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*Research article*

# Dynamics of a general model of nonlinear difference equations and its applications to LPA model

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**Abstract:** In this study, we investigate the qualitative properties of solutions to a general model of difference equations (DEs), which includes the flour beetle model as a particular case. We investigate local and global stability and boundedness, as well as the periodic behavior of the solutions to this model. Moreover, we present some general theorems that help study the periodicity of solutions to the DEs. The presented numerical examples support the finding and illustrate the behavior of the solutions for the studied model. A significant agricultural pest that is extremely resistant to insecticides is the flour beetle. Therefore, studying the qualitative characteristics of the solutions in this model greatly helps in understanding the behavior of this pest and how to resist it or benefit from it. By applying the general results to the flour beetle model, we clarify the conditions of global stability, boundedness, and periodicity.

**Keywords:** difference equations; stability and periodicity; global stability; boundedness; mathematical model

## 1. Introduction

In phenomena where the plurality of observations of a temporally changing variable are discrete, difference equations (DE) are utilized to explain how a phenomenon develops in real life. These equations therefore become crucial in mathematical models. Applications involving higher-order nonlinear DEs are very common. Furthermore, the DEs naturally arise as numerical solutions and discrete analogs to ordinary and functional differential equations that model a wide range of distinct biological, physical, and economic events, see [1–5].

The global asymptotic behavior, as a qualitative characteristic of solutions to linear DEs, has significant applications in a variety of fields, such as control theory, biology, neural networks, and many more. It is difficult to verify the global stability of solutions to a DE using numerical methods. As a result, many mathematicians and engineers are interested in the analytical study of those qualitative features because it is the only way to understand those properties (see [6–12]).

In [13], Sun and Xi investigated the stability of the DE

$$B_{m+1} = F\left(B_{m-k}, B_{m-l}\right), \tag{1.1}$$

where $k$ and $l$ are in $\mathbb{Z}$, $k < l$, and $F(s,t)$ is non-increasing in $s$ and non-decreasing in $t$. In [14], Kocic et al. studied the stability of

$$B_{m+1} = (1-\delta)\, B_m + \delta B_{m-k}\left(1 + \eta\left(1 - \left(\frac{B_{m-k}}{M}\right)^{\gamma}\right)\right)_{+}, \tag{1.2}$$

which describes the dynamics of baleen whales.

The study of general models of difference equations, despite its difficulty compared to the study of specific models, provides more results, creates new methods, and gives a more general overview of similar models. Abdelrahman et al. [15, 16] investigated the qualitative properties of the DE

$$B_{m+1} = \alpha B_{m-l} + \beta B_{m-k} + F\left(B_{m-l}, B_{m-k}\right), \tag{1.3}$$

where $\alpha, \beta \in [0,\infty)$, $F \in C\left((0,\infty)^2, (0,\infty)\right)$, and $F$ is homogenous with degree 0.

Recently, Moaaz et al. [17–19] examined the asymptotic features of the DEs

$$B_{m+1} = F\left(B_{m-l}, B_{m-k}\right) \tag{1.4}$$

and

$$B_{m+1} = \alpha B_{m-1}\mathbf{e}^{-F(B_m, B_{m-1})}. \tag{1.5}$$

**Definition 1.** *[20] The function $\phi$ is called a homothetic function if there exist functions $G : \mathbb{R} \to \mathbb{R}$ and $H : \mathbb{R}^2 \to \mathbb{R}$, such that G is a monotonic function, H is a homogenous function with degree $\kappa$, and $\phi(t,s) = G(H(t,s))$.*

In this work, we study the general DE

$$B_{m+1} = \alpha B_m + \beta B_{m-\ell}\Phi\left(B_m, B_{m-\ell}\right), \tag{E}$$

where $\Phi$ is a homothetic function, and

(H1) $m, \ell \in \mathbb{Z}^+$, $\alpha \in [0,\infty)$, $\beta$, $B_{-\ell}$, $B_{-\ell+1}$, ..., $B_0 \in (0,\infty)$.
(H2) $\Phi$ is a homothetic function, where $\Phi(t,s) = G(H(t,s))$ and $H$ is a homogenous function with degree $\kappa > 0$.

**Theorem 1.** *[21, Theorem 1.3.1] Suppose that $f_1, f_2 \in C\left([0,\infty)^2, [0,1)\right)$ satisfy*
(i) *$f_1$ and $f_2$ are non-increasing in each of their arguments*;
(ii) $f_1(u,u) > 0$ *for* $u \geq 0$;
(iii) $f_1(u,v) + f_2(u,v) < 1$ *for* $u, v \in (0,\infty)$.

*Then the zero equilibrium of the DE*

$$B_{n+1} = f_1\left(B_n, B_{n-1}\right)\, B_n + f_2\left(B_n, B_{n-1}\right)\, B_{n-1}, \quad B_{-1},\ B_0 > 0,$$

*is globally asymptotically stable.*

## 2. Behavior of solutions of (E)

Here, we define $\mathcal{F} : [0,\infty) \times [0,\infty) \to (0,\infty)$ by

$$\mathcal{F}(t,s) = \alpha t + \beta s \Phi(t,s). \tag{2.1}$$

A fixed point $p^*$, in $[0,\infty) \times [0,\infty)$, is called an equilibrium point (EP) of the DE $B_{m+1} = \mathcal{F}(B_m, B_{m-\ell})$. The study of any physical system's dynamics centers on the concept of EPs (states). All states (solutions) of a particular system tend to its equilibrium state, which is known from multiple applications in science.

