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http://www.aimspress.com/journal/era

ERA, 32(11): 6235–6257.
DOI: 10.3934/era.2024290
Received: 02 October 2024
Revised: 31 October 2024
Accepted: 06 November 2024
Published: 19 November 2024

Research article

Spatial behavior for the quasi-static heat conduction within the second gradient of type III

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Abstract: This article focused on investigating the spatial behavior of the quasi-static biharmonic conduction equation within the framework of type III of the second gradient in a two-dimensional cylindrical domain. The results of growth or decay estimates were established by using a second-order differential inequality. When the distance tends to infinity, the energy either grows exponentially or decays exponentially. The results showed that the Saint-Venant principle was also valid for the quasi-static biharmonic conduction equation.

Keywords: Phragmén-Lindelöf alternative; quasi-static heat conduction; Saint-Venant's principle; spatial behavior; biharmonic equation

1. Introduction

In [1], the authors studied a higher the order problem under the theory of the second gradient of type III. The thermal displacement θ satisfies $\dot{u} = \theta$. The evolution equation reads (also see [2])

$$c\ddot{u} + du = \mu_0 \Delta u + \mu_1 \Delta \dot{u} - d_0 \Delta^2 u - d_1 \Delta^2 \dot{u}, \tag{1.1}$$

where c is the inertia coefficient, μ_0 is the elastic coefficient, μ_1 is the time-dependent elastic coefficient, d_0 is the diffusion coefficient, d_1 is the time-dependent diffusion coefficient, and d is the damping coefficient. Δ represents the harmonic operator, and Δ^2 represents the biharmonic operator. They obtained the spatial decay estimates result for the energy in a semi-infinite cylinder. An a priori decay assumption for the solution was imposed at infinity. In [2], a uniqueness result for the basic boundary-initial-value problem is presented and an existence theorem is established for the boundary value problem.

In the present paper, we also study the spatial property for this type of equation. We delete the a priori decay assumptions for solution at infinity. We consider the quasi-static version of (1.1). It can be derived if we assume that the changes in the temperature are so slow that we can neglect the second-order time derivatives. In this case, the equation would become

$$\Delta u + \Delta \dot{u} - \Delta^2 u - \Delta^2 \dot{u} = u. \tag{1.2}$$

In recent years, the biharmonic equation has been used to describe the behavior of two-dimensional physical fields within a plane. It can represent many different physical phenomenas, including sound waves, electric fields, and magnetic fields. Many important applications are studied in applied mathematics and mechanics. The Saint-Venant principle is a fundamental concept in solid mechanics that has significant implications for the analysis and design of structures. The principle states that, for a sufficiently large distance from a localized load or boundary condition, the exact form and distribution of the load or condition become unimportant, and the stress and strain fields are governed only by the remote boundary conditions and the gross geometry of the body. In essence, Saint-Venant's principle allows engineers to simplify complex loading conditions and focus on the overall behavior of the structure, rather than the detailed nature of the loading. In order to obtain the Saint-Venant type results for the biharmonic equations, many studies and various methods have been proposed for researching the spatial behaviour for the solutions of the biharmonic equations in a semi-infinite strip in R^2. We mention the studies by Knowles [3, 4], Flavin [5], Flavin and Knops [6], and Horgan [7]. We note that some time-dependent problems concerning the biharmonic operator are considered in the literature. We mention the papers by Lin [8], Knops and Lupoli [9] in connection with the spatial behaviour of solutions for a fourth-order transformed problem associated with the slow flow of an incompressible viscous fluid along a semi-infinite strip. Then, Song in his paper [10, 11] improved the results obtained by Lin in [8] for the time-dependent Stokes flow.

Some papers have studied the Phragmén-Lindelöf type alternative results for various types of equations: Liu and Lin [12] studied the Phragmén-Lindelöf type alternative results for the time dependent flow. Lin and Payne [13] studied the Phragmén-Lindelöf results for the general heat equation. In [14], the authors studied the Phragmén-Lindelöf results for the harmonic functions. Some new Phragmén-Lindelöf results may be found in [15, 16]. Other results for the Saint-Venant principle may be found in [17].

We consider the problem on an unbounded region Ω_0 defined by

$$\Omega_0 := \{(x_1, x_2) \mid x_1 > 0, 0 \leq x_2 \leq h\},$$

where h is a fixed constant, and we introduce the notation

$$L_z = \{(x_1, x_2) \mid x_1 = z \geq 0, 0 \leq x_2 \leq h\}.$$

For the fourth-order differential equations, it is common to specify both Dirichlet and Neumann boundary conditions to give the value of the solution and the behavior of the normal derivative of the solution at the boundaries of the domain.

The initial boundary conditions are

$$u(x_1, 0, t) = 0 \quad x_1 > 0, t > 0, \tag{1.3}$$

$$u(x_1, h, t) = 0 \quad x_1 > 0, t > 0, \tag{1.4}$$

$$u(0, x_2, t) = g_1(x_2, t) \quad 0 \leq x_2 \leq h, t > 0, \tag{1.5}$$

$$u_{,1}(0, x_2, t) = g_2(x_2, t) \quad 0 \leq x_2 \leq h, t > 0, \tag{1.6}$$

and

$$u(x_1, x_2, 0) = 0 \quad 0 \leq x_2 \leq h, x_1 > 0, \tag{1.7}$$

where $g_1(x_2, t)$ and $g_2(x_2, t)$ are given functions.

In this paper, the spatial behavior of solutions of quasi-static heat conduction within the second gradient of type III is studied. Apart from paper [1], we have not found any research on the Saint-Venant principle in the context of quasi-static heat conduction with the biharmonic operator. Due to the complexity of deriving second-order differential inequalities, there is a scarcity of existing literature that utilizes this method to obtain Phragmén-Lindelöf type theorem. The results of growth or decay estimates are established associating some appropriate cross-sectional lines and area integral measures. Since the a priori decay assumptions may not always hold true in practical applications, we eliminate these assumptions in order to allow for a broader range of solutions that more accurately reflect the physical and mathematical complexities of the system being studied. The main difficulty in this paper is how to construct the energy expression without the assumption that the solution tends to zero at infinity. What is more, it is difficult to obtain the result that the energy expression can be bounded by its second-order differentiation. The method of the proof is based on a second-order differential inequality leading to an alternative of Phragmén-Lindelöf type in terms of an area measure of the amplitude in question. The Phragmén-Lindelöf theorem is particularly useful in the study of partial differential equations that arise in physics and engineering. For instance, in potential theory, it can be employed to derive bounds on solutions of Laplace's equation, which is fundamental in electrostatics and fluid dynamics. These bounds help in understanding the behavior of electric and magnetic fields, as well as fluid flow patterns. The estimation of spatial decay of solutions plays an important role in mathematics and physics, especially when analyzing solutions to partial differential equations. This estimation can help us understand the behavior of solutions at different spatial positions, especially the properties of solutions that are far from certain specific points or regions. The exponential growth of the solution tells us that at this point, the solution has great instability and may blow up. These results provide the theoretical bases for further in-depth research on the stability of solutions and numerical simulations.

In the present paper, we are concerned with the Phragmén-Lindelöf alternative for quasi-static heat conduction within the second gradient of type III in a semi-infinite channel. We formulate the energy expressions and derive a second order differential inequality, which is useful in deriving our main result in Section 2. In Section 3, we obtain the Phragmén-Lindelöf alternative results for the solution which can be seen as a version of Saint-Venant principle. The comma is used to indicate partial differentiation, and the differentiation with respect to the direction x_k is denoted as $,k$, thus $u_{,\alpha}$ denotes $\frac{\partial u}{\partial x_\alpha}$, and $u_{,t}$ denotes $\frac{\partial u}{\partial t}$. The usual summation convection is employed with repeated Greek subscripts α summed from 1 to 2. Hence, $u_{,\alpha\alpha} = \sum_{\alpha=1}^{2} \frac{\partial^2 u}{\partial x_\alpha^2}$.

