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https://www.aimspress.com/journal/era

ERA, 32(11): 6225–6234.
DOI: 10.3934/era.2024289
Received: 12 September 2024
Revised: 24 October 2024
Accepted: 13 November 2024
Published: 18 November 2024

Research article

Upper and lower bounds for the blow-up time of a fourth-order parabolic equation with exponential nonlinearity

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Abstract: This paper investigated the blow-up properties of solutions to the initial value problem for a fourth-order nonlinear parabolic equation with an exponential source term. By using an improved concavity method, we obtained upper and lower bound estimates for the blow-up time of the solution.

Keywords: fourth-order parabolic equation; exponential source term; Cauchy problem; blow-up; upper and lower bounds for the blow-up time

1. Introduction

In this paper, we consider the Cauchy problem for the following fourth-order nonlinear parabolic equation:

$$\begin{cases} \partial_t u + \Delta^2 u + u = f(u), & (t, x) \in (0, T) \times \mathbb{R}^4, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^4, \end{cases} \tag{1.1}$$

where $u_0 \in H^2(\mathbb{R}^4)$, and $f(u) \in C^1(R, R)$ is an exponential nonlinearity and satisfies the following assumptions:

(A1) $f(0) = 0$.

(A2) There exists $\nu_0 > 0$ such that for some constant $C_\delta > 0$ and any $\delta > 0$, we have

$$|f(s_1) - f(s_2)| \le C_\delta |s_1 - s_2| (e^{\nu_0(1+\delta)s_1^2} + e^{\nu_0(1+\delta)s_2^2}), \qquad s_1, s_2 \in \mathbb{R}.$$

(A3) There exists $\vartheta > 0$ such that

$$uf(u) \ge (2 + \vartheta)F(u),$$

where $F(u) := \displaystyle\int_0^u f(\varsigma)d\varsigma$.

The fourth-order parabolic equation can describe various physical phenomena, such as phase transitions, thin film, and lubrication theories. Specifically, it describes the evolution process of nanoscale thin film epitaxial growth [1–6]. Currently, many authors have studied the initial-boundary value problems associated with fourth-order parabolic equations and proved the existence of global solutions and the blow-up behavior of solutions for this equation [7–9].

Ishiwata et al. [10] considered the initial value problem for the following nonlinear parabolic equation with exponential terms

$$\begin{cases} \partial_t u - \Delta u = \pm u(e^{u^2-1}), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^2. \end{cases} \tag{1.2}$$

They established the existence and uniqueness of a local solution in $H^1(\mathbb{R}^2)$ for problem (1.2) and demonstrated that the solution with negative energy blows up in finite time. Subsequently, Saanouni [11, 12] extended this result to $2n$-dimensional space and generalized the nonlinear term to a general exponential nonlinearity. The specific result is as follows: let $u_0 \in H^n(\mathbb{R}^{2n})$ and the nonlinear term f satisfies certain. Then, there exists a unique maximal solution $u \in \mathbb{C}([0,T^*), H^n(\mathbb{R}^{2n}))$, and when u_0 belongs to the unstable set, the solution blows up in finite time. In addition, Ishiwata et al. [13] studied the following Cauchy problem:

$$\begin{cases} \partial_t u = \Delta u - u + \lambda f(u), & (t,x) \in (0,T) \times \mathbb{R}^2, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^2, \end{cases} \tag{1.3}$$

where $\lambda > 0$ and $f(u) = 2\alpha_0 u e^{\alpha_0 u^2}$. Utilizing the contraction mapping principle, they obtained the local existence and uniqueness of solutions as $0 < \lambda < \frac{1}{2\alpha_0}$. Meanwhile, they gave the blow-up properties of the solutions by applying the concavity method. Wang and Qian [14] used an improved concavity method to prove the blow-up of solutions with arbitrarily high initial energy and provided upper and lower bounds for the blow-up time.