Now, we assume $\mathcal{F}(p^*, p^*) = p^*$ and look for the positive EPs. Then,

$$p^* = \alpha p^* + \beta p^* \Phi(p^*, p^*).$$

Thus, we obtain

$$((1-\alpha) - \beta\Phi(p^*, p^*))\, p^* = 0.$$

Since $\Phi = G(H)$ and $H$ is a homogenous function with degree $\kappa$, we obtain

$$\Phi(p^*, p^*) = G(H(p^*, p^*)) = G\left((p^*)^\kappa H(1,1)\right) = \frac{1-\alpha}{\beta}. \tag{2.2}$$

Hence,

$$p^* = \left(\frac{\lambda}{H(1,1)}\right)^{1/\kappa}, \tag{2.3}$$

under the conditions $\kappa \neq 0$, $H(1,1) > 0$ and

$$\lambda := G^{-1}\left(\frac{1-\alpha}{\beta}\right) > 0. \tag{2.4}$$

Thus, EPs of (E) are *zero* and the positive value defined in (2.3).

### *2.1. Stability of EPs*

This section is concerned with determining the criteria that guarantee that the EP of DE (E) is locally asymptotically stable (LAS), or globally asymptotically stable (GAS).

Determining the behavior of solutions of DEs near EPs is one of the main objectives when studying them. It is easy to verify that

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{F}(t,s) &= \alpha + \beta s \Phi_t(s,t) \\ &= \alpha + \beta s G'(H(t,s))\, H_t(t,s), \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} \frac{\partial}{\partial s}\mathcal{F}(t,s) &= \beta \Phi(t,s) + \beta s \Phi_s(t,s) \\ &= \beta G(H(t,s)) + \beta s G'(H(t,s))\, H_s(t,s). \end{aligned} \tag{2.6}$$

By substituting for $(t, s) = (0, 0)$ in (2.5) and (2.6), we have that the linearized equation about the trivial EP is

$$z_{m+1} = \alpha\, z_m + \beta\, G(0)\, z_{m-\ell}. \tag{2.7}$$

When $\ell = 1$, employing [21, Theorem 1.1.1], we get that the trivial EP of Eq (E) is (see Figure 1):

(a) LAS and sink if $\alpha < 1 - \beta\, G(0) < 2$;
(b) unstable and repeller if $\alpha > 1$ and $\alpha < |1 - \beta\, G(0)|$;
(c) saddle point if $\alpha + 4\beta\, G(0) > 0$ and $\alpha > |1 - \beta\, G(0)|$;
(d) nonhyperbolic point if $\alpha = |1 - \beta\, G(0)|$.

From Figure 1, it is easy to see the areas where the different types of stability and instability occur. We also notice that LAS occurs when the values of $\alpha$ and $\widehat{\beta} := \beta G(0)$ lie below the straight line $\alpha + \widehat{\beta} = 1$.

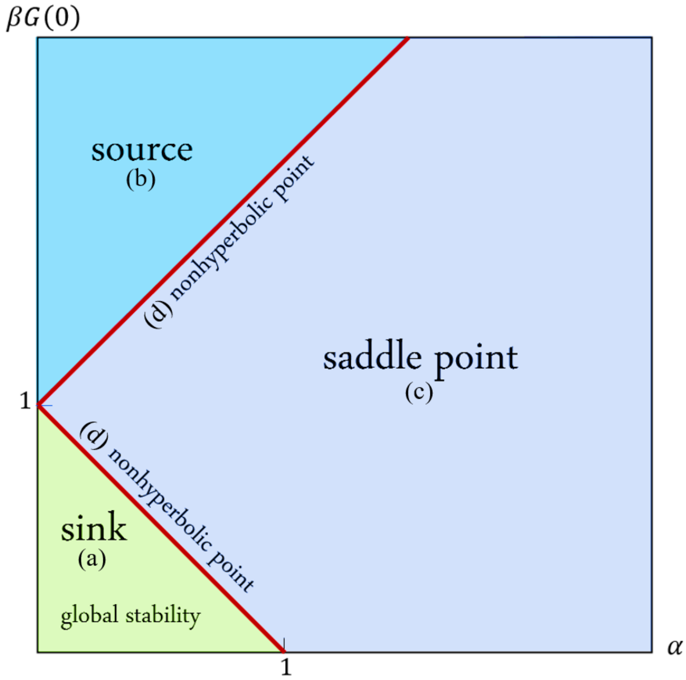



**Figure 1.** The topological classification for zero EP of (E).

Whereas when $\ell > 1$, the trivial EP of Eq (E) is LAS if $\alpha + \beta\,|G(0)| < 1$. On the other hand, the linearized equation about the positive EP is

$$z_{m+1} = L_1\, z_m + L_2\, z_{m-\ell}, \tag{2.8}$$

where

$$L_1 := \alpha + \beta \frac{H_t(1,1)}{H(1,1)} \lambda G'(\lambda),$$

and

$$L_2 := 1 - \alpha + \beta \frac{H_s(1,1)}{H(1,1)} \lambda G'(\lambda).$$

By using the Clark criterion (see [14, Theorem 1.3.7]), we get that the positive EP of (E) is asymptotically stable if

$$|L_1| + |L_2| < 1. \tag{2.9}$$

Next, we study the global stability of EPs of Eq (E).

