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Research article

Boundedness and large time behavior of a signal-dependent motility system with nonlinear indirect signal production

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Abstract: In this paper, we study a chemotaxis system with nonlinear indirect signal production

$$\begin{cases} u_t = \Delta\left(\gamma\left(v\right)u\right) + ru - \mu u^l, & x \in \Omega, t > 0, \\ v_t = \Delta v - v + w^{\beta}, & x \in \Omega, t > 0, \\ w_t = -\delta w + u, & x \in \Omega, t > 0, \end{cases}$$

under homogeneous Neumann boundary conditions in a smooth bounded domain $\Omega \subset \mathbb{R}^n (n \geq 2)$, where the parameters r, μ, β, $\delta > 0$, and $l > 1$, the motility function $\gamma \in C^3([0,\infty))$, $\gamma(v) > 0$ is bounded, $\gamma^{'}(v) < 0$, and $\frac{\gamma^{'}(v)}{\gamma(v)}$ is bounded. We show that if $\frac{l}{\beta} > \frac{n}{2}$, the system has a unique global classical solution. Moreover, the solution exponentially converges to $((\frac{r}{\mu})^{\frac{1}{l-1}}, (\frac{1}{\delta})^{\beta}(\frac{r}{\mu})^{\frac{\beta}{l-1}}, \frac{1}{\delta}(\frac{r}{\mu})^{\frac{1}{l-1}})$ in the large time limit under some extra hypotheses.

Keywords: chemotaxis; global existence; signal-dependent motility; nonlinear indirect signal production; logistics source

1. Introduction

The chemotaxis models, introduced by Keller and Segel in 1970 [1], have cast a long and profound shadow across the disciplines of mathematics and biology alike. Based on the biological background, the cells move toward the chemical signal, which is secreted by the cells themselves, and many researchers have studied the chemotaxis-production system

$$\begin{cases} u_t = \Delta u - \chi\nabla\cdot(u\nabla v) + f(u) & x \in \Omega, t > 0, \\ v_t = \Delta u - v + u, & x \in \Omega, t > 0, \end{cases} \tag{1.1}$$

where $u(x,t)$ denotes the density of cells and $v(x,t)$ signifies the concentration of the chemical signal.

The cross-diffusion term $-\chi\nabla\cdot(u\nabla v)$ means the cells are moving toward the high concentration of chemical signal. Moreover, $f(u)$ is the logistic source; it represents the rate of the cells reproduction and death. Many particular cases and derivatives of this system have been successfully investigated up to now (see the surveys [2–6] and references therein for details).

Furthermore, in order to explain more complex biological phenomena, some researchers have proposed the following models with signal-dependent motility [7–9]:

$$\begin{cases} u_t = \Delta(\gamma(v)u) + f(u) & x\in\Omega, t>0,\\ v_t = \Delta u - v + u, & x\in\Omega, t>0. \end{cases} \tag{1.2}$$

The model was developed based on an experimental study of Escherichia coli (E. coli), which revealed the formation of a stripe pattern through a mechanism known as "self-trapping". Here, $u(x,t)$ signifies the density of E. coli, while $v(x,t)$ denotes the concentration of acyl-homoserine lactone(AHL), which secreted by E. coli. The motility function $\gamma(v)$ is a sufficiently smooth and positive function with the property $\gamma'(v)\le 0$. Since the first equation of (1.2) can be rewritten as $u_t = \nabla\cdot(\gamma(v)\nabla u) + \nabla\cdot(\gamma'(v)u\nabla v) + f(u)$, it can be regarded as a chemotaxis model of Keller-Segel type, where both the diffusion rate of the cells and the chemotactic sensitivity depend nonlinearly on the concentration of the chemical signal. When $f(u)\equiv 0$, if the positive function $\gamma(v)$ is bounded, Tao and Winkler [10] demonstrated that (1.2) admits a global classical solution in two dimensions and global weak solutions in higher dimensional settings. For the particular cases $\gamma(v)=\frac{c_0}{v^k}$, $\gamma(v)=e^{-v}$, or $\gamma(v)=\frac{1}{c_0+v^k}$, the existence of global solutions to (1.2) has been detected in [11–16], respectively.

In the presence of the logistic source (i.e. $f(u)=\lambda u-\mu u^l$), Jin et al. [17] established the existence of a global classical solution to (1.2) in two-dimensional settings in the case $l=2$; when $l>2$, many interesting results on the existence of global classical solutions to (1.2) have been demonstrated by Lv and Wang in [18–20]. Furthermore, when the second equation in (1.2) is replaced by $v_t=\Delta u - v + u^\beta$, Tao and Fang [21] showed that the system (1.2) has a global classical solution for $n\ge 2$ and $\frac{l}{\beta}>\frac{n+2}{2}$. Similarly, under the same conditions, the system with nonlinear signal consumption has also been studied by Tian and Xie in [22].

In the classical Keller-Segel model, the chemical signal is directly produced by the cells themselves. However, signal generation is often a complex process, which may involve external factors or the interplay of multiple signals generated through diverse mechanisms. Inspired by the spread and aggregative behaviors of the mountain pine beetle (MPB) in forest habitats, Strohm et al. [23] proposed a chemotaxis-growth system with indirect signal generation

$$\begin{cases} u_t = \Delta(u) - \chi\nabla\cdot(u\nabla v) + \mu(u-u^l), & x\in\Omega, t>0,\\ v_t = \Delta v - v + w, & x\in\Omega, t>0,\\ w_t = -\delta w + u, & x\in\Omega, t>0. \end{cases} \tag{1.3}$$

Here, $u(x,t)$ and $w(x,t)$ denote the density of flying MPB and nesting MPB, respectively. $v(x,t)$ stands for the concentration of MPB pheromone, which is secreted only by those nesting MPB. When $l=2$, Hu and Tao [24] employed the coupled L^p estimate method to demonstrate that, under sufficiently regular initial conditions, model (1.3) admits a unique global smooth solution in three-dimensional spaces. Similar results were considered in higher dimensions [25]. For $l>\frac{n}{2}$, Li and Tao [26] established the existence of a classical solution for model (1.3). Additionally, when $l=\frac{n}{2}$, Ren and

Liu [27] confirmed the existence of a global bounded classical solution to (1.3) under the critical parameter condition. For more related research, readers can refer to [28–30] etc.