Han [8] discussed the initial-boundary value problem for the following fourth-order parabolic equation:

$$\begin{cases} u_t + \Delta^2 u = k(t) f(u), & (x,t) \in \Omega \times (0,T), \\ u = \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T), \\ u(x,0) = u_0, & x \in \Omega. \end{cases} \tag{1.4}$$

By using the differential inequalities, he proved that under specific initial value conditions, the solution to this problem blows up in finite time, and derived upper and lower bounds for the blow-up time. When $k(t) = 1$, $f(s) = |s|^{p-1}s$, $1 < p < 2^*-1$ for $n > 4$ and $1 < p < +\infty$ for $n \leq 4$, where $2^* = \frac{2n}{n-4}$. Besides, Philippin [15] gave an upper and a lower bound by using differential inequalities method when $f(u) = |u|^{p-1}u$. Han [7] studied the equation

$$u_t + \Delta^2 u - div(|\nabla u|^{p-2}\nabla u) = |u|^{q-1}u,$$

where the p-Laplacian diffusion term is present. He established the global well-posedness and finite-time blow-up of solutions to problem (1.4) by applying the potential well method, initially introduced by Payne and Sattinger [16] for studying the global existence of solutions for nonlinear hyperbolic equations, and further developed by others [17–20]. For the following fourth-order semilinear quasi-linear parabolic equation containing a strong damping term and a nonlocal source term,

$$u_t - \alpha\Delta u_t - \Delta u + \Delta^2 u = |u|^{p-1}u - \frac{1}{|\Omega|}\int_\Omega |u|^{p-1}udx. \tag{1.5}$$

Polat [21] obtained blow-up results for the solutions and showed lower bound estimates for the blow-up time.

Inspired by the above research, this paper considers the blow-up properties of solutions to the initial value problem for a fourth-order parabolic equation with exponential terms. By using an improved concavity method, we establish the upper bound for the blow-up time of the solution when the initial value u_0 belongs to the unstable set, i.e.,

$$T^* \leq 2^{\frac{3\vartheta+2}{2\vartheta}} \frac{\vartheta\left(\frac{C_1}{C_2}\right)^{\frac{2(1+\vartheta)}{\vartheta}}}{2\sqrt{C_1}}\left(1-\left(1+\left(\frac{C_1}{C_2}\right)^{\frac{2(1+\vartheta)}{\vartheta}}\psi(0)\right)^{-\frac{1}{\vartheta}}\right),$$

where the specific indicators of this will be given in Theorem 2. Simultaneously, when the initial value u_0 satisfies $\|\Delta u_0\|^2 \leq h$ and $\|u_0\|^2 \leq l$, where h and l are constants, we provide the lower bound for the blow-up time of the solution, i.e.,

$$T^* \geq \frac{(M+\frac{l}{2})^{-\delta^2}}{C_\delta C\delta^2}\left(\frac{32\pi^2}{\nu_0}(1-\delta)\right)^{\frac{1}{1+\delta^2}},$$

where the specific indicators of this will be given in Theorem 3. This complements the results in [12].

The structure of the paper is as follows. In Section 2, we give some preliminaries. Section 3 presents the upper bound for the blow-up time of problem (1.1). In Section 4, we focus on the lower bound for the blow-up time of problem (1.1).

2. Preliminaries

For simplicity, we use $\|\cdot\|_p$ and $\|\cdot\|$ to denote the norms in $L^p(\mathbb{R}^4)$ and $L^2(\mathbb{R}^4)$, respectively. The constant C appearing in this paper may vary from line to line.

Define the functionals in $H^2(\mathbb{R}^4))$ as follows:

$$K(u) := \frac{1}{2}\|\Delta u\|^2 + \frac{1}{2}\|u\|^2 - \int_{R^4} F(u)dx, \tag{2.1}$$

$$S(u) := \|\Delta u\|^2 + \|u\|^2 - \int_{R^4} uf(u)dx. \tag{2.2}$$

Let

$$m := \inf\left\{K(v) : v \in H^2(\mathbb{R}^4)\backslash\{0\}, S(v) = 0\right\}, \tag{2.3}$$

and define the stable and unstable sets as follows:

$$W := \left\{v \in H^2(\mathbb{R}^4) : K(v) < m, S(v) \geq 0\right\}, \tag{2.4}$$

$$V := \left\{ v \in H^2(\mathbb{R}^4) : K(v) < m, S(v) < 0 \right\}. \tag{2.5}$$

