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Research article

An existence result for a new coupled system of differential inclusions involving with Hadamard fractional orders

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Abstract: In this paper, we introduce a new coupled system of differential inclusions involving with Hadamard fractional orders. By applying a fixed point theorem for three operators containing $x \in AxBx + Cx$ in Banach algebras, we get an existence result for the discussed system via multi-valued maps in a Banach space. An example is provided to support the validation of the theoretical result achieved.

Keywords: coupled system; differential inclusion; Hadamard fractional order; multi-valued maps; fixed-point theorem

1. Introduction

Nowadays, fractional calculus (FC) is a very important field in applied mathematics. Fractional differential equations, as an important part of FC, have been invented by mathematicians as a pure branch of mathematics. As a result, FC has been rapidly developed and has many important applications in various applied sciences. Some models involving with fractional order are more realistic and practical than previous integer-order models [1–5]. More recent developments on fractional differential equations can be found in [6–10] and the references therein. In a large part of the literature, many works involve either the Riemann-Liouville derivative or Caputo derivative. Besides, there is also one important concept of Hadamard fractional derivative (HFD for short), which was first introduced by Hadamard in 1892 [12]. For more information about the HFD and integral, see [13–16].

When solving numerous real-life problems, researchers always construct differential equations and discuss their properties. In fact, some systems, such as economics and biology, involve certain macro changes; in this case, instead of differential equations, differential inclusions are considered and they can describe the uncertainty of the system itself. Differential inclusion systems, as a meaningful model for describing uncertainty in human society, have attracted the enthusiasm and interest of many scholars. In addition, differential inclusions play an important role in various fields [11, 12]. In this article, we are devoted to investigating fractional differential inclusion problems with Dirichlet boundary conditions. In research over the past few decades, mathematicians have been using many different methods and techniques to study fractional differential inclusion problems, and some good results concerning the solvability were obtained. For example, in [13], Benchohora and Ntouyas studied the solvability of a periodic boundary problem for first-order differential inclusions. Dhage [15] proved some existence theorems for hyperbolic differential inclusions in Banach algebras. Papageorgiou and Staicu studied second-order differential inclusions by establishing a method of upper-lower solutions in [16]. Moreover, Chang and Nieto extended the study to a fractional differential inclusion by using the Bohnenblust-Karlin's fixed point theorem in [17]. In addition, fractional differential inclusion for different types of single equations with some different boundary conditions was researched in [14].

The characterization of uncertainty in differential inclusion systems is often illustrated by set-valued mapping in mathematics. Fixed point theory for multi-valued mappings is an important and hot tool in set-valued analysis, which has several applications. Many of the well-known and useful fixed point theorems of single-valued mappings, such as those of Banach, Schaefer, and Schauder, have been extended to multi-valued mappings in Banach spaces. Naturally, the case of extending the Krasnoselskii fixed point theorem to set-valued mapping has also been obtained in literature; we refer the interested readers to [23, 24].

In [25], the authors proposed a fractional boundary value problem with the generalized Riemann-Liouville fractional derivative:

$$\begin{cases} D^{\alpha}\omega(\tau) \in F(\tau, \omega(\tau)), \quad \tau \in [0,1],\ \alpha \in (1,2), \\ \omega(0) = 0, \\ \omega(1) = mI_{0^+}^{\mu_1} h_1(\xi, \omega(\xi)) + nI_{0^+}^{\mu_2} h_2(\eta, \omega(\eta)), \end{cases}$$

where $1 < \alpha < 2$, $m, n \geq 0$, $\mu_1, \mu_2 \geq 1$, $0 < \xi, \eta \leq 1$, $h_j : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function for $1 \leq j \leq 2$, and $F : [0,1] \times \mathbb{R} \to \mathscr{P}(\mathbb{R})$ is a multi-valued function. The existence and uniqueness results were obtained by using the iterative method. In [26], by using the fixed point technique, the authors obtain a new solution for the generalized system of fractional the q-differential inclusions involving p-Laplacian operator and sequential derivatives.

As we know, compared with some of the previous articles, fractional differential inclusions under boundary conditions were mainly focused on a single equation. However, there are relatively few studies on fractional differential coupled inclusion systems. Our aim is to obtain an existence result for a new coupled system of differential inclusions involving Hadamard fractional order. That is, we investigate the following system

$$\begin{cases} {}^{H}D^{\alpha_1}\left(\frac{(\xi(\tau) - g_1(\tau,\xi(\tau),\eta(\tau))}{f_1(\tau,\xi(\tau),\eta(\tau))}\right) \in \mathrm{G}_1(\tau, \xi(\tau), \eta(\tau)), \quad \tau \in (1,e), \\ {}^{H}D^{\alpha_2}\left(\frac{(\eta(\tau) - g_2(\tau,\xi(\tau),\eta(\tau))}{f_2(\tau,\xi(\tau),\eta(\tau))}\right) \in \mathrm{G}_2(\tau, \xi(\tau), \eta(\tau)), \quad \tau \in (1,e), \\ \xi(1) = \xi(e) = 0, \quad \eta(1) = \eta(e) = 0, \end{cases} \tag{1.1}$$

where ${}^{H}D^{\alpha_1}, {}^{H}D^{\alpha_2}$ represent the HFDs of orders α_1, α_2, and $\alpha_1 \in (1,2]$, $\alpha_2 \in (1,2]$; f_1, $f_2 \in C([1,e] \times \mathbb{R}^2, \mathbb{R}\backslash\{0\})$, $g_1, g_2 \in C([1,e] \times \mathbb{R}^2, \mathbb{R})$ and satisfy $g_i(1,0,0) = 0, (i = 1,2)$,

$G_1, G_2 : [1, e] \times \mathbb{R}^2 \to \mathscr{P}(\mathbb{R})$ are the multi-valued maps, where $\mathscr{P}(\mathbb{R})$ denotes the set composed of all nonempty subsets of $\mathbb{R}$. By utilizing a fixed point theorem about several operators containing $x \in AxBx + Cx$ in Banach algebras, the existence result of solutions for (1.1) is derived via multi-valued maps in a normed space.

We give simple arrangements. In Section 2, some needed preliminary concepts and lemmas are reviewed. Section 3 proves an existence result for (1.1). In Section 4, an example is provided to verify our theoretical result. Finally, in Section 5, we conclude with a comprehensive description of the findings that are shown.

The main contributions of our work are as follows:

1) The coupled system of differential inclusions involving Hadamard fractional order is first proposed.

2) We derive sufficient conditions for the existence of solutions to (1.1), and the method is a fixed point theorem for three operators of Schaefer type.

3) The existence of solutions for the system is obtained.

4) The application is demonstrated through an example of coupled fractional differential inclusions.

2. Materials and methods

2.1. Material for fractional calculus

For a measurable function $y : [1, e] \to \mathbb{R}$ which is Lebesgue integrable, all such functions define a Banach space $L^1([1, e], \mathbb{R})$ normed by $\|y\|_{L^1} = \int_1^e |y(\tau)| d\tau$.

Definition 2.1 ([4, 5]) For an integrable function $h : [1, +\infty) \to \mathbb{R}$, the Hadamard fractional integral of order $q > 0$ is defined as

$$ {}^{H}I^{q} h(\tau) = \frac{1}{\Gamma(q)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{q-1} \frac{h(s)}{s} ds, $$

provided that the integral exists.

Definition 2.2 ([4, 5]) For an integrable function $h : [1, +\infty) \to \mathbb{R}$, the HFD of order $q > 0$ is defined as:

$$ {}^{H}D^{q} h(\tau) = \frac{1}{\Gamma(n-q)} \left(\tau \frac{d}{d\tau}\right)^n \int_1^\tau \left(\log \frac{\tau}{s}\right)^{n-q-1} \frac{h(s)}{s} ds, \quad n-1 < q < n, $$

where $n = [q] + 1$, $[q]$ is the smallest integer greater than or equal to q, and $\log(\cdot) = \log_e(\cdot)$.