To the best of our knowledge, the following chemotaxis production system with both signal-dependent motility and nonlinear indirect signal production only has very little research so far:

$$\begin{cases} u_t = \Delta(\gamma(v)u) + ru - \mu u^l, & x \in \Omega, t > 0, \\ v_t = \Delta v - v + w^\beta, & x \in \Omega, t > 0, \\ w_t = -\delta w + u, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \upsilon} = \frac{\partial v}{\partial \upsilon} = 0, & x \in \partial\Omega, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), w(x,0) = w_0(x), & x \in \Omega, \end{cases} \tag{1.4}$$

where $\Omega \subset R^n$ is a smooth bounded domain. The initial data u_0, v_0, and w_0 satisfy

$$u_0 \in C^0(\overline{\Omega}), v_0 \in W^{1,\infty}(\Omega), w_0 \in W^{1,\infty}(\Omega), u_0 \geq 0, v_0 \geq 0, w_0 \geq 0, \tag{1.5}$$

and

$$\gamma \in C^3([0,\infty)), \gamma(v) > 0 \text{ is bounded }, \gamma'(v) < 0 \text{ and } \frac{\gamma'(v)}{\gamma(v)} \text{ is bounded }. \tag{1.6}$$

When the signal production is linear (i.e., $\beta = 1$), the dynamical behaviors of (1.4) were investigated in [31]. Motivated by the aforementioned works, we continue to explore the global dynamics for (1.4) with nonlinear signal production, that is, $\beta \not\equiv 1$. The purpose of our paper is to clarify the global existence and large time behavior of (1.4) under the conditions

$$r, \mu, \beta, \delta > 0, l > 1 \text{ and } \frac{l}{\beta} > \frac{n}{2}. \tag{1.7}$$

Our main results are as follows.

Theorem 1.1. *Let $\Omega \in R^n (n \geq 2)$ be a bounded domain with smooth boundary. Assume that the initial data (u_0, v_0, w_0), the motility function γ, and the parameters satisfy (1.5), (1.6), and (1.7), respectively. Then, model (1.4) possesses a global bounded classical solution (u, v, w) in the sense that*

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} + \|v(\cdot,t)\|_{W^{1,\infty}(\Omega)} + \|w(\cdot,t)\|_{L^\infty(\Omega)} \leq C \quad \text{for all} \quad t > 0,$$

where C is a positive constant independent of t.

Remark 1.1. *Theorem 1.1 shows that the propagation of signal is much weaker than the death of the cells, i.e., $\beta < \frac{2}{n}l$ is conducive to ensuring the existence of global bounded classical solutions to (1.4). Inspired by this, it is interesting to consider whether there is a critical exponent a^*. That is, if $\beta < a^* l$, (1.4) has a global solution, while blow-up occurs when $\beta > a^* l$. However, for (1.4), the exact value of a^* remains unknown.*

Theorem 1.2. *Let $\Omega \in R^n (n \geq 2)$ be a bounded domain with smooth boundary. Assume that the initial data (u_0, v_0, w_0), the motility function γ, and the parameters satisfy (1.5), (1.6), and (1.7), respectively. Moreover, $l \geq 2$ and $0 < \beta \leq 1$, then there exist some positive constants τ, C, T, and $\mu_0 > 0$ such that if $\mu > \mu_0$,*

$$\| u - (\frac{r}{\mu})^{\frac{1}{l-1}} \|_{L^\infty(\Omega)} + \| v - (\frac{1}{\delta})^\beta (\frac{r}{\mu})^{\frac{\beta}{l-1}} \|_{L^\infty(\Omega)} \| + \| w - \frac{1}{\delta}(\frac{r}{\mu})^{\frac{1}{l-1}} \|_{L^\infty(\Omega)} \leq Ce^{-\tau t}$$

for all $t > T$.

The structure of the paper is as follows. In Section 2, we address the local existence of solution to (1.1) and some preliminary estimates which are essential for proving Theorem 1.1. In Section 3, we prove the global existence of the solution to (1.4) by using a priori estimates, some important inequalities, and the standard Alikakos-Moser iteration. In Section 4, we study the large time behavior of system (1.4) with the aid of a Lyapunov function.

2. Preliminaries

In order to prove the main result, we will introduce some useful lemmata. Initially, we begin by establishing the local existence of solution, which can be referenced in [32].

Lemma 2.1. *(Local existence) Let $\Omega \in R^n (n \geq 2)$ be a bounded domain with a smooth boundary. Assume that the initial data (u_0, v_0, w_0), the motility function γ, and the parameters satisfy the conditions (1.5), (1.6), and (1.7), respectively. Then, there exists $T_{\max} \in (0, \infty]$ and a uniquely determined non-negative triple of functions (u, v, w)*

$$u \in C^0(\overline{\Omega} \times [0, T_{max})) \bigcap C^{2,1}(\overline{\Omega} \times (0, T^{max})),$$

$$v \in \bigcap_{\theta > n} C^0([0, T_{\max}); W^{1,\theta}(\Omega)) \bigcap C^{2,1}(\overline{\Omega} \times (0, T_{max})),$$

$$w \in C^0(\overline{\Omega} \times [0, T_{\max})) \bigcap C^{0,1}(\overline{\Omega} \times (0, T_{\max})),$$

which solves (1.4) in the classical sense. If $T_{\max} < \infty$, we have

$$\lim_{t \to T_{max}} sup(\|u(.,t)\|_{L^\infty(\Omega)} + \|v(.,t)\|_{W^{1,\infty}(\Omega)} + \|w(.,t)\|_{L^\infty(\Omega)}) = \infty.$$

In addition, we also need to utilize the $L^p - L^q$ estimate.

Lemma 2.2. *([5] Lemma 1.3) Let $(e^{t\Delta})_{t \geq 0}$ be the Neumann heat semigroup in Ω, and $\lambda_1 > 0$ denote the first nonzero eigenvalue of $-\Delta$ in Ω under the homogeneous Neumann boundary condition. Then, we obtain the following estimates with positive constants k_1, k_2 depending only on Ω.*
(i) If $1 \leq q \leq p \leq \infty$, then

$$\| \nabla e^{t\Delta} u \|_{L^p(\Omega)} \leq k_1 (1 + t^{-\frac{1}{2} - \frac{n}{2}(\frac{1}{q} - \frac{1}{p})}) e^{-\lambda_1 t} \| u \|_{L^q(\Omega)} \tag{2.1}$$

for all $u \in L^q(\Omega)$ and $t > 0$.
(ii) If $2 \leq p < \infty$, then

$$\| \nabla e^{t\Delta} u \|_{L^p(\Omega)} \leq k_2 e^{-\lambda_1 t} \| \nabla u \|_{L^p(\Omega)} \tag{2.2}$$

for all $u \in W^{1,p}(\Omega)$ and $t > 0$.