2. The definitions of the energy functions

In this part, we will derive some energy expressions that are useful in deriving our results. The definitions of the energy functions can be divided into the following lemmas. In the following discussions, ω is an arbitrary positive constant, and we will give some restriction later. A is an area element on $x_1 - x_2$ plane. $dA = dx_2 d\xi$.

Lemma 2.1: Let u be the classical solution of Eq (1.2) and satisfy the initial boundary value problems (1.3)–(1.7), we define a function

$$\begin{aligned} E_1(z,t) = &\frac{1}{2}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\eta}^2 \mathrm{d}x_2 \mathrm{d}\eta - \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\eta} u_{,11} \mathrm{d}x_2 \mathrm{d}\eta \\ &+ \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\alpha\eta} u_{,\alpha\eta} \mathrm{d}x_2 \mathrm{d}\eta - \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\eta} u_{,11\eta} \mathrm{d}x_2 \mathrm{d}\eta. \end{aligned} \tag{2.1}$$

$E(z, t)$ can also be expressed as

$$\begin{aligned} E_1(z,t) = &\frac{\omega}{2}\int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta)(z-\xi) u_{,\alpha} u_{,\alpha} \mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_0^z \int_{L_\xi} \exp(-\omega t) u_{,\alpha} u_{,\alpha} \mathrm{d}A \\ &+ \int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta)(z-\xi) u_{,\alpha\eta} u_{,\alpha\eta} \mathrm{d}A\mathrm{d}\eta + \frac{\omega}{2}\int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta)(z-\xi) u_{,\alpha\beta} u_{,\alpha\beta} \mathrm{d}A\mathrm{d}\eta \\ &+ \frac{1}{2}\int_0^z \int_{L_\xi} \exp(-\omega t)(z-\xi) u_{,\alpha\beta} u_{,\alpha\beta} \mathrm{d}A + \int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta)(z-\xi) u_{,\alpha\beta\eta} u_{,\alpha\beta\eta} \mathrm{d}A\mathrm{d}\eta \\ &- 2\int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta) u_{,\alpha\eta} u_{,1\alpha} \mathrm{d}A\mathrm{d}\eta - \int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta)(z-\xi) u_{,\eta} u \mathrm{d}A\mathrm{d}\eta \\ &- \int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta) u_{,\eta} u_{,1} \mathrm{d}A\mathrm{d}\eta + f_1(z,t), \end{aligned} \tag{2.2}$$

where $f_1(z, t)$ will be defined later.

Proof: From (1.2), we have the equality

$$0 = \int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta)(z-\xi) u_{,\eta}(u_{,\alpha\alpha} + \dot{u}_{,\alpha\alpha} - u_{,\alpha\alpha\beta\beta} - \dot{u}_{,\alpha\alpha\beta\beta} - u) \mathrm{d}A\mathrm{d}\eta. \tag{2.3}$$

We now begin to deal with items on the right side of (2.3). Integrating by parts and using the initial-boundary conditions (1.3)–(1.7), we can obtain

$$\begin{aligned}
&\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\eta}u_{,\alpha\alpha}\mathrm{d}A\mathrm{d}\eta\\
&=-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\alpha\eta}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}u_{,1}\mathrm{d}A\mathrm{d}\eta\\
&-z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta \qquad (2.4)\\
&=-\frac{\omega}{2}\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta-\frac{1}{2}\int_0^z\int_{L_\xi}\exp(-\omega t)(z-\xi)u_{,\alpha}u_{,\alpha}\mathrm{d}A\\
&+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}u_{,1}\mathrm{d}A\mathrm{d}\eta-z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}$$

The second term on the right side of (2.3) can be expressed as

$$\begin{aligned}
&\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\eta}u_{,\alpha\alpha\eta}\mathrm{d}A\mathrm{d}\eta\\
&=-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}u_{,1\eta}\mathrm{d}A\mathrm{d}\eta\\
&-z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta \qquad (2.5)\\
&=-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta\\
&-\frac{1}{2}\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta-z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}$$

The third term on the right side of (2.3) can be expressed as

$$\begin{aligned}
&-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\eta}u_{,\alpha\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta\\
&=\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\alpha\eta}u_{,\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}u_{,1\beta\beta}\mathrm{d}A\mathrm{d}\eta\\
&+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta\\
&=-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\alpha\beta\eta}u_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta\\
&-z\int_0^t\int_{L_0}\exp(-\omega\eta))u_{,\alpha\eta}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta))u_{,\beta\eta}u_{,1\beta}\mathrm{d}A\mathrm{d}\eta\\
&-\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u_{,11}\mathrm{d}x_2\mathrm{d}\eta+\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,11}\mathrm{d}x_2\mathrm{d}\eta
\end{aligned}$$

$$
\begin{aligned}
&+ z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&= -\frac{\omega}{2}\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta - \frac{1}{2}\int_0^z\int_{L_\xi}\exp(-\omega t)(z-\xi)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}A \\
&+ 2\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta - z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\alpha\eta}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta \\
&- \int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u_{,11}\mathrm{d}x_2\mathrm{d}\eta + \int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,11}\mathrm{d}x_2\mathrm{d}\eta \\
&+ z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
\tag{2.6}
$$

The fourth term on the right side of (2.3) can be expressed as

$$
\begin{aligned}
&- \int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\eta}u_{,\alpha\alpha\beta\beta\eta}\mathrm{d}A\mathrm{d}\eta \\
&= \int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\alpha\eta}u_{,\alpha\beta\beta\eta}\mathrm{d}A\mathrm{d}\eta - \int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}u_{,1\beta\beta\eta}\mathrm{d}A\mathrm{d}\eta \\
&+ z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,1\beta\beta\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&= -\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\alpha\beta\eta}u_{,\alpha\beta\eta}\mathrm{d}A\mathrm{d}\eta + \int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,1\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\
&- z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\alpha\eta}u_{,1\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta + \int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,\beta\eta}u_{,1\beta\eta}\mathrm{d}A\mathrm{d}\eta \\
&- \int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u_{,11\eta}\mathrm{d}x_2\mathrm{d}\eta + \int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,11\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,1\beta\beta\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&= -\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\alpha\beta\eta}u_{,\alpha\beta\eta}\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&- \frac{1}{2}\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta - z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\alpha\eta}u_{,1\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\beta\eta}u_{,\beta\eta}\mathrm{d}x_2\mathrm{d}\eta - \frac{1}{2}\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\beta\eta}u_{,\beta\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&- \int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u_{,11\eta}\mathrm{d}x_2\mathrm{d}\eta + \int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,11\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,1\beta\beta\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
\tag{2.7}
$$

If we define new expressions $E_1(z,t)$ and $f_1(z,t)$ as

$$\begin{aligned}E_1(z,t)=&\frac{\omega}{2}\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\int_0^z\int_{L_\xi}\exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}A\\&+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta+\frac{\omega}{2}\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta\\&+\frac{1}{2}\int_0^z\int_{L_\xi}\exp(-\omega t)(z-\xi)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}A+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\alpha\beta\eta}u_{,\alpha\beta\eta}\mathrm{d}A\mathrm{d}\eta\\&-2\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,\eta}u_{,1}\mathrm{d}A\mathrm{d}\eta\\&-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,\eta}u\mathrm{d}A\mathrm{d}\eta+f_1(z,t),\end{aligned}\tag{2.8}$$

and

$$\begin{aligned}f_1(z,t)=&z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2}\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta\\&-z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\alpha\eta}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta\\&-\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,11}\mathrm{d}x_2\mathrm{d}\eta-z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta\\&+\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta\\&-\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,11\eta}\mathrm{d}x_2\mathrm{d}\eta-z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\eta}u_{,1\beta\beta\eta}\mathrm{d}x_2\mathrm{d}\eta.\end{aligned}\tag{2.9}$$

