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Research article

A construction of Shatz strata in the polystable G_2-bundles moduli space using Hecke curves

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Abstract: Let X be a compact Riemann surface of genus $g \geq 2$ and $M(G_2)$ be the moduli space of polystable principal G_2-bundles over X. The Harder-Narasimhan types of the bundles induced a stratification of the moduli space $M(G_2)$ called Shatz stratification. In this paper, a description of the Shatz strata of the unstable locus of $M(G_2)$ corresponding to certain family of Harder-Narasimhan types (specifically, those of the form $(\lambda, \mu, 0, -\mu, -\lambda)$ with $\mu < \lambda \leq 0$) was given. For this purpose, a family of vector bundles was constructed in which a 3-form and a 2-form were defined so that it was proved that they were strictly polystable principal G_2-bundles. From this, it was proved that, when the genus of X was $g \geq 12$, these Shatz strata were the disjoint union of a family of G_2-Hecke curves in $M(G_2)$ that will be constructed along the paper. Therefore, the presented results provided an advance in the knowledge of the geometry of $M(G_2)$ through the study of its Shatz strata and presented a methodological innovation, by using Hecke curves for this study.

Keywords: principal bundle; moduli space; G_2; Shatz stratification; Hecke curve

1. Introduction

The group G_2 is a simple complex Lie group of exceptional type which is defined to be the group of automorphisms of a complex vector space $V \cong \mathbb{C}^7$, which preserves certain nondegenerate skew-symmetric 3-form and also a nondegenerate symmetric 2-form. The group G_2 has been recently studied because of its interest in many fields such as geometry, physics, or dynamical systems. In theoretical physics, for instance, M-theory suggests that the structure of the universe can be described in terms of a 7-dimensional manifold M which admits a holonomy whose group is a real form of G_2 [1]. This shows the growing interest in studying geometric structures related to G_2. There are also many relevant papers, like [2], which uses G_2 strongly in the context of integrable dynamical systems.

The main focus of this research is on principal G_2-bundles over a complex projective curve. Let X be a compact Riemann surface of genus $g \geq 2$. A principal G_2-bundle over X is a holomorphic complex vector bundle E of rank 7 equipped with a holomorphic nondegenerate skew-symmetric globally-defined trilinear form Ω. The bundle E will be also equipped with a holomorphic nondegenerate globally-defined symmetric bilinear form ω. There is a suitable notion of polystability for principal G_2-bundles, according to Ramanathan's framework [3–5] that allows the construction of the moduli space $M(G_2)$ of polystable principal G_2-bundles over X. This is a coarse moduli scheme of complex dimension $14(g-1)$.

The deepening of the study of the geometry of moduli spaces of principal bundles has been aided by the understanding of the stratifications of these moduli spaces. In this article it is relevant to consider the Shatz stratification [6], which is defined from the Harder-Narasimhan filtrations. Any vector bundle V over X admits a filtration into vector sub-bundles

$$0 = V_0 \subset V_1 \subset \cdots \subset V_{k-1} \subset V_k = V$$

such that V_j/V_{j-1} is semi-stable and the slopes of the successive quotients are decreasing, that is, if $\mu(V_j/V_{j-1})$ denotes the slope of V_j/V_{j-1} for $j = 1, \ldots, k$, then $\mu(V_j/V_{j-1}) > \mu(V_{j+1}/V_j)$ for $j = 1, \ldots, k-1$ (recall that the slope of a vector bundle is the quotient of its degree and its rank). Given a vector bundle V of rank r, its Harder-Narasimhan filtration induces the so-called *Harder-Narasimhan type*, which is a vector $(\mu_1, \ldots, \mu_r)$ given by the slopes of the quotients of the sub-bundles of the Harder-Narasimhan filtration, where each μ_j is repeated as many times as the rank of V_j/V_{j-1} specifies. The Shatz stratification is then defined by the equality of the Harder-Narasimhan type.