Furthermore, the subsequent inequality is crucial for our proof.

Lemma 2.3. *For all $q > 1$, there exists $k_3 = k_3(q) > 0$ such that*

$$\| \nabla e^{t\Delta} \phi \|_{L^q(\Omega)} \leq k_3 \| \nabla \phi \|_{L^\infty(\Omega)} \tag{2.3}$$

for all $\phi \in W^{1,\infty}(\Omega)$ and $t > 0$.

Proof. We can easily obtain (2.3) by Lemma 2.2 and Hölder's inequality. □

Finally, we introduce the lemma related to the comparison principle, as referred to in [33].

Lemma 2.4. *Let $T > 0$, $t_0 \in (0, T)$, $a > 0$, and $b > 0$. Assume that $y : [0, T) \to [0, \infty)$ is uniformly continuous and satisfies*

$$y'(t) + ay(t) \le h(t) \quad \text{for a.e.} \quad t \in (0, T),$$

where h is a nonnegative function in $L^l_{loc}([0, T))$ satisfying

$$\int_t^{t+t_0} h(s)ds \le b \quad \text{for all} \quad t \in [0, T - t_0).$$

Then, we obtain

$$y(t) \le \max\left\{y(0) + b, \frac{b}{at_0} + 2b\right\} \quad \text{for all} \quad t \in (0, T).$$

3. Global existence and boundedness

In this section, we aim to demonstrate the global existence and boundedness of the classical solution of (1.4). At first, it is essential to verify the L^1 boundedness of $u(x, t)$.

Lemma 3.1. *Let (1.5), (1.6), and (1.7) hold, then there exist constants $C_1 > 0$ and $C_2 > 0$ such that*

$$\int_\Omega u \le C_1 \tag{3.1}$$

for all $t \in (0, T_{\max})$ and

$$\int_t^{t+\tau} \int_\Omega u^l \le C_2 \tag{3.2}$$

for all $t \in (0, T_{\max} - \tau)$, where $\tau := \min\left\{1, \frac{1}{2}T_{\max}\right\}$.

Proof. Integrating the first equation of (1.4), we have

$$\begin{aligned} \frac{d}{dt}\int_\Omega u &= r\int_\Omega u - \mu\int_\Omega u^l \\ &\le r\int_\Omega u - \mu|\Omega|^{1-l}\left(\int_\Omega u\right)^l \end{aligned} \tag{3.3}$$

for all $t \in (0, T_{\max})$. Subsequently, utilizing an ODE comparison argument leads us to deduce (3.1). Integrating (3.3) from t to $t + \tau$, we obtain

$$\int_t^{t+\tau} \frac{d}{dt}\int_\Omega u = r\int_t^{t+\tau}\int_\Omega u - \mu\int_t^{t+\tau}\int_\Omega u^l \tag{3.4}$$

Therefore, we have

$$\begin{aligned} \int_t^{t+\tau}\int_\Omega u^l &= \frac{r}{\mu}\int_t^{t+\tau}\int_\Omega u + \frac{1}{\mu}\int_\Omega u(., t) - \frac{1}{\mu}\int_\Omega u(., t+\tau) \\ &\le \frac{r}{\mu}\int_t^{t+\tau}\int_\Omega u + \frac{1}{\mu}\int_\Omega u(., t) \\ &\le \frac{C_1}{\mu}(r\tau + 1) := C_2 \end{aligned} \tag{3.5}$$

for all $t \in (0, T_{\max} - \tau)$, where $\tau := \min\left\{1, \frac{1}{2}T_{\max}\right\}$. Thus, we get (3.2) □

Due to (3.1), we can obtain the L^q boundedness of w.

Lemma 3.2. *If* $1 \le q \le l$ *and* $\delta > 0$*, then there exists a constant* $C_3 > 0$ *such that*

$$\int_\Omega w^q(\cdot, t) \le C_3 \tag{3.6}$$

for all $t \in (0, T_{\max})$.

Proof. Multiplying the w-equation of (1.4) by qw^{q-1}, we obtain

$$\frac{d}{dt}\int_\Omega w^q = -q\delta \int_\Omega w^q + q\int_\Omega uw^{q-1} \tag{3.7}$$

for all $t \in (0, T_{\max})$. Now, we will prove (3.6) in two cases: $q = 1$ and $q > 1$.
If $q = 1$, we can get

$$\frac{d}{dt}\int_\Omega w = -\delta \int_\Omega w + \int_\Omega u \tag{3.8}$$

for all $t \in (0, T_{\max})$. Combining (3.1) and an ODE comparison argument, we can obtain (3.6).
If $q > 1$, by using Young's inequality, we can find there exists a constant $c_1 > 0$ such that

$$\frac{d}{dt}\int_\Omega w^q \le -\frac{q\delta}{2}\int_\Omega w^q + c_1\int_\Omega u^q \tag{3.9}$$

for all $t \in (0, T_{\max})$, where $c_1 = (\frac{2(q-1)}{\delta q})^{q-1}$. Combining Lemma 2.4 and (3.2), we complete the proof of Lemma 3.2. □

Based on Lemma 3.2, we can get the L^1 boundedness of v.

Lemma 3.3. *Let* $\Omega \subset R^n$ *be a bounded domain with smooth boundary. Assume that the initial data* (u_0, v_0, w_0)*, the motility function* γ*, and the parameters satisfy the conditions (1.5), (1.6), and (1.7), respectively. Then, we have*

$$\int_\Omega v \le C_4 \tag{3.10}$$

for all $t \in (0, T_{\max})$*, where* C_4 *is some positive constant.*

Proof. We will also prove (3.10) in two cases: $0 < \beta < 1$ and $1 \le \beta < \frac{2}{n}l \le l$.
In case of $0 < \beta < 1$, integrating the second equation of (1.4), combining Hölder's inequality and (3.6), we can obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega v + \int_\Omega v &= \int_\Omega w^\beta \\ &\le (\int_\Omega w^q)^{\frac{\beta}{q}} \cdot (\int_\Omega 1^{\frac{q}{q-\beta}})^{\frac{q-\beta}{q}} \\ &\le C^{\frac{\beta}{q}} \cdot |\Omega|^{\frac{q-\beta}{q}}\end{aligned} \tag{3.11}$$

for all $t \in (0, T_{\max})$. The ODE comparsion argument leads to (3.10).
In case of $1 \le \beta < \frac{2}{n}l \le l$, integrating the second equation of (1.4) and combining (3.6), we have

$$\frac{d}{dt}\int_\Omega v + \int_\Omega v = \int_\Omega w^\beta \le C \tag{3.12}$$

for all $t \in (0, T_{\max})$. Thus, we get (3.10). □

To prove the global existence and boundedness of the classical solution to (1.4), we need to calculate the boundedness of $\|\nabla v(., t)\|_{L^q(\Omega)}$.