A combination of (2.3)–(2.9) gives

$$\begin{aligned}E_1(z,t)=&\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta-\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u_{,11}\mathrm{d}x_2\mathrm{d}\eta\\&+\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta-\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u_{,11\eta}\mathrm{d}x_2\mathrm{d}\eta.\end{aligned}\tag{2.10}$$

Lemma 2.2: Let u be the classical solution of Eq (1.2) and satisfy the initial boundary value problems (1.3)–(1.7), we define a function

$$\begin{aligned}E_2(z,t)=&\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta-\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1}^2\mathrm{d}x_2\mathrm{d}\eta\\&+\left(\frac{1}{2}+\frac{\omega}{2}\right)\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta+\left(\frac{1}{2}+\frac{\omega}{2}\right)\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta\\&-(\omega+1)\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta-\left(\frac{1}{2}+\frac{\omega}{2}\right)\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\beta}u_{,1\beta}\mathrm{d}x_2\mathrm{d}\eta\end{aligned}$$

$$
\begin{aligned}
&+(\omega+1)\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,11}^2\mathrm{d}x_2\mathrm{d}\eta+\frac{\omega}{2}\int_{L_z}\exp(-\omega t)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\\
&+\frac{\omega}{2}\int_{L_z}\exp(-\omega t)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}x_2-\omega\int_{L_z}\exp(-\omega t)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\\
&-\frac{\omega}{2}\int_{L_z}\exp(-\omega t)u_{,1\beta}u_{,1\beta}\mathrm{d}x_2+\omega\int_{L_z}\exp(-\omega t)u_{,11}^2\mathrm{d}x_2\\
&+\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta-\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta\\
&+\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta-\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\eta}^2\mathrm{d}x_2\mathrm{d}\eta\\
&+\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\beta\eta}u_{,\alpha\beta\eta}\mathrm{d}x_2\mathrm{d}\eta\\
&-\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta-\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta\\
&-\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\beta\eta}u_{,1\beta\eta}\mathrm{d}x_2\mathrm{d}\eta+\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,11\eta}^2\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
\tag{2.11}
$$

$E(z,t)$ can also be expressed as

$$
\begin{aligned}
E_2(z,t)={}&\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\int_0^z\int_{L_\xi}\exp(-\omega t)(z-\xi)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta\\
&+(\omega+1)\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha\beta}u_{,1\alpha\beta}\mathrm{d}A\mathrm{d}\eta+\omega\int_0^z\int_{L_\xi}\exp(-\omega t)u_{,1\alpha\beta}u_{,1\alpha\beta}\mathrm{d}A\\
&+\frac{\omega}{2}\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\int_0^z\int_{L_\xi}\exp(-\omega t)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A\\
&+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha\beta\eta}u_{,1\alpha\beta\eta}\mathrm{d}A\mathrm{d}\eta\\
&-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,12}u_{,2\eta}\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta\\
&-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\eta}u_{,11}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,11}u\mathrm{d}A\mathrm{d}\eta\\
&-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,11\eta}u\mathrm{d}A\mathrm{d}\eta+f_2(z,t),
\end{aligned}
\tag{2.12}
$$

where $f_2(z,t)$ will be defined later.

Proof: From (1.2), we have the equality

$$
\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,11}(u_{,\alpha\alpha}+u_{,\alpha\alpha\eta}-u_{,\alpha\alpha\beta\beta}-u_{,\alpha\alpha\beta\beta\eta}-u)\mathrm{d}A\mathrm{d}\eta=0.
\tag{2.13}
$$

The first term on the right side of (2.13) can be expressed as

$$
\begin{aligned}
&\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,11}u_{,\alpha\alpha}\mathrm{d}A\mathrm{d}\eta\\
&=-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,11\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,11}u_{,1}\mathrm{d}A\mathrm{d}\eta\\
&-z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,11}u_{,1}\mathrm{d}x_2\mathrm{d}\eta\\
&=\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta\\
&+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,11}u_{,1}\mathrm{d}A\mathrm{d}\eta\\
&-z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,11}u_{,1}\mathrm{d}x_2\mathrm{d}\eta\\
&=\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta-\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta\\
&+\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1}^2\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2}\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta\\
&-\frac{1}{2}\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1}^2\mathrm{d}x_2\mathrm{d}\eta+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta\\
&-z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,11}u_{,1}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
\tag{2.14}
$$

The second term on the right side of (2.13) can be expressed as

$$
\begin{aligned}
&\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,11}u_{,\alpha\alpha\eta}\mathrm{d}A\mathrm{d}\eta\\
&=-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,11\alpha}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,11}u_{,1\eta}\mathrm{d}A\mathrm{d}\eta\\
&-z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,11}u_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta\\
&=\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha}u_{,1\alpha\eta}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta\\
&+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\alpha}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,11}u_{,1\eta}\mathrm{d}A\mathrm{d}\eta\\
&-z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,11}u_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta\\
&=\frac{1}{2}\int_0^z\int_{L_\xi}\exp(-\omega t)(z-\xi)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,12}u_{,2\eta}\mathrm{d}A\mathrm{d}\eta\\
&+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,12}u_{,2\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
\tag{2.15}
$$

The third term on the right side of (2.13) can be expressed as

$$
\begin{aligned}
&-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,11}u_{,\alpha\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta\\
&=\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,11\alpha}u_{,\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,11}u_{,1\beta\beta}\mathrm{d}A\mathrm{d}\eta\\
&+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,11}u_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta\\
&=-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha}u_{,1\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha}u_{,\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta\\
&-z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\alpha}u_{,\alpha\beta\beta}\mathrm{d}x_2\mathrm{d}\eta+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\beta1}u_{,1\beta}\mathrm{d}A\mathrm{d}\eta\\
&-\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,11}u_{,11}\mathrm{d}x_2\mathrm{d}\eta+\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,11}^2\mathrm{d}x_2\mathrm{d}\eta+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,11}u_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta\\
&=\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha\beta}u_{,1\alpha\beta}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha1}\mathrm{d}A\mathrm{d}\eta\\
&+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha1}\mathrm{d}x_2\mathrm{d}\eta-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha\beta}u_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta\\
&+\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta-\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta\\
&-z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\alpha}u_{,\alpha\beta\beta}\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\beta}u_{,1\beta}\mathrm{d}x_2\mathrm{d}\eta\\
&-\frac{1}{2}\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\beta}u_{,1\beta}\mathrm{d}x_2\mathrm{d}\eta-\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,11}u_{,11}\mathrm{d}x_2\mathrm{d}\eta\\
&+\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,11}^2\mathrm{d}x_2\mathrm{d}\eta+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,11}u_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta\\
&=\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha\beta}u_{,1\alpha\beta}\mathrm{d}A\mathrm{d}\eta-\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta\\
&+\frac{1}{2}\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta-\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta\\
&+\frac{1}{2}\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta+\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta\\
&-\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\beta}u_{,1\beta}\mathrm{d}x_2\mathrm{d}\eta\\
&-\frac{1}{2}\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\beta}u_{,1\beta}\mathrm{d}x_2\mathrm{d}\eta-\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,11}^2\mathrm{d}x_2\mathrm{d}\eta\\
&+\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,11}^2\mathrm{d}x_2\mathrm{d}\eta+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha1}\mathrm{d}x_2\mathrm{d}\eta\\
&-z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\alpha}u_{,\alpha\beta\beta}\mathrm{d}x_2\mathrm{d}\eta+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,11}u_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
\tag{2.16}
$$

