0))) - (g((\alpha(x_0+h)) - g((\alpha(x_0)))g(f_\alpha'^{\oplus}(x_0))| = 0.$$

By the continuity of $g(\alpha(x))$, we have

$$\lim_{h\to 0}|g(f(x_0+h)) - g(f(x_0))| = 0.$$

It implies

$$\lim_{h\to 0} f(x_0+h) = f(x_0).$$

Hence, f is pseudo-continuous on $[c,d]$.

Theorem 4.2. Let $\oplus$ be continuous and strictly increasing, α be a nondecreasing function, and f_1 and f_2 be two $\alpha-$pseudo-differentiable functions on $[c,d]$. Then the following statements hold for any $\lambda,\ \lambda_1,\ \lambda_2 \in [a,b]$.

1) $\lambda_1 \otimes f_1 \oplus \lambda_2 \otimes f_2$ is $\alpha-$pseudo-differentiable on $[c,d]$ and

$$\frac{d_\alpha^{\oplus}(\lambda_1 \otimes f_1 \oplus \lambda_2 \otimes f_2)}{dx} = \lambda_1 \otimes \frac{d_\alpha^{\oplus} f_1}{dx} \oplus \lambda_2 \otimes \frac{d_\alpha^{\oplus} f_2}{dx};$$

2) $f_1 \otimes f_2$ is $\alpha-$pseudo-differentiable on $[c,d]$ and

$$\frac{d_\alpha^{\oplus}(f_1 \otimes f_2)}{dx} = \frac{d_\alpha^{\oplus} f_1}{dx} \otimes f_2 \oplus f_1 \otimes \frac{d_\alpha^{\oplus} f_2}{dx};$$

3) $\frac{d_\alpha^{\oplus}\lambda}{dx} = \mathbf{0}$.

Proof. 1) Since f_1 and f_2 are $\alpha-$pseudo-differentiable, we have

$$\begin{aligned}
&\frac{d_\alpha^{\oplus}(\lambda_1 \otimes f_1 \oplus \lambda_2 \otimes f_2)}{dx} \\
=&\lim_{h\to 0}\{[(\lambda_1 \otimes f_1(x+h) \oplus \lambda_2 \otimes f_2(x+h)) \ominus (\lambda_1 \otimes f_1(x) \oplus \lambda_2 \otimes f_2(x))] \oslash [\alpha(x+h) \ominus \alpha(x)]\} \\
=&\lim_{h\to 0}\{[\lambda_1 \otimes (f_1(x+h) \ominus f_1(x)) \oplus \lambda_2 \otimes (f_2(x+h) \ominus f_2(x))] \oslash [\alpha(x+h) \ominus \alpha(x)]\} \\
=&\lambda_1 \otimes \lim_{h\to 0}[(f_1(x+h) \ominus f_1(x)) \oslash (\alpha(x+h) \ominus \alpha(x)] \oplus \lambda_2 \otimes \lim_{h\to 0}[(f_2(x+h) \ominus f_2(x)) \oslash (\alpha(x+h) \ominus \alpha(x)] \\
=&\lambda_1 \otimes \frac{d_\alpha^{\oplus} f_1}{dx} \oplus \lambda_2 \otimes \frac{d_\alpha^{\oplus} f_2}{dx}.
\end{aligned}$$

According to Definition 4.1, $\lambda_1 \otimes f_1 \oplus \lambda_2 \otimes f_2$ is $\alpha-$pseudo-differentiable on $[c, d]$ and

$$\frac{d^{\oplus}(\lambda_1 \otimes f_1 \oplus \lambda_2 \otimes f_2)}{dx} = \lambda_1 \otimes \frac{d_\alpha^{\oplus} f_1}{dx} \oplus \lambda_2 \otimes \frac{d_\alpha^{\oplus} f_2}{dx}.$$

