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Research article

Blow-up of solutions for a time fractional biharmonic equation with exponentional nonlinear memory

Yuchen Zhu*

College of Mathematics & Statistics, Nanjing University of Information Science & Technology, Nanjing 210044, China

* **Correspondence:** Email: zhuyc913@163.com.

Abstract: In the paper, we focus on the local existence and blow-up of solutions for a time fractional nonlinear equation with biharmonic operator and exponentional nonlinear memory in an Orlicz space. We first establish a $L^p - L^q$ estimate for solution operators of a time fractional nonlinear biharmonic equation, and obtain bilinear estimates for mild solutions. Then, based on the contraction mapping principle, we establish the local existence of mild solutions. Moreover, by using the test function method, we obtain the blow-up result of solutions.

Keywords: fractional biharmonic equation; exponentional nonlinear memory; blow-up; local existence

1. Introduction

Fractional differential equations have garnered significant interest owing to their extensive utilization across various scientific and engineering disciplines [1, 2]. Fractional differential equations serve as a modeling tool for anomalous diffusion processes, characterize Hamiltonian chaos, and various other phenomena, as detailed in the references [3–5]. In recent years, more and more papers study the properties of solutions for fractional differential equations; see [6–11] and references therein. For example, in [7], the local well-posedness and existence of blow-up solutions for a fourth-order Schrödinger equation with combined power-type nonlinearities were established by applying Banach's fixed point theorem, iterative method, modified Strichartz estimates, and variational analysis theory for dynamical systems. Zhang et al. [9] proved the local and global well-posedness for a higher order nonlinear dispersive equation with the initial data in the Sobolev space $H^s(\mathbb{R})$ by using the Fourier restriction norm method, Tao's [k, Z]-multiplier method, and the contraction mapping principle.

In this paper, we consider the local existence and blow-up of solutions to the following fractional biharmonic equation

$$\begin{cases} \partial_{0|t}^{\alpha}u + \Delta^2 u = J_{0|t}^{1-\gamma}(e^u), \quad t > 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x) \quad x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $1 < \alpha < 2$, $0 < \gamma < 1$, $\partial_{0|t}^{\alpha}$ is α order Caputo fractional derivative, Δ^2 denotes the biharmonic operator, and $J_{0|t}^{1-\gamma}(e^u)$ is the left Riemann-Liouville fractional integral of order $1-\gamma$ for e^u, defined by

$$J_{0|t}^{1-\gamma}(e^u) = \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} e^{u(s)} ds,$$

and the initial data $u_0, u_1 \in \exp L_0^p(\mathbb{R}^n)$, where $\exp L_0^p(\mathbb{R}^n)$ is the so-called Orlicz space, and its definition will be presented in Section 2.

There are many papers that studied equations in the Orlicz space. For example, Ioku [12] derived the global solutions to the following problem

$$\begin{cases} \partial_t u - \Delta u = f(u), \quad t > 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.2}$$

where u_0 is small enough in $\exp L^2(\mathbb{R}^n)$ and

$$|f(u) - f(v)| \leq C|u-v|(|u|^{m-1}e^{\lambda|u|^p} + |v|^{m-1}e^{\lambda|v|^p}), \quad f(0) = 0, \tag{1.3}$$

for $u, v \in \mathbb{R}$ with $m = 1 + \frac{4}{n}$. Then, Ioku et al. [13] obtained the results of local existence of (1.2) in $\exp L_0^2(\mathbb{R}^2)$, which is a subspace of $\exp L^2(\mathbb{R}^2)$ if

$$|f(u) - f(v)| \leq C|u-v|(e^{\lambda|u|^p} + e^{\lambda|v|^p}), \quad f(0) = 0, \tag{1.4}$$

for $u, v \in \mathbb{R}$. When the nonlinearity $f(u) = |u|^{\frac{4}{n}} u e^{u^2}$, Furioli et al. [14] derived the asymptotic behavior and decay estimates for global solutions of (1.2) in $\exp L^2(\mathbb{R}^n)$.

In [15], Majdoub et al. studied the following biharmonic equation

$$\begin{cases} \partial_t u + \Delta^2 u = f(u), \quad t > 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n. \end{cases} \tag{1.5}$$

They studied the local existence of solutions in $\exp L_0^2(\mathbb{R}^n)$ and the global existence of solutions when u_0 is small enough in $\exp L^2(\mathbb{R}^n)$. Later, [16, 17] obtained the local solutions in $\exp L_0^p(\mathbb{R}^n)$ and the global solutions of (1.2) $\exp L^p(\mathbb{R}^n)$. In [18], the authors generalized [16, 17] to the case of fractional laplacian.