Lemma 2.1 Let $\zeta_1,\ \zeta_2 \in C([1, e], \mathbb{R})$, $f_1,\ f_2 \in C([1, e] \times \mathbb{R}^2, \mathbb{R}\backslash\{0\})$, $g_1, g_2 \in C([1, e] \times \mathbb{R}^2, \mathbb{R})$, and satisfy $g_i(1, 0, 0) = 0, (i = 1, 2)$. Then, the integral solution of the Hadamard fractional differential system

$$ \begin{cases} {}^{H}D^{\alpha_1} \left(\frac{(\xi(\tau) - g_1(\tau, \xi(\tau), \eta(\tau))}{f_1(\tau, \xi(\tau), \eta(\tau))}\right) = \zeta_1(\tau), \quad \tau \in (1, e),\ \alpha_1 \in (1, 2], \\ {}^{H}D^{\alpha_2} \left(\frac{(\eta(\tau) - g_2(\tau, \xi(\tau), \eta(\tau))}{f_2(\tau, \xi(\tau), \eta(\tau))}\right) = \zeta_2(\tau), \quad \tau \in (1, e),\ \alpha_2 \in (1, 2], \\ \xi(1) = \xi(e) = 0, \quad \eta(1) = \eta(e) = 0 \end{cases} \tag{2.1} $$

is given by

$$\begin{cases} \xi(\tau) = f_1(\tau,\xi(\tau),\eta(\tau))\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_1^\tau\left(\log\frac{\tau}{s}\right)^{(\alpha_1-1)}\frac{\zeta_1(s)}{s}ds - \frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_1-1)} \\ \qquad -\frac{(\log\tau)^{(\alpha_1-1)}}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{(\alpha_1-1)}\frac{\zeta_1(s)}{s}ds\Bigg) + g_1(\tau,\xi(\tau),\eta(\tau)), \quad \tau\in(1,e), \\ \eta(\tau) = f_2(\tau,\xi(\tau),\eta(\tau))\Bigg(\frac{1}{\Gamma(\alpha_2)}\int_1^\tau\left(\log\frac{\tau}{s}\right)^{(\alpha_2-1)}\frac{\zeta_2(s)}{s}ds - \frac{g_2(e,0,0)}{f_2(e,0,0)}(\log\tau)^{(\alpha_2-1)} \\ \qquad -\frac{(\log\tau)^{(\alpha_2-1)}}{\Gamma(\alpha_2)}\int_1^e\left(\log\frac{e}{s}\right)^{(\alpha_2-1)}\frac{\zeta_2(s)}{s}ds\Bigg) + g_2(\tau,\xi(\tau),\eta(\tau)), \quad \tau\in(1,e). \end{cases} \tag{2.2}$$

Proof. According to the formula in [14] Chapter 9, the solution of (2.1) can be formulated in the following manner:

$$\begin{cases} \xi(\tau) = f_1(\tau,\xi(\tau),\eta(\tau))\left(\frac{1}{\Gamma(\alpha_1)}\int_1^\tau\left(\log\frac{\tau}{s}\right)^{(\alpha_1-1)}\frac{\zeta_1(s)}{s}ds + a_1(\log\tau)^{(\alpha_1-1)} + a_2(\log\tau)^{(\alpha_1-2)}\right) \\ \qquad + g_1(\tau,\xi(\tau),\eta(\tau)), \\ \eta(\tau) = f_2(\tau,\xi(\tau),\eta(\tau))\left(\frac{1}{\Gamma(\alpha_2)}\int_1^\tau\left(\log\frac{\tau}{s}\right)^{(\alpha_2-1)}\frac{\zeta_2(s)}{s}ds + b_1(\log\tau)^{(\alpha_2-1)} + b_2(\log\tau)^{(\alpha_2-2)}\right) \\ \qquad + g_2(\tau,\xi(\tau),\eta(\tau)), \end{cases} \tag{2.3}$$

where $a_1, a_2, b_1, b_2 \in \mathbb{R}$. By using the Dirichlet boundary conditions $\xi(1) = \xi(e) = 0, \eta(1) = \eta(e) = 0$ in (2.1), $g_i(1,0,0) = 0$, $(i = 1,2)$, we obtain

$$\begin{aligned} a_2 = 0, \quad a_1 &= -\tfrac{g_1(e,0,0)}{f_1(e,0,0)} - \tfrac{1}{\Gamma(\alpha_1)}\int_1^e\left(\log\tfrac{e}{s}\right)^{(\alpha_1-1)}\tfrac{\zeta_1(s)}{s}ds, \\ b_2 = 0, \quad b_1 &= -\tfrac{g_2(e,0,0)}{f_2(e,0,0)} - \tfrac{1}{\Gamma(\alpha_2)}\int_1^e\left(\log\tfrac{e}{s}\right)^{(\alpha_2-1)}\tfrac{\zeta_2(s)}{s}ds. \end{aligned}$$

By substituting a_1, a_2, b_1, b_2 into (2.3), we can get (2.2). □

Definition 2.3 The solution of (1.1) is used to define a pair of functions (ξ,η) that satisfy the following conditions:
(1) There exists a pair of function $(\zeta_1,\zeta_2) \in L^1([1,e],\mathbb{R}) \times L^1([1,e],\mathbb{R})$ with $\zeta_1 \in \mathrm{G}_1(\tau,\xi(\tau),\eta(\tau))$ and $\zeta_2 \in \mathrm{G}_2(\tau,\xi(\tau),\eta(\tau))$ satisfying ${}^HD^{\alpha_1}\left(\frac{\xi(\tau)-g_1(\tau,\xi(\tau),\eta(\tau))}{f_1(\tau,\xi(\tau),\eta(\tau))}\right) = \zeta_1(\tau)$, ${}^HD^{\alpha_2}\left(\frac{\eta(\tau)-g_2(\tau,\xi(\tau),\eta(\tau))}{f_2(\tau,\xi(\tau),\eta(\tau))}\right) = \zeta_2(\tau)$, for almost every on $[1,e]$;
(2) $\xi(1) = \xi(e) = 0, \eta(1) = \eta(e) = 0$.

2.2. *Material for multi-valued maps*

Next, an introduction is provided to fundamental concepts concerning normed spaces and multi-valued maps.

Let $X = C([1,e],\mathbb{R}) = \{\xi : \xi : [1,e] \to \mathbb{R}\, \textit{is continuous}\}$, and the norm $\|\xi\| = \sup_{\tau\in[1,e]}|\xi(\tau)|$. Then, X is a Banach space. With respect to a suitable multiplication " $\cdot$ " defined by $(\xi,\eta)(\tau) = \xi(\tau)\cdot\eta(\tau)$ for $\xi,\eta \in X$, the aforementioned entity X will be regarded as a Banach algebra.

For the product space $\Pi = X \times X$ under the norm $\|(\xi,\eta)\| = \|\xi\| + \|\eta\|$, it can also be demonstrated that the space Π is a Banach space. Further, with respect to a suitable multiplication " $\cdot$ " defined by

$((\xi,\eta)\cdot(\bar{\xi},\bar{\eta}))(\tau) = (\xi,\eta)(\tau)\cdot(\bar{\xi},\bar{\eta})(\tau) = (\xi(\tau)\bar{\xi}(\tau),\eta(\tau)\bar{\eta}(\tau))$ for $(\xi,\eta),(\bar{\xi},\bar{\eta}) \in \Pi$, the product space Π will be regarded as a Banach algebra. For further information regarding the results of the product space Π, please see [20, 21].

Now, we cover some basic and useful knowledge about multi-valued mappings. $\mathscr{P}(\Pi)$ denotes the set composed of all nonempty subsets of Π (see [22]):

$$\mathscr{P}_{cl}(\Pi) = \{\mathscr{F} \in \mathscr{P}(\Pi) : \mathscr{F} \; is\, closed\}, \quad \mathscr{P}_b(\Pi) = \{\mathscr{F} \in \mathscr{P}(\Pi) : \mathscr{F} \; is\, bounded\},$$
$$\mathscr{P}_{cp}(\Pi) = \{\mathscr{F} \in \mathscr{P}(\Pi) : \mathscr{F} \; is\, compact\}, \quad \mathscr{P}_{cp,cv}(\Pi) = \{\mathscr{F} \in \mathscr{P}(\Pi) : \mathscr{F} \; is\, compact\, and\, convex\}.$$

Definition 2.4 In the study of a multi-valued map $G : \Pi \to \mathscr{P}_{cl}(\Pi)$, if $G(\xi,\eta)$ is convex (closed) for $(\xi,\eta) \in \Pi$, then it is called convex (closed) valued.

Definition 2.5 It is our contention that the map G is bounded on bounded sets, if $G(\mathbb{B}) = \cup_{(\xi,\eta)\in\mathbb{B}} G(\xi,\eta)$ is bounded in Ξ for any bounded set $\mathbb{B}$ of Π (i.e., $\sup_{(\xi,\eta)\in\mathbb{B}}\{\|(x,y)\| : (x,y) \in G(\xi,\eta)\} < \infty$).

Definition 2.6 The map G is defined as an upper semi-continuous (*u.s.c.*) map on Π: if for each $(\xi,\eta) \in \Pi$, the set $G(\xi,\eta)$ is a nonempty closed subset of Π, and if for each open set $\mathbb{B}$ of Π containing $G(\xi,\eta)$, there exists an open neighborhood O of (ξ,η) such that $G(O) \subset \mathbb{B}$.