The fourth term on the right side of (2.13) can be expressed as

$$\begin{aligned}
&-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,11}u_{,\alpha\alpha\beta\beta\eta}\mathrm{d}A\mathrm{d}\eta\\
&=\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,11\eta}u_{,\alpha\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta-\omega\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,11}u_{,\alpha\alpha\beta\beta}\mathrm{d}A\mathrm{d}\eta\\
&-\int_0^z\int_{L_\xi}\exp(-\omega t)(z-\xi)u_{,11}u_{,\alpha\alpha\beta\beta}\mathrm{d}A.
\end{aligned}\tag{2.17}$$

From (1.2), we also have the equality

$$\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,11\eta}(u_{,\alpha\alpha}+u_{,\alpha\alpha\eta}-u_{,\alpha\alpha\beta\beta}-u_{,\alpha\alpha\beta\beta\eta}-u)\mathrm{d}A\mathrm{d}\eta=0.\tag{2.18}$$

The first term on the right side of (2.18) can be expressed as

$$\begin{aligned}
&\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,11\eta}u_{,\alpha\alpha}\mathrm{d}A\mathrm{d}\eta\\
&=-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,11\alpha\eta}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,11\eta}u_{,1}\mathrm{d}A\mathrm{d}\eta\\
&+z\int_0^t\int_{L_0}\exp(-\omega\eta)(z-\xi)u_{,11\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta\\
&=\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha\eta}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta\\
&+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\eta}u_{,11}\mathrm{d}A\mathrm{d}\eta\\
&+\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta-\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta\\
&+z\int_0^t\int_{L_0}\exp(-\omega\eta)(z-\xi)u_{,11\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta\\
&=\frac{\omega}{2}\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\int_0^z\int_{L_\xi}\exp(-\omega t)(z-\xi)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A\\
&+\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta-\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta\\
&+\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta\\
&-\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\eta}u_{,11}\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta\\
&-\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,1\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta+z\int_0^t\int_{L_0}\exp(-\omega\eta)u_{,11\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}\tag{2.19}$$

The second term on the right side of (2.18) can be expressed as

$$
\begin{aligned}
&\int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta)(z-\xi)u_{,11\eta}u_{,\alpha\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\
&= -\int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta)(z-\xi)u_{,11\alpha\eta}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta + \int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta)u_{,11\eta}u_{,1\eta}\mathrm{d}A\mathrm{d}\eta \\
&- z\int_0^t \int_{L_0} \exp(-\omega\eta)u_{,11\eta}u_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&= \int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}A\mathrm{d}\eta - \int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta)u_{,1\alpha\eta}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\
&+ z\int_0^t \int_{L_0} \exp(-\omega\eta)u_{,1\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta + \frac{1}{2}\int_0^t \int_{L_z} \exp(-\omega\eta)u_{,1\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
&- \frac{1}{2}\int_0^t \int_{L_0} \exp(-\omega\eta)u_{,1\eta}^2\mathrm{d}x_2\mathrm{d}\eta - z\int_0^t \int_{L_0} \exp(-\omega\eta)u_{,11\eta}u_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&= \int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta)(z-\xi)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}A\mathrm{d}\eta - \frac{1}{2}\int_0^t \int_{L_z} u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{1}{2}\int_0^t \int_{L_0} \exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta + z\int_0^t \int_{L_0} \exp(-\omega\eta)u_{,1\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{1}{2}\int_0^t \int_{L_z} \exp(-\omega\eta)u_{,1\eta}^2\mathrm{d}x_2\mathrm{d}\eta - \frac{1}{2}\int_0^t \int_{L_0} \exp(-\omega\eta)u_{,1\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
&- z\int_0^t \int_{L_0} \exp(-\omega\eta)u_{,11\eta}u_{,1\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
\tag{2.20}
$$

The third term on the right side of (2.18) can be expressed as

$$
\begin{aligned}
&-\int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta)(z-\xi)u_{,11\eta}u_{,\alpha\alpha\beta\beta\eta}\mathrm{d}A\mathrm{d}\eta \\
&= \int_0^t \int_0^z \int_{L_\xi} \exp(-\omega\eta)(z-\xi)u_{,1\alpha\beta\eta}u_{,1\alpha\beta\eta}\mathrm{d}A\mathrm{d}\eta - \frac{1}{2}\int_0^t \int_{L_z} \exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{1}{2}\int_0^t \int_{L_0} \exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta - \frac{1}{2}\int_0^t \int_{L_z} \exp(-\omega\eta)u_{,\alpha\beta\eta}u_{,\alpha\beta\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{1}{2}\int_0^t \int_{L_0} \exp(-\omega\eta)u_{,\alpha\beta\eta}u_{,\alpha\beta\eta}\mathrm{d}x_2\mathrm{d}\eta + \int_0^t \int_{L_z} \exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&- \int_0^t \int_{L_0} \exp(-\omega\eta)u_{,\alpha\eta}u_{,1\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta + \frac{1}{2}\int_0^t \int_{L_z} \exp(-\omega\eta)u_{,1\beta\eta}u_{,1\beta\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&- \frac{1}{2}\int_0^t \int_{L_0} \exp(-\omega\eta)u_{,1\beta\eta}u_{,1\beta\eta}\mathrm{d}x_2\mathrm{d}\eta - \int_0^t \int_{L_z} \exp(-\omega\eta)u_{,11\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
&+ \int_0^t \int_{L_0} \exp(-\omega\eta)u_{,11\eta}^2\mathrm{d}x_2\mathrm{d}\eta + z\int_0^t \int_{L_0} \exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha1\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&- z\int_0^t \int_{L_0} \exp(-\omega\eta)u_{,1\alpha\eta}u_{,\alpha\beta\beta\eta}\mathrm{d}x_2\mathrm{d}\eta + z\int_0^t \int_{L_0} \exp(-\omega\eta)u_{,11\eta}u_{,1\beta\beta\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
\tag{2.21}
$$

We define a new function $E_2(z,t)$ as

$$\begin{aligned}
E_2(z,t) &= \frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta - \frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1}^2\mathrm{d}x_2\mathrm{d}\eta \\
&+ \left(\frac{1}{2}+\frac{\omega}{2}\right)\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta + \left(\frac{1}{2}+\frac{\omega}{2}\right)\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&- (\omega+1)\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta - \left(\frac{1}{2}+\frac{\omega}{2}\right)\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\beta}u_{,1\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ (\omega+1)\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,11}^2\mathrm{d}x_2\mathrm{d}\eta + \frac{\omega}{2}\int_{L_z}\exp(-\omega t)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2 \\
&+ \frac{\omega}{2}\int_{L_z}\exp(-\omega t)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}x_2 - \omega\int_{L_z}\exp(-\omega t)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2 \\
&- \frac{\omega}{2}\int_{L_z}\exp(-\omega t)u_{,1\beta}u_{,1\beta}\mathrm{d}x_2 + \omega\int_{L_z}\exp(-\omega t)u_{,11}^2\mathrm{d}x_2 \qquad (2.22) \\
&+ \int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta - \int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta - \frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\eta}^2\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta + \frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\beta\eta}u_{,\alpha\beta\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&- \int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta - \frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&- \frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\beta\eta}u_{,1\beta\eta}\mathrm{d}x_2\mathrm{d}\eta + \int_0^t\int_{L_z}\exp(-\omega\eta)u_{,11\eta}^2\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}$$