2) Since f_1 and f_2 are $\alpha-$pseudo-differentiable, we have

$$\begin{aligned}
&\frac{d_\alpha^{\oplus}(f_1 \otimes f_2)}{dx} \\
=&\lim_{h\to 0}\{[f_1(x+h) \otimes f_2(x+h) \ominus f_1(x) \otimes f_2(x)] \oslash [\alpha(x+h) \ominus \alpha(x)]\} \\
=&\lim_{h\to 0}\{[f_1(x+h) \otimes f_2(x+h) \ominus f_1(x) \otimes f_2(x) \ominus f_1(x) \otimes f_2(x+h) \oplus f_1(x) \otimes f_2(x+h)] \oslash [\alpha(x+h) \ominus \alpha(x)]\} \\
=&\lim_{h\to 0}\{[(f_1(x+h) \ominus f_1(x)) \otimes f_2(x+h) \oplus f_1(x) \otimes (f_2(x+h) \ominus f_2(x))] \oslash [\alpha(x+h) \ominus \alpha(x)]\} \\
=&\lim_{h\to 0}[(f_1(x+h) \ominus f_1(x)) \oslash (\alpha(x+h) \ominus \alpha(x))] \otimes f_2(x) \\
&\oplus f_1(x) \otimes \lim_{h\to 0}[(f_2(x+h) \ominus f_2(x)) \oslash (\alpha(x+h) \ominus \alpha(x))] \\
=&\frac{d_\alpha^{\oplus} f_1}{dx} \otimes f_2 \oplus f_1 \otimes \frac{d_\alpha^{\oplus} f_2}{dx}.
\end{aligned}$$

According to Definition 4.1, $f_1 \otimes f_2$ is $\alpha-$pseudo-differentiable on $[c, d]$ and

$$\frac{d_\alpha^{\oplus}(f_1 \otimes f_2)}{dx} = \frac{d_\alpha^{\oplus} f_1}{dx} \otimes f_2 \oplus f_1 \otimes \frac{d_\alpha^{\oplus} f_2}{dx}.$$

3) $\frac{d_\alpha^{\oplus}\lambda}{dx} = \lim\limits_{h\to 0}(\lambda \ominus \lambda) \oslash (\alpha(x+h) \ominus \alpha(x)) = \mathbf{0}$.

Theorem 4.3. Let $\oplus$ be strictly increasing and continuous, α be a nondecreasing function, and $f : [c, d] \to [a, b]$. Let f be $\alpha-$pseudo-differentiable and the generator $g(\alpha)$ of $\oplus$ be differentiable on $[a, b]$. Then

$$\frac{d_\alpha^{\oplus} f(x)}{dx} = g^{-1}\left(\frac{dg(f(x))}{dg(\alpha(x))}\right).$$

Proof.

$$\begin{aligned}
\frac{d_\alpha^{\oplus} f(x)}{dx} &= \lim_{h\to 0}[(f(x+h) \ominus f(x)) \oslash (\alpha(x+h) \ominus \alpha(x))] \\
&= \lim_{h\to 0} g^{-1}\left(\frac{g(f(x+h)) - g(f(x))}{g(\alpha(x+h)) - g(\alpha(x))}\right) \\
&= g^{-1}\left(\lim_{h\to 0} \frac{g(f(x+h)) - g(f(x))}{h} \cdot \frac{h}{g(\alpha(x+h)) - g(\alpha(x))}\right) \\
&= g^{-1}\left([g(f)]' / g'(\alpha(x))\right) \\
&= g^{-1}\left(\frac{dg(f(x))}{dg(\alpha(x))}\right).
\end{aligned}$$

Remark 4.6. Obviously, if $\alpha(x) = x$, $f : [c, d] \to [a, b]$ be pseudo-differentiable [12], and the generator g of $\oplus$ be differentiable on $[a, b]$. Then

$$\frac{d^{\oplus} f(x)}{dx} = g^{-1}\left(\frac{dg(f(x))}{dg(x)}\right).$$

Remark 4.7. However, in [9, 15, 16], the g-derivative is directly defined as follows:

$$\frac{d^{\oplus} f(x)}{dx} = g^{-1}\left(\frac{dg(f(x))}{dx}\right).$$

However, it may be more natural to define the integral following the method proposed in this research according to Definition 4.1 and further obtain Theorem 4.2.