Definition 2.7 The map G is defined as a completely continuous map when the graph $G(\mathbb{B})$ is relatively compact for all bounded subsets $\mathbb{B}$ of Π.

Definition 2.8 A multi-valued map $G : [1,e] \times \mathbb{R}^2 \to \mathscr{P}(\mathbb{R})$ is called L^1-Carathéodory when
(i) $\tau \to G(\tau,\xi,\eta)$ is measurable for each $(\xi,\eta) \in \mathbb{R}\times\mathbb{R}$;
(ii) $(\xi,\eta) \to G(\tau,\xi,\eta)$ is upper semi-continuous for a.e. $\tau \in [1,e]$;
(iii) it can be shown that a function exists $\omega_r \in L^1([1,e],\mathbb{R}^+)$ such that

$$\|G(\tau,\xi,\eta)\| = \sup\{|x| : x \in G(\tau,\xi,\eta)\} \le \omega_r(\tau),$$

for $\xi,\eta \in \mathbb{R}$ with $|\xi| + |\eta| \le r$ and for a.e. $\tau \in [1,e]$.

With each $(\xi,\eta) \in \Pi$, state the set of selections of $G_{\xi\eta} = (G_{1,\xi\eta}, G_{2,\xi\eta})$ are given as follows:

$$G_{1,\xi\eta} := \{v_1 \in L^1([1,e],\mathbb{R}) : v_1(t) \in G_1(\tau,\xi(\tau),\eta(\tau)), \quad for\, a.e.\, \tau \in [1,e]\},$$
$$G_{2,\xi\eta} := \{v_2 \in L^1([1,e],\mathbb{R}) : v_2(t) \in G_2(\tau,\xi(\tau),\eta(\tau)), \quad for\, a.e.\, \tau \in [1,e]\}.$$

For two normed spaces X, Y and a multi-valued map $G : X \to \mathscr{P}(Y)$, we define $G_r(G) = \{(\xi,\eta) \in X\times Y, \eta \in G(\xi)\}$ as a graph of G and review two important lemmas.

Lemma 2.2 ([18]) If $G : X \to \mathscr{P}_{cl}(Y)$ is u.s.c., then $G_r(G) = \{(\xi,\eta) \in X\times Y, \eta \in G(x)\}$ is a closed subset of $X\times Y$; i.e., for every sequence $\{\xi_n\}_{n\in N} \subset X$ and $\{\eta_n\}_{n\in N} \subset Y$, if when $n \to \infty$, $\xi_n \to \xi_*$, $\eta_n \to \eta_*$, and $\eta_n \in G(\xi_n)$, then $\eta_* \in G(\xi_*)$. Conversely, if G is completely continuous and has a closed graph, then it is u.s.c.

Lemma 2.3 ([19]) X is a Banach space. Let $G : [0,T]\times X \to \mathscr{P}_{cp,cv}(X)$ be an L^1-Carathéodory set-valued map and $\Theta : L^1([0,T];X) \to C([0,T];X)$ be a linear continuous mapping. Subsequently, the operator

$$\Theta \circ S_{G,\xi} : C([0,T];X) \to P_{cp,cv}(C([0,T];X))$$

$$\xi \mapsto (\Theta \circ S_G)(\xi) = \Theta(S_{G,\xi})$$

is a closed graph operator in $C([0,T];X) \times C([0,T];X)$.

3. Results

Now, we can consider the system (1.1). Our method is based on the following two lemmas.

Lemma 3.1 ([17]) Let X be a Banach algebra, $\Omega \subset X$ be a nonempty, closed convex, and bounded subset. Operators $A, C : X \to X$ and $B : \Omega \to X$ satisfy:
(a) A and C are both Lipschitzian, and the corresponding Lipschitz constants are denoted by δ and ρ;
(b) B is a completely continuous map (i.e. is compact and continuous);
(c) $x = AxBy + Cx \Rightarrow x \in \Omega$ for $y \in \Omega$;
(d) $\delta M + \rho < 1$, where $M = \|B(\Omega)\| = \sup\{\|Bx\| : x \in \Omega\}$.
Then, $AxBx + Cx = x$ has a solution in Ω.

Lemma 3.2 ([22]) Let X be a Banach algebra, $\Omega \subset X$ be a nonempty, closed convex, and bounded subset. And, A, $C : X \to X$ are two single-valued and $B : \Omega \to \mathscr{P}_{cp,cv}(X)$ is multi-valued operator, satisfying:
(a) A and C are both Lipschitzian, and the corresponding Lipschitz constants are denoted by δ and ρ;
(b) B is compact and upper semi-continuous;
(c) $\delta M + \rho < 1/2$, where $M = \|B(\Omega)\|$.
Then, either:
(i) $x \in AxBx + Cx$ has a solution;
or,
(ii) the set $\Phi = \{x \in X | \mu x \in AxBx + Cx, \mu > 1\}$ is unbounded.

Next, we introduce the following assumptions:

($H1$) $f_i : [1,e] \times \mathbb{R}^2 \to \mathbb{R}\backslash\{0\}$ are continuous, and the constants $L_i > 0$ exist and are satisfied for

$$|f_i(\tau,\xi,\eta) - f_i(\tau,\tilde{\xi},\tilde{\eta})| \le L_i[|\xi - \tilde{\xi}| + |\eta - \tilde{\eta}|],\ i = 1,2,$$

a.e. $\tau \in [1,e]$, $\forall \xi,\eta,\tilde{\xi},\tilde{\eta} \in \mathbb{R}$;

($H2$) Functions $g_i : [1,e] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous, and satisfy $g_i(1,0,0) = 0$, ($i = 1,2$). The constants $K_i > 0$ exist and are satisfied for

$$|g_i(t,\xi,\eta) - g_i(t,\tilde{\xi},\tilde{\eta}) \le K_i[|\xi - \tilde{\xi}| + |\eta - \tilde{\eta}|],\ i = 1,2,$$

a.e. $\tau \in [1,e]$, $\forall \xi,\eta,\tilde{\xi},\tilde{\eta} \in \mathbb{R}$;

$(H3)$ Multi-valued maps $\mathrm{G}_i : [1,e]\times\mathbb{R}^2 \to \mathscr{P}(\mathbb{R})$ are L^1-Carathéodory and have nonempty compact and convex values, $i = 1, 2$;

$(H4)$ There exists a real number $r > 0$ such that

$$r > \frac{\frac{2F_{10}}{\Gamma(\alpha_1)}\|\omega_{1r}\|_{L^1} + \frac{2F_{20}}{\Gamma(\alpha_2)}\|\omega_{2r}\|_{L^1} + 2G_{10} + 2G_{20}}{1 - \delta\left(\frac{2}{\Gamma(\alpha_1)}\|\omega_{1r}\|_{L^1} + \frac{2}{\Gamma(\alpha_2)}\|\omega_{2r}\|_{L^1} + \frac{G_{10}}{F_{10}} + \frac{G_{20}}{F_{20}}\right) - \rho},$$

where

$$\delta\left(\frac{2}{\Gamma(\alpha_1)}\|\omega_{1r}\|_{L^1} + \frac{2}{\Gamma(\alpha_2)}\|\omega_{2r}\|_{L^1} + \frac{G_{10}}{F_{10}} + \frac{G_{20}}{F_{20}}\right) + \rho < 1/2,$$

$\delta = L_1 + L_2$, $\rho = K_1 + K_2$, $F_{10} = \sup_{\tau\in[1,e]}|f_1(\tau,0,0)|$, $F_{20} = \sup_{\tau\in[1,e]}|f_2(\tau,0,0)|$, $G_{10} = \sup_{\tau\in[1,e]}|g_1(\tau,0,0)|$, $G_{20} = \sup_{\tau\in[1,e]}|g_2(\tau,0,0)|$; here, $\omega_{1r}(\tau)$ and $\omega_{2r}(\tau)$ are provided in the aforementioned Definition 2.8.

Theorem 3.1 Suppose that $(H1) - (H4)$ are satisfied. Then, system (1.1) has at least one solution on $[1,e] \times [1,e]$.