This construction can be extended to principal G-bundles for a semi-simple complex Lie group G. Indeed, Ramanathan [7] proved that each principal G-bundle V admits a single reduction V_P to a parabolic subgroup P of G, called *canonical reduction*, such that the extension of the structure group to the Levi factor of P is semi-stable and the degree of the line bundle $\chi_* V_P$, where χ is an anti-dominant character of P, is ≥ 0. This concept of Harder-Narasimhan filtration corresponds to the one proposed by Atiyah and Bott [8], who considered the Harder-Narasimhan filtration of the adjoint bundle of a given principal G-bundle, the equivalence having been proved by Anchouche, Azad, and Biswas [9]. Also, a Shatz stratification can be defined for other geometric objects, including Higgs bundles [10].

In the case of principal G_2-bundles, Beckers, Hussin, and Winternitz [2] proved that G_2 admits two parabolic subgroups, P_1 and P_2, and a Borel subgroup, B. Notice that the homogeneous spaces G_2/P_1, G_2/P_2, and G_2/B parametrize flags of a given principal G_2-bundle V over X corresponding to the reductions of structure group to the corresponding parabolic subgroups. From this, it follows that there are three basic Harder-Narasimhan types for the strictly polystable G_2-bundles, which can be described in this way: $(\lambda, 0, -\lambda)$ for $\lambda < 0$ and with an associated isotropic vector sub-bundle of rank 1; $(\lambda, 0, -\lambda)$ for $\lambda < 0$ and with a corresponding rank-2 isotropic sub-bundle; and $(\lambda, \mu, 0, -\mu, -\lambda)$ for $\mu < \lambda \leq 0$, with two associated isotropic sub-bundles, of ranks 1 and 2, respectively (Section 3). Here, the Shatz strata given by Harder-Narasimhan types of the last form are considered. In particular, the main objective is giving a description of these strata as the union of certain G_2-Hecke curves that will be constructed.

In this work, certain Hecke curves will be constructed for the moduli space of polystable principal G_2-bundles over X that lie in the unstable locus of the moduli space (Section 5). Hecke transformations and Hecke curves have gained great importance in research in algebraic geometry because they have been extensively used in several contexts [11, 12]. For example, they provide a suitable context for giving a precise description of the group of automorphisms of the moduli space of vector bundles over a Riemann surface [13] and, thus, give a new proof of the classical result of Kouvidakis and Pantev [14]. Moreover, Hecke curves have also been used to compute the group of automorphisms of the moduli spaces of orthogonal or symplectic bundles [15]. In the present paper, Hecke curves will allow us to deepen the knowledge of the geometry of the moduli space of G_2-bundles through the study of its subvarieties, in particular, of the Shatz strata described above. The construction of these G_2-Hecke curves will be done by generalizing the construction made in [15] for orthogonal and symplectic bundles. It is worth noting that the use of Hecke curves for the study of the Shatz strata constitutes a methodological novelty of the present study. The techniques employed are strongly linked to the geometry of the Lie group G_2, so the paper is focused on the moduli of G_2-bundles whose interest is supported by the preceding literature, as already mentioned. Specifically, it will be proved that the maps defining the G_2-Hecke curves considered are generically injective (Proposition 5.1) in the case when $g \geq 5$ (the assumption on the genus of X will be necessary due to technical constraints). Finally, in the main result of the paper (Theorem 5.1), it is proved that, under the technical assumption that $g \geq 12$, the G_2-Hecke curves constructed are disjoint and cover the whole Shatz strata given by Harder-Narasimhan types of the form $(\lambda, \mu, 0, -\mu, -\lambda)$ for $\mu < \lambda \leq 0$.

Theorem. *The G_2-Hecke curves defined in* (5.2) *fall into the Shatz strata defined by the Harder-Narasimhan types of the form $(\lambda, \mu, 0, -\mu, -\lambda)$ for $\mu < \lambda \leq 0$. Moreover, if the genus of the compact Riemann surface is $g \geq 12$, then, for different choices of L_x, the corresponding G_2-Hecke curves are disjoint. Finally, the union of the G_2-Hecke curves is the union of all the abovementioned Shatz strata of the moduli space of polystable principal G_2-bundles over X.*

The paper is structured as follows. Section 2 is devoted to the presentation of some essential questions on the group G_2 that will be necessary throughout the article, such as its parabolic subgroups and the flags that they induce on the vector space where the group is represented. In Section 3, the moduli space of polystable principal G_2-bundles over a compact Riemann surface is presented, with special emphasis on the description of the possible Harder-Narasimhan types of a given G_2-bundle. The actual construction of the G_2-Hecke curves is done in Sections 4 and 5. The bundles involved are constructed in Section 4, where it is proved that those bundles that are constructed are truly G_2-bundles, and the construction of the G_2-Hecke curves and the main result of the paper will be performed in Section 5.