Lemma 3.4. *Let conditions (1.5), (1.6), and (1.7) hold. Then, for all*

$$q \in \begin{cases} \left[1, \frac{nl}{n\beta - l}\right) & \frac{l}{\beta} \le n, \\ [1, \infty] & \frac{l}{\beta} > n, \end{cases}$$

there exists $C_5 = C_5(q) > 0$ *such that*

$$\|\nabla v(., t)\|_{L^q(\Omega)} \le C_5 \tag{3.13}$$

for all $t \in (0, T_{\max})$.

Proof. Applying the variation-of-constants formula for v, we have

$$v(., t) = e^{t(\Delta-1)} v_0 + \int_0^t e^{(t-s)(\Delta-1)} w^\beta(., s) ds \tag{3.14}$$

for all $t \in (0, T_{\max})$. Without losing the generality, we suppose $q > \frac{l}{\beta}$. Combining Lemma 2.2, Lemma 2.3, and Hölder's inequality, we can find a constant $c_1 > 0$ such that

$$\begin{aligned} \|\nabla v(., t)\|_{L^q(\Omega)} &\le \|\nabla e^{t(\Delta-1)} v_0\|_{L^q(\Omega)} + \int_0^t \|\nabla e^{(t-s)(\Delta-1)} w^\beta(., s)\|_{L^q(\Omega)} ds \\ &\le c_1 \|\nabla v_0\|_{L^\infty(\Omega)} + \int_0^t \|\nabla e^{(t-s)(\Delta-1)} w^\beta(., s)\|_{L^q(\Omega)} ds \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &\int_0^t \|\nabla e^{(t-s)(\Delta-1)} w^\beta(., s)\|_{L^q(\Omega)} ds \\ &\le k_1 \int_0^t \left(1 + (t-s)^{-\frac{1}{2} - \frac{n}{2}(\frac{\beta}{l} - \frac{1}{q})}\right) e^{-\lambda_1 (t-s)} \|w^\beta(., s)\|_{L^{\frac{l}{\beta}}(\Omega)} ds \\ &= k_1 \int_0^t \left(1 + (t-s)^{-\frac{1}{2} - \frac{n}{2}(\frac{\beta}{l} - \frac{1}{q})}\right) e^{-\lambda_1 (t-s)} \|w(., s)\|^\beta_{L^l(\Omega)} ds \end{aligned} \tag{3.16}$$

for all $t \in (0, T_{\max})$. Due to (3.6), we can choose c_1 fulfilling

$$\|w(., s)\|^\beta_{L^l(\Omega)} \le c_1 \tag{3.17}$$

for all $t \in (0, T_{\max})$.

If $\frac{l}{\beta} \le n$, we have

$$\begin{aligned} -\frac{1}{2} - \frac{n}{2}\left(\frac{\beta}{l} - \frac{1}{q}\right) &> -\frac{1}{2} - \frac{n}{2}\left(\frac{\beta}{l} - \frac{n\beta - l}{nl}\right) \\ &= -1 \end{aligned} \tag{3.18}$$

If $\frac{l}{\beta} > n$, we have

$$-\frac{1}{2}-\frac{n}{2}\left(\frac{\beta}{l}-\frac{1}{q}\right) > \frac{1}{2}-\frac{n}{2}\left(\frac{1}{n}-\frac{1}{q}\right) \geq -\frac{1}{2}-\frac{n}{2}\cdot\frac{1}{n} = -1 \tag{3.19}$$

Thus, combining (3.18) and (3.19), we can find a constant $c_2 > 0$ such that

$$\int_0^t \left(1+(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{\beta}{l}-\frac{1}{q})}\right)e^{-\lambda_1(t-s)}ds \leq c_2 \tag{3.20}$$

for all $t \in (0, T_{\max})$. Inserting (3.17) and (3.20) into (3.16), there exists a constant $c_3 > 0$ such that

$$\int_0^t \|\nabla e^{(t-s)(\Delta-1)}w^{\beta}(.,s)\|_{L^q(\Omega)}ds \leq c_3 \tag{3.21}$$

for all $t \in (0, T_{\max})$. Thus, combining (3.15) and (3.21), we can get (3.13). □

Owing to (3.13), we can use an Ehrling-type inequality to demonstrate the boundedness of v.

Lemma 3.5. *Suppose that (1.5), (1.6), and (1.7) are valid. Then there exists $C_6 > 0$ such that*

$$\|v(.,t)\|_{L^\infty(\Omega)} \leq C_6 \tag{3.22}$$

for all $t \in (0, T_{\max})$.

Proof. We see that $\frac{\beta}{l} < \frac{2}{n}$ ensures that $\frac{nl}{n\beta-l} > n$, which allows us to select $q_1 \in (n, \frac{nl}{n\beta-l})$. We can use Lemma 3.4 to choose a positive constant c_1 such that

$$\|\nabla v(.,t)\|_{L^{q_1}(\Omega)} \leq c_1 \tag{3.23}$$

for all $t \in (0, T_{\max})$. Combining (3.10), (3.23), and the Gagliardo-Nirenberg inequality, we can find some constants $c_2 > 0$ and $c_3 > 0$ such that

$$\|v(.,t)\|_{L^{q_1}(\Omega)} \leq c_2\|\nabla v(.,t)\|_{L^{q_1}(\Omega)}^{\frac{1-\frac{1}{q_1}}{\frac{1}{n}+1-\frac{1}{q_1}}}\|v(.,t)\|_{L^1(\Omega)}^{\frac{\frac{1}{n}}{\frac{1}{n}+1-\frac{1}{q_1}}} + c_2\|v(.,t)\|_{L^1(\Omega)} \leq c_3 \tag{3.24}$$

for all $t \in (0, T_{\max})$. Combining (3.23) and (3.24), we have

$$\|v(.,t)\|_{W^{1,q_1}(\Omega)} \leq c_4 \tag{3.25}$$

for all $t \in (0, T_{\max})$, where $c_4 := c_1 + c_3$. Consequently, by using the Sobolev embedding theorem, we can conclude (3.22). □

Drawing on Lemma 3.4 and a series of important inequalities, we estimate the L^p boundedness of u.