Definition 4.2. Let $f : [c, d] \to [a, b]$, if f has the $(n-1)$-th $\alpha-$pseudo-derivative, then the (n)-th $\alpha-$pseudo-derivative of f (if it exists) is defined as

$$\frac{d_{\alpha}^{(n)\oplus} f}{dx^n} = \frac{d^{\oplus}}{dx}\left(\frac{d_{\alpha}^{(n-1)\oplus} f}{dx^{n-1}}\right),\ n \geqslant 1.$$

Theorem 4.4. Let $f : [c, d] \to [a, b]$, $\oplus$ be strictly increasing and continuous, and α be a nondecreasing function. If f (n)-th $\alpha-$pseudo-differentiable on $[c, d]$ and the generator $g(\alpha)$ of $\oplus$ be (n)-th differentiable on $[a, b]$. Then

$$\frac{d_{\alpha}^{(n)\oplus} f(x)}{x^n} = g^{-1}\left(\frac{d_{\alpha}^{n} g(f(x))}{d[g(\alpha(x))]^n}\right),\ n \geqslant 0.$$

Proof. For $n = 0$, the theorem is obviously true.

Assume that the theorem is true for $n-1$, i.e.,

$$\frac{d_{\alpha}^{(n-1)\oplus} f(x)}{x^{n-1}} = g^{-1}\left(\frac{d_{\alpha}^{n-1} g(f(x))}{d[g(\alpha(x))]^{n-1}}\right),$$

then

$$\begin{aligned}
\frac{d_{\alpha}^{(n)\oplus} f(x)}{x^n} &= \frac{d_{\alpha}^{\oplus}}{dx}\left(\frac{d^{(n-1)\oplus} f}{dx^{n-1}}\right) \\
&= \frac{d_{\alpha}^{\oplus}}{dx}\left(g^{-1}\left(\frac{d^{n-1} g(f(x))}{d[g(\alpha(x))]^{n-1}}\right)\right) \\
&= g^{-1}\left(\frac{d}{dg(\alpha(x))}\left(\frac{d^{n-1} g(f(x))}{d[g(\alpha(x))]^{n-1}}\right)\right) \\
&= g^{-1}\left(\frac{d_{\alpha}^{n} g(f(x))}{d[g(\alpha(x))]^n}\right).
\end{aligned}$$

By mathematical induction, the proof is completed.

5. The definitions of the pseudo-Stieltjes integrability and its Newton–Leibniz formula

Definition 5.1. Let $\oplus$ be strictly increasing and continuous, α be a nondecreasing function, and $f(x)$ be a bounded function defined on $[c, d]$. If for any partition of $[c, d]$

$$P : c = x_0 < x_1 < x_2 < \cdots < x_n = d,$$

denote $\lambda = \max\limits_{1\leqslant i\leqslant n}(x_i \ominus x_{i-1})$, and if for any $\xi_i \in [x_{i-1}, x_i]$, the limit

$$\lim_{\lambda\to\mathbf{0}} \bigoplus_{i=1}^{n} f(\xi_i) \otimes (\alpha(x_i) \ominus \alpha(x_{i-1}))$$

exists, then $f(x)$ is said to be pseudo-Stieltjes integrable on $[c, d]$, and its pseudo-Stieltjes integral value equals to the limit value, denoted by $\int_{[c,d]}^{(\oplus,\otimes,\alpha)} f(x)d\alpha$.