**Theorem 2.** *Assume that* $\ell = 1$, $\Phi$ *has non-positive partial derivatives, and there is a real number* $c$ *such that* $0 < \Phi(t, s) \leq c$ *for all* $t, s \in [0, \infty)$. *Then the zero EP of (E) is GAS if* $\alpha > 0$ *and* $\alpha + c\beta < 1$.

*Proof.* Assume that $\mathcal{F}$ is defined as in (2.1). Now, we define the functions

$$\begin{aligned} f_0(t, s) &: = \alpha \\ f_1(t, s) &: = \beta\,\Phi(t, s). \end{aligned}$$

Then, Eq (E) takes the form

$$B_{m+1} = f_0(B_m, B_{m-1})\, B_m + f_1(B_m, B_{m-1})\, B_{m-1}.$$

Hence, it is easy to notice the following:

1) $f_0$ and $f_1$ are non-increasing with respect to $t$ and $s$.
2) $f_0(t, t) > 0$ for all $t \geq 0$.
3) Since $\alpha + c\beta < 1$, we have

$$\Phi(t, s) \leq c < \frac{1-\alpha}{\beta},$$

and so $\alpha + \beta\,\Phi(t, s) < 1$. Therefore,

$$f_0(t, s) + f_1(t, s) < 1 \text{ for all } t, s \in (0, \infty).$$

From Theorem 1, the *zero* EP of (E) is GAS. This completes the proof.

**Theorem 3.** *Suppose that* $\alpha \in (0, 1)$ *and there is a real number* $h$ *such that* $0 < \Phi(t, s) \leq h$. *If* $\alpha + \beta h < 1$, *then* $\lim_{m\to\infty} B_m = 0$.

*Proof.* We define the sequence $\left\{\widetilde{B}_m\right\}_{m=0}^{\infty}$ by

$$\widetilde{B}_m := \max\{B_{m-i},\ i = 0, 1, ..., \ell\}.$$

From (E), we see that

$$\begin{aligned} B_{m+1} &\leq \widetilde{B}_m\left[\alpha + \beta\Phi(B_m, B_{m-\ell})\right] \\ &\leq \widetilde{B}_m\left[\alpha + \beta h\right] \\ &\leq \widetilde{B}_m. \end{aligned} \tag{2.10}$$

Hence, the sequence $\left\{\widetilde{B}_m\right\}_{m=0}^{\infty}$ is nonincreasing, and so $\lim_{t\to\infty} \widetilde{B}_m = B_0 \geq 0$.

Suppose that $B_0 > 0$. Then, for all $\epsilon > 0$, there is a $M > 0$ such that $B_m \leq \widetilde{B}_m < B_0 + \epsilon$ for all $m > M$. It follows from (2.10) that

$$B_{m+1} < (B_0 + \epsilon)(\alpha + \beta h). \tag{2.11}$$

Since $\alpha + \beta h < 1$, there is a $\epsilon > 0$ small enough such that $\alpha + \beta h < 1 - 2\epsilon$, and so

$$(B_0 + \epsilon)(\alpha + \beta h) < B_0 - \epsilon,$$

which with (2.11) gives $B_{m+1} < B_0 - \epsilon$. Therefore, $\lim_{t\to\infty} B_m \leq B_0 - \epsilon$, a contradiction. Then, $B_0 = 0$ and $\lim_{t\to\infty} B_m = 0$. This completes the proof.

**Lemma 1.** *Suppose that $\Phi'(t,t) < 0$. Then, $\mathcal{F}(t,t)$ satisfies the negative feedback condition*

$$(t - p^*)(\mathcal{F}(t,t) - t) < 0 \tag{2.12}$$

*for $t \in \mathbb{R}^+ - \{p^*\}$.*

*Proof.* We define the function

$$\Theta(t) := \mathcal{F}(t,t) - t.$$

Then,

$$\begin{aligned} \Theta(t) &= (\alpha - 1)t + \beta t \Phi(t,t) \\ &= t[\alpha - 1 + \beta\Phi(t,t)]. \end{aligned}$$

It is easy to notice that $\Theta(t) = 0$ if and only if $t = 0$ or $p^*$. Now, assume $t < p^*$. Hence, $\Phi(t,t) > \Phi(p^*, p^*) = (1-\alpha)/\beta$, and then $\Theta(t) > 0$. Also, if $t > p^*$, then $\Theta(t) < 0$. Therefore, $\mathcal{F}(t,t)$ satisfies condition (2.12). This completes the proof.