Proof. With the aforementioned Lemma 2.1, we can obviously turn problem (1.1) into an operator fixed-point problem. Before that, we define the operator $\mathcal{N} : \Pi \to \mathscr{P}(\Pi)$ as $\mathcal{N}(\xi,\eta)(\tau) = (\mathcal{N}_1(\xi,\eta)(\tau), \mathcal{N}_2(\xi,\eta)(\tau))$, where

$$\begin{aligned}\mathcal{N}_1(\xi,\eta)(\tau) = \Bigg\{ & \mathfrak{h}_1 \in C([1,e],\mathbb{R}) : \mathfrak{h}_1(\tau) = f_1(\tau,\xi(\tau),\eta(\tau))\left(\frac{1}{\Gamma(\alpha_1)}\int_1^\tau \left(\log\frac{\tau}{s}\right)^{(\alpha_1-1)}\frac{v_1(s)}{s}ds\right. \\ & \left. -\frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_1-1)} - \frac{(\log\tau)^{(\alpha_1-1)}}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{(\alpha_1-1)}\frac{v_1(s)}{s}ds\right) \\ & +g_1(\tau,\xi(\tau),\eta(\tau)), v_1 \in G_{1,\xi\eta}\Bigg\},\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\mathcal{N}_2(\xi,\eta)(\tau) = \Bigg\{ & \mathfrak{h}_2 \in C([1,e],\mathbb{R}) : \mathfrak{h}_2(\tau) = f_2(\tau,\xi(\tau),\eta(\tau))\left(\frac{1}{\Gamma(\alpha_2)}\int_1^\tau \left(\log\frac{\tau}{s}\right)^{(\alpha_2-1)}\frac{v_2(s)}{s}ds\right. \\ & \left. -\frac{g_2(e,0,0)}{f_2(e,0,0)}(\log\tau)^{(\alpha_2-1)} - \frac{(\log\tau)^{(\alpha_2-1)}}{\Gamma(\alpha_2)}\int_1^e\left(\log\frac{e}{s}\right)^{(\alpha_2-1)}\frac{v_2(s)}{s}ds\right) \\ & +g_2(\tau,\xi(\tau),\eta(\tau)), v_2 \in G_{2,\xi\eta}\Bigg\}.\end{aligned} \tag{3.2}$$

Thereby, we define three operators, which are $\mathcal{A} = (\mathcal{A}_1, \mathcal{A}_2)$, $\mathcal{B} = (\mathcal{B}_1, \mathcal{B}_2)$, and $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$. Here, the mapping $\mathcal{A}_i$: $\Pi \to \Pi$ is given by

$$\mathcal{A}_i(\xi,\eta)(\tau) = f_i(\tau,\xi(\tau),\eta(\tau)), \quad \tau \in [1,e],\ i = 1,2,$$

and define $\mathcal{B}_i : \Pi \to \mathscr{P}(\Pi)$ as

$$\begin{aligned}\mathcal{B}_i(\xi,\eta)(\tau) = \Bigg\{ & \frac{1}{\Gamma(\alpha_i)}\int_1^\tau\left(\log\frac{\tau}{s}\right)^{(\alpha_i-1)}\frac{v_i(s)}{s}ds - \frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_i-1)} \\ & -\frac{(\log\tau)^{(\alpha_i-1)}}{\Gamma(\alpha_i)}\int_1^e\left(\log\frac{e}{s}\right)^{(\alpha_i-1)}\frac{v_i(s)}{s}ds, v_i \in G_{i,\xi\eta}\Bigg\}, \quad \tau \in [1,e],\ i = 1.2,\end{aligned} \tag{3.3}$$

and the C_i: $\Pi \to \Pi$ are given by

$$C_i(\xi,\eta)(\tau) = g_i(t,\xi(\tau),\eta(\tau)), \quad \tau \in [1,e],\ i = 1,2.$$

Observe that $\mathcal{N}_i(\xi,\eta) = \mathcal{A}_i(\xi,\eta)\mathcal{B}_i(\xi,\eta) + C_i(\xi,\eta), i = 1,2$. Then, the operator $\mathcal{N}$ can also be written as

$$\mathcal{N}(\xi,\eta) = (\mathcal{A}_1(\xi,\eta)\mathcal{B}_1(\xi,\eta) + C_1(\xi,\eta), \mathcal{A}_2(\xi,\eta)\mathcal{B}_2(\xi,\eta) + C_2(\xi,\eta)).$$

We need to show that the operators $\mathcal{A}$, $\mathcal{B}$, and C satisfy all the conditions of Lemma 3.2. For a clearer and more intuitive reading, we split the proof into several steps.

Step 1. We first show that Lemma 3.2(a) holds, i.e., we are going to prove A and C are both Lipschitzian, and the corresponding Lipschitz constants are denoted by δ and ρ. By $(H1)$, we have

$$\begin{aligned}|\mathcal{A}_i(\xi,\eta)(\tau) - \mathcal{A}_i(\tilde{\xi},\tilde{\eta})(\tau)| &= |f_i(\tau,\xi(\tau),\eta(\tau)) - f_i(\tau,\tilde{\xi}(\tau),\tilde{\eta}(\tau))|\\ &\le L_i[|\xi(\tau) - \tilde{\xi}(\tau)| + |\eta(\tau) - \tilde{\eta}(\tau)|]\\ &\le L_i[\|\xi - \tilde{\xi}\| + \|\eta - \tilde{\eta}\|], \quad \tau \in [1,e],\ i = 1,2.\end{aligned}$$

Hence, $\|\mathcal{A}_i(\xi,\eta) - \mathcal{A}_i(\tilde{\xi},\tilde{\eta})\| \le L_i[\|\xi - \tilde{\xi}\| + \|\eta - \tilde{\eta}\|], \quad i = 1,2$. Then,

$$\begin{aligned}\|\mathcal{A}(\xi,\eta) - \mathcal{A}(\tilde{\xi},\tilde{\eta})\| &= \|\mathcal{A}_1(\xi,\eta) - \mathcal{A}_1(\tilde{\xi},\tilde{\eta})\| + \|\mathcal{A}_2(\xi,\eta) - \mathcal{A}_2(\tilde{\xi},\tilde{\eta})\|\\ &\le L_1[\|\xi - \tilde{\xi}\| + \|\eta - \tilde{\eta}\|] + L_2[\|\xi - \tilde{\xi}\| + \|\eta - \tilde{\eta}\|]\\ &\le (L_1 + L_2)\|(\xi,\eta) - (\tilde{\xi},\tilde{\eta})\|.\end{aligned}$$

So, $\mathcal{A}$ is Lipschitzian on Π, and the Lipschitz constant is $\delta = L_1 + L_2$.

In the same way, from $(H2)$, we have

$$\begin{aligned}|C_i(\xi,\eta)(\tau) - C_i(\tilde{\xi},\tilde{\eta})(\tau)| &= |g_i(\tau,\xi(\tau),\eta(\tau)) - g_i(\tau,\tilde{\xi}(\tau),\tilde{\eta}(\tau))|\\ &\le K_i[|\xi(\tau) - \tilde{\xi}(\tau)| + |\eta(\tau) - \tilde{\eta}(\tau)|]\\ &\le K_i[\|\xi - \tilde{\xi}\| + \|\eta - \tilde{\eta}\|], \quad \tau \in [1,e],\ i = 1,2.\end{aligned}$$

Hence, $\|C_i(\xi,\eta) - C_i(\tilde{\xi},\tilde{\eta})\| \le K_i[\|\xi - \tilde{\xi}\| + \|\eta - \tilde{\eta}\|], \quad i = 1,2$. Thus,

$$\begin{aligned}\|C(\xi,\eta) - C(\tilde{\xi},\tilde{\eta})\| &= \|C_1(\xi,\eta) - C_1(\tilde{\xi},\tilde{\eta})\| + \|C_2(\xi,\eta) - C_2(\tilde{\xi},\tilde{\eta})\|\\ &\le K_1[\|\xi - \tilde{\xi}\| + \|\eta - \tilde{\eta}\|] + K_2[\|\xi - \tilde{\xi}\| + \|\eta - \tilde{\eta}\|]\\ &\le (K_1 + K_2)\|(\xi,\eta) - (\tilde{\xi},\tilde{\eta})\|.\end{aligned}$$

So, C is Lipschitzian on Π, and the Lipschitz constant is $\rho = K_1 + K_2$.

Step 2. We show that Lemma 3.2(b) holds, i.e., $\mathcal{B}$ is compact and u.s.c. on Π.