Lemma 3.6. *Assume that conditions (1.5), (1.6), and (1.7) exist. Then, for any $p \geq 2$, there exists a positive constant C_7 such that*

$$\int_\Omega u^p \leq C_7 \tag{3.26}$$

for all $t \in (0, T_{\max})$.

Proof. Multiplying the u-equation of (1.4) by u^{p-1}, integrating by parts in Ω, and using Young's inequality, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p &= -p(p-1)\int_\Omega u^{p-2}\gamma(v)|\nabla u|^2 - p(p-1)\int_\Omega u^{p-1}\gamma'(v)|\nabla u||\nabla v| \\ &\quad + rp\int_\Omega u^p - \mu p\int_\Omega u^{p+l-1} \\ &\leq -\frac{p(p-1)}{2}\int_\Omega u^{p-2}|\nabla u|^2\gamma(v) + \frac{p(p-1)}{2}\int_\Omega u^p\frac{|\gamma'(v)|^2}{\gamma(v)}|\nabla v|^2 \\ &\quad + rp\int_\Omega u^p - \mu p\int_\Omega u^{p+l-1}\end{aligned} \tag{3.27}$$

for all $t \in (0, T_{\max})$. Because of Lemma 3.5 and (1.6), we can find some positive constants c_1 and c_2 such that

$$\gamma(v) \geq c_1 \ \text{ and } \ \frac{|\gamma'(v)|^2}{\gamma(v)} \leq c_2 \tag{3.28}$$

for all $t \in (0, T_{\max})$. Inserting (3.28) into (3.27), and using Young's inequality, we can find some positive constants c_3, c_4, and c_5 such that

$$\begin{aligned}\frac{d}{dt}\int_\Omega u^p + c_3\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \int_\Omega u^p &\leq c_4\int_\Omega u^p|\nabla v|^2 + (rp+1)\int_\Omega u^p - \mu p\int_\Omega u^{p+l-1} \\ &\leq c_4\int_\Omega u^p|\nabla v|^2 - \frac{\mu p}{2}\int_\Omega u^{p+l-1} + c_5\end{aligned} \tag{3.29}$$

for all $t \in (0, T_{\max})$. Next, we will prove (3.26) in two cases.

In the case of $\frac{l}{\beta} > n$, for some positive constants c_6 and c_7, combining Lemma 3.4 and Young's inequality, we have

$$\begin{aligned}c_4\int_\Omega u^p|\nabla v|^2 &\leq {c_6}^2 c_4\int_\Omega u^p \\ &\leq \frac{\mu p}{4}\int_\Omega u^{p+l-1} + c_7\end{aligned} \tag{3.30}$$

for all $t \in (0, T_{\max})$. Inserting (3.30) into (3.29), we can obtain

$$\frac{d}{dt}\int_\Omega u^p + c_3\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \int_\Omega u^p + \frac{\mu p}{4}\int_\Omega u^{p+l-1} \leq c_5 + c_7 \tag{3.31}$$

for all $t \in (0, T_{\max})$, which completes the proof of (3.26).

In the case of $\frac{n}{2} < \frac{l}{\beta} \leq n$, fixing $q_0 \in (n, \frac{nl}{n\beta - l})$ and using Lemma 3.4, we can find a positive constants c_8 such that

$$||\nabla v(., t)||_{L^{q_0}(\Omega)} \leq c_8 \tag{3.32}$$

for all $t \in (0, T_{\max})$. Now, using Hölder's inequality, the Gagliardo-Nirenberg inequality, and (3.32), we can choose some positive constants c_9 and c_{10} such that

$$\begin{aligned} c_4 \int_\Omega u^p |\nabla v|^2 &\le c_4 \left(\int_\Omega |\nabla v|^{q_0} \right)^{\frac{2}{q_0}} \left(\int_\Omega u^{\frac{pq_0}{q_0-2}} \right)^{\frac{q_0-2}{q_0}} \\ &\le c_4 {c_8}^2 \|u^{\frac{p}{2}}\|^2_{L^{\frac{2q_0}{q_0-2}}(\Omega)} \\ &\le c_9 \left(\|\nabla u^{\frac{p}{2}}\|^{\frac{2n}{q_0}}{}_{L^2(\Omega)} \|u^{\frac{p}{2}}\|^{\frac{2(q_0-n)}{q_0}}{}_{L^2(\Omega)} + \|u^{\frac{p}{2}}\|^2{}_{L^2(\Omega)} \right) \\ &\le \frac{c_3}{2} \|\nabla u^{\frac{p}{2}}\|^2{}_{L^2(\Omega)} + \frac{\mu p}{4} \int_\Omega u^{p+l-1} + c_{10} \end{aligned} \tag{3.33}$$

for all $t \in (0, T_{\max})$. Therefore, (3.29) can be changed to

$$\frac{d}{dt} \int_\Omega u^p + \frac{c_3}{2} \int_\Omega |\nabla u^{\frac{p}{2}}|^2 + \int_\Omega u^p + \frac{\mu p}{4} \int_\Omega u^{p+l-1} \le c_5 + c_{10} \tag{3.34}$$

for all $t \in (0, T_{\max})$. Hence, we complete the proof of (3.26). □

To sum up, we can easily prove Theorem 1.1.