Theorem 5.1. Let $\oplus$ be strictly increasing and continuous, α be a nondecreasing function, and $f_1 : [c, d] \to [a, b]$, $f_2 : [c, d] \to [a, b]$. If f_1 and f_2 are pseudo-Stieltjes integrable on $[c, d]$. Then for $\lambda_1,\ \lambda_2 \in [a, b],\ \lambda_1 \otimes f_1 \oplus \lambda_2 \otimes f_2$ is also generalized integrable on $[c, d]$ and

$$\int_{[c,d]}^{(\oplus,\otimes,\alpha)} (\lambda_1 \otimes f_1 \oplus \lambda_2 \otimes f_2)d\alpha = \lambda_1 \otimes \int_{[c,d]}^{(\oplus,\otimes,\alpha)} f_1 d\alpha \oplus \lambda_2 \otimes \int_{[c,d]}^{(\oplus,\otimes,\alpha)} f_2 d\alpha.$$

Proof. For any partition of $[c, d]$

$$P : c = x_0 < x_1 < x_2 < \cdots < x_n = d$$

and for any $\xi_i \in [x_{i-1}, x_i]$, we have

$$\begin{aligned}
&\bigoplus_{i=1}^{n} (\lambda_1 \otimes f_1(\xi_i) \oplus \lambda_2 \otimes f_2(\xi_i)) \otimes (\alpha(x_i) \ominus \alpha(x_{i-1}))\\
=&\bigoplus_{i=1}^{n} (\lambda_1 \otimes f_1(\xi_i) \otimes (\alpha(x_i) \ominus \alpha(x_{i-1})) \oplus \lambda_2 \otimes f_2(\xi_i) \otimes (\alpha(x_i) \ominus \alpha(x_{i-1})))\\
=&\lambda_1 \otimes \left(\bigoplus_{i=1}^{n} f_1(\xi_i) \otimes (\alpha(x_i) \ominus \alpha(x_{i-1}))\right) \oplus \lambda_2 \otimes \left(\bigoplus_{i=1}^{n} f_2(\xi_i) \otimes (\alpha(x_i) \ominus \alpha(x_{i-1}))\right).
\end{aligned}$$

Let $\lambda = \max\limits_{1 \leqslant i \leqslant n}(x_i \ominus x_{i-1}) \to \mathbf{0}$, since f_1 and f_2 are pseudo-Stieltjes integrable on $[c, d]$, we have

$$\begin{aligned}
&\lim_{\lambda \to \mathbf{0}} \bigoplus_{i=1}^{n} (\lambda_1 \otimes f_1(\xi_i) \oplus \lambda_2 \otimes f_2(\xi_i)) \otimes (\alpha(x_i) \ominus \alpha(x_{i-1}))\\
=&\lambda_1 \otimes \left(\lim_{\lambda \to \mathbf{0}} \bigoplus_{i=1}^{n} f_1(\xi_i) \otimes (\alpha(x_i) \ominus \alpha(x_{i-1}))\right) \oplus \lambda_2 \otimes \left(\lim_{\lambda \to \mathbf{0}} \bigoplus_{i=1}^{n} f_2(\xi_i) \otimes (\alpha(x_i) \ominus \alpha(x_{i-1}))\right)\\
=&\lambda_1 \otimes \int_{[c,d]}^{(\oplus,\otimes,\alpha)} f_1 d\alpha \oplus \lambda_2 \otimes \int_{[c,d]}^{(\oplus,\otimes,\alpha)} f_2 d\alpha.
\end{aligned}$$

According to Definition 5.1, $\lambda_1 \otimes f_1 \oplus \lambda_2 \otimes f_2$ is pseudo-Stieltjes integrable on $[c, d]$ and

$$\int_{[c,d]}^{(\oplus,\otimes,\alpha)} (\lambda_1 \otimes f_1 \oplus \lambda_2 \otimes f_2)dx = \lambda_1 \otimes \int_{[c,d]}^{(\oplus,\otimes,\alpha)} f_1 d\alpha \oplus \lambda_2 \otimes \int_{[c,d]}^{(\oplus,\otimes,\alpha)} f_2 d\alpha.$$

Theorem 5.2. Let $\oplus$ be strictly increasing and continuous, α be a nondecreasing function, and $f(x)$ pseudo-Stieltjes integrable on $[c, d]$. Then

$$\int_{[c,d]}^{(\oplus,\otimes,\alpha)} f d\alpha = g^{-1}\left(\int_c^d g(f(x))dg(\alpha(x))\right),$$

when the right part is meaningful.