The maximal existence time of the solution $u(t, x)$ to problem (1.1) is defined as

$$T^* := \sup \left\{ T > 0 : u \in \mathbb{C}([0, T]; H^2(\mathbb{R}^4)) \right\} \in (0, +\infty].$$

Lemma 1 ([22]). *For any $\alpha \in (0, 32\pi^2)$, there exists $C(\alpha) > 0$ such that*

$$\int_{R^4} (e^{\alpha u^2} - 1)dx \le C(\alpha)\|u\|^2, \quad for\ any\ u \in H^2(\mathbb{R}^4)\ with\ \|\Delta u\| \le 1,$$

and the above inequality is false if $\alpha > 32\pi^2$.

Lemma 2 ([11]). *For any $t \in (0, T)$, we have*

$$\frac{\partial}{\partial t} K(u) = -\|\partial_t u\|^2, \tag{2.6}$$

$$\frac{1}{2}\frac{\partial}{\partial t}\|u\|^2 = -S(u). \tag{2.7}$$

Theorem 1 (Theorem 2.5, [12]). *Let $u \in \mathbb{C}([0, T^*), H^2(\mathbb{R}^4))$, f satisfies the assumptions (A1)–(A3), and $u_0 \in H^2(\mathbb{R}^4)$. If $u(t_0) \in V$ for some $t_0 \in [0, T^*)$, then the solution u blows up in the sense of the L^2 norm, i.e., $\lim_{t \to T^*} \|u(t)\| = +\infty$.*

3. Upper bound for the blow-up time

In this section, we will discuss the upper bound for the blow-up time of the solution. To prove the main result, we present the following lemma.

Lemma 3 (Lemma 4.2, [23]). *If $\psi(t)$ is a non-increasing function on $[0, +\infty]$ and satisfies*

$$[\psi'(t)]^2 \ge a + b\psi(t)^{2+\frac{1}{k}}, \qquad \forall t \ge 0,$$

where $a, b > 0$ are constants, then there exists a finite time $T^* > 0$ such that

$$\lim_{t \to T^{*-}} \psi(t) = 0,$$

where

$$T^* \le 2^{\frac{3k+1}{2k}} \frac{k(\frac{a}{b})^{2+\frac{1}{k}}}{\sqrt{a}} \Big(1 - \Big(1 + (\frac{a}{b})^{2+\frac{1}{k}}\psi(0)\Big)^{-\frac{1}{2k}}\Big).$$

Theorem 2. *Let $u \in \mathbb{C}([0, T^*), H^2(\mathbb{R}^4))$ and $u_0 \in H^2(\mathbb{R}^4)$. If $u(t_0) \in V$, for some $t_0 \in [0, T^*)$, then an upper bound for the blow-up time of the solution u is*

$$T^* \le 2^{\frac{3\vartheta+2}{2\vartheta}} \frac{\vartheta\Big(\frac{C_1}{C_2}\Big)^{\frac{2(1+\vartheta)}{\vartheta}}}{2\sqrt{C_1}} \Big(1 - \Big(1 + \Big(\frac{C_1}{C_2}\Big)^{\frac{2(1+\vartheta)}{\vartheta}} \psi(0)\Big)^{-\frac{1}{\vartheta}}\Big),$$

where

$$\psi(0) = (T^*\|u_0\|^2 + b\mu^2)^{-\frac{\vartheta}{2}}, C_1 = (\psi'(0))^2 - C_2(\psi(0))^{\frac{2(1+\vartheta)}{\vartheta}},$$

$$C_2 = [-2(2+\vartheta)(\alpha\gamma - \beta^2) + 2(2+\vartheta)K(u_0) + 2(1+\vartheta)b] \cdot \frac{\vartheta^2}{1+\vartheta}.$$

Proof. The blow-up of the solution is given in Theorem 1. Next, we will prove the upper bound for the blow-up time. We first define the auxiliary functional

$$Q(t) := \int_0^t \|u(s)\|^2 ds + (T^* - t)\|u_0\|^2 + b(t+\mu)^2, \qquad \forall t \in [0, T^*) \tag{3.1}$$

where $b > 0$ and $\mu > 0$.