**Lemma 2.** *Suppose that $\alpha \in (0,1)$, $\lambda > 0$, and there is a real number $c$ such that $s\Phi(t,s) \leq c$. Then*

$$\limsup_{m\to\infty} B_m \leq \frac{c\beta}{(1-\alpha)}. \tag{2.13}$$

*Proof.* From (E), we have

$$\begin{aligned} B_{m+1} &= \alpha B_m + \beta B_{m-\ell}\Phi(B_m, B_{m-\ell}) \\ &\leq \alpha B_m + c\beta. \end{aligned}$$

It is easy to verify that the sequence on the right side has the solution

$$\alpha^{m+1} B_0 + c\beta \frac{1 - \alpha^{m+1}}{1 - \alpha}.$$

Then

$$B_{m+1} \leq \alpha^{m+1} B_0 + c\beta \frac{1 - \alpha^{m+1}}{1 - \alpha}.$$

Taking $\limsup$ as $m \to \infty$, we arrive at (2.13). The proof is complete.

*2.2. Existence of periodic solutions*

We create, in the following, a criterion to ensure that there are two-cycle periodic solutions to Eq (E).

**Theorem 4.** *Suppose that $\ell$ is odd. DE (E) has a prime period two solution if and only if there is a real number $\gamma > 0$ such that*

$$\alpha\gamma + \beta G\left(\frac{H(\gamma,1)}{H(1,\gamma)} G^{-1}\left(\frac{\gamma-\alpha}{\beta\gamma}\right)\right) = 1, \tag{2.14}$$

*and this solution is $\{S_m\}_{m=-2}^{\infty}$ where*

$$S_m := \begin{cases} \gamma H^{-1/\kappa}(1,\gamma)\left[G^{-1}\left(\frac{\gamma-\alpha}{\beta\gamma}\right)\right]^{1/\kappa} & \text{for } m \text{ even;} \\ H^{-1/\kappa}(\gamma,1)\left[G^{-1}\left(\frac{1-\alpha\gamma}{\beta}\right)\right]^{1/\kappa} & \text{for } m \text{ odd.} \end{cases}$$

*Proof.* Suppose that (E) has a prime period two solution $\{..., l, k, l, k, l, k, ...\}$. It follows from (E) that

$$l = \alpha k + \beta l \Phi(k,l), \tag{2.15}$$

$$k = \alpha l + \beta k \Phi(l,k). \tag{2.16}$$

From (2.15), we obtain

$$\begin{aligned} 1 - \alpha\frac{k}{l} &= \beta G(H(k,l)) \\ &= \beta G\left(l^{\kappa} H\left(\frac{k}{l}, 1\right)\right), \end{aligned}$$

and so,

$$l^{\kappa} = \frac{1}{H(\gamma,1)} G^{-1}\left(\frac{1-\alpha\gamma}{\beta}\right). \tag{2.17}$$

Similarly, from (2.16), we obtain

$$l^{\kappa} = \frac{1}{H(1,\gamma)} G^{-1}\left(\frac{\gamma-\alpha}{\beta\gamma}\right). \tag{2.18}$$

From (2.17) and (2.18), we arrive at (2.14).

Conversely, we assume that (2.14) holds. Now, we choose

$$B_{-1} = \frac{1}{H^{1/\kappa}(\gamma,1)}\left[G^{-1}\left(\frac{1-\alpha\gamma}{\beta}\right)\right]^{1/\kappa} \text{ and } B_0 = \frac{\gamma}{H^{1/\kappa}(1,\gamma)}\left[G^{-1}\left(\frac{\gamma-\alpha}{\beta\gamma}\right)\right]^{1/\kappa},$$

for $\gamma \in \mathbb{R}^{+}$. Hence, from (2.14), we find

$$\begin{aligned} \Phi(B_0, B_{-1}) &= G(H(B_0, B_{-1})) \\ &= G\left(H\left(\frac{\gamma}{H^{1/\kappa}(1,\gamma)}\left[G^{-1}\left(\frac{\gamma-\alpha}{\beta\gamma}\right)\right]^{1/\kappa}, \frac{1}{H^{1/\kappa}(\gamma,1)}\left[G^{-1}\left(\frac{1-\alpha\gamma}{\beta}\right)\right]^{1/\kappa}\right)\right) \\ &= G\left(\frac{1}{H(\gamma,1)} G^{-1}\left(\frac{1-\alpha\gamma}{\beta}\right) H(\gamma,1)\right) \\ &= \frac{1-\alpha\gamma}{\beta}. \end{aligned}$$

Then,

$$\begin{aligned} B_1 &= \alpha B_0 + \beta B_{-1}\Phi(B_0, B_{-1}) \\ &= \frac{\alpha\gamma}{H^{1/\kappa}(1,\gamma)}\left[G^{-1}\left(\frac{\gamma-\alpha}{\beta\gamma}\right)\right]^{1/\kappa} + \beta\frac{1-\alpha\gamma}{\beta}\frac{1}{H^{1/\kappa}(\gamma,1)}\left[G^{-1}\left(\frac{1-\alpha\gamma}{\beta}\right)\right]^{1/\kappa} \\ &= \frac{1}{H^{1/\kappa}(\gamma,1)}\left[G^{-1}\left(\frac{1-\alpha\gamma}{\beta}\right)\right]^{1/\kappa} \\ &= B_{-1}. \end{aligned}$$

Similarly, we have that $B_2 = B_0$. Therefore, Eq (E) has a prime period two solution.