Proof.

$$
\begin{aligned}
\int_{[c,d]}^{(\oplus,\otimes,\alpha)} f d\alpha &= \lim_{\lambda\to\mathbf{0}} \bigoplus_{i=1}^{n} f(\xi_i)\otimes(\alpha(x_i)\ominus\alpha(x_{i-1}))\\
&= \lim_{\lambda\to\mathbf{0}}[f(\xi_1)\otimes(\alpha(x_1)\ominus\alpha(x_0))\oplus f(\xi_2)\otimes(\alpha(x_2)\ominus\alpha(x_1))\oplus\cdots\oplus f(\xi_n)\otimes(\alpha(x_n)\ominus\alpha(x_{n-1}))]\\
&= \lim_{\lambda\to\mathbf{0}}\{g^{-1}[g(f(\xi_1))\cdot g(\alpha(x_1)\ominus\alpha(x_0))]\oplus g^{-1}[g(f(\xi_2))\cdot g(\alpha(x_2)\ominus\alpha(x_1))]\oplus\cdots\\
&\quad \oplus g^{-1}[g(f(\xi_n))\cdot g(\alpha(x_n)\ominus\alpha(x_{n-1}))]\}\\
&= \lim_{\lambda\to\mathbf{0}} g^{-1}[g(f(\xi_1))\cdot g(\alpha(x_1)\ominus\alpha(x_0)) + g(f(\xi_2))\cdot g(\alpha(x_2)\ominus\alpha(x_1)) + \cdots\\
&\quad + g(f(\xi_n))\cdot g(\alpha(x_n)\ominus\alpha(x_{n-1}))]\\
&= g^{-1}[\lim_{\lambda'\to 0}(g(f(\xi_1))\cdot(g(\alpha(x_1)) - g(\alpha(x_0))) + g(f(\xi_2))\cdot(g(\alpha(x_2)) - g(\alpha(x_1))) + \cdots\\
&\quad + g(f(\xi_n))\cdot(g(\alpha(x_n)) - g(\alpha(x_{n-1}))))]\\
&= g^{-1}\left(\int_c^d g(f(x))dg(\alpha(x))\right),
\end{aligned}
$$

where $\lambda' = \max_{1\leqslant i\leqslant n}|g(\alpha(x_i)) - g(\alpha(x_{i-1}))|$.

Remark 5.1. For $1 \leqslant i \leqslant n$, we have

$$
\begin{aligned}
x_i\ominus x_{i-1}\to\mathbf{0} &\iff d(x_i, x_{i-1})\to 0\\
&\iff |g(x_i) - g(x_{i-1})|\to 0\\
&\iff |g(\alpha(x_i)) - g(\alpha(x_{i-1}))|\to 0,
\end{aligned}
$$

therefore

$$\max_{1\leqslant i\leqslant n}(x_i\ominus x_{i-1})\to\mathbf{0}\iff \max_{1\leqslant i\leqslant n}|g(\alpha(x_i)) - g(\alpha(x_{i-1}))|\to 0,$$

namely,

$$\lambda\to\mathbf{0}\iff\lambda'\to 0.$$

Remark 5.2. In [9, 15, 16], the g-integral is directly defined as follows:

$$\int_{[c,d]}^{(\oplus,\otimes)} f dx = g^{-1}\left(\int_c^d g(f)dx\right).$$

However, it may be more natural to define the g-integral the way proposed in Definition 5.1 in this research and obtain Theorem 5.2. In addition, the definition of integral presented in this research is consistent with the definition of integral regarding a decomposable measure m proposed in [1, 2], i.e., $\int_{[c,d]}^{(\oplus,\otimes)} f dm = g^{-1}\left(\int_c^d g(f)dg\circ m\right)$.

Theorem 5.3. Let $\oplus$ be continuous and strictly increasing, α be a nondecreasing function, and f be continuous on $[c, d]$. Then we have

$$\frac{d_\alpha^\oplus}{dx}\left(\int_{[c,x]}^{(\oplus,\otimes,\alpha)} f(t)d\alpha\right) = f(x)$$

for any $x\in[c, d]$.