Proof of Theorem 1.1. With the help of Lemma 3.6 and a standard Alikakos-Moser iteration ([34] Lemma A.1), we can find a positive constant C_1 independent of t such that

$$\|u(., t)\|_{L^\infty(\Omega)} \le C_1 \tag{3.35}$$

for all $t \in (0, T_{\max})$. Applying the variation-of-constants formula for w, we conclude

$$w(\cdot, t) = e^{-\delta t} w_0 + \int_0^t e^{-\delta(t-s)} u(\cdot, s)\, \mathrm{d}s$$

for all $t \in (0, T_{\max})$, which implies that there exists $C_2 > 0$ such that

$$\|w(\cdot, t)\|_{L^\infty(\Omega)} \le C_2 \tag{3.36}$$

for all $t \in (0, T_{\max})$. Besides, by using the heat semigroup theorem on the v-equation of (1.4), we can find a constant $C_3 > 0$ such that

$$\|\nabla v(\cdot, t)\|_{L^\infty(\Omega)} \le C_3 \tag{3.37}$$

for all $t \in (0, T_{\max})$. Combining (3.22), we deduce that there exists a positive constant $C_4 > 0$ such that

$$\|v(\cdot, t)\|_{W^{1,\infty}(\Omega)} \le C_4 \tag{3.38}$$

for all $t \in (0, T_{\max})$. Thus, Theorem 1.1 is proved due to (3.35), (3.36), (3.38), and the extensibility criterion from Lemma 2.1. □

4. Large time behavior

In this section, we will construct a Lyapunov function, which will serve as the cornerstone in our proof of Theorem 1.2. First of all, we shall present a few auxiliary lemmas.

Lemma 4.1. *([21] Lemma 2.4) Let* $A > 0$*,* $B > 0$*, and* $0 < p < 1$*. Then,*

$$|A^p - B^p| \leq 2^{1-p} min\left\{A^{p-1}, B^{p-1}\right\} |A - B|$$

Lemma 4.2. *Assume that* (u, v, w) *is the classical solution of system (1.4) in Theorem 1.1. Then, there exist* $C > 0$ *and* $\delta \in (0, 1)$ *such that*

$$||u||_{C^{2+\delta,1+\frac{\delta}{2}}(\Omega\times[t,t+1])} \leq C \tag{4.1}$$

for all $t > 1$*.*

Proof. we rewrite the u-equation of (1.4) as follows:

$$u_t = \nabla \cdot \nabla(\gamma(v)u) + ru - \mu u^l = \nabla \cdot (\nabla u\gamma(v) + u\gamma^{'}(v)\nabla v) + ru - \mu u^l \tag{4.2}$$

According to (1.6), we can find some positive constants k_1, k_2, and k_3 such that

$$k_1 \leq \gamma(v) \leq k_2 \quad \text{and} \quad |\gamma^{'}(v)| \leq k_3 \tag{4.3}$$

for all $t \in (0, T_{\max})$. Obviously, Theorem 1.1 ensures that u, v, and ∇v are bounded. Now, by applying Young's inequality, we can find some positive constants c_1, c_2, and c_3

$$\begin{aligned} \nabla u \cdot (\nabla u\gamma(v) + u\gamma^{'}(v)\nabla v) &= |\nabla u|^2\gamma(v) + u\gamma^{'}(v)\nabla v\nabla u \\ &\geq k_1|\nabla u|^2 - k_3u|\nabla v||\nabla u| \\ &\geq \frac{k_1}{2}|\nabla u|^2 - \frac{k_3^2c_1}{2k_1} \end{aligned} \tag{4.4}$$

and

$$ru - \mu u^l \leq c_2 \tag{4.5}$$

as well as

$$\nabla u\gamma(v) + u\gamma^{'}(v)\nabla v \leq c_3 \tag{4.6}$$

From (4.4)–(4.6), according to Hölder's regularity, there exists a positive constant c_4, and we can deduce that

$$||u||_{C^{\delta,\frac{\delta}{2}}(\Omega\times[t,t+1])} \leq c_4 \tag{4.7}$$

for all $t > 1$. Thus, applying the standard parabolic Schauder theory [35], we can obtain (4.1). □

Lemma 4.3. *Assume that* (u, v, w) *is the global bounded classical solution of (1.4). Let (1.5), (1.6), and (1.7) hold. The energy functions defined by*

$$E(t) = \int_\Omega \left(u - u_* - u_* ln\frac{u}{u_*}\right) + \frac{B_1}{2}\int_\Omega (v - v_*)^2 + \frac{B_2}{2}\int_\Omega (w - w_*)^2 \tag{4.8}$$

with $u_* = (\frac{r}{\mu})^{\frac{1}{l-1}}$*,* $v_* = (\frac{1}{\delta})^\beta(\frac{r}{\mu})^{\frac{\beta}{l-1}}$*,* $w_* = \frac{1}{\delta}(\frac{r}{\mu})^{\frac{1}{l-1}}$*,* $B_1 = \frac{1}{4}ku_*$ *and* $B_2 = \delta\mu u_*^{l-2}$*, and*

$$F(t) = \left(\int_\Omega (u - u_*)^2 + \int_\Omega (v - v_*)^2 + \int_\Omega (w - w_*)^2\right) \tag{4.9}$$

for all $t > 0$. We have

$$E(t) \geq 0 \tag{4.10}$$

for all $t > 0$. When $l \geq 2$ and $0 < \beta \leq 1$, there exist some positive constants ε and μ_0 such that if $\mu > \mu_0$

$$\frac{d}{dt}E(t) \leq -\varepsilon F(t) \tag{4.11}$$

for all $t \geq 0$.

Proof. We note that

$$E(t) = A(t) + B(t) + C(t) \tag{4.12}$$

where

$$\begin{aligned} A(t) &:= \int_\Omega \left(u - u_* - u_* ln \frac{u}{u_*} \right), \\ B(t) &:= \frac{B_1}{2} \int_\Omega (v - v_*)^2, \\ C(t) &:= \frac{B_2}{2} \int_\Omega (w - w_*)^2. \end{aligned}$$