2. The group G_2

The simple complex Lie group G_2 is defined to be the group of complex automorphisms of the octonions [16]. Its Lie algebra, $\mathfrak{g}_2$, is the algebra of derivations of the octonions. The fundamental irreducible representation of G_2 has dimension 7, so G_2 can be understood as a subgroup of $\mathrm{SL}(7, \mathbb{C})$. Moreover, G_2 consists of orthogonal matrices for the canonical symmetric bilinear form ω on $\mathbb{C}^7$ for which the vectors $e_1, \ldots, e_7$ of the canonical basis form an orthonormal basis. Then the fundamental representation of G_2 induces an inclusion of groups $G_2 \hookrightarrow \mathrm{SO}(7, \mathbb{C})$ (and, of course, $\mathfrak{g}_2$ can also be understood as a sub-algebra of $\mathfrak{so}(7, \mathbb{C})$). If e^{ijk} denotes the wedge product of the vectors of the dual basis $e_i^* \wedge e_j^* \wedge e_k^*$, $i, j, k \in \{1, \ldots, 7\}$, then the expression

$$\Omega = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356}$$

defines a complex skew-symmetric 3-form on the vector space $\mathbb{C}^7$ on which G_2 is represented. The subgroup of $\mathrm{SL}(7, \mathbb{C})$ which fixes Ω, is exactly G_2. Therefore, it can be stated that

$$G_2 = \{g \in \mathrm{SL}(7, \mathbb{C}) \ : \ g^*\Omega = \Omega\}.$$

Then, G_2 is the group of automorphisms of a 7-dimensional complex vector space which fixes a nondegenerate symmetric 2-form and a nondegenerate skew-symmetric 3-form. Thus defined, the group G_2 is simply connected and centerless.

In [2] it is given a description of the parabolic subgroups of G_2. The group G_2 admits three parabolic subgroups. Two of them are maximal and the third is the intersection of the maximal ones, which is the Borel subgroup. One of the maximal parabolic subgroups, P_1, induces a filtration on the vector space $\mathbb{C}^7$ on which G_2 is represented of the form

$$0 \subset V_1 \subset V_1^{\perp} \subset \mathbb{C}^7,$$

where V_1 is a subspace of dimension 1 of $\mathbb{C}^7$ which is isotropic for both forms, ω and Ω, and the orthogonality with respect to ω is denoted by $\perp$. Recall that a subspace W of $\mathbb{C}^7$ is said to be isotropic for the trilinear form Ω if $\Omega(W, W, W) = 0$. Similarly, the other maximal parabolic subgroup, P_2, induces a filtration of the form

$$0 \subset V_2 \subset V_2^{\perp} \subset \mathbb{C}^7,$$

where V_2 has rank 2 and is also isotropic for ω and Ω. The last parabolic subgroup, $B = P_1 \cap P_2$, induces a filtration of the form

$$0 \subset V_1 \subset V_2 \subset V_2^{\perp} \subset V_1^{\perp} \subset \mathbb{C}^7,$$

where V_1 and V_2 are isotropic subspaces for ω and Ω with $\operatorname{rk} V_1 = 1$ and $\operatorname{rk} V_2 = 2$.

3. Principal G_2-bundles and Harder-Narasimhan types

Let X be a compact Riemann surface of genus $g \geq 2$. A principal G_2-bundle over X is a triple (V, Ω, ω), where V is a holomorphic vector bundle of rank 7 over X equipped with a globally-defined nondegenerate skew-symmetric holomorphic 3-form Ω and a globally-defined nondegenerate holomorphic symmetric 2-form ω. For simplicity, the principal G_2-bundles will be referred to by the name of the underlying vector bundle V.

From the abovementioned description of the parabolic subgroups of G_2, the following notions of stability and polystability of principal G_2-bundles are obtained, which are given in terms of filtrations of isotropic sub-bundles of the underlying vector bundles [17]. The notions of stability and polystability given here are clearly equivalent to those introduced by Subramanian [18].