**Theorem 5.** *Suppose that $\ell$ is even. DE (E) has a prime period two solution if and only if there exists a real number $\gamma > 0$ such that*

$$\alpha\gamma + \beta\gamma G\left(\gamma^{\kappa} G^{-1}\left(\frac{\gamma-\alpha}{\beta}\right)\right) = 1, \tag{2.19}$$

*and this solution is $\{S_m\}_{m=-2}^{\infty}$ where*

$$S_m := \begin{cases} H^{-1/\kappa}(1,1)\left[G^{-1}\left(\frac{1-\alpha\gamma}{\beta\gamma}\right)\right]^{1/\kappa} & \text{for } m \text{ even;} \\ H^{-1/\kappa}(1,1)\left[G^{-1}\left(\frac{\gamma-\alpha}{\beta}\right)\right]^{1/\kappa} & \text{for } m \text{ odd.} \end{cases}$$

*Proof.* Suppose that (E) has a prime period two solution $\{..., l, k, l, k, l, k, ...\}$. It follows from (E) that

$$l = \alpha k + \beta k\Phi(k,k) \tag{2.20}$$

$$k = \alpha l + \beta l\Phi(l,l). \tag{2.21}$$

From (2.20), we obtain

$$\frac{1}{\beta}\left(\frac{l}{k} - \alpha\right) = G\left(k^{\kappa} H(1,1)\right),$$

and then

$$k^{\kappa} = \frac{1}{H(1,1)} G^{-1}\left(\frac{1-\alpha\gamma}{\beta\gamma}\right). \tag{2.22}$$

Similarly, from (2.21), we obtain

$$l^{\kappa} = \frac{1}{H(1,1)} G^{-1}\left(\frac{\gamma-\alpha}{\beta}\right). \tag{2.23}$$

Combining (2.17) and (2.18), we arrive at (2.19).

We omitted the rest of the proof because it is similar to the proof of the previous theorem.

**Remark 1.** *Equations (1.2)–(1.5) are special cases of Eq (E). Therefore, we can obtain the results for stability and periodicity in [15–19] by imposing different forms for the function $\Phi$ and the parameters $\alpha$ and $\beta$.*

## 3. The flour beetle model

The genus Tribolium or Tenebrio of darkling beetles includes flour beetles. They are common laboratory animals because they are easy to keep. The flour beetles eat wheat and other grains, can endure even more radiation than cockroaches, and are designed to live in very arid settings [22]. They are a significant pest in the agriculture sector and have a high level of pesticide resistance. These insects are present all over the world and can infest food that has been preserved, which can alter the flavor of the food.

The larva, pupa, and adult stages of the flour beetle's life cycle are separated by approximately two weeks. The species also shows nonlinear interactions between life stages, such as moving stages cannibalizing non-moving stages. Population dynamics are made possible, which is highly interesting.

The "Larvae-Pupae-Adult" (LPA) model, which describes the dynamics of flour beetle population dynamics, is one of the most thoroughly studied and well-validated models in mathematical ecology. The following system of three DEs provides the model:

$$\begin{cases} L_{m+1} = kA_m \exp\left(-\mu_{ea}L_m - \mu_{el}A_m\right) \\ P_{m+1} = \left(1-\eta_l\right)L_m \\ A_{m+1} = P_m \exp\left(-\mu_{pa}A_m\right) + \left(1-\eta_a\right)A_m, \end{cases} \tag{3.1}$$

where $L_m$, $P_m$, and $A_m$ are the number of larvae at time $t$, the number of individuals in the "$P$ stage" (including non-feeding larvae, pupae, and callow adults) at time $t$, and the number of sexually mature adults at time $t$, respectively. $k$ is the rate at which an adult lays eggs. $\mu_{ea}$ and $\mu_{el}$ are cannibalism coefficients of eggs by larvae and eggs by adults, respectively. Pupae must escape cannibalism by adults $\mu_{pa}$ to become adults. $\eta_a$ and $\eta_l$ are the rates of adult death and naturally larval death, respectively.

In the simplified case when larval cannibalism of eggs is not present, i.e., $\mu_{el}=0$, we observe that the DE

$$B_{m+1} = \left(1-\eta_a\right)B_m + k\left(1-\eta_l\right)B_{m-2}\exp\left(-\mu_{ea}B_{m-2} - \mu_{pa}B_m\right), \tag{3.2}$$

represents system (3.1), where $\eta_a, \eta_l \in (0,1)$, $k>0$, $\mu_{pa}+\mu_{el}>0$, and

$$\begin{aligned} B_{-2} &= A_0; \\ B_{-1} &= P_0\exp\left(-\mu_{pa}A_0\right) + \left(1-\eta_a\right)A_0; \\ B_0 &= \left(1-\eta_l\right)L_0^{-\mu_{pa}A_1} + \left(1-\eta_a\right)A_1. \end{aligned}$$

Now, to apply the results of the previous section, we set that

$$\kappa = 1,\ \alpha = 1-\eta_a,\ \beta = k\left(1-\eta_l\right),\ G(t) = \mathrm{e}^{-t},\ \text{and } H(t,s) = \mu_{ea}\,s + \mu_{pa}\,t.$$

Then, we conclude that the EPs of model (3.2) are $p^*=0$ or

$$p^* = \frac{1}{\mu_{ea}+\mu_{pa}}\ln\left(\frac{k\left(1-\eta_l\right)}{\eta_a}\right),$$

which is positive if $k\left(1-\eta_l\right) > \eta_a$. Moreover, the stability behavior of EPs is as follows:

- The trivial EP of (3.2) is LAS if $\eta_a > k\left(1-\eta_l\right)$.