In [19], Tuan et al. studied the following fractional biharmonic equation

$$\begin{cases} \partial_{0|t}^{\alpha}u + \Delta^2 u = f(u), \quad t > 0, \quad x \in \mathbb{R}^n, \\ u(0,x) = u_0(x), \quad x \in \mathbb{R}^n, \end{cases} \tag{1.6}$$

where $0 < \alpha < 1$ and f satisfies (1.3). They first proved the generalized formula for the mild solution as well as the smoothing effects of resolvent operators by using the Fourier transform concept. Then, by some embeddings between the Orlicz space and the usual Lebesgue spaces, they obtained the global solutions and the blow-up solutions with the initial data $u_0 \in L^p(\mathbb{R}^n \cap C_0(\mathbb{R}^n))$. They also proved the local existence of mild solutions with $u_0 \in \exp L^p(\mathbb{R}^n)$ and the global well-posedness of mild solutions with $u_0 \in \exp L_0^p(\mathbb{R}^n)$. Later, In [20], Tuan et al. studied the case of fractional laplacian corresponding (1.6) and obtained the local solutions with initial data in $L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ and the global solutions with small initial data in an Orlicz space by using the Picard iteration method and some L^p-L^q estimates of fundamental solutions associated with the Mittag–Leffler function. In [21], the authors studied the solvability of the Cauthy problem of (1.6) with an irregular initial data u_0 and proved the presence of a strongly continuous analytic semigroup.

For time fractional diffusion-wave equation

$$\begin{cases} \partial_{0|t}^{\alpha} u - \Delta u = f(u), & x \in \mathbb{R}^n, \quad t > 0, 1 < \alpha < 2 \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x) & x \in \mathbb{R}^n, \end{cases} \tag{1.7}$$

Wang et al. [22] first obtained a nonlinear estimate and $L^p - L^q$ estimates for the nonlinearity and the solution operators, respectively, and then by applying the contraction mapping principle, they proved the local existence of solutions in $\exp L^2(\mathbb{R}^n)$ when the nonlinearity of (1.7) possesses an exponential growth. Furthermore, with some additional assumptions on the initial data, the authors proved the global existence of solutions in the high dimension case where $n \geq 3$.

For the type of the nonlinearity in (1.1), we also give an overview. Fino and Kirane [23] considered the following space fractional diffusion equation with a nonlinearity memory

$$\begin{cases} u_t + (-\Delta)^{\frac{\beta}{2}} u = J_{0|t}^{1-\alpha}(|u|^{p-1}u), & x \in \mathbb{R}^n, \quad t > 0, 0 < \alpha < 1 \\ u(0,x) = u_0(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.8}$$

where $0 < \beta \leq 2$. They proved the local and global well-posedness of solutions and studied the time blow-up profile in $C_0(\mathbb{R}^n)$. Ahmad et al. [24] considered another case that the nonlinearity in (1.8) is replaced by $J_{0|t}^{1-\alpha}(e^u)$. They proved the local well-posedness of solutions in $C_0(\mathbb{R}^n)$ and obtained the blow-up solutions with some conditions on the initial data. They also studied the time blow-up profile of the solutions. In [25], the authors generalized [24] to a time–space fractional equation and obtained similar results.

Motivated by the above papers, our purpose in this paper is to consider the local existence and blow-up of solutions for (1.1) involving the time fractional operator, the space biharmonic operator, and the nonlinearity of the form $J_{0|t}^{1-\gamma}(e^u)$ with $0 < \gamma < 1$ under the assumption that $u_0, u_1 \in \exp L_0^p(\mathbb{R}^n)$. Compared with [24, 25], our paper also obtains the local existence of the mild solutions and the blow-up result. However, we must emphasize that our paper is not a simple generalization of [24, 25]. We study the mild solutions in $\exp L_0^p(\mathbb{R}^n)$, while [24, 25] is in $C_0(\mathbb{R}^n)$. Moreover, there are many differences between the equation we study in this paper and the equation in [24, 25]. The equation study in this paper involves the time fractional operator with $1 < \alpha < 2$ and the space biharmonic operator while [24, 25] involves the time integer operator or fractional operator with $0 < \alpha < 1$ and the space fractional operator. Also, we do not set the parameter in the time fractional operator and the parameter in the nonlinearity as the same, which is different from [24, 25]. To the best of my knowledge, there are few papers to deal with fractional biharmonic equations with $1 < \alpha < 2$. Note that if $0 < \alpha < 1$, the estimating $L^p - L^q$ are available for the corresponding solution operator. However, some estimates of the form $L^p - L^q$ are not available on the domain $\mathbb{R}^n$ for the case of $1 < \alpha < 2$. So, the main difficulty is to establish the estimates of $L^p - L^q$ for the solution operator. By using the definition of Orlicz space and the $L^p - L^q$ estimate, we obtain bilinear estimates for both the nonlinear and linear parts within the representation of mild solutions. Then, based on the contraction mapping principle, we proved the local existence and uniqueness of mild solutions in $\exp L^p_0(\mathbb{R}^n)$. Finally, we obtain the blow-up result in $\exp L^p_0(\mathbb{R}^n)$ that when $u_0 \geq 0$, $u_0 \not\equiv 0$, $u_1 \equiv 0$, then the solutions of (1.1) will blow up in a finite time if $\frac{\alpha n}{4} - \alpha + \gamma \geq 0$.