(i) We demonstrate that the operator $\mathcal{B}$ has convex values. Let u_{11}, $u_{12} \in B_1(\xi,\eta)$, u_{21}, $u_{22} \in B_2(\xi,\eta)$. Then, there exist v_{11}, $v_{12} \in \mathrm{G}_{1,\xi\eta}$, v_{21}, $v_{22} \in \mathrm{G}_{2,\xi\eta}$ such that

$$\begin{aligned}u_{1j}(\tau) =& \frac{1}{\Gamma(\alpha_1)}\int_1^\tau \left(\log\frac{\tau}{s}\right)^{(\alpha_1-1)} \frac{v_{1j}(s)}{s}ds - \frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_1-1)}\\ &- \frac{(\log\tau)^{(\alpha_1-1)}}{\Gamma(\alpha_1)}\int_1^e \left(\log\frac{e}{s}\right)^{(\alpha_1-1)} \frac{v_{1j}(s)}{s}ds, \quad j = 1,2,\ \tau \in [1,e].\end{aligned}$$

$$u_{2j}(\tau) = \frac{1}{\Gamma(\alpha_2)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{(\alpha_2-1)} \frac{v_{2j}(s)}{s} ds - \frac{g_1(e,0,0)}{f_1(e,0,0)} (\log \tau)^{(\alpha_2-1)}$$
$$- \frac{(\log \tau)^{(\alpha_2-1)}}{\Gamma(\alpha_2)} \int_1^e \left(\log \frac{e}{s}\right)^{(\alpha_2-1)} \frac{v_{2j}(s)}{s} ds, \quad j = 1,2,\ \tau \in [1,e].$$

For any constant $0 \le \sigma \le 1$, we have

$$\sigma u_{11}(\tau) + (1-\sigma)u_{12}(\tau) = \frac{1}{\Gamma(\alpha_1)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{(\alpha_1-1)} \frac{\sigma v_{11}(s) + (1-\sigma)v_{12}(s)}{s} ds - \frac{g_1(e,0,0)}{f_1(e,0,0)} (\log \tau)^{(\alpha_1-1)}$$
$$- \frac{(\log \tau)^{(\alpha_1-1)}}{\Gamma(\alpha_1)} \int_1^e \left(\log \frac{e}{s}\right)^{(\alpha_1-1)} \frac{\sigma v_{11}(s) + 1 - \sigma v_{12}(s)}{s} ds,$$

$$\sigma u_{21}(\tau) + (1-\sigma)u_{22}(\tau) = \frac{1}{\Gamma(\alpha_2)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{(\alpha_2-1)} \frac{\sigma v_{21}(s) + (1-\sigma)v_{22}(s)}{s} ds - \frac{g_1(e,0,0)}{f_1(e,0,0)} (\log \tau)^{(\alpha_1-1)}$$
$$- \frac{(\log \tau)^{(\alpha_2-1)}}{\Gamma(\alpha_2)} \int_1^e \left(\log \frac{e}{s}\right)^{(\alpha_2-1)} \frac{\sigma v_{21}(s) + (1-\sigma)v_{22}(s)}{s} ds,$$

where $\bar{v}_1(\tau) = \sigma v_{11}(\tau) + (1-\sigma)v_{12}(\tau) \in \mathrm{G}_{1,\xi\eta}$, $\bar{v}_2(\tau) = \sigma v_{21}(\tau) + (1-\sigma)v_{22}(\tau) \in \mathrm{G}_{2,\xi\eta}$ for all $\tau \in [1,e]$. Therefore,

$$\sigma u_{11}(\tau) + (1-\sigma)u_{12}(\tau) \in B_1(\xi,\eta), \sigma u_{21}(\tau) + (1-\sigma)u_{22}(\tau) \in B_2(\xi,\eta),$$

$$B(\sigma u_{11}(\tau) + (1-\sigma)u_{12}(\tau), \sigma u_{21}(\tau) + (1-\sigma)u_{22}(\tau)) = \sigma B(u_{11}(\tau), u_{21}(\tau)) + (1-\sigma)B(u_{12}(\tau), u_{22}(\tau)) \in B(\xi,\eta).$$

Then, we obtain $\mathcal{B}(\xi,\eta)$ which is convex for each $(\xi,\eta) \in \Pi$. Then, operator B defines a multi-valued operator $B : \Pi \to \mathscr{P}_{cv}(\Pi)$.

(ii) We display that the operator $\mathcal{B}$ maps bounded sets into bounded sets in Π. Let $\Omega = \{(\xi,\eta) | \, \|(\xi,\eta)\| \le r, (\xi,\eta) \in \Pi\}$. Then, for each $p_i \in \mathcal{B}_i(\xi,\eta), i = 1,2$, there exist $v_i \in \mathrm{G}_{i,\xi\eta} (i = 1,2)$ such that

$$p_i(\tau) = \frac{1}{\Gamma(\alpha_i)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{(\alpha_i-1)} \frac{v_i(s)}{s} ds - \frac{g_1(e,0,0)}{f_1(e,0,0)} (\log \tau)^{(\alpha_i-1)}$$
$$\frac{(\log \tau)^{(\alpha_i-1)}}{\Gamma(\alpha_i)} \int_1^e \left(\log \frac{e}{s}\right)^{(\alpha_i-1)} \frac{v_i(s)}{s} ds, \tau \in [1,e].$$

From $(H3)$, we have

$$|\mathcal{B}_1(\xi,\eta)(\tau)| \le \left| \frac{1}{\Gamma(\alpha_1)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{(\alpha_1-1)} \frac{v_1(s)}{s} ds - \frac{g_1(e,0,0)}{f_1(e,0,0)} (\log \tau)^{(\alpha_1-1)} - \frac{(\log \tau)^{(\alpha_1-1)}}{\Gamma(\alpha_1)} \int_1^e \left(\log \frac{e}{s}\right)^{(\alpha_1-1)} \frac{v_1(s)}{s} ds \right|$$
$$\le \frac{1}{\Gamma(\alpha_1)} \int_1^\tau \left(\log \frac{\tau}{s}\right)^{(\alpha_1-1)} \frac{\omega_{1r}(s)}{s} ds + \frac{g_1(e,0,0)}{f_1(e,0,0)} (\log \tau)^{(\alpha_1-1)} + \frac{(\log \tau)^{(\alpha_1-1)}}{\Gamma(\alpha_1)} \int_1^e \left(\log \frac{e}{s}\right)^{(\alpha_1-1)} \frac{\omega_{1r}(s)}{s} ds$$
$$\le \frac{2}{\Gamma(\alpha_1+1)} \|\omega_{1r}\|_{L^1} + \frac{G_{10}}{F_{10}},$$

and, similarly,

$$|\mathcal{B}_2(\xi,\eta)(\tau)| \le \frac{2}{\Gamma(\alpha_2+1)} \|\omega_{2r}\|_{L^1} + \frac{G_{20}}{F_{20}}.$$

This implies that

$$\|\mathcal{B}(\xi,\eta)\| = \|\mathcal{B}_1(\xi,\eta)\| + \|\mathcal{B}_2(\xi,\eta)\| \le \frac{2}{\Gamma(\alpha_1+1)} \|\omega_{1r}\|_{L^1} + \frac{2}{\Gamma(\alpha_2+1)} \|\omega_{2r}\|_{L^1} + \frac{G_{10}}{F_{10}} + \frac{G_{20}}{F_{20}}.$$

Thus, $\mathcal{B}(\Pi)$ is uniformly bounded. Then, $\mathcal{B}$ defines a multi-valued operator $\mathcal{B}:\Pi\to\mathscr{P}_b(\Pi)$.
(iii) We show that the operator $\mathcal{B}$ maps bounded sets into equi-continuous sets in Π. Let $q_i\in\mathcal{B}_i(\xi,\eta)(i=1,2)$ for some $(\xi,\eta)\in\Omega$, where Ω is given as earlier. So, there exists $u_i\in \mathrm{G}_{i,\xi\eta}$, such that

$$\begin{aligned}q_i(\tau)=&\frac{1}{\Gamma(\alpha_i)}\int_1^\tau\left(\log\frac{\tau}{s}\right)^{(\alpha_i-1)}\frac{u_i(s)}{s}ds-\frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_i-1)}\\&-\frac{(\log\tau)^{(\alpha_i-1)}}{\Gamma(\alpha_i)}\int_1^e\left(\log\frac{e}{s}\right)^{(\alpha_i-1)}\frac{u_i(s)}{s}ds,\ i=1,2.\end{aligned}$$