A combination of (2.13)–(2.22) gives

$$\begin{aligned}
E_2(z,t) &= \int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_0^z\int_{L_\xi}\exp(-\omega t)(z-\xi)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta \\
&+ (\omega+1)\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha\beta}u_{,1\alpha\beta}\mathrm{d}A\mathrm{d}\eta + \omega\int_0^z\int_{L_\xi}\exp(-\omega t)u_{,1\alpha\beta}u_{,1\alpha\beta}\mathrm{d}A \\
&+ \frac{\omega}{2}\int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_0^z\int_{L_\xi}\exp(-\omega t)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A \\
&+ \int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}A\mathrm{d}\eta + \int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)(z-\xi)u_{,1\alpha\beta\eta}u_{,1\alpha\beta\eta}\mathrm{d}A\mathrm{d}\eta \qquad (2.23) \\
&- \int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,12}u_{,2\eta}\mathrm{d}A\mathrm{d}\eta + \int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta \\
&- \int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\eta}u_{,11}\mathrm{d}A\mathrm{d}\eta - \int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,11}u\mathrm{d}A\mathrm{d}\eta \\
&- \int_0^t\int_0^z\int_{L_\xi}\exp(-\omega\eta)u_{,11\eta}u\mathrm{d}A\mathrm{d}\eta + f_2(z,t),
\end{aligned}$$

with

$$
\begin{aligned}
f_2(z,t) =& \frac{1}{2}\int_0^t\int_{L_0} \exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta - \frac{1}{2}\int_0^t\int_{L_0} \exp(-\omega\eta)u_{,1}^2\mathrm{d}x_2\mathrm{d}\eta \\
&+ z\int_0^t\int_{L_0} \exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta - z\int_0^t\int_{L_0} \exp(-\omega\eta)u_{,11}u_{,1}\mathrm{d}x_2\mathrm{d}\eta \\
&+ z\int_0^t\int_{L_0} \exp(-\omega\eta)u_{,12}u_{,2\eta}\mathrm{d}x_2\mathrm{d}\eta + \left(\frac{1}{2}+\frac{\omega}{2}\right)\int_0^t\int_{L_0} \exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \left(\frac{1}{2}+\frac{\omega}{2}\right)\int_0^t\int_{L_0} \exp(-\omega\eta)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta - (\omega+1)\int_0^t\int_{L_0} \exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta \\
&- \left(\frac{1}{2}+\frac{\omega}{2}\right)\int_0^t\int_{L_0} \exp(-\omega\eta)u_{,1\beta}u_{,1\beta}\mathrm{d}x_2\mathrm{d}\eta + (\omega+1)\int_0^t\int_{L_0} \exp(-\omega\eta)u_{,11}^2\mathrm{d}x_2\mathrm{d}\eta \\
&+ (\omega+1)z\int_0^t\int_{L_0} \exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha1}\mathrm{d}x_2\mathrm{d}\eta + (\omega+1)z\int_0^t\int_{L_0} \exp(-\omega\eta)u_{,1\alpha}u_{,\alpha\beta\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ (\omega+1)z\int_0^t\int_{L_0} \exp(-\omega\eta)u_{,11}u_{,1\beta\beta}\mathrm{d}x_2\mathrm{d}\eta + \frac{1}{2}\int_{L_0} \exp(-\omega t)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2 \\
&+ \frac{1}{2}\int_{L_0} \exp(-\omega t)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}x_2 - \int_{L_0} \exp(-\omega t)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2 \\
&+ \frac{1}{2}\int_{L_0} \exp(-\omega t)u_{,1\beta}u_{,1\beta}\mathrm{d}x_2 - \int_{L_0} \exp(-\omega t)u_{,11}^2\mathrm{d}x_2 \\
&+ z\int_{L_0} \exp(-\omega t)u_{,1\alpha}u_{,1\alpha1}\mathrm{d}x_2 + z\int_{L_0} \exp(-\omega t)u_{,1\alpha}u_{,\alpha\beta\beta}\mathrm{d}x_2 \\
&+ z\int_{L_0} \exp(-\omega t)u_{,11}u_{,1\beta\beta}\mathrm{d}x_2.
\end{aligned} \tag{2.24}
$$

We define a new function

$$
E(z,t) = E_1(z,t) + E_2(z,t). \tag{2.25}
$$

From (2.2), we have

$$
\begin{aligned}
\frac{\partial^2 E_1(z,t)}{\partial z^2} =& \frac{\omega}{2}\int_0^t\int_{L_z} \exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta + \frac{1}{2}\int_{L_z} \exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2 \\
&+ \int_0^t\int_{L_z} \exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta + \frac{\omega}{2}\int_0^t\int_{L_z} \exp(-\omega\eta)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta \\
&+ \frac{1}{2}\int_{L_z} \exp(-\omega t)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}x_2 + \int_0^t\int_{L_z} \exp(-\omega\eta)u_{,\alpha\beta\eta}u_{,\alpha\beta\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&- 2\int_0^t\int_{L_z} \exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta - 2\int_0^t\int_{L_z} \exp(-\omega\eta)u_{,1\alpha1}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta \\
&- \int_0^t\int_{L_z} \exp(-\omega\eta)u_{,1\eta}u_{,1}\mathrm{d}x_2\mathrm{d}\eta - \int_0^t\int_{L_z} \exp(-\omega\eta)u_{,\eta}u_{,11}\mathrm{d}x_2\mathrm{d}\eta \\
&- \int_0^t\int_{L_z} \exp(-\omega\eta)u_{,\eta}u\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned} \tag{2.26}
$$

From (2.12), we obtain

$$
\begin{aligned}
\frac{\partial^2 E_2(z,t)}{\partial z^2} =& \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha} u_{,1\alpha} \mathrm{d}x_2 \mathrm{d}\eta + \frac{1}{2}\int_{L_z} \exp(-\omega t) u_{,1\alpha} u_{,1\alpha} \mathrm{d}x_2 \\
&+ (\omega+1)\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha\beta} u_{,1\alpha\beta} \mathrm{d}x_2 \mathrm{d}\eta + \omega \int_{L_z} \exp(-\omega\eta) u_{,1\alpha\beta} u_{,1\alpha\beta} \mathrm{d}x_2 \\
&+ \frac{\omega}{2}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha} u_{,1\alpha} \mathrm{d}x_2 \mathrm{d}\eta + \frac{1}{2}\int_{L_z} \exp(-\omega\eta) u_{,1\alpha} u_{,1\alpha} \mathrm{d}x_2 \\
&+ \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha\eta} u_{,1\alpha\eta} \mathrm{d}x_2 \mathrm{d}\eta + \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha\beta\eta} u_{,1\alpha\beta\eta} \mathrm{d}x_2 \mathrm{d}\eta \\
&- \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,112} u_{,2\eta} \mathrm{d}x_2 \mathrm{d}\eta - \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,12} u_{,12\eta} \mathrm{d}x_2 \mathrm{d}\eta \\
&+ \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha\eta} u_{,1\alpha} \mathrm{d}x_2 \mathrm{d}\eta + \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\alpha\eta} u_{,1\alpha 1} \mathrm{d}x_2 \mathrm{d}\eta \\
&- \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,11\eta} u_{,11} \mathrm{d}x_2 \mathrm{d}\eta - \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\eta} u_{,111} \mathrm{d}x_2 \mathrm{d}\eta \\
&- \int_0^t \int_{L_z} \exp(-\omega\eta) u u_{,11} \mathrm{d}x_2 \mathrm{d}\eta - \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,11\eta} u \mathrm{d}x_2 \mathrm{d}\eta.
\end{aligned} \tag{2.27}
$$

In the following discussions, we will use the following well-known Wirtinger-type inequality (see (3.1) in [12])

$$
\int_0^t \int_{L_z} \exp(-\omega\eta) u^2 \mathrm{d}x_2 \mathrm{d}\eta \le \frac{h^2}{\pi^2} \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\alpha} u_{,\alpha} \mathrm{d}x_2 \mathrm{d}\eta, \tag{2.28}
$$

and the Schwarz inequality

$$
\int_0^t \int_{L_z} ab \mathrm{d}x_2 \mathrm{d}\eta \le \frac{\epsilon}{2}\int_0^t \int_{L_z} a^2 \mathrm{d}x_2 \mathrm{d}\eta + \frac{1}{2\epsilon}\int_0^t \int_{L_z} b^2 \mathrm{d}x_2 \mathrm{d}\eta, \tag{2.29}
$$

with ϵ an arbitrary positive constant.