Definition 3.1. *Let (V, Ω, ω) be a principal G_2-bundle over the compact Riemann surface X. The principal G_2-bundle is stable (resp., semi-stable) if $\deg W < 0$ (resp., ≤ 0) for every rank 1 or rank 2 vector sub-bundle W, which is isotropic for Ω and ω. It is polystable if it admits a rank 1 or rank 2 and degree 0 isotropic vector sub-bundle as a direct summand.*

Since the center of G_2 is $Z(G_2) = \{1\}$, a principal G_2-bundle over X is said to be simple if it admits no other automorphisms than the identity map. Thus, the moduli space $M(G_2)$ of polystable principal G_2-bundles over X is an algebraic variety of dimension $14(g-1)$ which parametrizes isomorphism classes of polystable principal G_2-bundles over X, the subset $M_*(G_2)$ of stable and simple G_2-bundles being an open dense subset formed by smooth elements of $M(G_2)$ [19]. The moduli space $M(G_2)$ is also irreducible, since the group G_2 is simply connected [19].

It is relevant for this research to consider the Harder-Narasimhan filtration of a principal G_2-bundle. Given any vector bundle V over X, it admits a single filtration of vector sub-bundles $0 \subset V_1 \subset \cdots \subset V_k \subset V_k^{\perp} \subset \cdots \subset V_1^{\perp} \subset V$ characterized for the following relations of the slopes:

$$\mu(V/V_k) < \mu(V_k/V_{k-1}) < \cdots < \mu(V_2/V_1) < \mu(V_1),$$

where $\mu(W)$ denotes the slope of W for a sub-bundle W of V, that is, $\mu(W) = \frac{\deg W}{\operatorname{rk} W}$. Given the above Harder-Narasimhan filtration of V, the Harder-Narasimhan type of V is the vector $(\mu_1, \ldots, \mu_r, 0, -\mu_r, \ldots, -\mu_1)$ of the above slopes, with r being the rank of V. In particular, the components of the Harder-Narasimhan type satisfy $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_r \geq 0$. Moreover, each vector bundle V admits a well-defined Harder-Narasimhan type, so this defines a stratification of the moduli space of vector bundles, called Shatz stratification, whose strata are given by each possible Harder-Narasimhan type [6].

If V is a strictly polystable principal G_2-bundle over X, then the description of the parabolic subgroups of G_2 gives three possible Harder-Narasimhan filtrations, so there are three possibilities for the Harder-Narasimhan type:

- $0 \subset W \subset W^{\perp} \subset V$ with Harder-Narasimhan type of the form $(\lambda, 0, -\lambda)$ for some $\lambda < 0$, being the isotropic sub-bundle W corresponding to λ of rank 1;
- $0 \subset W \subset W^{\perp} \subset V$ with Harder-Narasimhan type of the form $(\lambda, 0, -\lambda)$ for some $\lambda < 0$, being the isotropic sub-bundle W corresponding to λ of rank 2;
- $0 \subset V_1 \subset V_2 \subset V_2^{\perp} \subset V_1^{\perp} \subset V$ with $\operatorname{rk} V_1 = 1$ and $\operatorname{rk} V_2 = 2$ and Harder-Narasimhan type of the form $(\lambda, \mu, 0, -\mu, -\lambda)$ for some $\mu < \lambda \leq 0$.

In summary, there are three different families of Harder-Narasimhan types for strictly polystable principal G_2-bundles over X (and also of Shatz strata), corresponding to the following vectors of slopes and ranks of the sub-bundles:

- $(\lambda, 0, -\lambda)$ for $\lambda < 0$ and such that the isotropic vector sub-bundle corresponding to λ has rank 1;
- $(\lambda, 0, -\lambda)$ for $\lambda < 0$ and such that the isotropic vector sub-bundle corresponding to λ has rank 2;
- $(\lambda, \mu, 0, -\mu, -\lambda)$ for $\mu < \lambda \leq 0$. In this case, the isotropic vector sub-bundles of the associated filtration of V always have ranks 1 and 2, respectively.