- The positive EP of (3.2) is LAS if

$$\left|1 - \eta_a - \delta\mu_{pa}\right| + |\eta_a - \delta\mu_{ea}| < 1, \tag{3.3}$$

where

$$\delta := \frac{\eta_a}{\mu_{ea} + \mu_{pa}} \ln\left(\frac{k(1-\eta_l)}{\eta_a}\right).$$

- There exists a constant $c = 1/(\mathrm{e}\mu_{ea})$ such that $s\Phi(t,s) \leq c$, and then

$$\lim_{m\to\infty}\sup B_m \leq \frac{k(1-\eta_l)}{\mathrm{e}\mu_{ea}\eta_a}. \tag{3.4}$$

- There exists a constant $h = 1$ such that $\Phi(t,s) \leq h$. Then $\lim_{m\to\infty} B_m = 0$, if $\eta_a > k(1-\eta_l)$.

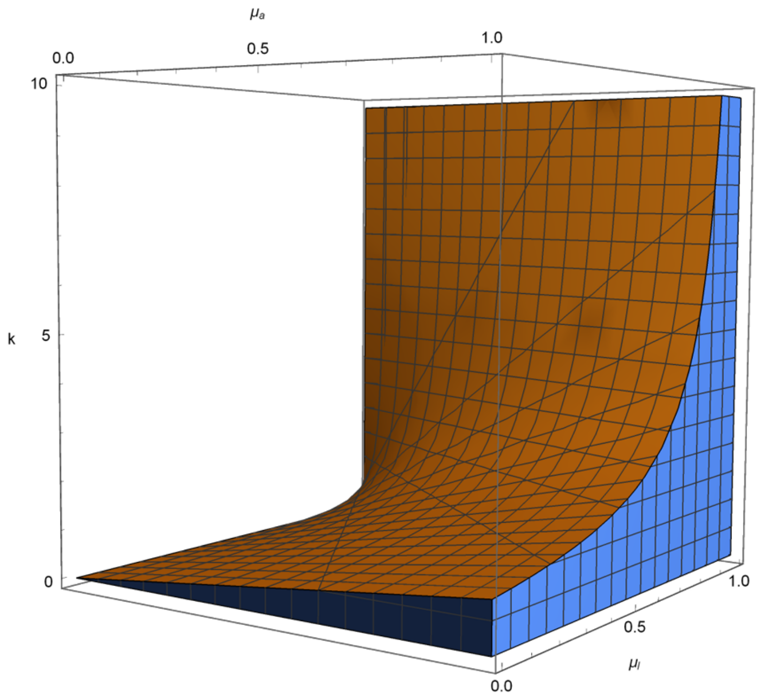



**Figure 2.** Local stability region of the zero EP of Model (3.2).

Figure 2 shows the region where local stability occurs for the zero EP of model (3.2). We note that the relationship between the ratio $\frac{\mu_a}{1-\mu_l}$ and $k$ (the number of eggs laid per adult per unit of time) is the one that controls the stability or instability of the zero EP of model (3.2). We notice that with the increase in rates of adult death and naturally larval death (approaching 1), the system approaches stability, and in the case of a decrease (approaching 0) of these rates, the system turns into instability. As for the instability points, they are of the saddle points if the ratio $\frac{\mu_a}{1-\mu_l}$ is less than the number of eggs laid per adult per unit of time, and of the nonhyperbolic points if this ratio is equal to $k$.

**Lemma 3.** *Assume that $\mathcal{F}(t,s) = \alpha t + \beta s \exp(-c_1 s - c_2 t)$. If $\beta < \mathrm{e}(1-\alpha)$ and $\beta c_2 < \mathrm{e}\,\alpha\, c_1$, then $\mathcal{F}_t$ and $\mathcal{F}_s$ are non-negative.*

*Proof.* From (2.5), we have $\mathcal{F}_t \geq \alpha - c_2\beta\, s \exp(-c_1 s) \geq \alpha - \frac{c_2\beta}{c_1 \mathrm{e}}$, which with the fact that $\beta c_2 < \mathrm{e}\,\alpha\, c_1$ gives $\mathcal{F}_t \geq 0$. Also, we have that $\mathcal{F}_s = \beta(1 - c_1 s)\exp(-c_1 s - c_2 t)$, which is positive by using (3.4) and the fact that $\beta < \mathrm{e}(1-\alpha)$. The proof is complete.

**Theorem 6.** *If*

$$\eta_a < k(1-\eta_l) < \min\left\{\mathrm{e}\eta_a, \mathrm{e}(1-\eta_a)\frac{\mu_{ea}}{\mu_{pa}}\right\}, \tag{3.5}$$

*then the positive EP of (3.2) is GAS.*

*Proof.* It is easy to notice that Lemmas 1 and 3 guarantee the conditions of global stability in Theorem 1.4.1 in [21].