We now begin to bound terms in (2.27). Using the Wirtinger-type inequality (2.28) and Schwarz inequality (2.29), we can obtain the following estimates:

$$
\left| 2\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha\eta} u_{,1\alpha} \mathrm{d}x_2 \mathrm{d}\eta \right| \le \epsilon_1 \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha\eta} u_{,1\alpha\eta} \mathrm{d}x_2 \mathrm{d}\eta + \frac{1}{\epsilon_1}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha} u_{,1\alpha} \mathrm{d}x_2 \mathrm{d}\eta,
$$

$$
\left| 2\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha 1} u_{,\alpha\eta} \mathrm{d}x_2 \mathrm{d}\eta \right| \le \epsilon_2 \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha 1} u_{,1\alpha 1} \mathrm{d}x_2 \mathrm{d}\eta + \frac{1}{\epsilon_2}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\alpha\eta} u_{,\alpha\eta} \mathrm{d}x_2 \mathrm{d}\eta,
$$

$$
\left| \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\eta} u_{,1} \mathrm{d}x_2 \mathrm{d}\eta \right| \le \epsilon_3 \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\eta} u_{,1\eta} \mathrm{d}x_2 \mathrm{d}\eta + \frac{1}{2\epsilon_3}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1}^2 \mathrm{d}x_2 \mathrm{d}\eta,
$$

$$\left|\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u_{,11}\mathrm{d}x_2\mathrm{d}\eta\right|\leq\frac{\epsilon_4}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2\epsilon_4}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,11}^2\mathrm{d}x_2\mathrm{d}\eta$$
$$\leq\frac{\epsilon_4}{2\lambda_1}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2\epsilon_4}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,11}^2\mathrm{d}x_2\mathrm{d}\eta,$$

$$\left|\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,112}u_{,2\eta}\mathrm{d}x_2\mathrm{d}\eta\right|\leq\frac{\epsilon_5}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,112}^2\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2\epsilon_5}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,2\eta}^2\mathrm{d}x_2\mathrm{d}\eta,$$

$$\left|\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,11}u_{,11\eta}\mathrm{d}x_2\mathrm{d}\eta\right|\leq\frac{\epsilon_6}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,11}^2\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2\epsilon_6}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,11\eta}^2\mathrm{d}x_2\mathrm{d}\eta,$$

$$\left|\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,2\eta}u_{,121}\mathrm{d}x_2\mathrm{d}\eta\right|\leq\frac{\epsilon_7}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,2\eta}^2\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2\epsilon_7}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,121}^2\mathrm{d}x_2\mathrm{d}\eta,$$

$$\left|\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\eta}u_{,111}\mathrm{d}x_2\mathrm{d}\eta\right|\leq\frac{\epsilon_8}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\eta}^2\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2\epsilon_8}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,111}^2\mathrm{d}x_2\mathrm{d}\eta,$$

where ϵ_1, ϵ_2, ϵ_3, ϵ_4, ϵ_5, ϵ_6, ϵ_7, and ϵ_8 are arbitrary positive constants; we will define them later. $\lambda_1=\frac{\pi}{h}$.

Using the Schwarz inequality (2.29) and the Wirtinger-type inequality (2.28) again, we can obtain

$$\left|\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,11}u\mathrm{d}x_2\mathrm{d}\eta\right|\leq\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u^2\mathrm{d}x_2\mathrm{d}\eta$$
$$\leq\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2}\frac{h^2}{\pi^2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta,$$

$$\left|\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,11\eta}u\mathrm{d}x_2\mathrm{d}\eta\right|\leq\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u^2\mathrm{d}x_2\mathrm{d}\eta$$
$$\leq\frac{1}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2}\frac{h^2}{\pi^2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta,$$

$$\left|\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}u\mathrm{d}x_2\mathrm{d}\eta\right|\leq\frac{\pi^2}{4h^2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\eta}^2\mathrm{d}x_2\mathrm{d}\eta+\frac{h^2}{\pi^2}\int_0^t\int_{L_z}\exp(-\omega\eta)u^2\mathrm{d}x_2\mathrm{d}\eta$$
$$\leq\frac{1}{4}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta+\frac{h^4}{\pi^4}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta.$$

If we choose $\epsilon_1=\frac{1}{4},\epsilon_2=4,\epsilon_3=\frac{1}{4},\epsilon_4=\frac{\lambda_1}{6},\epsilon_5=6,\epsilon_6=2,\epsilon_7=\frac{1}{6},\epsilon_8=\frac{1}{4}$, we have

$$
\begin{aligned}
\frac{\partial^2 E(z,t)}{\partial z^2} \geq & \left(\frac{\omega}{2} - \frac{3}{\lambda_1} - \frac{11}{2}\right)\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\alpha\beta} u_{,\alpha\beta} \mathrm{d}x_2 \mathrm{d}\eta \\
& + \left(\frac{\omega}{2} - 2 - \frac{h^2}{\pi^2} - \frac{h^4}{\pi^4}\right)\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\alpha} u_{,\alpha} \mathrm{d}x_2 \mathrm{d}\eta + \frac{1}{2}\int_{L_z} \exp(-\omega t) u_{,\alpha} u_{,\alpha} \mathrm{d}x_2 \\
& + \frac{1}{4}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\alpha\eta} u_{,\alpha\eta} \mathrm{d}x_2 \mathrm{d}\eta + \frac{1}{2}\int_{L_z} \exp(-\omega t) u_{,\alpha\beta} u_{,\alpha\beta} \mathrm{d}x_2 \\
& + \frac{1}{2}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\alpha\beta\eta} u_{,\alpha\beta\eta} \mathrm{d}x_2 \mathrm{d}\eta + \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha} u_{,1\alpha} \mathrm{d}x_2 \mathrm{d}\eta \\
& + \frac{1}{2}\int_{L_z} \exp(-\omega t) u_{,1\alpha} u_{,1\alpha} \mathrm{d}x_2 + (\omega - 11)\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha\beta} u_{,1\alpha\beta} \mathrm{d}x_2 \mathrm{d}\eta \\
& + \omega \int_{L_z} \exp(-\omega t) u_{,1\alpha\beta} u_{,1\alpha\beta} \mathrm{d}x_2 + \frac{\omega}{2}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha} u_{,1\alpha} \mathrm{d}x_2 \mathrm{d}\eta \\
& + \frac{1}{2}\int_{L_z} \exp(-\omega t) u_{,1\alpha} u_{,1\alpha} \mathrm{d}x_2 + \frac{1}{2}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha\eta} u_{,1\alpha\eta} \mathrm{d}x_2 \mathrm{d}\eta \\
& + \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha\beta\eta} u_{,1\alpha\beta\eta} \mathrm{d}x_2 \mathrm{d}\eta.
\end{aligned}
\tag{2.30}
$$