4. Construction of the bundles

A fundamental step in the construction process of the Hecke curves under consideration is the construction of the bundles that will compose these Hecke curves. In this section, the construction of these bundles is carried out and it is proved that, indeed, they are principal G_2-bundles by proving that they admit the forms that define this type of bundles, according to Section 3. Additionally, the principal G_2-bundles discussed are those whose Harder-Narasimhan type is of the form $(\lambda, \mu, 0, -\mu, -\lambda)$ for $\mu < \lambda \leq 0$, following the discussion at the end of that section, which are strictly polystable G_2-bundles.

Let V be a generic element of $M(G_2)$ equipped with a nondegenerate symmetric 2-form ω, a nondegenerate skew-symmetric 3-form Ω, and with Harder-Narasimhan type of the form $(\lambda, \mu, 0, -\mu, -\lambda)$ for $\mu < \lambda \leq 0$. The following construction closely follows the constructions given in [20] for vector bundles and in [15] for orthogonal and symplectic bundles. Specifically, the construction given in [15] is generalized to the case of G_2-bundles.

Take a point $x \in X$ and choose a line L_x of the Grassmannian of lines $\mathrm{Gr}(1, V_x)$ of the fiber of V over x such that L_x is isotropic for both ω and Ω. The sheaf V^{L_x} is defined to be the kernel of the composition $V \to V_x \to V_x/L_x^{\perp}$, where the orthogonality $\perp$ is taken with respect to ω and $L_x^{\perp}$ has co-dimension 1. By Hecke transformation, this defines the exact sequence

$$0 \to V^{L_x} \to V \to (V_x/L_x^{\perp}) \otimes \mathbb{C}_x \to 0, \tag{4.1}$$

where $\mathbb{C}_x$ denotes the skyscraper at x. By taking the dual of the Hecke transformation and noticing that $(V_x/L_x^{\perp})^*$ is canonically isomorphic to L_x, one obtains the exact sequence

$$0 \to L_x \to V_x^* \to \left(V^{L_x}\right)^*_x \to \mathbb{C} \to 0. \tag{4.2}$$

From this, a sheaf injection $\alpha_1 : \left(V^{L_x}\right)^*(-x) \to V^*$ is defined, which gives a new sheaf injection $\alpha_2 : \wedge^2 \left(V^{L_x}\right)^*(-2x) \to \wedge^2 V^*$. Then, sheaf surjections $\beta_1 = \alpha_1^* : V \to V^{L_x}(x)$ and $\alpha_2 = \beta_2^* : \wedge^2 V \to \wedge^2 V^{L_x}(2x)$ are also defined by taking the dual of the corresponding injections. If $q_\omega : V^* \to V$ and $Q_\Omega : \wedge^2 V^* \to V$ are the morphisms of vector bundles induced by the forms ω and Ω, respectively, then one may consider the compositions

$$\beta_1 \circ q_\omega \circ \alpha_1 : \left(V^{L_x}\right)^*(-x) \to V^* \to V \to V^{L_x}(x)$$

and

$$\beta_1 \circ Q_\Omega \circ \alpha_2 : \wedge^2 \left(V^{L_x}\right)^*(-2x) \to \wedge^2 V^* \to V \to V^{L_x}(x),$$

so a symmetric form given by $\mathrm{Sym}^2 \left(V^{L_x}\right)^* \to O_X(2x)$ and a skew-symmetric form given by $\wedge^3 \left(V^{L_x}\right)^* \to O_X(3x)$ are defined, since they come from a symmetric 2-form and a skew-symmetric 3-form, respectively, defined on V, where O_X denotes the trivial line bundle over X. Now, the restriction of the form $\mathrm{Sym}^2 \left(V^{L_x}\right)^* \to O_X(2x)$ to the fiber at x factors through $\mathrm{Sym}^2(\mathrm{Im}(\alpha_{1x})) = \mathrm{Sym}^2 L_x$ and the restriction of the form $\wedge^3 \left(V^{L_x}\right)^* \to O_X(3x)$ to the fiber at x factors through $\wedge^2(\mathrm{Im}(\alpha_{1x})) = \wedge^3 L_x$, being both zero $\mathrm{Sym}^2 L_x$ and $\wedge^3 L_x$, since L_x has been chosen to be isotropic for ω and Ω. This results in the definition of a symmetric map