The structure of this paper is outlined in the following manner. Section 2 presents some preliminaries. In Section 3, we state some properties and estimates of the related operators. In Section 4, we establish the local well-posedness of solutions for problem (1.1). In Section 5, we prove the blow-up of solutions to problem (1.1).

2. Preliminaries

First, we present the definition of Orlicz space on $\mathbb{R}^n$. Readers can refer to [26,27] for more details. Let us define a convex increasing function $w : \mathbb{R}^+ \to \mathbb{R}^+$ and

$$w(0) = 0 = \lim_{z\to 0^+} w(z), \lim_{z\to\infty} w(z) = \infty.$$

The Orlicz space

$$L^w(\mathbb{R}^n) = \left\{ u \in L^1_{loc}(\mathbb{R}^n) \;\middle|\; \int_{\mathbb{R}^n} w\Big(\frac{|u(x)|}{\lambda}\Big) dx < \infty,\ for\ some\ \lambda > 0 \right\},$$

with the norm

$$\|u\|_{L^w(\mathbb{R}^n)} = \inf \left\{ \lambda > 0 \;\middle|\; \int_{\mathbb{R}^n} w\Big(\frac{|u(x)|}{\lambda}\Big) dx \leq 1 \right\}.$$

We also denote another Orlicz space

$$L^w_0(\mathbb{R}^n) = \left\{ u \in L^1_{loc}(\mathbb{R}^n) \;\middle|\; \int_{\mathbb{R}^n} w\Big(\frac{|u(x)|}{\lambda}\Big) dx < \infty,\ for\ every\ \lambda > 0 \right\}.$$

It has been shown in [13] that $L^w_0(\mathbb{R}^n)$ is the closure of $C^\infty_0(\mathbb{R}^n)$ in $L^w(\mathbb{R}^n)$.
$(L^w(\mathbb{R}^n), \|\cdot\|_{L^w(\mathbb{R}^n)})$ and $(L^w_0(\mathbb{R}^n), \|\cdot\|_{L^w(\mathbb{R}^n)})$ are Banach spaces. Therefore, we can easily get that $L^w(\mathbb{R}^n)$ is $\exp L^p(\mathbb{R}^n)$ and $L^w_0(\mathbb{R}^n)$ is $\exp L^p_0(\mathbb{R}^n)$ if $w(z) = e^{z^p}, 1 \leq p < \infty$. Moreover, for $u \in L^w$ and $S = \|u\|_{L^w(\mathbb{R}^n)} > 0$, by the definition of the infimum, we can easily obtain that

$$\left\{ \lambda > 0 \;\middle|\; \int_{\mathbb{R}^n} w\Big(\frac{|u(x)|}{\lambda}\Big) dx \leq 1 \right\} = [S, \infty[.$$

Then, we present two Lemmas involving Orlicz space and Lebesgue space.

Lemma 2.1. *[16] For $1 \leq q \leq p$, we have the embedding $L^q(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \hookrightarrow \exp L^p_0(\mathbb{R}^n) \hookrightarrow \exp L^p(\mathbb{R}^n)$ and the estimate*

$$\|u\|_{\exp L^p(\mathbb{R}^n)} \leq \frac{1}{(\ln 2)^{\frac{1}{p}}} (\|u\|_{L^q} + \|u\|_{L^\infty}). \tag{2.1}$$

Lemma 2.2. *[16] For $1 \leq p \leq q < \infty$, we have the embedding $\exp L^p(\mathbb{R}^n) \hookrightarrow L^q(\mathbb{R}^n)$ and the estimate*

$$\|u\|_q \leq \left(\Gamma\Big(\frac{q}{p} + 1\Big)\right)^{\frac{1}{q}} \|u\|_{\exp L^p(\mathbb{R}^n)}. \tag{2.2}$$

Next, we present some properties concerning the fractional derivatives and integrals. For more details, readers can refer to [1, 28].