By (3.1), we have

$$Q'(t) = \|u(t)\|^2 - \|u_0\|^2 + 2b(t+\mu) = \int_0^t \frac{d}{ds}\|u(s)\|^2 ds + 2b(t+\mu), \tag{3.2}$$

$$Q''(t) = \frac{d}{dt}\|u(t)\|^2 + 2b. \tag{3.3}$$

According to (3.3), Lemma 2, and assumption (A3), we get

$$\begin{aligned}
\frac{1}{2}Q''(t) =& \frac{1}{2}\frac{d}{dt}\|u(t)\|^2 + b = -S(u(t)) + b \\
=& -\|\Delta u(t)\|^2 - \|u(t)\|^2 + \int_{R^4} u(t) f(u(t)) dx + b \\
\geq& -\|\Delta u(t)\|^2 - \|u(t)\|^2 + \int_{R^4} (2+\vartheta) F(u(t)) dx + b \\
\geq& -(2+\vartheta) K(u(t)) + \frac{\vartheta}{2}(\|\Delta u(t)\|^2 + \|u(t)\|^2) + b \\
\geq& -(2+\vartheta) K(u(t)) + \frac{\vartheta}{2}\|u(t)\|^2 + b \\
=& -(2+\vartheta) K(u_0) + (2+\vartheta)\int_0^t \|\partial_s u(s)\|^2 ds + \frac{\vartheta}{2}\|u(t)\|^2 + b.
\end{aligned} \tag{3.4}$$

Choosing $b > (2+\vartheta)K(u_0)$, we get $Q''(t) > 0$ for any $t \in [0, T^*)$. Thus, $Q'(t)$ is monotonically increasing with respect to t on $[0, T^*)$. Since $Q'(0) = 2b\mu > 0$, it follows that $Q(t)$ is monotonically increasing on $[0, T^*)$. Furthermore, by $Q(0) = T^*\|u_0\|^2 + b\mu^2 > 0$, we have $Q(t) > 0$, $\forall t \in [0, T^*)$.

In addition, combining (3.1), (3.2), and (3.4), we obtain

$$Q(t) \geq \int_0^t \|u(s)\|^2 ds + b(t+\mu)^2 := \alpha, \tag{3.5}$$

$$Q'(t) = 2\left(\frac{1}{2}\int_0^t \frac{d}{ds}\|u(s)\|^2 ds + b(t+\mu)\right) := 2\beta, \tag{3.6}$$

$$\begin{aligned}
Q''(t) \geq& -2(2+\vartheta)K(u_0) + 2(2+\vartheta)\Big(\int_0^t \|\partial_s u(s)\|^2 ds + b\Big) - (2(2+\vartheta) - 2)b \\
:=& -2(2+\vartheta)K(u_0) + 2(2+\vartheta)\gamma - 2(1+\vartheta)b.
\end{aligned} \tag{3.7}$$

For any $z \in \mathbb{R}$, we have

$$\alpha z^2 - 2\beta z + \gamma \geq \int_0^t (\|z u(s)\| - \|\partial_s u(s)\|)^2 ds + b(z(t+\mu) - 1)^2 \geq 0,$$

thus $\alpha\gamma - \beta^2 \geq 0$.

From (3.5)–(3.7), we get

$$
\begin{aligned}
&Q(t)Q''(t) - \frac{2+\vartheta}{2}[Q'(t)]^2 \\
&\geq \alpha[-2(2+\vartheta)K(u_0) + 2(2+\vartheta)\gamma - 2(1+\vartheta)b] - \frac{2+\vartheta}{2}\cdot 4\beta^2 \\
&\geq 2(2+\vartheta)(\alpha\gamma - \beta^2) - (2(2+\vartheta)K(u_0) + 2(1+\vartheta)b)\, Q(t).
\end{aligned} \tag{3.8}
$$