For any $t_1,t_2\in[1,e]$ and $t_1<t_2$, we have

$$\begin{aligned}|q_1(t_1)-q_1(t_2)|\leq&\frac{\|\omega_{1r}\|_{L^1}}{\Gamma(\alpha_1)}\left|\int_1^{t_1}\left(\log\frac{t_1}{s}\right)^{\alpha_1-1}\frac{1}{s}ds-\int_1^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha_1-1}\frac{1}{s}ds\right|\\&+\frac{g_1(e,0,0)}{f_1(e,0,0)}\left|(\log t_1)^{\alpha_1-1}-(\log t_2)^{\alpha_1-1}\right|\\&+\frac{\|\omega_{1r}\|_{L^1}}{\Gamma(\alpha_1)}\left|(\log t_1)^{\alpha_1-1}-(\log t_2)^{\alpha_1-1}\right|\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{1}{s}ds\\&\leq\frac{\|\omega_{1r}\|_{L^1}}{\Gamma(\alpha_1)}\left|\int_1^{t_1}\left[\left(\log\frac{t_1}{s}\right)^{\alpha_1-1}-\left(\log\frac{t_2}{s}\right)^{\alpha_1-1}\right]\frac{1}{s}ds\right|\\&+\frac{\|\omega_{1r}\|_{L^1}}{\Gamma(\alpha_1)}\left|\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha_1-1}\frac{1}{s}ds\right|\\&+\frac{\|\omega_{1r}\|_{L^1}}{\Gamma(\alpha_1)}\left|(\log t_1)^{\alpha_1-1}-(\log t_2)^{\alpha_1-1}\right|\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_1-1}\frac{1}{s}ds\\&+\frac{G_{10}}{F_{10}}\left|(\log t_1)^{\alpha_1-1}-(\log t_2)^{\alpha_1-1}\right|\end{aligned}\tag{3.4}$$

and

$$\begin{aligned}|q_2(t_1)-q_2(t_2)|\leq&\frac{\|\omega_{2r}\|_{L^1}}{\Gamma(\alpha_2)}\left|\int_1^{t_1}\left[\left(\log\frac{t_1}{s}\right)^{\alpha_2-1}-\left(\log\frac{t_2}{s}\right)^{\alpha_2-1}\right]\frac{1}{s}ds\right|\\&+\frac{\|\omega_{2r}\|_{L^1}}{\Gamma(\alpha_2)}\left|\int_{t_1}^{t_2}\left(\log\frac{t_2}{s}\right)^{\alpha_1-1}\frac{1}{s}ds\right|\\&+\frac{\|\omega_{2r}\|_{L^1}}{\Gamma(\alpha_2)}\left|(\log t_1)^{\alpha_1-1}-(\log t_2)^{\alpha_2-1}\right|\int_1^e\left(\log\frac{e}{s}\right)^{\alpha_2-1}\frac{1}{s}ds\\&+\frac{G_{20}}{F_{20}}\left|(\log t_1)^{\alpha_2-1}-(\log t_2)^{\alpha_2-1}\right|.\end{aligned}\tag{3.5}$$

Note that the right-hand side of the two inequalities (3.4) and (3.5) go to zero for arbitrary $(\xi,\eta)\in\Omega$ as $t_2\to t_1$.

So, as $t_2\to t_1$, we have

$$\|\mathcal{B}_1(\xi,\eta)(t_1)-\mathcal{B}_1(\xi,\eta)(t_2)\|\to0,\|\mathcal{B}_2(\xi,\eta)(t_1)-\mathcal{B}_2(\xi,\eta)(t_2)\|\to0.$$

Therefore, $\mathcal{B}_1$ and $\mathcal{B}_2$ are equi-continuous. Also, note that $\|\mathcal{B}(\xi,\eta)\|=\|\mathcal{B}_1(\xi,\eta)\|+\|\mathcal{B}_2(\xi,\eta)\|$, so, as $t_2\to t_1$,

$$\|\mathcal{B}(\xi,\eta)(t_1)-\mathcal{B}(\xi,\eta)(t_2)\|\to0.$$

So, B is equi-continuous.

From $(ii)-(iii)$ and the Arzelá-Ascoli theorem, we have $\mathcal{B}:\Pi\to\mathscr{P}(\Pi)$ is completely continuous. Thus, $\mathcal{B}$ defines a compact multi-valued operator $\mathcal{B}:\Pi\to\mathscr{P}_{cp}(\Pi)$.
(iv) We claim that $\mathcal{B}$ has a closed graph. Let $(\xi_n,\eta_n)\to(\xi_*,\eta_*)$ as $n\to\infty$, $(h_{1n},h_{2n})\in\mathcal{B}(\xi_n,\eta_n)$ and $(h_{1n},h_{2n})\to(h_{1*},h_{2*})$ as $n\to\infty$. Then, we need to prove that $(h_{1*},h_{2*})\in B(\xi_*,\eta_*)$, i.e., $h_{1*}\in\mathcal{B}_1(\xi_*,\eta_*)$, $h_{2*}\in\mathcal{B}_2(\xi_*,\eta_*)$. Due to $h_{1n}\in\mathcal{B}_1(\xi_n,\eta_n)$, $h_{2n}\in\mathcal{B}_2(\xi_n,\eta_n)$, there are $v_{1n}\in \mathrm{G}_{1,\xi\eta}$, $v_{2n}\in \mathrm{G}_{2,\xi\eta}$ such that

$$\begin{aligned}h_{1n}(\tau)=&\frac{1}{\Gamma(\alpha_1)}\int_1^\tau\left(\log\frac{\tau}{s}\right)^{(\alpha_1-1)}\frac{v_{1n}(s)}{s}ds-\frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_1-1)}\\&-\frac{(\log\tau)^{(\alpha_1-1)}}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{(\alpha_1-1)}\frac{v_{1n}(s)}{s}ds,\quad\tau\in[1,e]\end{aligned}$$

and

$$\begin{aligned}h_{2n}(\tau)=&\frac{1}{\Gamma(\alpha_2)}\int_1^\tau\left(\log\frac{\tau}{s}\right)^{(\alpha_2-1)}\frac{v_{2n}(s)}{s}ds-\frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_2-1)}\\&-\frac{(\log\tau)^{(\alpha_2-1)}}{\Gamma(\alpha_2)}\int_1^e\left(\log\frac{e}{s}\right)^{(\alpha_2-1)}\frac{v_{2n}(s)}{s}ds,\quad\tau\in[1,e].\end{aligned}$$

Thus, it suffices to show that there are $v_{1*}\in \mathrm{G}_{1,\xi_*\eta_*}$, $v_{2*}\in \mathrm{G}_{2,\xi_*\eta_*}$, such that for each $\tau\in[1,e]$,

$$\begin{aligned}h_{1*}(\tau)=&\frac{1}{\Gamma(\alpha_1)}\int_1^\tau\left(\log\frac{\tau}{s}\right)^{(\alpha_1-1)}\frac{v_{1*}(s)}{s}ds-\frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_1-1)}\\&-\frac{(\log\tau)^{(\alpha_1-1)}}{\Gamma(\alpha_1)}\int_1^e\left(\log\frac{e}{s}\right)^{(\alpha_1-1)}\frac{v_{1*}(s)}{s}ds,\quad\tau\in[1,e]\end{aligned}$$

and

$$\begin{aligned}h_{2*}(\tau)=&\frac{1}{\Gamma(\alpha_2)}\int_1^\tau\left(\log\frac{\tau}{s}\right)^{(\alpha_2-1)}\frac{v_{2*}(s)}{s}ds-\frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_2-1)}\\&-\frac{(\log\tau)^{(\alpha_2-1)}}{\Gamma(\alpha_2)}\int_1^e\left(\log\frac{e}{s}\right)^{(\alpha_2-1)}\frac{v_{2*}(s)}{s}ds,\quad\tau\in[1,e].\end{aligned}$$

Let us take the linear operator $\Gamma=(\Gamma_1,\Gamma_2)$, where $\Gamma_i:L^1([1,e],\mathbb{R})\to C([1,e],\mathbb{R})$ are given by:

$$\Gamma_i(v_i)(\tau)=\frac{1}{\Gamma(\alpha_i)}\int_1^\tau\left(\log\frac{\tau}{s}\right)^{(\alpha_i-1)}\frac{v_i(s)}{s}ds-\frac{(\log\tau)^{(\alpha_i-1)}}{\Gamma(\alpha_i)}\int_1^e\left(\log\frac{e}{s}\right)^{(\alpha_i-1)}\frac{v_i(s)}{s}ds,\quad i=1,2.$$

From Lemma 2.3, it follows that $\Gamma\circ S_{\mathrm{G}}$ is a closed graph operator, and from the definition of Γ, one has that for $f_{in}\in\Gamma_i\circ \mathrm{G}_{i,\xi\eta}$, $f_{in}\to f_{i*}$ there exists $v_{i*}\in \mathrm{G}_{i,\xi\eta}$, $(i=1,2)$ such that

$$f_{i*}=\frac{1}{\Gamma(\alpha_i)}\int_1^\tau(\log\frac{\tau}{s})^{(\alpha_i-1)}\frac{v_{i*}(s)}{s}ds-\frac{(\log\tau)^{(\alpha_i-1)}}{\Gamma(\alpha_i)}\int_1^e(\log\frac{e}{s})^{(\alpha_i-1)}\frac{v_{i*}(s)}{s}ds,\quad i=1,2,$$

where

$$f_{in}=\frac{1}{\Gamma(\alpha_i)}\int_1^\tau(\log\frac{\tau}{s})^{(\alpha_i-1)}\frac{v_{in}(s)}{s}ds-\frac{(\log\tau)^{(\alpha_i-1)}}{\Gamma(\alpha_i)}\int_1^e(\log\frac{e}{s})^{(\alpha_i-1)}\frac{v_{in}(s)}{s}ds,\quad i=1,2.$$