Let $f \in C^2([0,T])$, $\alpha \in (1,2)$. Then the Caputo fractional derivative of order α can be written as

$$\partial^\alpha_{0|t} f(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} f''(s)ds.$$

The left-sided and right-sided Riemann–Liouville fractional derivative of order α are defined by

$$D^\alpha_{0|t} f(t) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_0^t (t-s)^{1-\alpha} f(s)ds.$$

$$D^\alpha_{t|T} f(t) = \frac{1}{\Gamma(2-\alpha)} \frac{d^2}{dt^2} \int_t^T (s-t)^{1-\alpha} f(s)ds.$$

Let $f, g \in C^2([0,T])$, then if $D^\alpha_{0|t} f, D^\alpha_{0|t} g$ exist and are continuous, we have the formula of integration by parts

$$\int_0^T g(t) D^\alpha_{0|t} f(t)dt = -\int_0^T f(t) D^\alpha_{t|T} g(t)dt.$$

For given $T > 0$ and $\eta >> 1$, if we put

$$\varphi_1(t) = (1 - \frac{t}{T})^\eta_+,$$

then for $\alpha > 0$, we have

$$D^\alpha_{t|T} \varphi_1(t) = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\alpha+1)} T^{-\alpha} (1 - \frac{t}{T})^{\eta-\alpha}_+, \tag{2.3}$$

and

$$\int_0^T D^\alpha_{t|T} \varphi_1(t)dt = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\alpha+2)} T^{1-\alpha}. \tag{2.4}$$

The following are Riemann–Liouville fractional integrals:

$$J^\alpha_{0|t} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)ds, J^\alpha_{t|T} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^T (s-t)^{\alpha-1} f(s)ds.$$

The operators $J^\alpha_{0|t}$ and $J^\alpha_{t|T}$ are bounded on $L^p((0,T))$ ($1 \le p \le +\infty$). $J^\alpha_{0|t} J^\beta_{0|t} f = J^{\alpha+\beta}_{0|t} f$ and $J^\alpha_{t|T} J^\beta_{t|T} f = J^{\alpha+\beta}_{t|T} f$ if $f \in L^1((0,T))$.

The Mittag–Leffler function is defined for complex $z \in \mathbb{C}$ as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^\infty \frac{z^k}{\Gamma(\alpha k + \beta)},\ \alpha, \beta \in \mathbb{C},\ Re(\alpha) > 0.$$

It satisfies

$$J^{2-\alpha}_{0|t}(t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha)) = t E_{\alpha,2}(\lambda t^\alpha)\ for\ \lambda \in \mathbb{C},\ 1 < \alpha < 2,$$
$$J^{2-\alpha}_{0|t}(t^{\alpha-2} E_{\alpha,\alpha-1}(\lambda t^\alpha)) = E_{\alpha,1}(\lambda t^\alpha)\ for\ \lambda \in \mathbb{C},\ 1 < \alpha < 2,$$
$$J^{\alpha-1}_{0|t}(E_{\alpha,1}(\lambda t^\alpha)) = t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha)\ for\ \lambda \in \mathbb{C},\ 1 < \alpha < 2,$$
$$\frac{d}{dt}[t^{\alpha-1} E_{\alpha,\alpha}(\lambda t^\alpha)] = t^{\alpha-2} E_{\alpha,\alpha-1}(\lambda t^\alpha)\ for\ \lambda \in \mathbb{C},\ 1 < \alpha < 2,$$
$$J^1_{0|t}(E_{\alpha,1}(\lambda t^\alpha)) = t E_{\alpha,2}(\lambda t^\alpha)\ for\ \lambda \in \mathbb{C},\ 1 < \alpha < 2.$$

3. Some properties and estimates of operators

This section mainly presents properties and estimates of solution operators.

First, we state the definition. Ley $\mathcal{F}^{-1}$ denote the Fourier inverse transform. For any $u \in L^p(\mathbb{R}^n),\ p \geq 1$, we define

$$\begin{cases} X_{\alpha,1}(t)u(x) = \int_{\mathbb{R}^n} \mathcal{F}^{-1}_{\xi\to y}[E_{\alpha,1}(-t^\alpha|\xi|^4)](y)u(x-y)dy, \\ X_{\alpha,2}(t)u(x) = \int_{\mathbb{R}^n} \mathcal{F}^{-1}_{\xi\to y}[t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha|\xi|^4)](y)u(x-y)dy, \\ \bar{X}_{\alpha,2}(t)u(x) = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-\tau)^{1-\alpha}X_{\alpha,2}(\tau)u(x)d\tau, \end{cases}$$

which is a generalization of the operator semigroup.

Remark 3.1. *From Proposition 2.1 in [21], we can get Δ^2 is a sectorial operator in $L^p(\mathbb{R}^n)$, so the above definition is equivalent to Definition 3.1 in [29].*

Then, combining [6] Theorems 3.1 and 3.2 and Remark 1.6, one has the following lemma that will help us prove the continuity of solution operators.

