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Research article

The criteria for automorphisms on finite-dimensional algebras

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Abstract: In this paper, we will establish a criterion for automorphisms of finite-dimensional algebras. As an application, we will describe all automorphisms of the single-parameter generalized quaternion algebra. Additionally, we will obtain all automorphisms of Sweedler's 4-dimensional Hopf algebra.

Keywords: Hopf algebra; matrix; automorphism; quaternion

1. Introduction

The study of algebraic automorphisms on different algebraic systems is a classic direction in algebra. Usually, it is very difficult to determine the automorphisms of an algebra. How to describe the automorphisms in an algebra is still an open problem. A well-studied example is the automorphism group of an incidence algebra [1, 2]. Andruskiewitsch and Dumas studied the algebra automorphisms and Hopf algebra automorphisms of the positive part of the quantum enveloping algebra of simple complex finite-dimensional Lie algebras in [3]. For more works on the algebra automorphisms of other algebras, please refer to [4–10].

The purpose of this paper is to find an effective method to determine the automorphisms of finite-dimensional algebras. Using the method, we not only describe all automorphisms of low-dimensional algebras, but also identify some good automorphisms of high-dimensional algebras.

The paper is organized as follows:

In Section 2, we establish a criterion for automorphisms of finite-dimensional algebras. In Section 3, as an application, we give all automorphisms of the single-parameter generalized quaternion algebra. As special cases of the single-parameter generalized quaternion algebra, all automorphisms on the semi-quaternion algebra and the split semi-quaternion algebra are given. Since Sweedler's 4-dimensional Hopf algebra, as an algebra, is a split semi-quaternion algebra, all algebraic automorphisms in Sweedler's 4-dimensional Hopf algebra are described.

2. The criteria for automorphisms on finite-dimensional algebras

Throughout the paper, $\mathbb{R}$ denotes the real number field. All algebras are over $\mathbb{R}$, and linear refers to $\mathbb{R}$-linear. Given a matrix M, M^T denotes the transpose of M.

Let

$$\eta_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \eta_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \cdots, \eta_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}$$

be the standard basis of $\mathbb{R}^n$.

Let A be a finite-dimensional algebra with unit 1 and generators $g_1, g_2, \cdots, g_s$ which are subject to certain relationships. Assume that $\{\alpha_1 = 1, \alpha_2, \cdots, \alpha_n\}$ is a basis for A. Using the relationships among the g_i, we have

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} (\alpha_1, \alpha_2, \cdots, \alpha_n) = \mathbf{U}_1\alpha_1 + \mathbf{U}_2\alpha_2 + \cdots + \mathbf{U}_n\alpha_n, \tag{2.1}$$

where each $\mathbf{U_i}$ is a $n \times n$ digital matrix. By dividing $\mathbf{U}_i$ into block matrices by the columns, we obtain

$$\mathbf{U}_i = ({}_i\gamma_1, {}_i\gamma_2, \cdots, {}_i\gamma_n), \ {}_i\gamma_j = \begin{pmatrix} {}_iu_{1j} \\ {}_iu_{2j} \\ \vdots \\ {}_iu_{nj} \end{pmatrix}.$$

Construct the following matrices

$$\mathbf{W}_i = ({}_1\gamma_i, {}_2\gamma_i, \cdots, {}_n\gamma_i), i = 1, 2, \cdots, n.$$

Definition 2.1. With the matrices $\mathbf{U}_i (i = 1, 2, \cdots, n)$ as above. We call $\mathbf{W}_i (i = 1, 2, \cdots, n)$ the matrix induced from the i-th columns of $\{U_j\}_{j=1}^n$.