Let

$$
\psi(t) := (Q(t))^{-\frac{\vartheta}{2}}, \tag{3.9}
$$

and then

$$
\psi'(t) = -\frac{\vartheta}{2}(Q(t))^{-\frac{2+\vartheta}{2}}Q'(t), \tag{3.10}
$$

$$
\begin{aligned}
\psi''(t) =& -\frac{\vartheta}{2}(Q(t))^{-\frac{2+\vartheta}{2}-1}[Q(t)Q''(t) - \frac{2+\vartheta}{2}(Q'(t))^2] \\
&\leq \vartheta(\psi(t))^{\frac{2+\vartheta}{2}}[-2(2+\vartheta)(\alpha\gamma - \beta^2) + 2(2+\vartheta)K(u_0) + 2(1+\vartheta)b].
\end{aligned} \tag{3.11}
$$

From $Q(t) > 0$ and $Q'(t) > 0$, we have $\psi'(t) < 0$. Multiplying both sides of (3.11) by $\psi'(t)$ and integrating from 0 to t, we obtain

$$
[\psi'(t)]^2 \geq C_1 + C_2(\psi(t))^{\frac{2(1+\vartheta)}{\vartheta}},
$$

where

$$
C_1 = (\psi'(0))^2 - C_2(\psi(0))^{\frac{2(1+\vartheta)}{\vartheta}},
$$

$$
C_2 = \vartheta[-2(2+\vartheta)(\alpha\gamma - \beta^2) + 2(2+\vartheta)K(u_0) + 2(1+\vartheta)b]\cdot\frac{\vartheta}{1+\vartheta} > 0.
$$

Next, we are going to prove $C_1 > 0$.

$$
\begin{aligned}
C_1 =&(\psi'(0))^2 - \vartheta\left[-2(2+\vartheta)(\alpha\gamma - \beta^2) + 2(2+\vartheta)K(u_0) + 2(1+\vartheta)b\right] \\
&\cdot\frac{\vartheta}{1+\vartheta}(\psi(0))^{\frac{2(1+\vartheta)}{\vartheta}} \\
=&(-\frac{\vartheta}{2})^2(T^*\|u_0\|^2 + b\mu^2)^{-(2+\vartheta)}(2b\mu)^2 - \vartheta[-2(2+\vartheta)(\alpha\gamma - \beta^2) \\
&+ 2(2+\vartheta)K(u_0) + 2(1+\vartheta)b]\cdot\frac{\vartheta}{1+\vartheta}(T^*\|u_0\|^2 + b\mu^2)^{-(1+\vartheta)} \\
=&\vartheta^2(T^*\|u_0\|^2 + b\mu^2)^{-(2+\vartheta)}[b^2\mu^2 - \frac{1}{1+\vartheta}(-2(2+\vartheta)(\alpha\gamma - \beta^2) \\
&+ 2(2+\vartheta)K(u_0) + 2(1+\vartheta)b)(T^*\|u_0\|^2 + b\mu^2)].
\end{aligned}
$$

Since $T^*\|u_0\|^2 + b\mu^2 > 0$, when μ is large enough, we have $C_1 > 0$.
Therefore, according to Lemma 3, we have

$$
T^* \leq 2^{\frac{3\vartheta+2}{2\vartheta}}\frac{\vartheta(\frac{C_1}{C_2})^{\frac{2(1+\vartheta)}{\vartheta}}}{2\sqrt{C_1}}\Big(1 - \Big(1 + \Big(\frac{C_1}{C_2}\Big)^{\frac{2(1+\vartheta)}{\vartheta}}\psi(0)\Big)^{-\frac{1}{\vartheta}}\Big).
$$

4. Lower bound for the blow-up time

In this section, we will establish a lower bound for the blow-up time of the solution to problem (1.1).