Lemma 3.1. *$X_{\alpha,1}(t), t^{1-\alpha}X_{\alpha,2}(t), t^{-1}\bar{X}_{\alpha,2}(t)$ is bounded linear operators on $L^p(\mathbb{R}^n),\ p \geq 1$ and $t \to X_{\alpha,1}(t), t \to t^{1-\alpha}X_{\alpha,2}(t), t \to t^{-1}\bar{X}_{\alpha,2}(t)$ is continuous function from $\mathbb{R}^+$ to $L^p(\mathbb{R}^n)$.*

The following theorem is $L^p - L^q$ estimates for $X_{\alpha,1}(t)u$ and $X_{\alpha,2}(t)u$, which plays a great role in deriving the estimates for solution operators in Orlicz spaces.

Theorem 3.1. *Let $1 < p \leq q \leq \infty$, $p < \infty$. Then, there exists a positive constant C such that for $t > 0$, the following assertions are satisfied.*

(i) *If $\frac{n}{p} - \frac{n}{q} < 4$, for $u \in L^p(\mathbb{R}^n)$ we have*

$$\|X_{\alpha,1}(t)u\|_{L^q} \leq Ct^{\frac{\alpha n}{4}(\frac{1}{q}-\frac{1}{p})}\|u\|_{L^p}. \tag{3.1}$$

(ii) *If $\frac{n}{p} - \frac{n}{q} < 8$, for $u \in L^p(\mathbb{R}^n)$ we have*

$$\|X_{\alpha,2}(t)u\|_{L^q} \leq Ct^{\alpha-1+\frac{\alpha n}{4}(\frac{1}{q}-\frac{1}{p})}\|u\|_{L^p}. \tag{3.2}$$

Proof. If one lets the operator $A = \Delta^2$ in [29] Lemma 3.3 and then uses the same method, we can obtain the following estimates; for any $u \in L^p(\mathbb{R}^n)$,

$$\|X_{\alpha,1}(t)u\|_{L^p} \leq C\|u\|_{L^p}\ ,\ \|\Delta^2 X_{\alpha,1}(t)u\|_{L^p} \leq Ct^{-\alpha}\|u\|_{L^p}, \tag{3.3}$$

$$\|X_{\alpha,2}(t)u\|_{L^p} \leq Ct^{\alpha-1}\|u\|_{L^p}\ ,\ \|\Delta^2 X_{\alpha,2}(t)u\|_{L^p} \leq Ct^{-1}\|u\|_{L^p}. \tag{3.4}$$

Then, using the Gagliardo–Nirenberg inequality, we obtain

$$\|X_{\alpha,1}(t)u\|_{L^q} \leq C\|\Delta^2 X_{\alpha,1}(t)u\|^a_{L^p}\|X_{\alpha,1}(t)u\|^{1-a}_{L^p},$$

where $a \in [0,1)$ and $\frac{1}{q} = a(\frac{1}{p} - \frac{4}{n}) + \frac{1-a}{p}$. Therefore, by (3.3) we obtain

$$\|X_{\alpha,1}(t)u\|_{L^q} \leq Ct^{-a\alpha}\|u\|^a_{L^p}\|u\|^{1-a}_{L^p} = Ct^{\frac{\alpha n}{4}(\frac{1}{q}-\frac{1}{p})}\|u\|_{L^p}.$$

We omit the proof of (ii), which is similar to that of (i). □

Remark 3.2. *For $1 < \alpha < 2,\ 1 \le p = q \le \infty$, we have better estimates. In fact, noting that the Fourier transform evaluated at $\xi = 0$ equals the integral of the function, we can get the following estimates:*

$$\|\mathcal{F}^{-1}[E_{\alpha,1}(-t^\alpha|\xi|^4)]\|_{L^1} = E_{\alpha,1}(-t^\alpha 0^4) = E_{\alpha,1}(0) = 1,$$

$$\|\mathcal{F}^{-1}[t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha|\xi|^4)]\|_{L^1} = t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha 0^4) = t^{\alpha-1}E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}.$$

Then, for any $u \in L^p$, using Young's convolution inequality, we have

$$\|X_{\alpha,1}(t)u\|_{L^p} = \|\mathcal{F}^{-1}[E_{\alpha,1}(-t^\alpha|\xi|^4)] * u\|_{L^p} \le \|u\|_{L^p}, \tag{3.5}$$

$$\|X_{\alpha,2}(t)u\|_{L^p} = \|\mathcal{F}^{-1}[t^{\alpha-1}E_{\alpha,\alpha}(-t^\alpha|\xi|^4)] * u\|_{L^p} \le \frac{1}{\Gamma(\alpha)}t^{\alpha-1}\|u\|_{L^p}. \tag{3.6}$$

Next, we give a proposition that is important for the proof of the local existence of mild solutions in the next sections.