Lemma 2.2. *Let $\mathcal{P}$ be a linear transformation on A, and*

$$P = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = (\xi_1, \xi_2, \cdots, \xi_n),$$

the matrix of $\mathcal{P}$ with respect to the basis $\alpha_1, \alpha_2, \cdots, \alpha_n$. Then we have

$$\mathcal{P}(\alpha_i)\mathcal{P}(\alpha_j) = (\alpha_1, \alpha_2, \cdots, \alpha_n) \begin{pmatrix} \xi_j & & & \\ & \xi_j & & \\ & & \ddots & \\ & & & \xi_j \end{pmatrix}^T \mathbf{C}^T \xi_i, \tag{2.2}$$

for all $i, j = 1, 2, \cdots, n$, where

$$\mathbf{C} = (\mathbf{U}_1, \mathbf{U}_2, \cdots, \mathbf{U}_n).$$

Proof. For all i, j, since

$$\begin{aligned}
\mathcal{P}(\alpha_i)\mathcal{P}(\alpha_j) &= \xi_i^T \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} (\alpha_1, \alpha_2, \cdots, \alpha_n)\xi_j \\
&= \xi_i^T \left(\mathbf{U}_1\alpha_1 + \mathbf{U}_2\alpha_2 + \cdots + \mathbf{U}_n\alpha_n\right)\xi_j \\
&= \xi_i^T \mathbf{U}_1 \xi_j \alpha_1 + \xi_i^T \mathbf{U}_2 \xi_j \alpha_2 + \cdots + \xi_i^T \mathbf{U}_n \xi_j \alpha_n \\
&= \xi_i^T \mathbf{C} \begin{pmatrix} \xi_j & & & \\ & \xi_j & & \\ & & \ddots & \\ & & & \xi_j \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix},
\end{aligned}$$

it follows that (2.2) holds.

Example 2.3. Recall that a single-parameter quaternion q is an expression of the form

$$q = a_0 + a_1 e_1 + a_2 e_2 + a_3 e_3,$$

where a_0, a_1, a_2, a_3 are real numbers and e_1, e_2, e_3 satisfy the following equalities:

$$e_1^2 = -\mu, e_2^2 = 0, e_3^2 = 0, e_1 e_2 = e_3 = -e_2 e_1, e_2 e_3 = 0 = -e_3 e_2, e_3 e_1 = \mu e_2 = -e_1 e_3,$$

where $0 \neq \mu \in \mathbb{R}$. The set of single-parameter quaternions is denoted by $\mathbb{H}_\mu$ [8], and $\mathbb{H}_\mu$ is an associative algebra. We call $\mathbb{H}_\mu$ an algebra of single-parameter quaternions (or single-parameter quaternion algebra).

Observe that $\mathbb{H}_\mu$ is a 4-dimensional algebra with basis $1, e_1, e_2, e_3$. By computing

$$\begin{aligned}
\begin{pmatrix} 1 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} (1, e_1, e_2, e_3) &= \begin{pmatrix} 1 & e_1 & e_2 & e_3 \\ e_1 & -\mu & e_3 & -\mu e_2 \\ e_2 & -e_3 & 0 & 0 \\ e_3 & \mu e_2 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} 1 + \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} e_1 \\
&\quad + \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\mu \\ 1 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \end{pmatrix} e_2 + \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} e_3.
\end{aligned}$$

We have the corresponding $\mathbf{U}_i (i = 1, 2, 3, 4)$ as follows:

$$\mathbf{U}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\mu & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{U}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{U}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\mu \\ 1 & 0 & 0 & 0 \\ 0 & \mu & 0 & 0 \end{pmatrix},$$

$$\mathbf{U}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we have

$$\mathbf{C} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & -\mu & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let $\mathcal{P}$ be a linear transformation on $\mathbb{H}_\mu$, and

$$P = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix} = (\xi_1, \xi_2, \xi_3, \xi_4),$$

the matrix of $\mathcal{P}$ with respect to the basis $\alpha_1 = 1, \alpha_2 = e_1, \alpha_3 = e_2, \alpha_4 = e_3$. For instance, we aim to compute $\mathcal{P}(e_1)\mathcal{P}(e_2)$, i.e., $\mathcal{P}(\alpha_2)\mathcal{P}(\alpha_3)$. Since