$$\omega^{L_x} : \left(V^{L_x}\right)^* \to V^{L_x}(x) \tag{4.3}$$

and a skew-symmetric 3-form

$$\Omega^{L_x} : \wedge^2 \left(V^{L_x}\right)^* \to V^{L_x}(x). \tag{4.4}$$

Let now W_x be a rank 2 subspace of $\left(V^{L_x}\right)^*_x$ which is isotropic for ω^{L_x} and Ω^{L_x} defined in (4.3) and (4.4), respectively, and with $L_x \subset W_x$. Let $\overline{V}^{W_x}$ be the bundle obtained by taking the Hecke transformation

$$0 \to \overline{V}^{W_x} \to \left(V^{L_x}\right)^* \to \left((V^{L_x})^*_x / W_x^{\perp}\right) \otimes \mathbb{C}_x \to 0, \tag{4.5}$$

and let $V^{W_x} = \left(\overline{V}^{W_x}\right)^*$.

Proposition 4.1. *Let V be a polystable principal G_2-bundle over X, ω be its nondegenerate symmetric 2-form, and Ω be its nondegenerate skew-symmetric 3-form. Let L_x be a subspace of V_x of dimension 1 isotropic for ω and Ω and W_x be a subspace of $\left(V^{L_x}\right)^*_x$, defined in* (4.2)*, isotropic for ω^{L_x} and Ω^{L_x}, defined in* (4.3) *and* (4.4)*, respectively. Then, the bundle V^{W_x} defined in* (4.5) *admits a globally defined nondegenerate symmetric 2-form ω^{W_x} induced by ω^{L_x} and a globally defined nondegenerate skew-symmetric 3-form Ω^{W_x} induced by Ω^{L_x}.*

Proof. Take the compositions

$$\overline{V}^{W_x} \to \left(V^{L_x}\right)^* \stackrel{\omega^{L_x}}{\to} V^{L_x} \to \left(\overline{V}^{W_x}\right)^*(x)$$

and

$$\wedge^2 \overline{V}^{W_x} \to \wedge^2 \left(V^{L_x}\right)^* \stackrel{\Omega^{L_x}}{\to} V^{L_x} \to \left(\overline{V}^{W_x}\right)^*(x).$$

This defines forms $\mathrm{Sym}^2\, \overline{V}^{W_x} \to O_X(x)$ and $\wedge^3 \overline{V}^{W_x} \to O_X(x)$, as adapted from [15, Lemma 4.6]. Since the constructed forms factor through $\mathrm{Im}\left(\overline{V}^{W_x} \to \left(V^{L_x}\right)^*\right) = W_x^{\perp}$ (as in [15, Lemma 4.6]) and $\mathrm{Im}\left(\wedge^2 \overline{V}^{W_x} \to \wedge^2 \left(V^{L_x}\right)^*\right) = \wedge^2 W_x^{\perp}$, respectively, and $\ker \omega_x^{L_x} = W_x$ and $\ker \Omega_x^{L_x} = \wedge^2 W_x$, these forms vanish along the fiber at x, since W_x is maximal isotropic. Then they define forms $\omega^{W_x} : \mathrm{Sym}^2\, \overline{V}^{W_x} \to O_X$ and $\Omega^{W_x} : \wedge^3 \overline{V}^{W_x} \to O_X$. Of course, this induces forms on V^{W_x}. □

Remark. Notice that, from (4.2), the subspace W_x may be understood as a subspace of V_x^* which contains L_x.

5. Construction of the Hecke curves

In this section, the G_2-Hecke curves of $M(G_2)$ introduced are constructed from the G_2-bundles obtained in Section 4. In addition, the main result of the paper is proved, which states that the union of all constructed G_2-Hecke curves completes the strata of the Shatz stratification of $M(G_2)$ defined by the Harder-Narasimhan types of the form $(\lambda, \mu, 0, -\mu, -\lambda)$ with $\mu < \lambda \leq 0$.