Proposition 3.1. (i) *If $1 \le p < \infty$, then for $t > 0,\ u \in \exp L^p$, we have*

$$\|X_{\alpha,1}(t)u\|_{\exp L^p} \le \|u\|_{\exp L^p}, \tag{3.7}$$

$$\|X_{\alpha,2}(t)u\|_{\exp L^p} \le \frac{1}{\Gamma(\alpha)}t^{\alpha-1}\|u\|_{\exp L^p}. \tag{3.8}$$

(ii) *If $\frac{n}{q} - \frac{n}{p} < 4,\ 1 < q \le p < \infty$, then for $t > 0,\ u \in L^q$, we have*

$$\|X_{\alpha,1}(t)u\|_{\exp L^p} \le Ct^{-\frac{\alpha n}{4q}}[\ln(t^{-\frac{\alpha n}{4}} + 1)]^{-\frac{1}{p}}\|u\|_{L^q}, \tag{3.9}$$

$$\|X_{\alpha,2}(t)u\|_{\exp L^p} \le Ct^{\alpha-1-\frac{\alpha n}{4q}}[\ln(t^{-\frac{\alpha n}{4}} + 1)]^{-\frac{1}{p}}\|u\|_{L^q}. \tag{3.10}$$

(iii) *If $4r > n,\ 1 \le q \le p < \infty,\ 1 < r < \infty$, then for $t > 0,\ u \in L^r \cap L^q$, we have*

$$\|X_{\alpha,1}(t)u\|_{\exp L^p} \le \frac{1}{(\ln 2)^{\frac{1}{p}}}(Ct^{-\frac{\alpha n}{4r}}\|u\|_{L^r} + \|u\|_{L^q}), \tag{3.11}$$

$$\|X_{\alpha,2}(t)u\|_{\exp L^p} \le \frac{1}{(\ln 2)^{\frac{1}{p}}}(Ct^{\alpha-1-\frac{\alpha n}{4r}}\|u\|_{L^r} + \frac{1}{\Gamma(\alpha)}t^{\alpha-1}\|u\|_{L^q}). \tag{3.12}$$

Proof. (i) Let $\lambda > 0$, by (3.5) and Taylor expansion, it follows

$$\begin{aligned}\int_{\mathbb{R}^n}\Big(\exp\Big(\frac{|X_{\alpha,1}(t)u|^p}{\lambda^p}\Big) - 1\Big)dx &= \sum_{k=1}^{\infty}\frac{\|X_{\alpha,1}(t)u\|_{L^{pk}}^{pk}}{k!\lambda^{pk}}\\ &\le \sum_{k=1}^{\infty}\frac{\|u\|_{L^{pk}}^{pk}}{k!\lambda^{pk}}\\ &= \int_{\mathbb{R}^n}\Big(\exp\Big(\frac{|u|^p}{\lambda^p}\Big) - 1\Big)dx.\end{aligned}$$

Then

$$\left\{\lambda > 0 \,\middle|\, \int_{\mathbb{R}^n} \left(\exp\left(\frac{|u|^p}{\lambda^p}\right) - 1 \right) dx \le 1 \right\}$$
$$\subseteq \left\{\lambda > 0 \,\middle|\, \int_{\mathbb{R}^n} \left(\exp\left(\frac{|X_{\alpha,1}(t)u|^p}{\lambda^p}\right) - 1 \right) dx \le 1 \right\},$$

and so

$$\begin{aligned} \|X_{\alpha,1}(t)u\|_{\exp L^p} &= \inf\left\{\lambda > 0 \,\middle|\, \int_{\mathbb{R}^n} \left(\exp\left(\frac{|X_{\alpha,1}(t)u|^p}{\lambda^p}\right) - 1 \right) dx \le 1 \right\} \\ &\le \inf\left\{\lambda > 0 \,\middle|\, \int_{\mathbb{R}^n} \left(\exp\left(\frac{|u|^p}{\lambda^p}\right) - 1 \right) dx \le 1 \right\} \\ &= \|u\|_{\exp L^p}. \end{aligned}$$

Similarly, using (3.6) and Taylor expansion, we can obtain

$$\|X_{\alpha,2}(t)u\|_{\exp L^p} \le \frac{1}{\Gamma(\alpha)} t^{\alpha-1} \|u\|_{\exp L^p}.$$

(ii) For $\lambda > 0$, by (3.1) and Taylor expansion, one obtain

$$\begin{aligned} \int_{\mathbb{R}^n} \left(\exp\left(\frac{|X_{\alpha,1}(t)u|^p}{\lambda^p}\right) - 1 \right) dx &= \sum_{k=1}^{\infty} \frac{\|X_{\alpha,1}(t)u\|_{L^{pk}}^{pk}}{k!\lambda^{pk}} \\ &\le \sum_{k=1}^{\infty} \frac{C^{pk} t^{\frac{\alpha n}{4}(\frac{1}{pk}-\frac{1}{q})pk} \|u\|_{L^q}^{pk}}{k!\lambda^{pk}} \\ &= t^{\frac{\alpha n}{4}} \left(\exp\left(\frac{C t^{-\frac{\alpha n}{4q}} \|u\|_{L^q}}{\lambda}\right)^p - 1 \right). \end{aligned}$$