$$\begin{pmatrix} a_{13} & 0 & 0 & 0 \\ a_{23} & 0 & 0 & 0 \\ a_{33} & 0 & 0 & 0 \\ a_{43} & 0 & 0 & 0 \\ 0 & a_{13} & 0 & 0 \\ 0 & a_{23} & 0 & 0 \\ 0 & a_{33} & 0 & 0 \\ 0 & a_{43} & 0 & 0 \\ 0 & 0 & a_{13} & 0 \\ 0 & 0 & a_{23} & 0 \\ 0 & 0 & a_{33} & 0 \\ 0 & 0 & a_{43} & 0 \\ 0 & 0 & 0 & a_{13} \\ 0 & 0 & 0 & a_{23} \\ 0 & 0 & 0 & a_{33} \\ 0 & 0 & 0 & a_{43} \end{pmatrix}^T \mathbf{C}^T \begin{pmatrix} a_{12} \\ a_{22} \\ a_{32} \\ a_{42} \end{pmatrix} = \begin{pmatrix} a_{12}a_{13} - a_{22}a_{23}\mu \\ a_{13}a_{22} + a_{12}a_{23} \\ -a_{22}a_{43}\mu + a_{23}a_{42}\mu + a_{13}a_{32} + a_{12}a_{33} \\ -a_{23}a_{32} + a_{22}a_{33} + a_{13}a_{42} + a_{12}a_{43} \end{pmatrix}$$

by Lemma 2.2, we have

$$\mathcal{P}(e_1)\mathcal{P}(e_2) = (1, e_1, e_2, e_3) \begin{pmatrix} a_{12}a_{13} - a_{22}a_{23}\mu \\ a_{13}a_{22} + a_{12}a_{23} \\ -a_{22}a_{43}\mu + a_{23}a_{42}\mu + a_{13}a_{32} + a_{12}a_{33} \\ -a_{23}a_{32} + a_{22}a_{33} + a_{13}a_{42} + a_{12}a_{43} \end{pmatrix}.$$

Theorem 2.4. *With notation as shown in Lemma 2.2. Then $\mathcal{P}$ is an automorphism if and only if $\xi_1 = \eta_1$ and the matrix P is an invertible matrix and satisfies the following equation:*

$$(\mathbf{W}_1, \mathbf{W}_2, \cdots, \mathbf{W}_n)\begin{pmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & P \end{pmatrix}^T = P^T\mathbf{C}\begin{pmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & P \end{pmatrix}\mathbf{K}, \tag{2.3}$$

where $\mathbf{C}$ *is shown in Lemma 2.2 and*

$$\mathbf{K} = \begin{pmatrix} \eta_1 & 0 & \cdots & 0 & \eta_2 & 0 & \cdots & 0 & \cdots & \eta_n & 0 & \cdots & 0 \\ 0 & \eta_1 & \cdots & 0 & 0 & \eta_2 & \cdots & 0 & \cdots & 0 & \eta_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta_1 & 0 & 0 & \cdots & \eta_2 & \cdots & 0 & 0 & \cdots & \eta_n \end{pmatrix}.$$

Proof. From (2.1), it follows that

$$\alpha_i\alpha_j = (\alpha_1, \alpha_2, \cdots, \alpha_n)\begin{pmatrix} {}_1u_{ij} \\ {}_2u_{ij} \\ \vdots \\ {}_nu_{ij} \end{pmatrix}, \forall i, j = 1, 2, \cdots, n. \tag{2.4}$$

For a fixed j, by using (2.4), we have

$$\mathcal{P}(\alpha_i\alpha_j) = (\alpha_1, \alpha_2, \cdots, \alpha_n)P\begin{pmatrix} {}_1u_{ij} \\ {}_2u_{ij} \\ \vdots \\ {}_nu_{ij} \end{pmatrix}.$$