We let $\varphi : (0, \infty) \to R$ be defined by

$$\varphi(x) := x - u_* - u_* ln \frac{x}{u_*}, \quad x > 0.$$

Due to φ is convex with $\varphi(u_*) = \varphi^{'}(u_*) = 0$, so $\varphi(x) \geq 0$ for all $x > 0$, we have $E(t) \geq 0$. Using the first equation in (1.4) and Young's inequality, we can obtain

$$\begin{aligned} \frac{d}{dt}A(t) &= \int_\Omega u_t \left(1 - \frac{u_*}{u}\right) \\ &= -\mu \int_\Omega (u - u_*) \left(u^{l-1} - \frac{r}{\mu}\right) - u_* \int_\Omega \gamma(v) \frac{|\nabla u|^2}{u^2} - u_* \int_\Omega \frac{\gamma^{'}(v)}{u} |\nabla u||\nabla v| \\ &= -\mu \int_\Omega (u - u_*) \left(u^{l-1} - u_*^{l-1}\right) - u_* \int_\Omega \gamma(v) \frac{|\nabla u|^2}{u^2} - u_* \int_\Omega \frac{\gamma^{'}(v)}{u} |\nabla u||\nabla v| \\ &\leq \frac{1}{4} u_* \int_\Omega \frac{|\gamma^{'}(v)|^2}{\gamma(v)} |\nabla v|^2 - \mu \int_\Omega (u - u_*) \left(u^{l-1} - u_*^{l-1}\right) \end{aligned} \tag{4.13}$$

for all $t > 0$. Accoring to hypothesis (1.6), we can choose $k > 0$, fulfilling

$$\frac{|\gamma^{'}(v)|^2}{\gamma(v)} \leq k \tag{4.14}$$

for all $t > 0$. With the help of the elementary inequality: if $\zeta \geq 1$, then for all $x \geq 0$, $y \geq 0$, and $x \neq y$, we can see that

$$\frac{x^\zeta - y^\zeta}{x - y} \geq y^{\zeta - 1} \tag{4.15}$$

Hence, since $l \geq 2$, combining (4.13)–(4.15), we have

$$\frac{d}{dt}A(t) \leq \frac{1}{4}u_*k\int_\Omega |\nabla v|^2 - \mu u_*^{l-2}\int_\Omega (u-u_*)^2 \tag{4.16}$$

for all $t > 0$. We use the second equation in (1.4) and Young's inequality to obtain

$$\begin{aligned}\frac{d}{dt}B(t) &= B_1\int_\Omega (v-v_*)v_t \\ &= B_1\int_\Omega (v-v_*)(\triangle v - v + w^\beta) \\ &= -B_1\int_\Omega |\nabla v|^2 - B_1\int_\Omega (v-v_*)^2 + B_1\int_\Omega (v-v_*)(w^\beta - v_*) \\ &\leq -B_1\int_\Omega |\nabla v|^2 - \frac{B_1}{2}\int_\Omega (v-v_*)^2 + \frac{B_1}{2}\int_\Omega (w^\beta - v_*)^2\end{aligned} \tag{4.17}$$

for all $t > 0$. Also, using the third equation in (1.4) and Young's inequality, we have

$$\begin{aligned}\frac{d}{dt}C(t) &= B_2\int_\Omega (w-w_*)w_t \\ &= B_2\int_\Omega (w-w_*)(-\delta w + u) \\ &= -\delta B_2\int_\Omega (w-w_*)^2 + B_2\int_\Omega (w-w_*)(u-\delta w_*) \\ &\leq -\frac{\delta}{2}B_2\int_\Omega (w-w_*)^2 + \frac{B_2}{2\delta}\int_\Omega (u-\delta w_*)^2 \\ &= -\frac{\delta}{2}B_2\int_\Omega (w-w_*)^2 + \frac{B_2}{2\delta}\int_\Omega (u-u_*)^2\end{aligned} \tag{4.18}$$

for all $t > 0$. Next, we will prove (4.11).
Let

$$\mu_0 := \left(\frac{k}{4^\beta \delta^{2\beta} r^{\frac{l-2\beta-1}{l-1}}}\right)^{\frac{l-1}{2\beta}},$$

and $\mu > \mu_0$, then

$$\begin{aligned}&\frac{1}{2}(\delta B_2 - B_1 4^{1-\beta}\delta^{2-2\beta}u_*^{2\beta-2}) \\ &= \frac{1}{2}\left(\delta^2\mu u_*^{l-2} - k4^{-\beta}\delta^{2-2\beta}u_*^{2\beta-1}\right) \\ &= \frac{1}{2}\left(\delta^2\mu\left(\frac{r}{\mu}\right)^{\frac{l-2}{l-1}} - \frac{kr^{\frac{2\beta-1}{l-1}}\delta^{2-2\beta}}{4^\beta\mu^{\frac{2\beta-1}{l-1}}}\right) > 0\end{aligned}$$

Since $0 < \beta \leq 1$, using Lemma 4.1, we have

$$\begin{aligned}\frac{B_1}{2}\int_\Omega (w^\beta - w_*^\beta)^2 &\leq \frac{B_1}{2}2^{2(1-\beta)}w_*^{2(\beta-1)}\int_\Omega (w-w_*)^2 \\ &= \frac{B_1}{2}4^{1-\beta}\delta^{2-2\beta}u_*^{2\beta-2}\int_\Omega (w-w_*)^2\end{aligned} \tag{4.19}$$

for all $t > 0$. Combining (4.16)–(4.19), we can get

$$\begin{aligned}\frac{d}{dt}E(t) &\le \left(\frac{1}{4}u_*k - B_1\right)\int_\Omega |\nabla v|^2 - \left(\mu u_*^{l-2} - \frac{B_2}{2\delta}\right)\int_\Omega (u-u_*)^2 - \frac{B_1}{2}\int_\Omega (v-v_*)^2 \\ &- \frac{1}{2}\left(\delta B_2 - B_1 4^{1-\beta}\delta^{2-2\beta}u_*^{2\beta-2}\right)\int_\Omega (w-w_*)^2 \\ &\le -\varepsilon\left(\int_\Omega (u-u_*)^2 + \int_\Omega (v-v_*)^2 + \int_\Omega (w-w_*)^2\right)\end{aligned}$$

with $\mu > \mu_0$ and $\varepsilon = min\left\{\frac{1}{2}\mu u_*^{l-2}, \frac{1}{8}ku_*, \frac{1}{2}\left(\delta^2\mu u_*^{l-2} - k4^{-\beta}\delta^{2-2\beta}u_*^{2\beta-1}\right)\right\}$, for all $t > 0$. So, we complete the proof of Lemma 4.3. □

In the following discussion, let $\mu > \mu_0$, $l \ge 2$, and $0 < \beta \le 1$ hold, where μ_0 is defined in Lemma 4.3.

Proof of Theorem 1.2. Building upon the functional inequality (4.11), the proof of Theorem 1.2 can be approached in the same way as in [36]. To avoid redundancy, we do not recount the entire proof here. However, for the reader's convenience, we outline the main ideas of the proof.