Theorem 3. *Let* $u \in \mathbb{C}([0,T^*), H^2(\mathbb{R}^4))$*,* $f \in \mathbb{C}^2(\mathbb{R},\mathbb{R})$ *satisfies assumptions (A1) and (A2), and* $u_0 \in H^2(\mathbb{R}^4)$*. Let* $0 < h < \frac{32\pi^2}{\nu_0}$ *and* $l > 0$*. If* u_0 *satisfies* $\|\Delta u_0\|^2 \leq h$ *and* $\|u_0\|^2 \leq l$*, then the lower bound for the blow-up time is given by*

$$T^* \geq \frac{(M+\frac{l}{2})^{-\delta^2}}{C_\delta C\delta^2}\Big(\frac{32\pi^2}{\nu_0}(1-\delta)\Big)^{\frac{1}{1+\delta^2}}.$$

Proof. Let

$$\varphi(t) := \frac{1}{2}\int_{R^4} u^2(x,t)dx. \tag{4.1}$$

From Theorem 1, we have

$$\lim_{t\to T^*}\varphi(t) = +\infty. \tag{4.2}$$

According to (A1), (A2), and (4.1), we obtain

$$\begin{aligned}
\varphi^{'}(t) =&\frac{1}{2}\frac{d}{dt}\int_{R^4} u^2(x,t)dx = -\|\Delta u(t)\|^2 - \|u(t)\|^2 + \int_{R^4} u(t)f(u(t))dx\\
\leq& \int_{R^4} u(t)f(u(t))dx \leq \int_{R^4} |u(t)f(u(t))|dx \leq C_\delta \int_{R^4} u^2(e^{\nu_0(1+\delta)u^2(s)}+1)dx\\
\leq& C_\delta \int_{R^4} u^2(e^{\nu_0(1+\delta)u^2(s)}-1)dx + C_\delta\int_{R^4} 2u^2dx\\
=&2C_\delta\|u\|^2 + C_\delta\int_{R^4} u^2(e^{\nu_0(1+\delta)u^2(s)}-1)dx.
\end{aligned}$$

We write (1.1) as the equivalent integral form

$$u(t) = e^{-\Delta^2 t}u_0 + \int_0^t e^{-(t-s)\Delta^2}(f(u(s)) - u(s))ds. \tag{4.3}$$

From $0 < h < \frac{32\pi^2}{\nu_0}$, we know that there exists $\delta > 0$ such that $h = \frac{32\pi^2}{\nu_0}(1-\delta)$.
Defining the following set

$$\Gamma = \Big\{u \in L^\infty((0,T), H^2(\mathbb{R}^4)) : \sup_{t\in[0,T]}\|\Delta u(t)\|^2 \leq \frac{32\pi^2}{\nu_0}(1-\frac{\delta}{2}), \sup_{t\in[0,T]}\|u(t)\|^2 \leq 2l\Big\}. \tag{4.4}$$

According to the Hölder inequality, we have

$$\int_{R^4} u^2(e^{\nu_0(1+\delta)u^2(s)}-1)dx \leq \|u(t)\|_{2p}^2\left(\int_{R^4}(e^{\nu_0 q(1+\delta)u^2(s)}-1)dx\right)^{\frac{1}{q}},$$

where $p, q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$. Let $q = 1+\delta^2$, and then by the Trudinger-Moser inequality, we can obtain

$$\begin{aligned}&\Big(\int_{R^4}(e^{\nu_0 q(1+\delta)u^2(s)} - 1)dx\Big)^{\frac{1}{q}} = \Big(\int_{R^4}\Big(e^{\frac{32\pi^2}{h}(1-\delta)(1+\delta)^2(1+\delta)u^2(s)} - 1\Big)dx\Big)^{\frac{1}{1+\delta^2}}\\ =&\Big(\int_{R^4}\Big(e^{32\pi^2(1-\delta^4)(\frac{u(s)}{\sqrt{h}})^2} - 1\Big)dx\Big)^{\frac{1}{1+\delta^2}} \le C\Big\|\frac{u(s)}{\sqrt{h}}\Big\|^{\frac{2}{1+\delta^2}} \le C\Big(\frac{32\pi^2}{\nu_0}(1-\delta)\Big)^{-\frac{1}{1+\delta^2}}\|u\|^2.\end{aligned} \tag{4.5}$$