Since $\mathcal{P}(\alpha_i\alpha_j) = \mathcal{P}(\alpha_i)\mathcal{P}(\alpha_j)$, for all i, it follows that

$$\begin{pmatrix} \xi_j & & & \\ & \xi_j & & \\ & & \ddots & \\ & & & \xi_j \end{pmatrix}^T \mathbf{C}^T\xi_i = P\begin{pmatrix} {}_1u_{ij} \\ {}_2u_{ij} \\ \vdots \\ {}_nu_{ij} \end{pmatrix},$$

which is equivalent to

$$\begin{pmatrix} \xi_j & & & \\ & \xi_j & & \\ & & \ddots & \\ & & & \xi_j \end{pmatrix}^T \mathbf{C}^T P = P\begin{pmatrix} {}_1u_{1j} & {}_1u_{2j} & \cdots & {}_1u_{nj} \\ {}_2u_{1j} & {}_2u_{2j} & \cdots & {}_2u_{nj} \\ \vdots & \vdots & \vdots & \vdots \\ {}_nu_{1j} & {}_nu_{2j} & \cdots & {}_nu_{nj} \end{pmatrix} = P\mathbf{W}_j^T.$$

Thus

$$
\begin{aligned}
&(\mathbf{W}_1, \mathbf{W}_2, \cdots, \mathbf{W}_n)\begin{pmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & P \end{pmatrix}^T \\
= \quad & P^T\mathbf{C}\begin{pmatrix} \xi_1 & 0 & \cdots & 0 & \xi_2 & 0 & \cdots & 0 & \cdots & \xi_n & 0 & \cdots & 0 \\ 0 & \xi_1 & \cdots & 0 & 0 & \xi_2 & \cdots & 0 & \cdots & 0 & \xi_n & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \xi_1 & 0 & 0 & \cdots & \xi_2 & \cdots & 0 & 0 & \cdots & \xi_n \end{pmatrix} \\
= \quad & P^T\mathbf{C}\begin{pmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & P \end{pmatrix}\mathbf{K}.
\end{aligned}
$$

From $\mathcal{P}(1) = 1$, we obtain $\xi_1 = \eta_1$. The proof is completed.

Example 2.5. Let $\mathbf{U}_i(i = 1, 2, 3, 4)$ and $\mathbf{C}$ be given in Example 2.3. By Definition 2.1, we can obtain the desired $\mathbf{W}_i(i = 1, 2, 3, 4)$ as follows:

$$
\mathbf{W}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathbf{W}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & \mu & 0 \end{pmatrix}, \mathbf{W}_3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

$$
\mathbf{W}_4 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\mu & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

Thus, we have

$$
(\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\mu & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}
$$

and

$$\mathbf{K} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

3. Application to $\mathbb{H}_\mu$

In this section, we consider the application of Theorem 2.4 to $\mathbb{H}_\mu$. To describe all automorphisms on $\mathbb{H}_\mu$, we need to determine the matrices P that satisfy the condition (2.3). Let

$$P = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}.$$

Since

$$\begin{aligned} & \text{LHS of (2.3)} \\ = \ & (\mathbf{W}_1, \mathbf{W}_2, \mathbf{W}_3, \mathbf{W}_4) \begin{pmatrix} P & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix}^T \\ = \ & \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & -\mu & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -\mu & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \mu & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
&\begin{pmatrix}
1 & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & a_{32} & a_{33} & a_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & a_{42} & a_{43} & a_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & a_{32} & a_{33} & a_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & a_{42} & a_{43} & a_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{32} & a_{33} & a_{34} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{42} & a_{43} & a_{44} & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a_{12} & a_{13} & a_{14}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{22} & a_{23} & a_{24}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{32} & a_{33} & a_{34}\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{42} & a_{43} & a_{44}
\end{pmatrix}^T\\
= \;& (\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3, \mathbf{M}_4),
\end{aligned}
$$