We choose ω large enough, we have

$$
\begin{aligned}
\frac{\partial^2 E(z,t)}{\partial z^2} \geq & \frac{\omega}{4}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\alpha\beta} u_{,\alpha\beta} \mathrm{d}x_2 \mathrm{d}\eta \\
& + \frac{\omega}{4}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\alpha} u_{,\alpha} \mathrm{d}x_2 \mathrm{d}\eta + \frac{1}{2}\int_{L_z} \exp(-\omega t) u_{,\alpha} u_{,\alpha} \mathrm{d}x_2 \\
& + \frac{1}{4}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\alpha\eta} u_{,\alpha\eta} \mathrm{d}x_2 \mathrm{d}\eta + \frac{1}{2}\int_{L_z} \exp(-\omega t) u_{,\alpha\beta} u_{,\alpha\beta} \mathrm{d}x_2 \\
& + \frac{1}{2}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\alpha\beta\eta} u_{,\alpha\beta\eta} \mathrm{d}x_2 \mathrm{d}\eta + \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha} u_{,1\alpha} \mathrm{d}x_2 \mathrm{d}\eta \\
& + \frac{1}{2}\int_{L_z} \exp(-\omega t) u_{,1\alpha} u_{,1\alpha} \mathrm{d}x_2 + \frac{\omega}{2}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha\beta} u_{,1\alpha\beta} \mathrm{d}x_2 \mathrm{d}\eta \\
& + \omega \int_{L_z} \exp(-\omega t) u_{,1\alpha\beta} u_{,1\alpha\beta} \mathrm{d}x_2 + \frac{\omega}{2}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha} u_{,1\alpha} \mathrm{d}x_2 \mathrm{d}\eta \\
& + \frac{1}{2}\int_{L_z} \exp(-\omega t) u_{,1\alpha} u_{,1\alpha} \mathrm{d}x_2 + \frac{1}{2}\int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha\eta} u_{,1\alpha\eta} \mathrm{d}x_2 \mathrm{d}\eta \\
& + \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,1\alpha\beta\eta} u_{,1\alpha\beta\eta} \mathrm{d}x_2 \mathrm{d}\eta.
\end{aligned}
\tag{2.31}
$$

Using the similar procedure, we can also obtain

$$
\frac{\partial^2 E(z,t)}{\partial z^2} \leq \omega \int_0^t \int_{L_z} \exp(-\omega\eta) u_{,\alpha\beta} u_{,\alpha\beta} \mathrm{d}x_2 \mathrm{d}\eta
$$

$$
\begin{aligned}
&+\omega\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2}\int_{L_z}\exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}x_2\\
&+\frac{3}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta+\frac{1}{2}\int_{L_z}\exp(-\omega t)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}x_2\\
&+\frac{3}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,\alpha\beta\eta}u_{,\alpha\beta\eta}\mathrm{d}x_2\mathrm{d}\eta+\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta\\
&+\frac{1}{2}\int_{L_z}\exp(-\omega t)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2+\frac{3\omega}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha\beta}u_{,1\alpha\beta}\mathrm{d}x_2\mathrm{d}\eta \qquad (2.32)\\
&+\omega\int_{L_z}\exp(-\omega t)u_{,1\alpha\beta}u_{,1\alpha\beta}\mathrm{d}x_2+\frac{\omega}{2}\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2\mathrm{d}\eta\\
&+\frac{1}{2}\int_{L_z}\exp(-\omega t)u_{,1\alpha}u_{,1\alpha}\mathrm{d}x_2+\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}x_2\mathrm{d}\eta\\
&+\int_0^t\int_{L_z}\exp(-\omega\eta)u_{,1\alpha\beta\eta}u_{,1\alpha\beta\eta}\mathrm{d}x_2\mathrm{d}\eta.
\end{aligned}
$$

Combining (2.1), (2.11), (2.25), and (2.31), we easily obtain

$$|E(z,t)|\le k\frac{\partial^2E(z,t)}{\partial z^2}, \qquad (2.33)$$

with k is a computable positive constant.

3. Phragmén-Lindelöf alternative results

In this section, we will derive the Phragmén-Lindelöf alternative results. We will discuss the two cases for $\frac{\partial}{\partial z}E(z,t)>0$ or $\frac{\partial}{\partial z}E(z,t)\le 0$.

Case 1: For any fixed $\bar{t}$, we suggest that there exists a $z_0\ge 0$ such that $\frac{\partial}{\partial z}E(z_0,\bar{t})>0$.

From (2.31), we have $\frac{\partial^2}{\partial z^2}E(z,t_0)\ge 0$. We thus obtain the result

$$\frac{\partial}{\partial z}E(z,\bar{t})\ge\frac{\partial}{\partial z}E(z_0,\bar{t})>0, \qquad (3.1)$$

for each $z\ge z_0$.

We can also obtain

$$E(z,\bar{t})\ge E(z_0,\bar{t})+\frac{\partial}{\partial z}E(z_0,\bar{t})(z-z_0), \qquad (3.2)$$

for all $z\ge z_0$.

From (3.2), we can obtain the result that $E(z,\bar{t})$ must eventually become positive.

Combining (3.1) and (3.2), we have the result that there must exist a $z_1\ge z_0$ such that $\frac{\partial}{\partial z}E(z_1,\bar{t})>0$ and $E(z_1,\bar{t})>0$.

Inequality (2.33) can be rewritten as

$$\left\{\mathrm{e}^{-\bar{k}z}\left[\frac{\partial}{\partial z}E(z,\bar{t})+\bar{k}E(z,\bar{t})\right]\right\}'\ge 0, \qquad (3.3)$$

or

$$\left\{e^{\bar{k}z}\left[\frac{\partial}{\partial z}E(z,\bar{t})-\bar{k}E(z,\bar{t})\right]\right\}'\geq 0, \tag{3.4}$$

with $\bar{k}=\frac{1}{\sqrt{k}}$.

Integrating (3.3) and (3.4), we have the following results:

$$\frac{\partial}{\partial z}E(z,\bar{t})+\bar{k}E(z,\bar{t})\geq e^{\bar{k}(z-z_0)}\left[\frac{\partial}{\partial z_1}E(z_1,\bar{t})+\bar{k}E(z_1,\bar{t})\right], \tag{3.5}$$

$$\frac{\partial}{\partial z}E(z,\bar{t})-\bar{k}E(z,\bar{t})\geq e^{-\bar{k}(z-z_0)}\left[\frac{\partial}{\partial z}E(z_1,\bar{t})-\bar{k}E(z_1,\bar{t})\right]. \tag{3.6}$$

We obtain for any $z\geq z_1$

$$\frac{\partial}{\partial z}E(z,\bar{t})\geq\frac{\partial}{\partial z}E(z_1,\bar{t})\frac{e^{\bar{k}(z-z_1)}+e^{-\bar{k}(z-z_1)}}{2}+\bar{k}E(z_1,\bar{t})\frac{e^{\bar{k}(z-z_1)}-e^{-\bar{k}(z-z_1)}}{2}. \tag{3.7}$$

Integrating (2.32) from z_1 to z, we have

$$\begin{aligned}
\frac{\partial E(z,t)}{\partial z}-\frac{\partial E(z_1,t)}{\partial z}\leq{}&\omega\int_0^t\int_{z_1}^z\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta\\
&+\omega\int_0^t\int_{z_1}^z\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\int_{z_1}^z\int_{L_\xi}\exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}A\\
&+\frac{3}{2}\int_0^t\int_{z_1}^z\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta+\frac{1}{2}\int_{z_1}^z\int_{L_\xi}\exp(-\omega t)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}A\\
&+\frac{3}{2}\int_0^t\int_{z_1}^z\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\beta\eta}u_{,\alpha\beta\eta}\mathrm{d}A\mathrm{d}\eta+\int_0^t\int_{z_1}^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta\\
&+\frac{1}{2}\int_{z_1}^z\int_{L_\xi}\exp(-\omega t)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A+\frac{3\omega}{2}\int_0^t\int_{z_1}^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha\beta}u_{,1\alpha\beta}\mathrm{d}A\mathrm{d}\eta\\
&+\omega\int_{z_1}^z\int_{L_\xi}\exp(-\omega t)u_{,1\alpha\beta}u_{,1\alpha\beta}\mathrm{d}A+\frac{\omega}{2}\int_0^t\int_{z_1}^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta\\
&+\frac{1}{2}\int_{z_1}^z\int_{L_\xi}\exp(-\omega t)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A+\int_0^t\int_{z_1}^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}A\mathrm{d}\eta\\
&+\int_0^t\int_{z_1}^z\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha\beta\eta}u_{,1\alpha\beta\eta}\mathrm{d}A\mathrm{d}\eta\\
={}&F(z,t).
\end{aligned} \tag{3.8}$$

Inserting (3.8) into (3.7), we obtain

$$\lim_{z\to\infty}\left\{e^{-\bar{k}z}F(z,\bar{t})\right\}\geq C_1(\bar{t}), \tag{3.9}$$

where $C_1(\bar{t})=\frac{1}{2}e^{-\bar{k}z_1}\left[\frac{\partial}{\partial z}E(z_1,\bar{t})+\bar{k}E(z_1,\bar{t})\right]$.