Suppose that G_2 is represented in a 7-dimensional vector space $\mathbb{C}^7$ and preserves a symmetric 2-form ω and a skew-symmetric 3-form Ω of $\mathbb{C}^7$. It is well-known that G_2 admits two maximal parabolic subgroups, P_1 and P_2, which corresponds to the choices of a subspace of dimension 1 or dimension 2, respectively, of $\mathbb{C}^7$ isotropic for both ω and Ω. That is, they correspond to flags of the form $0 \subset U \subset U^{\perp} \subset \mathbb{C}^7$, where U is a subspace of dimension 1 or 2, respectively, $\perp$ is taken with respect to ω, and U

is also isotropic for Ω [16, 18]. The Borel subgroup B of G_2 may be understood as the intersection of the two maximal parabolic subgroups of G_2, and gives flags of the form

$$0 \subset U_1 \subset U_2 \subset U_2^{\perp} \subset U_1^{\perp} \subset \mathbb{C}^7, \tag{5.1}$$

where U_1 and U_2 have dimensions 1 and 2, respectively, and are also isotropic for Ω. The G_2 Grassmannian of dimension r subspaces of $\mathbb{C}^7$ (for $r = 1, 2$) is composed by subspaces of dimension r isotropic for ω and Ω, and is isomorphic to the homogeneous space $F_r = G_2/P_r$ [21], which is, in turn, isomorphic to $\mathbb{P}^4$ [22]. Flags of the form (5.1) are parametrized by the homogeneous space $F_{1,2} = G_2/B$, which is isomorphic to $\mathbb{P}^5$. The morphism $G_2/B \to G_2/P_2$ admits a copy of $\mathbb{P}^1$ as kernel, which parametrizes subspaces of dimension 1 of the corresponding subspaces of dimension 2.

Fix a point $x \in X$ and a generic element V of $M(G_2)$ with Harder-Narasimhan type of the form $(\lambda, \mu, 0, -\mu, -\lambda)$ for $\mu < \lambda \leq 0$, as in Section 4. Let V^{L_x} be the sheaf defined in (4.1). Consider the map $u_{L_x} : \ker\left(F_{1,2}\left(V_x^{L_x}\right) \to F_2\left(V_x^{L_x}\right)\right) \cong \mathbb{P}^1 \to F_{1,2}\left(V_x^{L_x}\right) \cong \mathbb{P}^5$. For each $t \in \mathbb{P}^1$, call $u_{L_x}(t) = W_{t,x}$, which is a subspace of $V_x^{L_x}$ of dimension 2 isotropic for the corresponding 2-form and 3-form. If L_x is the fiber at x of a line sub-bundle L of V, then $W_{t,x}$ is the fiber at x of a sub-bundle W_t of rank 2 of $\left(\overline{V}^{L_t}\right)^*$ defined in (4.2).

Then, the assign

$$\mathbb{P}^1 \ni t \mapsto \left(V^{W_{t,x}}, \omega^{W_{t,x}}, \Omega^{W_{t,x}}\right), \tag{5.2}$$

where $V^{W_{t,x}}$ is defined in (4.5) and $\omega^{W_{t,x}}$ and $\Omega^{W_{t,x}}$ are defined in Proposition 4.1, gives a rational curve in $M(G_2)$ by Proposition 4.1. These curves will be called *G_2-Hecke curves* in $M(G_2)$.

Proposition 5.1. *Let X be a compact Riemann surface of genus $g \geq 5$. Then the map defined in* (5.2) *given by a G_2-Hecke curve is generically injective.*

Proof. Take $s, t \in \mathbb{P}^1$ such that $s \neq t$ but $V^{W_{s,x}} \cong V^{W_{t,x}}$. This gives two linearly independent generic isomorpisms between V^{L_x} and $V^{W_{t,x}}$. These automorphisms are, in particular, orthogonal isomorphisms, so this contradicts [15, Lemma 4.2], where it is proved that, in a more general situation, $\dim H^0\left(X, \left(V^{L_x}\right)^* \otimes V^{W_{t,x}}\right) = 1$ in the case when $g \geq 5$. □

Theorem 5.1. *The G_2-Hecke curves defined in* (5.2) *fall into the Shatz strata defined by the Harder-Narasimhan types of the form $(\lambda, \mu, 0, -\mu, -\lambda)$ for $\mu < \lambda \leq 0$. Moreover, if the genus of the compact Riemann surface is $g \geq 12$, then, for different choices of L_x as in* (4.1)*, the corresponding G_2-Hecke curves are disjoint. Finally, the union of the G_2-Hecke curves is the union of all the abovementioned Shatz strata of $M(G_2)$.*