**Corollary 1.** *Model (3.2) has a prime period two solution if and only if there exists a positive real number $\gamma$ such that*

$$\left(\frac{k(1-\eta_l)}{\gamma - 1 + \eta_a}\right)^{\gamma} = \frac{k(1-\eta_l)\gamma}{1-(1-\eta_a)\gamma},$$

*and this solution is $\{S_m\}_{m=-2}^{\infty}$ where*

$$S_m := \begin{cases} \frac{1}{\mu_{ea}+\mu_{pa}} \ln\left(\frac{k(1-\eta_l)\gamma}{1-(1-\eta_a)\gamma}\right) & \text{for } m \text{ even;} \\ \frac{1}{\mu_{ea}+\mu_{pa}} \ln\left(\frac{k(1-\eta_l)}{\gamma-1+\eta_a}\right) & \text{for } m \text{ odd.} \end{cases}$$

**Remark 2.** *Kuang and Cushing [23] and Brozak et al. [24] presented the global stability results for model (3.1). The previous global stability results agree with the results presented in [23, 24] but the results in [23, 24] do not provide any information about the periodicity of the solutions of model (3.1).*

## 4. Numerical simulation examples

Here, we provide some numerical examples that support the results in the previous sections.

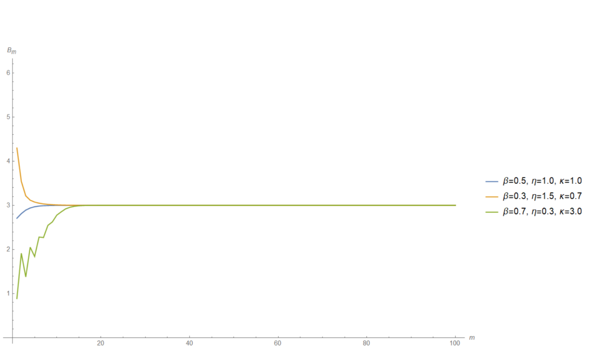
**Example 1.** *Consider the Baleen Whales model*

$$B_{m+1} = (1-\beta)B_m + \beta B_{m-\ell}\left(1 + \eta\left(1 - \left(\frac{B_{m-\ell}}{M}\right)^{\kappa}\right)\right)_+, \tag{4.1}$$

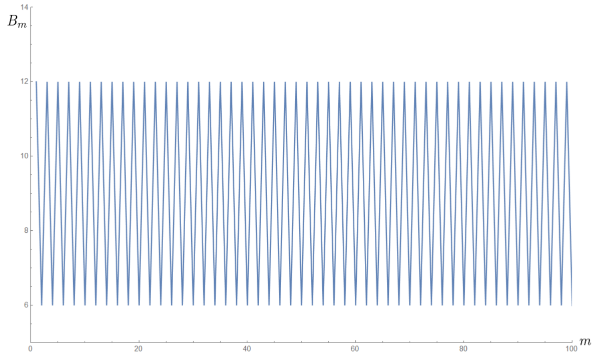
*where $\beta \in (0,1)$, and $\eta$, $M \in (0,\infty)$. Note that $\Phi(t,s) = G(H(t,s))$, where*

$$H(t,s) = \left(\frac{s}{M}\right)^{\kappa} \quad \text{and} \quad G(u) = (1 + \eta(1-u))_+ .$$

*The positive EP of (4.1) $p^* = M$ is asymptotically stable if $\kappa\eta < 2$, see Figure 3a. We notice in this example that the EP is not affected by the parameters $\kappa$, $\beta$, and $\eta$, which means that the EP will remain constant no matter how the values of these parameters change, as is clear from Figure 3a. In Figure 3b, we note that Eq (4.1) has the prime period two solution $\{..., 6, 12, 6, 12, ...\}$ when $\kappa = 1$, $\ell = 2$, $\beta = 0.5$, $\eta = 5$, and $M = 10$, as Theorem 5 indicates. We also notice that any slight change in the initial conditions causes a defect in the periodic solution of the equation.*




**(a)** Stable solutions to (4.1)




**(b)** A periodic solution to (4.1)

**Figure 3.** Some numerical solutions to Eq (4.1).

**Example 2.** *Consider the DE*

$$B_{m+1} = \alpha B_m + \frac{\beta B_{m-\ell}}{a + B_m B_{m-\ell}}, \tag{4.2}$$

*where* $\beta, a \in (0, \infty)$ *and* $\alpha,\ B_{-\ell},\ B_{-\ell+1},\ \ldots, B_0 \in [0, \infty)$. *Note that* $\Phi(t, s) = G(H(t, s))$, *where*

$$H(t, s) = ts \quad \text{and} \quad G(u) = \frac{1}{a + u}.$$

*Zero EP of Eq (4.2) is LAS if* $\alpha a + \beta < a$. *Moreover, it is easy to notice that* $0 < \Phi(t, s) \le \frac{1}{a}$. *Then, it follows from Theorem 2 that zero EP of (4.2) is GAS if* $\alpha a + \beta < a$. *Figure 4a shows a set of stable solutions to Eq (4.2). We notice that stability is not affected by the initial data as long as the condition* $\alpha a + \beta < a$ *is satisfied.*