Case 2: For every $z\geq 0$, $\frac{\partial}{\partial z}E(z,\bar{t})\leq 0$, we suggest there exists a $z_0>0$, such that $E(z_0,\bar{t})\leq 0$.

Since $\frac{\partial}{\partial z}E(z,\bar{t}) \leq 0$, we obtain

$$E(z,\bar{t}) \leq E(z_0,\bar{t}), \tag{3.10}$$

for all $z \geq z_0$.

Form (2.33), we have

$$\frac{\partial}{\partial z}E(z,\bar{t}) - \frac{\partial}{\partial z}E(z_0,\bar{t}) \geq -\frac{1}{k}E(z_0,\bar{t})(z-z_0). \tag{3.11}$$

For z large enough, we have $\frac{\partial E}{\partial z}E(z,\bar{t})$ can not remain nonposotive, this is a contradict to $\frac{\partial}{\partial z}E(z,\bar{t}) \leq 0$.

We thus have the following result:

$$If \ \ \frac{\partial}{\partial z}E(z,\bar{t}) \leq 0, \ \ then \ \ E(z,\bar{t}) \geq 0, \ \ for \ \ all \ \ z \geq 0. \tag{3.12}$$

We now integrating (3.6) from 0 to z,

$$-\frac{\partial}{\partial z}E(z,\bar{t}) + \bar{k}E(z,\bar{t}) \leq C_2(\bar{t})\mathrm{e}^{-\bar{k}z}, \tag{3.13}$$

where $C_2(\bar{t}) = -\frac{\partial}{\partial z}E(0,\bar{t}) + \bar{k}E(0,\bar{t})$.

Since $\frac{\partial}{\partial z}E(z,\bar{t}) \geq 0$ and $E(z,\bar{t}) \geq 0$ for all $z \geq 0$, we have

$$E(z,\bar{t}) \ \text{ and } \ -\frac{\partial E}{\partial z}E(z,\bar{t}) \ \text{ decay exponentially as } \ z \to \infty.$$

From (2.31), we have

$$\begin{aligned}
-\frac{\partial}{\partial z}E(z,\bar{t}) \geq{}& \frac{\omega}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}A\mathrm{d}\eta \\
&+\frac{\omega}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha}u_{,\alpha}\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha}u_{,\alpha}\mathrm{d}A \\
&+\frac{1}{4}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\eta}u_{,\alpha\eta}\mathrm{d}A\mathrm{d}\eta + \frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,\alpha\beta}u_{,\alpha\beta}\mathrm{d}A \\
&+\frac{1}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,\alpha\beta\eta}u_{,\alpha\beta\eta}\mathrm{d}A\mathrm{d}\eta + \int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta \\
&+\frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A + \frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha\beta}u_{,1\alpha\beta}\mathrm{d}A\mathrm{d}\eta \\
&+\omega\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,1\alpha\beta}u_{,1\alpha\beta}\mathrm{d}A + \frac{\omega}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A\mathrm{d}\eta \\
&+\frac{1}{2}\int_z^\infty\int_{L_\xi}\exp(-\omega t)u_{,1\alpha}u_{,1\alpha}\mathrm{d}A + \frac{1}{2}\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha\eta}u_{,1\alpha\eta}\mathrm{d}A\mathrm{d}\eta \\
&+\int_0^t\int_z^\infty\int_{L_\xi}\exp(-\omega\eta)u_{,1\alpha\beta\eta}u_{,1\alpha\beta\eta}\mathrm{d}A\mathrm{d}\eta \\
={}& G(z,t).
\end{aligned} \tag{3.14}$$

Inserting (3.14) into (3.13), we have

$$G(z,\bar{t}) \leq C_2(\bar{t})e^{-\bar{k}z}. \tag{3.15}$$

Summarizing all the above results, we get the following results.

Theorem: If u is the classical solution of the initial boundary value problem (1.2)–(1.7), we can obtain the following results: either the energy function $F(z,t)$ defined in (3.8) satisfies

$$\lim_{z\to\infty}\left\{e^{-\bar{k}z}F(z,\bar{t})\right\} \geq C_1(\bar{t}), \tag{3.16}$$

or the energy function $G(z,t)$ defined in (3.14) satisfies

$$G(z,\bar{t}) \leq C_2(\bar{t})e^{-\bar{k}z}. \tag{3.17}$$

Our work presents significant results concerning the spatial growth and decay estimates of solutions to a particular equation, captured by inequalities (3.16) and (3.17), respectively. Inequality (3.16) demonstrates that the solution can grow exponentially as the distance from the entry section tends to infinity, while inequality (3.17) reveals that the solution can decay exponentially under the same conditions. These findings are analogous to the Saint-Venant principle and provide valuable insights into the behavior of solutions in the context of our study. A key contribution of this work is the establishment of these spatial growth and decay estimates, which are crucial for understanding the behavior of solutions in unbounded domains. These estimates have potential applications in various fields, such as engineering, physics, and beyond, where understanding the behavior of solutions to partial differential equations is essential. When exploring the quasi-static version of Eq (1.1), our primary focus is on the behavior of the equation under static or nearly static conditions. However, when considering the second-order derivative term of time, i.e., when the equation becomes dynamic, its complexity and difficulty in solving increase significantly. This is because, compared to the first-order derivative (typically representing velocity or rate of change), the second-order derivative (representing acceleration or the rate of change of the rate of change) introduces more dynamic characteristics and potential oscillatory behavior. In physics and engineering, controlling energy through second-order derivatives is indeed a challenge. Traditional energy methods often rely on first or lower-order derivatives to define and solve problems. Therefore, when the equation includes a second-order derivative term, we need to seek new mathematical tools and methods to effectively handle this dynamic behavior. The method of differential-integral inequalities provides us with a possible solution. This method combines the ideas of differentiation and integration, and considers inequality constraints, thereby enabling more flexible handling of complex equations containing higher-order derivatives. Through this method, we may be able to capture the dynamic behavior in the equation and find solutions that meet our needs. To further illustrate our findings, we will present a numerical simulation of the solution to this equation. This simulation will provide a visual representation of the solution and its behavior as the distance from the entry section varies. In addition to our current results, there are several promising directions for future research. One area of interest is the structural stability of the equation in an unbounded domain. Understanding the stability of solutions to this equation is crucial for developing robust and reliable numerical methods and for ensuring the accuracy of solutions in practical applications. We plan to investigate this topic in a subsequent paper, leveraging the insights and methods developed in our current work.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

The work was supported by Guangdong Natural Science foundation (Grant #2023A1515012044), Scientific research Foundation of Guangzhou Huashang College (Grant #2024HSTS09).

Conflict of interest

The authors declare there is no conflicts of interest.

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