Proof. From Theorems 4.2 and 5.2, we have for any $x \in [c, d]$

$$\begin{aligned}\frac{d_\alpha^\oplus}{dx}\left(\int_{[c,x]}^{(\oplus,\otimes,\alpha)} f(t)d\alpha\right) &= g^{-1}\left(\frac{dg\left(\int_{[c,x]}^{(\oplus,\otimes,\alpha)} f(t)d\alpha\right)}{dg(\alpha(x))}\right)\\ &= g^{-1}\left(\frac{d(\int_c^x g(f(t))dg(\alpha(t)))}{dg(\alpha(x))}\right)\\ &= g^{-1}(g(f(x)))\\ &= f(x),\end{aligned}$$

where the fundamental theorems of calculus are used.

Theorem 5.4. (Newton–Leibniz formula) Let $\oplus$ be strictly increasing and continuous, and α be a nondecreasing function. If $\frac{d_\alpha^\oplus f}{dt}$ is continuous on $[c, d]$. Then we have for any $x \in [c, d]$

$$\int_{[c,x]}^{(\oplus,\otimes,\alpha)} \frac{d_\alpha^\oplus f}{dt} d\alpha = f(\alpha(x)) \ominus f(\alpha(c))$$

for any $x \in [c, d]$.

Proof. According to Theorems 4.2 and 5.2, for any $x \in [c, d]$, we have

$$\begin{aligned}\int_{[c,x]}^{(\oplus,\otimes,\alpha))} \frac{d_\alpha^\oplus f(t)}{dt} d\alpha &= g^{-1}\left(\int_c^x g\left(\frac{d_\alpha^\oplus f(t)}{dt}\right) dg(\alpha(t))\right)\\ &= g^{-1}\left(\int_c^x \frac{dg(f(t))}{dg(\alpha(t))} dg(\alpha(t))\right)\\ &= g^{-1}\left(\int_c^x dg(f(t))\right)\\ &= g^{-1}(g(f(x)) - g(f(c)))\\ &= f(x) \ominus f(c).\end{aligned}$$

6. Applications in the discussion of the nonlinear differential equations

Compared to the case where α is discontinuous, due to the application of Theorem 4.1 in this article, the involved α is not only continuous but also differentiable. Our example demonstrates that the proposed derivative and integral applications have certain practical value in a sense, as they can transform complex nonlinear calculus equations into simple calculus equations containing only newly defined derivatives and integrals.

Example 6.1. Considering the following first-order ordinary differential equation:

$$\ln y' + y - 2x^s - (s-1)\ln x - \ln s = 0, \tag{6.1}$$

where $s \in [0, +\infty)$.

Let $\alpha(x) = x^s$, and construct $x \oplus y = \ln(e^x + e^y)$, $x \otimes y = x + y$, and $g(x) = e^x$. Capitalize Eq (6.1) can have the following form:

$$\frac{d_\alpha^\oplus y}{dx} = x^s.$$

Pseudo-Stieltjes integrates the preceding equation correspondingly, and we have

$$\begin{aligned} y &= \int^{(\oplus,\otimes,\alpha)} x^s dx \oplus C_1 \\ &= g^{-1}\left(\int g(x^s) dg(x^s) + g(C_1)\right) \\ &= \ln\left(\frac{e^{2x^s}}{2} + C\right), \end{aligned}$$

where $C = e^{-C_1}$. That is to say, an ordinary differential equation (6.1) has a solution $y = \ln\left(\frac{e^{2x^s}}{2} + C\right)$.

Example 6.2. Consider the following ordinary differential equation:

$$((\frac{y}{2})')^{\frac{1}{p}} y^{1-\frac{1}{p}} - x^{(s+2-\frac{1}{p})} = 0. \tag{6.2}$$

Where $p > 0$ and $s \in [0, +\infty)$.