where

$$
\mathbf{M}_1 = \begin{pmatrix} 1 & 0 & 0 & 0\\ a_{12} & a_{22} & a_{32} & a_{42}\\ a_{13} & a_{23} & a_{33} & a_{43}\\ a_{14} & a_{24} & a_{34} & a_{44} \end{pmatrix}, \mathbf{M}_2 = \begin{pmatrix} a_{12} & a_{22} & a_{32} & a_{42}\\ -\mu & 0 & 0 & 0\\ -a_{14} & -a_{24} & -a_{34} & -a_{44}\\ a_{13}\mu & a_{23}\mu & a_{33}\mu & a_{43}\mu \end{pmatrix},
$$

$$
\mathbf{M}_3 = \begin{pmatrix} a_{13} & a_{23} & a_{33} & a_{43}\\ a_{14} & a_{24} & a_{34} & a_{44}\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{M}_4 = \begin{pmatrix} a_{14} & a_{24} & a_{34} & a_{44}\\ a_{13}(-\mu) & a_{23}(-\mu) & a_{33}(-\mu) & a_{43}(-\mu)\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{pmatrix},
$$

and

$$
\begin{aligned}
&\text{RHS of (2.3)}\\
= \;& P^T\mathbf{C}\begin{pmatrix} P & 0 & 0 & 0\\ 0 & P & 0 & 0\\ 0 & 0 & \ddots & 0\\ 0 & 0 & 0 & P \end{pmatrix}\mathbf{K}\\
= \;& \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14}\\ 0 & a_{22} & a_{23} & a_{24}\\ 0 & a_{32} & a_{33} & a_{34}\\ 0 & a_{42} & a_{43} & a_{44} \end{pmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1\\ 0 & -\mu & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -\mu & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mu & 0 & 1 & 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$
\begin{pmatrix}
1 & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{32} & a_{33} & a_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a_{42} & a_{43} & a_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_{32} & a_{33} & a_{34} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a_{42} & a_{43} & a_{44} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a_{12} & a_{13} & a_{14} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{22} & a_{23} & a_{24} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{32} & a_{33} & a_{34} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{42} & a_{43} & a_{44} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & a_{12} & a_{13} & a_{14} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{22} & a_{23} & a_{24} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{32} & a_{33} & a_{34} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{42} & a_{43} & a_{44}
\end{pmatrix}
$$

$$
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

$$
= \quad (\mathbf{N}_1, \mathbf{N}_2, \mathbf{N}_3, \mathbf{N}_4),
$$

where

$$
\mathbf{N}_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
a_{12} & a_{22} & a_{32} & a_{42} \\
a_{13} & a_{23} & a_{33} & a_{43} \\
a_{14} & a_{24} & a_{34} & a_{44}
\end{pmatrix},
$$

$$
\mathbf{N}_2 = \begin{pmatrix}
a_{12} & a_{22} & a_{32} & a_{42} \\
a_{12}^2 - a_{22}^2\mu & 2a_{12}a_{22} & 2a_{12}a_{32} & 2a_{12}a_{42} \\
a_{12}a_{13} - a_{22}a_{23}\mu & a_{13}a_{22} + a_{12}a_{23} & -a_{23}a_{42}\mu + a_{22}a_{43}\mu + a_{13}a_{32} + a_{12}a_{33} & a_{23}a_{32} - a_{22}a_{33} + a_{13}a_{42} + a_{12}a_{43} \\
a_{12}a_{14} - a_{22}a_{24}\mu & a_{14}a_{22} + a_{12}a_{24} & -a_{24}a_{42}\mu + a_{22}a_{44}\mu + a_{14}a_{32} + a_{12}a_{34} & a_{24}a_{32} - a_{22}a_{34} + a_{14}a_{42} + a_{12}a_{44}
\end{pmatrix},
$$