Step 1. First, by taking $E(t)$ and $F(t)$ as defined in Lemma 4.3, and integrating (4.11) from 1 to t, we deduce

$$E(t) + \varepsilon\int_1^t F(s)ds \le E(1) \tag{4.20}$$

for all $t > 1$. Since $E(t)$ is nonnegative by Lemma 4.3, this entails that $\int_1^\infty F(s)ds$ is finite. According to the definition (4.9) of F, we have

$$\int_1^\infty\int_\Omega (u-u_*)^2 < \infty, \quad \int_1^\infty\int_\Omega (v-v_*)^2 < \infty \quad \text{and} \quad \int_1^\infty\int_\Omega (w-w_*)^2 < \infty \tag{4.21}$$

The weak convergence information (4.21) along with uniform Hölder's bounds of solutions implies

$$\| u - (\frac{r}{\mu})^{\frac{1}{l-1}} \|_{L^\infty(\Omega)} + \| v - (\frac{1}{\delta})^\beta (\frac{r}{\mu})^{\frac{\beta}{l-1}} \|_{L^\infty(\Omega)} \| + \| w - \frac{1}{\delta}(\frac{r}{\mu})^{\frac{1}{l-1}} \|_{L^\infty(\Omega)} \to 0 \quad as \quad t\to\infty. \tag{4.22}$$

Step 2. Based on L'Hôpital's rule, we can obtain

$$\lim_{u\to u_*}\frac{u-u_*-u_*ln\frac{u}{u_*}}{(u-u_*)^2} = \lim_{u\to u_*}\frac{1-\frac{u_*}{u}}{2(u-u_*)} = \frac{1}{2u_*}$$

According to (4.22), we can pick a positive constant t_0 such that

$$\frac{1}{4u_*}\int_\Omega (u-u_*)^2 \le \int_\Omega\left(u-u_*-u_*ln\frac{u}{u_*}\right) \le \frac{1}{u_*}\int_\Omega (u-u_*)^2 \tag{4.23}$$

for all $t > t_0$.

Step 3. In order to estimate the rate of convergence in (4.22), combining (4.11) and (4.23), then there exists a constant $C_1 > 0$ such that

$$\frac{d}{dt}E(t) \le -\varepsilon F(t) \le -C_1E(t) \tag{4.24}$$

for all $t > t_0$. (4.24) means there exist some positive constants C_2 and k such that

$$E(t) \leq C_2 e^{-kt} \tag{4.25}$$

for all $t > t_0$. From the definitions of $E(t)$ and $F(t)$, (4.23) and (4.25) allow us to choose a constant $C_3 > 0$ such that

$$\left(\int_\Omega (u-u_*)^2 + \int_\Omega (v-v_*)^2 + \int_\Omega (w-w_*)^2 \right) \leq C_3 e^{-kt} \tag{4.26}$$

for all $t > t_0$.

Step 4. By using Lemma 4.2 and the Gagliardo-Nirenberg inequality, we get

$$\|\phi\|_{L^\infty(\Omega)} \leq C_{GN} \|\phi\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \|\phi\|_{L^2(\Omega)}^{\frac{2}{n+2}}$$

for all $\phi \in W^{1,\infty}(\Omega)$. So, we can find some constants $C_4 > 0$ and $C_5 > 0$ such that

$$\begin{aligned} \|u(.,t) - u_*\|_{L^\infty(\Omega)} &\leq C_4 \|u(.,t) - u_*\|_{W^{1,\infty}(\Omega)}^{\frac{n}{n+2}} \|u(.,t) - u_*\|_{L^2(\Omega)}^{\frac{2}{n+2}} \\ &\leq C_5 \|u(.,t) - u_*\|_{L^2(\Omega)}^{\frac{2}{n+2}} \end{aligned} \tag{4.27}$$

for all $t > t_0$. Together with (4.26), we can find some positive constants C_6 and λ such that

$$\|u(.,t) - u_*\|_{L^\infty(\Omega)} \leq C_6 e^{-\lambda t} \tag{4.28}$$

for all $t > t_0$. Similarly, according to (3.38) and the Gagliardo-Nirenberg inequality, we can find a constant $C_7 > 0$ such that

$$\|v(.,t) - v_*\|_{L^\infty(\Omega)} \leq C_7 e^{-\lambda t} \tag{4.29}$$

for all $t > t_0$.

Applying the ODE theorem for the third equation of (1.4), we have

$$\begin{aligned} w(\cdot,t) &= e^{-\delta(t-t_0)} w(.,t_0) + \int_{t_0}^t e^{-\delta(t-s)} u(\cdot,s)\, \mathrm{d}s \\ &= e^{-\delta(t-t_0)} w(.,t_0) + \int_{t_0}^t e^{-\delta(t-s)} (u(\cdot,s) - u_*)\, \mathrm{d}s + \int_{t_0}^t e^{-\delta(t-s)} u_*\, \mathrm{d}s \\ &= e^{\delta t_0} w(.,t_0) e^{-\delta t} + \int_{t_0}^t e^{-\delta(t-s)} (u(\cdot,s) - u_*)\, \mathrm{d}s + \frac{u_*}{\delta} e^{-\delta t} (e^{\delta t} - e^{\delta t_0}) \end{aligned} \tag{4.30}$$

for all $t > t_0$. From (1.5), (4.28), and (4.30), there exist some positive constants C_8, C_9, and C_{10} such that

$$\begin{aligned} \|w - w_*\|_{L^\infty(\Omega)} &\leq (e^{\delta t_0} \|w(.,t_0)\|_{L^\infty(\Omega)} + e^{\delta t_0} w_*) e^{-\delta t} + \int_{t_0}^t e^{-\delta(t-s)} \|u(\cdot,s) - u_*\|_{L^\infty(\Omega)}\, \mathrm{d}s \\ &\leq C_8 e^{-\delta t} + C_9 e^{-\lambda t} \\ &\leq C_{10} e^{-\tau t} \end{aligned} \tag{4.31}$$

for all $t > t_0$, where $\tau := min\{\delta, \lambda\}$. Combining (4.28), (4.29), and (4.31), we complete the proof of Theorem 1.2. □

In summary, this paper establishes the global boundedness and stability of the steady-state solution for a chemotactic system with nonlinear indirect signal production in a bounded domain, defined under a specific parameter range. This contrasts with previous studies on chemotactic systems of this nature that utilize linear signal production. Our next goal is to extend these results to heterogeneous environments (see for example [37]), drawing on concepts from this work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare that there is no conflict of interest.

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