Then, $(x_n, y_n) \to (x_*, y_*)$,

$$h_{1n} = f_{1n} - \frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_1-1)} \to f_{1*} - \frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_1-1)} = h_{1*},$$

$$h_{2n} = f_{2n} - \frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_2-1)} \to f_{2*} - \frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_2-1)} = h_{2*}.$$

So, there exists $v_{1*} \in \mathrm{G}_{1,\xi_*\eta_*}$, $v_{2*} \in \mathrm{G}_{2,\xi_*\eta_*}$, such that

$$\begin{aligned} h_{1*}(\tau) =& \frac{1}{\Gamma(\alpha_1)}\int_1^\tau \left(\log\frac{\tau}{s}\right)^{(\alpha_1-1)} \frac{v_{1*}(s)}{s}ds - \frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_1-1)} \\ &- \frac{(\log\tau)^{(\alpha_1-1)}}{\Gamma(\alpha_1)}\int_1^e \left(\log\frac{e}{s}\right)^{(\alpha_1-1)} \frac{v_{1*}(s)}{s}ds, \quad \tau\in[1,e] \end{aligned}$$

and

$$\begin{aligned} h_{2*}(\tau) =& \frac{1}{\Gamma(\alpha_2)}\int_1^\tau \left(\log\frac{\tau}{s}\right)^{(\alpha_2-1)} \frac{v_{2*}(s)}{s}ds - \frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_2-1)} \\ &- \frac{(\log\tau)^{(\alpha_2-1)}}{\Gamma(\alpha_2)}\int_1^e \left(\log\frac{e}{s}\right)^{(\alpha_2-1)} \frac{v_{2*}(s)}{s}ds, \quad \tau\in[1,e]. \end{aligned}$$

This means that $(h_{1*}, h_{2*}) \in \mathcal{B}(\xi_*, \eta_*)$.

Note that $\mathcal{B}: \Pi \to \mathscr{P}(\Pi)$ is completely continuous, thus it follows from Lemma 2.2 that the operator $\mathcal{B}$ is u.s.c. operator on Π.

Step 3. We show that Lemma 3.2(c) holds. From $(H4)$, we have $M = \|\mathcal{B}(\Omega)\| = \|\mathcal{B}(\xi,\eta)\| = \|\mathcal{B}_1(\xi,\eta)\| + \|\mathcal{B}_2(\xi,\eta)\| \leq (2/\Gamma(\alpha_1))\|\omega_{1r}\|_{L^1} + (2/\Gamma(\alpha_2))\|\omega_{2r}\|_{L^1} + G_{10}/F_{10} + G_{20}/F_{20}$ for $(\xi,\eta)\in\Omega$ and $\delta = L_1 + L_2$, $\rho = K_1 + K_2$.

At this point, we have completed the proof of all the conditions in the Lemma 3.2, which means that either Lemma 3.2(i) or Lemma 3.2(ii) holds. Finally, we demonstrate that Lemma 3.2(ii) is not satisfied.

Let $\Phi = \{(x,y)\in\Pi | \mu(x,y) \in (\mathcal{A}_1(x,y)\mathcal{B}_1(x,y) + C_1(x,y), \mathcal{A}_2(x,y)\mathcal{B}_2(x,y) + C_2(x,y))\}$ and $(x,y)\in\Pi$ be arbitrary. Then, for $\mu > 1$, $\mu(x,y) \in (\mathcal{A}_1(x,y)\mathcal{B}_1(x,y) + C_1(x,y), \mathcal{A}_2(u,v)\mathcal{B}_2(u,v) + C_2(x,y))$, there exists $(\psi_1,\psi_2) \in (\mathrm{G}_{1,xy}, \mathrm{G}_{2,xy})$ such that, for any $\mu > 1$, we have

$$\begin{aligned} x(\tau) =& \mu^{-1} f_1(\tau, x(\tau), y(\tau))\Bigg(\frac{1}{\Gamma(\alpha_1)}\int_1^\tau \left(\log\frac{\tau}{s}\right)^{(\alpha_1-1)} \frac{\psi_1(s)}{s}ds - \frac{g_1(e,0,0)}{f_1(e,0,0)}(\log\tau)^{(\alpha_1-1)} \\ &- \frac{(\log\tau)^{(\alpha_1-1)}}{\Gamma(\alpha_1)}\int_1^e \left(\log\frac{e}{s}\right)^{(\alpha_1-1)} \frac{\psi_1(s)}{s}ds\Bigg) + \mu^{-1} g_1(\tau, x(\tau), y(\tau)), \end{aligned}$$

and

$$\begin{aligned} y(\tau) =& \mu^{-1} f_2(\tau, x(\tau), y(\tau))\Bigg(\frac{1}{\Gamma(\alpha_2)}\int_1^\tau \left(\log\frac{\tau}{s}\right)^{(\alpha_2-1)} \frac{\psi_2(s)}{s}ds - \frac{g_2(e,0,0)}{f_2(e,0,0)}(\log\tau)^{(\alpha_2-1)} \\ &- \frac{(\log\tau)^{(\alpha_2-1)}}{\Gamma(\alpha_2)}\int_1^e \left(\log\frac{e}{s}\right)^{(\alpha_2-1)} \frac{\psi_2(s)}{s}ds\Bigg) + \mu^{-1} g_2(\tau, x(\tau), y(\tau)), \end{aligned}$$

for all $\tau \in [1, e]$. Therefore,

$$
\begin{aligned}
|x(\tau)| \leq & \mu^{-1}|f_1(\tau, x(\tau), y(\tau))| \left(\frac{1}{\Gamma(\alpha_1)} \int_1^\tau \left(\log \frac{\tau}{s} \right)^{(\alpha_1 - 1)} \frac{|\psi_1(s)|}{s} ds + \frac{g_1(e, 0, 0)}{f_1(e, 0, 0)} (\log \tau)^{(\alpha_1 - 1)} \right. \\
& \left. - \frac{(\log \tau)^{(\alpha_1 - 1)}}{\Gamma(\alpha_1)} \int_1^e \left(\log \frac{e}{s} \right)^{(\alpha_1 - 1)} \frac{|\psi_1(s)|}{s} ds \right) + \mu^{-1} |g_1(\tau, x(\tau), y(\tau))| \\
\leq & [|f_1(\tau, x(\tau), y(\tau)) - f_1(\tau, 0, 0)| + |f_1(\tau, 0, 0)|] \left(\frac{1}{\Gamma(\alpha_1)} \int_1^\tau \left(\log \frac{\tau}{s} \right)^{(\alpha_1 - 1)} \frac{|\psi_1(s)|}{s} ds \right. \\
& \left. + \frac{g_1(e, 0, 0)}{f_1(e, 0, 0)} (\log \tau)^{(\alpha_1 - 1)} - \frac{(\log \tau)^{(\alpha_1 - 1)}}{\Gamma(\alpha_1)} \int_1^e \left(\log \frac{e}{s} \right)^{(\alpha_1 - 1)} \frac{|\psi_1(s)|}{s} ds \right) \\
& + |g_1(\tau, x(\tau), y(\tau)) - g_1(\tau, 0, 0)| + |g_1(\tau, 0, 0)| \\
\leq & [L_1 r + F_{10}] \left[\frac{2}{\Gamma(\alpha_1)} \|\omega_{1r}\|_{L^1} + \frac{G_{10}}{F_{10}} \right] + [K_1 r + G_{10}],
\end{aligned}
$$

and

$$
|y(t)| \leq [L_2 r + F_{20}] \left[\frac{2}{\Gamma(\alpha_2)} \|\omega_{2r}\|_{L^1} + \frac{G_{20}}{F_{20}} \right] + [K_2 r + G_{20}] .
$$