$$\mathbf{N}_3 = \begin{pmatrix} a_{13} & a_{23} & a_{33} & a_{43} \\ a_{12}a_{13} - a_{22}a_{23}\mu & a_{13}a_{22} + a_{12}a_{23} & -a_{22}a_{43}\mu + a_{23}a_{42}\mu + a_{13}a_{32} + a_{12}a_{33} & -a_{23}a_{32} + a_{22}a_{33} + a_{13}a_{42} + a_{12}a_{43} \\ a_{13}^2 - a_{23}^2\mu & 2a_{13}a_{23} & 2a_{13}a_{33} & 2a_{13}a_{43} \\ a_{13}a_{14} - a_{23}a_{24}\mu & a_{14}a_{23} + a_{13}a_{24} & -a_{24}a_{43}\mu + a_{23}a_{44}\mu + a_{14}a_{33} + a_{13}a_{34} & a_{24}a_{33} - a_{23}a_{34} + a_{14}a_{43} + a_{13}a_{44} \end{pmatrix},$$

$$\mathbf{N}_4 = \begin{pmatrix} a_{14} & a_{24} & a_{34} & a_{44} \\ a_{12}a_{14} - a_{22}a_{24}\mu & a_{14}a_{22} + a_{12}a_{24} & -a_{22}a_{44}\mu + a_{24}a_{42}\mu + a_{14}a_{32} + a_{12}a_{34} & -a_{24}a_{32} + a_{22}a_{34} + a_{14}a_{42} + a_{12}a_{44} \\ a_{13}a_{14} - a_{23}a_{24}\mu & a_{14}a_{23} + a_{13}a_{24} & -a_{23}a_{44}\mu + a_{24}a_{43}\mu + a_{14}a_{33} + a_{13}a_{34} & -a_{24}a_{33} + a_{23}a_{34} + a_{14}a_{43} + a_{13}a_{44} \\ a_{14}^2 - a_{24}^2\mu & 2a_{14}a_{24} & 2a_{14}a_{34} & 2a_{14}a_{44} \end{pmatrix},$$

we have $\mathbf{M}_i = \mathbf{N}_i (i = 1, 2, 3, 4)$, and obtain the following system of equations:

$$a_{22}^2\mu - a_{12}^2 - \mu = 0, \tag{a1}$$

$$-2a_{12}a_{22} = 0, \tag{a2}$$

$$-2a_{12}a_{32} = 0, \tag{a3}$$

$$-2a_{12}a_{42} = 0, \tag{a4}$$

$$a_{22}a_{23}\mu - a_{12}a_{13} + a_{14} = 0, \tag{a5}$$

$$-a_{13}a_{22} - a_{12}a_{23} + a_{24} = 0, \tag{a6}$$

$$-a_{23}a_{42}\mu + a_{22}a_{43}\mu - a_{13}a_{32} - a_{12}a_{33} + a_{34} = 0, \tag{a7}$$

$$a_{23}a_{32} - a_{22}a_{33} - a_{13}a_{42} - a_{12}a_{43} + a_{44} = 0, \tag{a8}$$

$$a_{13}(-\mu) + a_{22}a_{24}\mu - a_{12}a_{14} = 0, \tag{a9}$$

$$-a_{23}\mu - a_{14}a_{22} - a_{12}a_{24} = 0, \tag{a10}$$

$$-a_{33}\mu - a_{24}a_{42}\mu + a_{22}a_{44}\mu - a_{14}a_{32} - a_{12}a_{34} = 0, \tag{a11}$$

$$-a_{43}\mu + a_{24}a_{32} - a_{22}a_{34} - a_{14}a_{42} - a_{12}a_{44} = 0, \tag{a12}$$

$$a_{22}a_{23}\mu - a_{12}a_{13} - a_{14} = 0, \tag{a13}$$

$$-a_{13}a_{22} - a_{12}a_{23} - a_{24} = 0, \tag{a14}$$

$$a_{23}a_{42}\mu - a_{22}a_{43}\mu - a_{13}a_{32} - a_{12}a_{33} - a_{34} = 0, \tag{a15}$$