Proof. For the first part, notice that, with the notation of Section 4 and the beginning of Section 5, the Harder-Narasimhan filtration of the G_2-bundle $V^{W_{t,x}}$ is of the form

$$0 \subset L \subset W \subset W^{\perp} \subset L^{\perp} \subset V^{W_{t,x}},$$

so it falls into the strata of the Shatz stratification of $M(G_2)$ corresponding to Harder-Narasimhan types of the form $(\lambda, \mu, 0, -\mu, -\lambda)$ for $\mu < \lambda \leq 0$, which fall into the unstable locus of $M(G_2)$.

For the second part, an adaptation to G_2 of the arguments of [15, Lemma 4.7] is made. Take different subspaces L_x, say, $L_x^1 \neq L_x^2$, with the notation of Section 4. Take V_k to be any point of the G_2-Hecke curve corresponding to L_x^k for $k = 1, 2$ and suppose that there exists an isomorphism $f : V_1 \to V_2$. Let $V^{L_x^1 \cap L_x^2}$ be the intersection $V^{L_x^1} \cap V^{L_x^2}$. One has that $\deg V^{L_x^1 \cap L_x^2} = \deg V - 2 = -2$, so two generic isomorphisms $V^{L_x^1 \cap L_x^2} \cong V_2$ are defined. The first one by composition $V^{L_x^1 \cap L_x^2} \subset V^{L_x^1} \subset V_1 \xrightarrow{f} V_2$, and the second one by composition $V^{L_x^1 \cap L_x^2} \subset V^{L_x^2} \subset V_2$. By [15, Lemma 4.2], the above generic isomorphisms must coincide, under the assumption of $g \geq 12$, which is necessary for [15, Lemma 4.2] to be applied. On the other hand, the image of the restriction of the dual map $V_2^* \to \left(V^{L_x^1 \cap L_x^2}\right)^*$ to the fiber at x falls into $\ker\left(\left(V^{L_x^1}\right)_x^* \to \left(V^{L_x^1 \cap L_x^2}\right)_x^*\right)$ and into $\ker\left(\left(V^{L_x^2}\right)_x^* \to \left(V^{L_x^1 \cap L_x^2}\right)_x^*\right)$, so $V_2 \cong V$, which is a contradiction, so such an isomorphism f does not exist and, therefore, the G_2-Hecke curves are disjoint.

The last part of the statement follows simply by noting that each G_2-Hecke curve contains the bundle V, so every G_2-bundle of the abovementioned Shatz strata is in a G_2-Hecke curve. □

6. Conclusions

There are three different families of Harder-Narasimhan types that define the Shatz strata of the moduli space $M(G_2)$ of polystable principal G_2-bundles over a compact Riemann surface X: pairs of the form $(\lambda, -\lambda)$ for $\lambda < 0$ and with associated sub-bundle of rank 1; pairs of the form $(\lambda, -\lambda)$ for $\lambda < 0$ with corresponding sub-bundle of rank 2; and pairs of the form $(\lambda, \mu, 0, -\mu, -\lambda)$ for $\mu < \lambda \leq 0$. These Harder-Narasimhan types, which have been described along the paper, come from the various possible reductions of a principal G_2-bundle over X to a parabolic subgroup of G_2, that define the Harder-Narasimhan filtration of the bundle. It has been proved that, when the genus g of X satisfies $g \geq 12$, the Shatz strata corresponding to Harder-Narasimhan types of the third form are disjoint unions of certain families of G_2-Hecke curves that have been constructed for the purposes of the research. Moreover, it has been proved that the maps that define the G_2-Hecke curves are generically injective. These findings provide new insights that enhance the understanding of the geometry of the moduli space of G_2-bundles through the analysis of its Shatz strata. Likewise, the methodological approach of using Hecke curves for the description of the abovementioned strata, constitutes an original technique that could be applicable to moduli spaces of bundles with other structure groups. However, the program used here makes strong use of the orthogonal structure of the group G_2, so it is not directly applicable to any semi-simple or reductive groups. Such an extension would require the development of new techniques, which is an interesting line of future research.

Use of AI tools declaration

The author declares they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflicts of interest.

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