And thus,

$$
\begin{aligned}
\|(x, y)\| \leq & \delta r \left(\frac{2}{\Gamma(\alpha_1)} \|\omega_{1r}\|_{L^1} + \frac{2}{\Gamma(\alpha_2)} \|\omega_{2r}\|_{L^1} + \frac{G_{10}}{F_{10}} + \frac{G_{20}}{F_{20}} \right) + \left(\frac{2F_{10}}{\Gamma(\alpha_1)} \|\omega_{1r}\|_{L^1} + \frac{2F_{20}}{\Gamma(\alpha_2)} \|\omega_{2r}\|_{L^1} + G_{10} + G_{20} \right) \\
& + (\rho r + G_{10} + G_{20}),
\end{aligned}
$$

where F_{i0} and $\omega_{ir} (r = 1, 2)$ are defined in $(H4)$. Then, if $\|(x, y)\| \geq r$, we have

$$
r \leq \frac{\frac{2F_{10}}{\Gamma(\alpha_1)} \|\omega_{1r}\|_{L^1} + \frac{2F_{20}}{\Gamma(\alpha_2)} \|\omega_{2r}\|_{L^1} + 2G_{10} + 2G_{20}}{1 - \delta \left(\frac{2}{\Gamma(\alpha_1)} \|\omega_{1r}\|_{L^1} + \frac{2}{\Gamma(\alpha_2)} \|\omega_{2r}\|_{L^1} + \frac{G_{10}}{F_{10}} + \frac{G_{20}}{F_{20}} \right) - \rho}.
$$

Therefore, Lemma 3.2(ii) is not satisfied by $(H4)$. Then, there exists $(\bar{x}, \bar{y}) \in \Pi$ such that

$$
(\bar{x}, \bar{y}) = (\mathcal{A}_1(\bar{x}, \bar{y}) \mathcal{B}_1(\bar{x}, \bar{y}) + C_1(\bar{x}, \bar{y}), \mathcal{A}_2(\bar{x}, \bar{y}) \mathcal{B}_2(\bar{x}, \bar{y}) + C_2(\bar{x}, \bar{y})).
$$

That is, operator $\mathcal{N}$ has a fixed point, which is a solution of system (1.1). So, system (1.1) has at least one solution on $[1, e] \times [1, e]$. □

4. An example

An example is given to illustrate the above theoretical result.

Example 4.1 We consider the following system of Hadamard fractional coupled differential inclusions

$$
\begin{cases}
{}^H D^{1.5} \left(\frac{\xi(\tau) - 0.1 e^{1-\tau} (\cos \xi(\tau) + \cos \eta(\tau))}{0.1 e^{1-\tau} (\cos \xi(\tau) + \cos \eta(\tau) + 2)} \right) \in \mathrm{G}_1(\tau, \xi(\tau), \eta(\tau)), \quad \tau \in (1, e), \\
{}^H D^{1.25} \left(\frac{\eta(\tau) - 0.1 (\arctan \xi(\tau) + \arctan \eta(\tau))}{0.1 (\arctan \xi(\tau) + \arctan \eta(\tau) + 3)} \right) \in \mathrm{G}_2(\tau, \xi(\tau), \eta(\tau)), \quad \tau \in (1, e), \\
\xi(1) = \xi(e) = 0, \quad \eta(1) = \eta(e) = 0,
\end{cases}
\tag{4.1}
$$

where $\alpha_1 = 1.5$, $\alpha_2 = 1.25$. The following is a formula for the multi-valued mapping $\mathrm{G}_i : [1,e]\times\mathbb{R}^2 \to \mathbb{R}$ $(i = 1,2)$:

$$\tau \mapsto \mathrm{G}_1(\tau,\xi(\tau),\eta(\tau)) = \left\{\frac{|\xi(\tau)|^3}{10(|\xi(\tau)|^3+|\eta(\tau)|^3+3)}, \frac{|\sin\xi(\tau)|}{20(|\sin\xi(\tau)|+|\sin\eta(\tau)|+1)}+\frac{1}{20}\right\},$$

and

$$\tau \mapsto \mathrm{G}_2(\tau,\xi(\tau),\eta(\tau)) = \left\{\frac{|\xi(\tau)|^3}{18(|\xi(\tau)|^3+|\eta(\tau)|^3+2)}+\frac{1}{18}, \frac{|\sin\xi(\tau)|}{10(|\sin\xi(\tau)|+|\sin\eta(\tau)|+3)}\right\}.$$

Compared with $(H1)$, we have

$$\begin{aligned}|f_1(\tau,\xi(\tau),\eta(\tau)) - f_1(\tau,\tilde{\xi}(\tau),\tilde{\eta}(\tau))| &= \left|0.1e^{1-\tau}(\cos\xi(\tau)+\cos\eta(\tau)+2) - 0.1e^{1-\tau}(\cos\tilde{\xi}(\tau)+\cos\tilde{\eta}(\tau)+2)\right| \\ &\le 0.1e^{1-\tau}\left[\left|\cos\xi(\tau)-\cos\tilde{\xi}(\tau)\right| + |\cos\eta(\tau)-\cos\tilde{\eta}(\tau)|\right], \quad \tau\in[1,e].\end{aligned}$$

So, $L_1 = 0.1$, $L_2 = 0.1$ with $\delta = 0.2$.
Compared with $(H2)$, we get

$$\begin{aligned}|f_2(\tau,\xi(\tau),\eta(\tau)) - f_2(\tau,\tilde{\xi}(\tau),\tilde{\eta}(\tau))| &= \left|0.1(\arctan\xi(\tau)+\arctan\eta(\tau)+3) - 0.1(\arctan\tilde{\xi}(\tau)+\arctan\tilde{\eta}(\tau)+3)\right| \\ &\le 0.1\left[\left|\arctan\xi(\tau)-\arctan\tilde{\xi}(\tau)\right| + |\arctan\eta(\tau)-\arctan\tilde{\eta}(\tau)|\right], \quad \tau\in[1,e].\end{aligned}$$

So, $K_1 = 0.1$, $K_2 = 0.1$ with $\rho = 0.2$.

For $v_1 \in \mathrm{G}_1$, $v_2 \in \mathrm{G}_2$, and arbitrary $(x,y)\in\mathbb{R}^2$, we have

$$|v_1| \le \max\left\{\frac{|x|^3}{10(|x|^3+|y|^3+3)}, \frac{|\sin x|}{20(|\sin x|+|\sin y|+1)}+\frac{1}{20}\right\} \le \frac{1}{10},$$

and

$$|v_2| \le \max\left\{\frac{|x|^3}{18(|x|^3+|y|^3+2)}+\frac{1}{18}, \frac{|\sin x|}{10(|\sin x|+|\sin y|+3)}\right\} \le \frac{1}{9}.$$

Then,

$$\|\mathrm{G}_1(\tau,x,y)\| = \sup\{|v_1| : v_1 \in G_1(\tau,x,y)\} \le \frac{1}{10} = \omega_{1r}(t),\ (x,y)\in\mathbb{R}^2,$$

$$\|\mathrm{G}_2(\tau,x,y)\| = \sup\{|v_2| : v_2 \in G_2(\tau,x,y)\} \le \frac{1}{9} = \omega_{2r}(t),\ (x,y)\in\mathbb{R}^2.$$

Clearly, from our calculation, $\|\omega_{1r}\|_{L^1} = \frac{e-1}{10}$, $\|\omega_{2r}\|_{L^1} = \frac{e-1}{9}$, $F_{10} = 0.2$, $F_{20} = 0.3$, $G_{10} = 0$, $G_{20} = 0$. Hence,

$$\delta\left(\frac{2}{\Gamma(\alpha_1)}\|\omega_{1r}\|_{L^1} + \frac{2}{\Gamma(\alpha_2)}\|\omega_{2r}\|_{L^1} + \frac{G_{10}}{F_{10}} + \frac{G_{20}}{F_{20}}\right) + \rho \approx 0.43 < 1/2,$$

and

$$r > \frac{\frac{2F_{10}}{\Gamma(\alpha_1)}\|\omega_{1r}\|_{L^1} + \frac{2F_{20}}{\Gamma(\alpha_2)}\|\omega_{2r}\|_{L^1} + 2G_{10} + 2G_{20}}{1-\delta\left(\frac{2}{\Gamma(\alpha_1)}\|\omega_{1r}\|_{L^1} + \frac{2}{\Gamma(\alpha_2)}\|\omega_{2r}\|_{L^1} + \frac{G_{10}}{F_{10}} + \frac{G_{20}}{F_{20}}\right) - \rho} \approx 0.48.$$

Consequently, all the assumptions of Theorem 3.1 are satisfied. Hence, by Theorem 3.1, system (4.1) has at least one solution on $[1,e]\times[1,e]$. □

5. Conclusions

We studied the existence of a solution for the new system (1.1) involving Hadamard coupled fractional differential inclusions equipped with Dirichlet boundary conditions. The results are obtained by combing fractional calculus, multi-valued analysis, and the multi-valued fixed point theorem for three operators of Schaefer type. One of the main objectives is to contribute to the growth of fractional calculus and to enrich the study as part of the mathematical analysis related to fractional differential inclusions.

Authors' contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript. Chengbo Zhai: Actualization, methodology, formal analysis, validation, investigation, initial draft and was a major contributor in writing the manuscript. Lili Zhang: Actualization, methodology, formal analysis, validation, investigation and initial draft.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there is no conflicts of interest.

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