$$-a_{23}a_{32} + a_{22}a_{33} - a_{13}a_{42} - a_{12}a_{43} - a_{44} = 0, \tag{a16}$$

$$a_{23}^2\mu - a_{13}^2 = 0, \tag{a17}$$

$$-2a_{13}a_{23} = 0, \tag{a18}$$

$$-2a_{13}a_{33} = 0, \tag{a19}$$

$$-2a_{13}a_{43} = 0, \tag{a20}$$

$$a_{23}a_{24}\mu - a_{13}a_{14} = 0, \tag{a21}$$

$$-a_{14}a_{23} - a_{13}a_{24} = 0, \tag{a22}$$

$$-a_{24}a_{43}\mu + a_{23}a_{44}\mu - a_{14}a_{33} - a_{13}a_{34} = 0, \tag{a23}$$

$$a_{24}a_{33} - a_{23}a_{34} - a_{14}a_{43} - a_{13}a_{44} = 0, \tag{a24}$$

$$a_{13}\mu + a_{22}a_{24}\mu - a_{12}a_{14} = 0, \tag{a25}$$

$$a_{23}\mu - a_{14}a_{22} - a_{12}a_{24} = 0, \tag{a26}$$

$$a_{33}\mu + a_{24}a_{42}\mu - a_{22}a_{44}\mu - a_{14}a_{32} - a_{12}a_{34} = 0, \tag{a27}$$

$$a_{43}\mu - a_{24}a_{32} + a_{22}a_{34} - a_{14}a_{42} - a_{12}a_{44} = 0, \tag{a28}$$

$$a_{23}a_{24}\mu - a_{13}a_{14} = 0, \tag{a29}$$

$$-a_{14}a_{23} - a_{13}a_{24} = 0, \tag{a30}$$

$$a_{24}a_{43}\mu - a_{23}a_{44}\mu - a_{14}a_{33} - a_{13}a_{34} = 0, \tag{a31}$$

$$-a_{24}a_{33} + a_{23}a_{34} - a_{14}a_{43} - a_{13}a_{44} = 0, \tag{a32}$$

$$a_{24}^2\mu - a_{14}^2 = 0, \tag{a33}$$

$$-2a_{14}a_{24} = 0, \tag{a34}$$

$$-2a_{14}a_{34} = 0, \tag{a35}$$

$$-2a_{14}a_{44} = 0. \tag{a36}$$

All automorphisms on $\mathbb{H}_\mu$ can be described as follows:

Theorem 3.1. *Each automorphism $\mathcal{P}$ on $\mathbb{H}_\mu$ has one of the following forms:*

(i) $\mathcal{P}(1) = 1$, $\mathcal{P}(e_1) = -e_1 + ae_2 + de_3$, $\mathcal{P}(e_2) = be_2 + \frac{c}{\mu}e_3$, $\mathcal{P}(e_3) = ce_2 - be_3$,
(ii) $\mathcal{P}(1) = 1$, $\mathcal{P}(e_1) = e_1 + ae_2 + de_3$, $\mathcal{P}(e_2) = be_2 - \frac{c}{\mu}e_3$, $\mathcal{P}(e_3) = ce_2 + be_3$,

where a, b, c, d are parameters and $\frac{\mu b^2+c^2}{\mu} \neq 0$.

Proof. From (a6) and (a14), we obtain $a_{24} = 0$. Therefore, from (a33), we have $a_{14} = 0$. Taking $a_{24} = 0$ and $a_{14} = 0$ in (a25) and (a26) yields $a_{13} = a_{23} = 0$.