Let $\alpha(x) = x^2$. By constructing $x \oplus y = (x^p + y^p)^{\frac{1}{p}}$, $x \otimes y = xy$, then $g(x) = x^p$, and Eq (6.2) has the following form:

$$\frac{d_\alpha^\oplus y}{dx} = x^s.$$

Pseudo-Stieltjes integrates the preceding equation correspondingly, and we have

$$\begin{aligned} y &= \int^{(\oplus,\otimes,\alpha)} x^s dx \oplus C_1 \\ &= g^{-1}\left(\int g(x^s) dg(x^2) + g(C_1)\right) \\ &= \left(\frac{2}{s+2} x^{p(s+2)} + C\right)^{1/p}. \end{aligned}$$

That is to say ordinary differential equation (6.2) has solution $y = \left(\frac{2}{s+2} x^{p(s+2)} + C\right)^{1/p}$.

Example 6.3. Consider the following differential equation:

$$\lambda(1+\lambda)^y y' - 2x(1+\lambda)^{2x^2} - 2x(1+\lambda)^{x^2} = 0. \tag{6.3}$$

where $\lambda > 0$.

Let $\alpha(x) = x^2$, $x \oplus y = \frac{\ln((1+\lambda)^x + (1+\lambda)^y - 1))}{\ln(1+\lambda)}$. Then $g(x) = \frac{(1+\lambda)^x - 1}{\lambda}$, $\lambda > 0$, and Eq (6.3) can be represented as follows:

$$\frac{d_\alpha^\oplus y}{dx} = x^2.$$

Pseudo-Stieltjes integrates the preceding equation correspondingly, and we have

$$\begin{aligned} y &= \int^{(\oplus,\otimes,\alpha)} x^2 dx \oplus C_1 \\ &= g^{-1}\left(\int g(x^2)dg(x^2) + g(C_1)\right) \\ &= g^{-1}\left(\frac{1}{2}g^2(x^2) + g(C_1)\right) \\ &= \frac{\ln\left(((1+\lambda)^{x^2}-1)^2/2\lambda + C\right)}{\ln(1+\lambda)}. \end{aligned}$$

That is to say ordinary differential equation (6.3) has solution $y = \frac{\ln\left(((1+\lambda)^{x^2}-1)^2/2\lambda+C\right)}{\ln(1+\lambda)}$.

Example 6.4. Considering the following first-order nonlinear integro-differential equation:

$$y'e^y = sx^{s-1}e^{x^s+1} + sx^{s-1}e^{x^s}\int_0^x sx^{s-1}e^{x^s}e^y \mathrm{d}x \tag{6.4}$$

where $s \in [0, +\infty)$.

Let $\alpha(x) = x^s$, and construct $x \oplus y = \ln(e^x + e^y)$, $x \otimes y = x + y$, and $g(x) = e^x$. Capitalize Eq (6.4) can have the following form:

$$\frac{d_\alpha^\oplus y}{dx} = 1 \oplus \int_{[0,x]}^{(\oplus,\otimes,\alpha)} y d\alpha.$$

By definitions of $\alpha-$pseudo-derivative and pseudo-Stieltjes integral, we have $y = e^{x^s}$. It is easy to calculate that

$$\frac{d_\alpha^\oplus e^{x^s}}{dx} = e^{x^s},$$

$$1 \oplus \int_{[0,x]}^{(\oplus,\otimes,\alpha)} e^{x^s} d\alpha = e^{x^s}.$$

That is to say, the integro-differential equation (6.4) has solution $y = e^{x^s}$.

7. Conclusions

We present the concepts of the $\alpha-$pseudo-differentiability and the pseudo-Stieltjes integrability, and also present the characteristic theorems and the transformation theorems. According to the transformation theorem between the $\alpha-$pseudo-derivative and the classical derivative and the transformation theorem between the pseudo-Stieltjes integration and the classical Stieltjes integration, the calculation methods and formulas for $\alpha-$pseudo-derivative and pseudo-Stieltjes integration are explored. Further, Newton–Leibniz formula is also obtained. At last, the obtained results provide a framework for analyzing nonlinear differential equations.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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