*The positive EP of (4.2) is*

$$p^* = \sqrt{\frac{\beta}{1 - \alpha} - a}, \quad \text{where } a\alpha + \beta > a,$$

*which is LAS if*

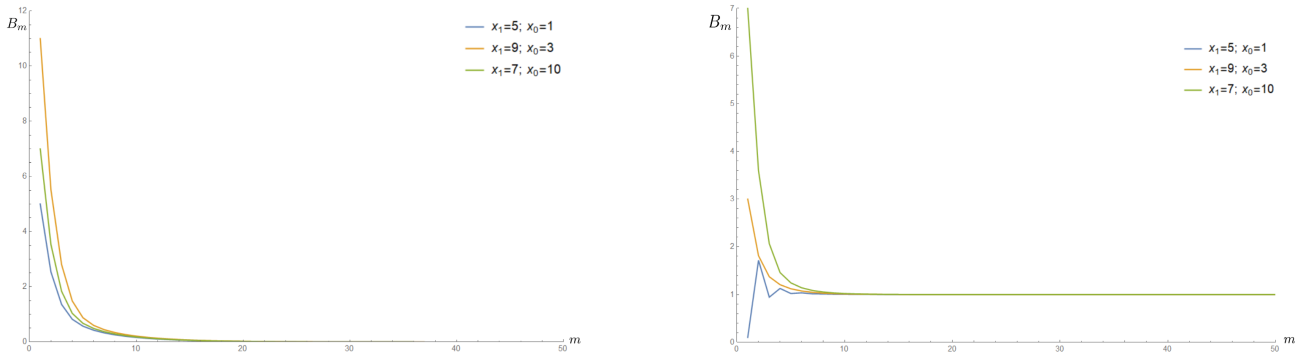
$$\left| \alpha - \beta \frac{\lambda}{(a + \lambda)^2} \right| + \left| 1 - \alpha - \beta \frac{\lambda}{(a + \lambda)^2} \right| < 1.$$

*In Figure 4b, we see that some solutions to Eq (4.2) are stable with different initial data; this is consistent with condition (2.9).*

**Example 3.** *Consider the flour beetle model (3.2) when* $\eta_a = \eta_l = 0.5$, *and* $\mu_{ea} = \mu_{pa} = 1$, *namely,*

$$B_{m+1} = \frac{1}{2} B_m + \frac{1}{2} k\, B_{m-2} \exp\left(-B_{m-2} - B_m\right). \tag{4.3}$$

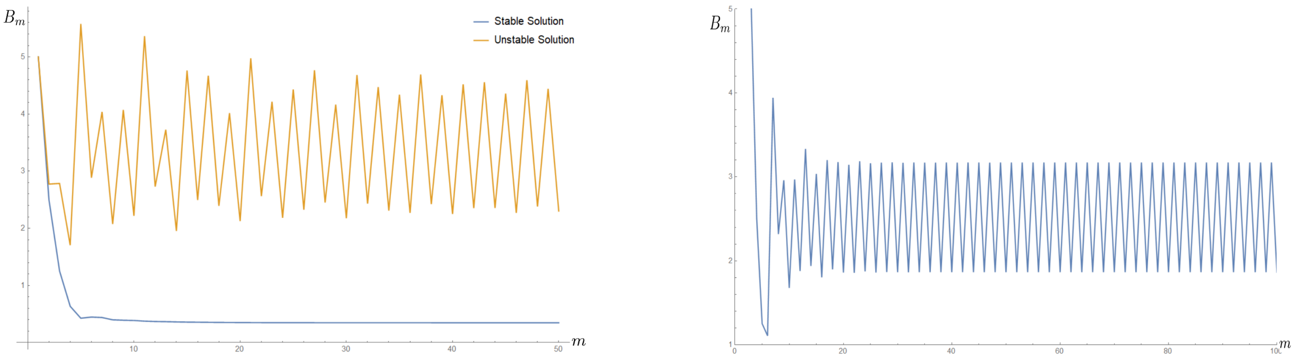
*Figure 5a shows two solutions of (3.2), one of which is stable (when* $k = 2$*) and the other is unstable (when* $k = 300$*); this supports the validity of condition (3.3). Figure 5b shows a periodic solution of Eq (4.3) when* $k = 100$*, as Theorem 5 indicates.*




**(a)** Stable solutions to (4.2) (zero EP)

**(b)** Stable solutions to (4.2) (positive EP)

**Figure 4.** Some numerical solutions to Eq (4.2).




**(a)** Stable and unstable solutions to (4.3)

**(b)** A periodic solution to (4.3)

**Figure 5.** Some numerical solutions to Eq (4.3).

## 5. Conclusions

In biology, DEs are frequently employed as mathematical representations of actual biological phenomena. In this work, we examined the qualitative characteristics of solutions to a general model of DEs, with the flour beetle model serving as a particular example. We studied the periodic behavior of the solutions to this model, as well as boundedness and stability. The numerical examples that are provided validate the findings and show how the solutions for the model under study behave. As an interesting future research point, we propose to study the oscillatory and bifurcation behavior of the general model studied. We also hope in the future to be able to obtain similar results that include cases where $\kappa$ is negative, as well as to study the existence of periodic solutions with period three and the global stability of periodic solutions.

## Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The authors declare there are no conflicts of interest.

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