If $a_{12} = 0$, then, from (a1), we have $a_{22} = 1$ or -1. If $a_{22} = 1$, then the system of Eqs (a1)–(a36) is equivalent to the following system of equations

$$a_{43}\mu + a_{34} = 0, a_{44} - a_{33} = 0.$$

Thus, we obtain the desired P as follows:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} \\ 0 & a_{42} & -\frac{a_{34}}{\mu} & a_{33} \end{pmatrix},$$

which is just the (ii) of Theorem 3.1. If $a_{22} = -1$, then we get the (i) of Theorem 3.1.

If $a_{12} \neq 0$, from (a34)–(a36), one has $a_{34} = a_{44} = 0$, which makes P degenerate.

If $\mu = 1$, then $\mathbb{H}_\mu$ is the algebra of semi-quaternions. If $\mu = -1$, then $\mathbb{H}_\mu$ is the algebra of split semi-quaternions. By applying Theorem 3.1 to these special cases, we have the following:

Corollary 3.2. *Each automorphism $\mathcal{P}$ on the algebra of semi-quaternions has one of the following forms:*

(i) $\mathcal{P}(1) = 1, \mathcal{P}(e_1) = -e_1 + ae_2 + de_3, \mathcal{P}(e_2) = be_2 + ce_3, \mathcal{P}(e_3) = ce_2 - be_3,$
(ii) $\mathcal{P}(1) = 1, \mathcal{P}(e_1) = e_1 + ae_2 + de_3, \mathcal{P}(e_2) = be_2 - ce_3, \mathcal{P}(e_3) = ce_2 + be_3,$

where a, b, c, d *are parameters and* $b^2 + c^2 \neq 0$.

Corollary 3.3. *Each automorphism* $\mathcal{P}$ *on the algebra of split semi-quaternions has one of the following forms:*

(i) $\mathcal{P}(1) = 1, \mathcal{P}(e_1) = -e_1 + ae_2 + de_3, \mathcal{P}(e_2) = be_2 - ce_3, \mathcal{P}(e_3) = ce_2 - be_3,$
(ii) $\mathcal{P}(1) = 1, \mathcal{P}(e_1) = e_1 + ae_2 + de_3, \mathcal{P}(e_2) = be_2 + ce_3, \mathcal{P}(e_3) = ce_2 + be_3,$

where a, b, c, d *are parameters and* $c^2 - b^2 \neq 0$.

Corollary 3.3 gives us an extra surprise. Using Corollary 3.3, we can determine all automorphisms on Sweedler's 4-dimensional Hopf algebra. First, recall that Sweedler algebra $\mathbb{H}_4$ is generated by two elements g and v subject to

$$g^2 = 1, v^2 = 0, gv + vg = 0.$$

The comultiplication, antipode, and counit of $\mathbb{H}_4$ are given by

$$\Delta(g) = g \otimes g, \Delta(v) = g \otimes v + v \otimes 1, \varepsilon(g) = 1, \varepsilon(v) = 0, S(g) = g, S(v) = -gv.$$

Note that the dimension of $\mathbb{H}_4$ is four with $1, g, v, gv$ forming a basis for $\mathbb{H}_4$. By setting $e_1 = g, e_2 = v, e_3 = gv$, we see that $\mathbb{H}_4$ as an algebra is a split semi-quaternion algebra. Thus, by Corollary 3.3, we have the following result.

Corollary 3.4. *Each automorphism* $\mathcal{P}$ *on Sweedler's 4-dimensional Hopf algebra has one of the following forms:*

(i) $\mathcal{P}(1) = 1, \mathcal{P}(g) = -g + av + dgv, \mathcal{P}(v) = bv - cgv, \mathcal{P}(gv) = cv - bgv,$
(ii) $\mathcal{P}(1) = 1, \mathcal{P}(g) = g + av + dgv, \mathcal{P}(v) = bv + cgv, \mathcal{P}(gv) = cv + bgv,$

where a, b, c, d *are parameters and* $c^2 - b^2 \neq 0$.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The authors declare there are no conflicts of interest.

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