From the Gagliardo-Nirenberg inequality for $q \ge 2$, $\|u\|^q_{L^q} \le C\|\Delta u\|^{q-2}\|u\|^2$, and we know

$$\|u(t)\|^2_{2p} \le C\|\Delta u(t)\|^{\frac{2}{1+\delta^2}}\|u\|^{\frac{2\delta^2}{1+\delta^2}} \le C\|u\|^{2\delta^2}. \tag{4.6}$$

Combining (4.5) and (4.6), we have

$$\int_{R^4} u^2(e^{\nu_0(1+\delta)u^2(s)} - 1)dx \le C\|u\|^{2(1+\delta^2)}\Big(\frac{32\pi^2}{\nu_0}(1-\delta)\Big)^{-\frac{1}{1+\delta^2}}. \tag{4.7}$$

Therefore

$$\begin{aligned}\varphi'(t) \le& C_\delta\Big(2\cdot 2l + C\Big(\frac{32\pi^2}{\nu_0}(1-\delta)\Big)^{-\frac{1}{1+\delta^2}}\|u\|^{2(1+\delta^2)}\Big)\\ \le& C_\delta\Big(4l + C\Big(\frac{32\pi^2}{\nu_0}(1-\delta)\Big)^{-\frac{1}{1+\delta^2}}(\varphi(t))^{1+\delta^2}\Big)\\ \le& C_\delta C\Big(\frac{32\pi^2}{\nu_0}(1-\delta)\Big)^{-\frac{1}{1+\delta^2}}\Big(M + \varphi(t)\Big)^{1+\delta^2},\end{aligned}$$

where $M = \frac{4l}{C}\Big(\frac{32\pi^2}{\nu_0}(1-\delta)\Big)^{\frac{1}{1+\delta^2}}$.

Thus, we get

$$\varphi'(t)\Big[C_\delta C\Big(\frac{32\pi^2}{\nu_0}(1-\delta)\Big)^{-\frac{1}{1+\delta^2}}(M + \varphi(t))^{1+\delta^2}\Big]^{-1} \le 1. \tag{4.8}$$

Integrating both sides of (4.8) with respect to t and letting $\theta = \varphi(s)$, we obtain

$$t \ge \frac{1}{C_\delta C}\Big(\frac{32\pi^2}{\nu_0}(1-\delta)\Big)^{\frac{1}{1+\delta^2}}\int_{\varphi(0)}^{\varphi(t)}\Big(M + \theta\Big)^{-(1+\delta^2)}d\theta. \tag{4.9}$$

Taking the limit as $t \to T^*$ on both sides of (4.9) and combining with (4.2), we acquire

$$\begin{aligned}T^* \ge& \frac{1}{C_\delta C}\Big(\frac{32\pi^2}{\nu_0}(1-\delta)\Big)^{\frac{1}{1+\delta^2}}\int_{\varphi(0)}^{+\infty}\Big(M + \theta\Big)^{-(1+\delta^2)}d\theta\\ =& \frac{1}{C_\delta C\delta^2}\Big(\frac{32\pi^2}{\nu_0}(1-\delta)\Big)^{\frac{1}{1+\delta^2}}\Big(M + \frac{\|u_0\|^2}{2}\Big)^{-\delta^2}\\ \ge& \frac{\Big(M + \frac{l}{2}\Big)^{-\delta^2}}{C_\delta C\delta^2}\Big(\frac{32\pi^2}{\nu_0}(1-\delta)\Big)^{\frac{1}{1+\delta^2}}.\end{aligned} \tag{4.10}$$

The proof of Theorem 3 is complete.

5. Conclusions

In this paper, we study the blow-up properties of solutions to the initial value problem for a fourth-order parabolic equation with exponential terms. By using an improved concavity method, we establish the upper bound for the blow-up time of the solution when the initial value u_0 belongs to the unstable set. Simultaneously, when the initial value u_0 satisfies $||\Delta u_0||^2 \leq h$ and $||u_0||^2 \leq l$, where h and l are constants, we provide the lower bound for the blow-up time of the solution.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Conflict of interest

The authors declare there are no conflicts of interest.

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