As

$$t^{\frac{\alpha n}{4}} \left(\exp\left(\frac{C t^{-\frac{\alpha n}{4q}} \|u\|_{L^q}}{\lambda}\right)^p - 1 \right) \le 1$$

is equal to

$$\lambda \ge C t^{-\frac{\alpha n}{4q}} (\ln(t^{-\frac{\alpha n}{4}} + 1))^{-\frac{1}{p}} \|u\|_{L^q},$$

then

$$\left\{\lambda > 0 \,\middle|\, \lambda \in [C t^{-\frac{\alpha n}{4q}} (\ln(t^{-\frac{\alpha n}{4}} + 1))^{-\frac{1}{p}} \|u\|_{L^q}; \infty[\right\}$$
$$\subseteq \left\{\lambda > 0 \,\middle|\, \int_{\mathbb{R}^n} \left(\exp\left(\frac{|X_{\alpha,1}(t)u|^p}{\lambda^p}\right) - 1 \right) dx \le 1 \right\};$$

whereupon

$$\begin{aligned} \|X_{\alpha,1}(t)u\|_{\exp L^p} &= \inf\left\{\lambda > 0 \,\middle|\, \int_{\mathbb{R}^n} \left(\exp\left(\frac{|X_{\alpha,1}(t)u|^p}{\lambda^p}\right) - 1 \right) dx \le 1 \right\} \\ &\le \inf\left\{\lambda > 0 \,\middle|\, \lambda \in [C t^{-\frac{\alpha n}{4q}} (\ln(t^{-\frac{\alpha n}{4}} + 1))^{-\frac{1}{p}} \|u\|_{L^q}; \infty[\right\} \end{aligned}$$

$$= Ct^{-\frac{\alpha n}{4q}}[\ln(t^{-\frac{\alpha n}{4}}+1)]^{-\frac{1}{p}}\|u\|_{L^q}.$$

Similarly, using (3.2) and Taylor expansion, we can obtain

$$\|X_{\alpha,2}(t)u\|_{\exp L^p} \leq Ct^{\alpha-1-\frac{\alpha n}{4q}}[\ln(t^{-\frac{\alpha n}{4}}+1)]^{-\frac{1}{p}}\|u\|_{L^q}.$$

(iii) We use (2.1) and obtain

$$\|X_{\alpha,1}(t)u\|_{\exp L^p} \leq \frac{1}{(\ln 2)^{\frac{1}{p}}}(\|X_{\alpha,1}(t)u\|_{L^q} + \|X_{\alpha,1}(t)u\|_{L^\infty}).$$

Using (3.1) and (3.5), we obtain

$$\|X_{\alpha,1}(t)u\|_{\exp L^p} \leq \frac{1}{(\ln 2)^{\frac{1}{p}}}(\|u\|_{L^q} + Ct^{-\frac{\alpha n}{4r}}\|u\|_{L^r}).$$

Similarly, using (2.1), (3.2) and (3.6), we can obtain

$$\|X_{\alpha,2}(t)u\|_{\exp L^p} \leq \frac{1}{(\ln 2)^{\frac{1}{p}}}(Ct^{\alpha-1-\frac{\alpha n}{4r}}\|u\|_{L^r} + \frac{1}{\Gamma(\alpha)}t^{\alpha-1}\|u\|_{L^q}).$$

□

We also need the following continuity results for proving local existence.

Proposition 3.2. *If $u \in \exp L_0^p(\mathbb{R}^n)$, then $X_{\alpha,1}(t)u, \bar{X}_{\alpha,2}(t)u \in C([0,T];\exp L_0^p(\mathbb{R}^n))$.*

Proof. By using the similar method in [19] Proposition 2.1, we stress that we can easily get the result that for $\alpha \in (1,2)$, $X_{\alpha,1}(t)u \in C([0,T];\exp L_0^p(\mathbb{R}^n))$. So, we only need to prove $\bar{X}_{\alpha,2}(t)u \in C([0,T];\exp L_0^p(\mathbb{R}^n))$.

Since $u \in \exp L_0^p(\mathbb{R}^n)$, there exists $\{u_n\}_{n\in\mathbb{N}} \subseteq C_0^\infty(\mathbb{R}^n)$ such that u_n converges to u in $\exp L^p(\mathbb{R}^n)$ norm. And therefore, for $t > 0, \bar{X}_{\alpha,2}(t)u_n$ converge to $\bar{X}_{\alpha,2}(t)u$. In fact, considering the definition of $\bar{X}_{\alpha,2}(t)u$, we can apply (3.8) and get

$$\begin{aligned}
\|\bar{X}_{\alpha,2}(t)u_n - \bar{X}_{\alpha,2}(t)u\|_{\exp L^p} &= \frac{1}{\Gamma(2-\alpha)}\left\|\int_0^t (t-\tau)^{1-\alpha}X_{\alpha,2}(\tau)(u_n-u)d\tau\right\|_{\exp L^p} \\
&\leq \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)}\int_0^t (t-\tau)^{1-\alpha}\tau^{\alpha-1}\|u_n-u\|_{\exp L^p}d\tau \\
&= \frac{\beta(\alpha,2-\alpha)}{\Gamma(\alpha)\Gamma(2-\alpha)}t\|u_n-u\|_{\exp L^p} \\
&\leq \frac{\beta(\alpha,2-\alpha)}{\Gamma(\alpha)\Gamma(2-\alpha)}T\|u_n-u\|_{\exp L^p}, \\
&\leq T\|u_n-u\|_{\exp L^p}
\end{aligned}$$

where β denotes the beta function. Therefore, when $n \to \infty$, we obtain

$$\|\bar{X}_{\alpha,2}(t)u_n - \bar{X}_{\alpha,2}(t)u\|_{\exp L^p} \leq T\|u_n-u\|_{\exp L^p} \to 0. \tag{3.13}$$

Next, for any $t_1, t_2 > 0$, we use the triangle inequality to obtain

$$\begin{aligned}
&\| \bar{X}_{\alpha,2}(t_2)u - \bar{X}_{\alpha,2}(t_1)u\|_{\exp L^p}\\
\leq &\| \bar{X}_{\alpha,2}(t_2)u_n - \bar{X}_{\alpha,2}(t_2)u\|_{\exp L^p} + \| \bar{X}_{\alpha,2}(t_1)u_n - \bar{X}_{\alpha,2}(t_1)u\|_{\exp L^p}\\
&+ \| \bar{X}_{\alpha,2}(t_2)u_n - \bar{X}_{\alpha,2}(t_1)u_n\|_{\exp L^p}.
\end{aligned}$$

Then, for the sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq C_0^\infty(\mathbb{R}^n)$, by applying Lemma 3.1, we can easily obtain

$$\begin{cases}
\lim\limits_{t_2\to t_1} \| \bar{X}_{\alpha,2}(t_2)u_n - \bar{X}_{\alpha,2}(t_1)u_n\|_{L^p} = 0,\\
\lim\limits_{t_2\to t_1} \| \bar{X}_{\alpha,2}(t_2)u_n - \bar{X}_{\alpha,2}(t_1)u_n\|_{L^\infty} = 0.
\end{cases}$$

By using the embedding (2.1), we have

$$\begin{aligned}
&\| \bar{X}_{\alpha,2}(t_2)u_n - \bar{X}_{\alpha,2}(t_1)u_n\|_{\exp L^p}\\
\leq &\frac{1}{(\ln 2)^{\frac{1}{p}}}(\| \bar{X}_{\alpha,2}(t_2)u_n - \bar{X}_{\alpha,2}(t_1)u_n\|_{L^p} + \| \bar{X}_{\alpha,2}(t_2)u_n - \bar{X}_{\alpha,2}(t_1)u_n\|_{L^\infty}) \to 0,
\end{aligned}$$

when $t_2 \to t_1$. Moreover, by (3.13), we immediately obtain

$$\begin{cases}
\lim\limits_{n\to\infty} \| \bar{X}_{\alpha,2}(t_1)u_n - \bar{X}_{\alpha,2}(t_1)u\|_{\exp L^p} = 0,\\
\lim\limits_{n\to\infty} \| \bar{X}_{\alpha,2}(t_2)u_n - \bar{X}_{\alpha,2}(t_2)u\|_{\exp L^p} = 0.
\end{cases}$$

Therefore, if we choose an appropriate n, then we can draw the desired conclusion of this proposition easily. □

Proposition 3.3. *If $f \in L^q((0,T), \exp L_0^p(\mathbb{R}^n))$, $1 \leq q \leq \infty$, then*

$$\int_0^t X_{\alpha,2}(t-\tau)f(\tau)d\tau \in C([0,T], \exp L_0^p(\mathbb{R}^n)).$$

Proof. Since $\alpha \in (1,2)$, the dominated convergence theorem tells us that the conclusion holds. □

4. Local existence

In this section, we establish the local existence and uniqueness of mild solutions to the problem (1.1). First, we define the mild solutions of (1.1).

Definition 4.1. *Given $u_0, u_1 \in \exp L_0^p(\mathbb{R}^n)$, $1 < \alpha < 2$, $0 < \gamma < 1$ and $T > 0$. Then u is a mild solution of* (1.1) *if $u \in C([0,T]; \exp L_0^p(\mathbb{R}^n))$ satisfying*

$$\begin{aligned}
u(t,x) = &\int_{\mathbb{R}^n} \mathcal{F}^{-1}_{\xi\to y}[E_{\alpha,1}(-t^\alpha|\xi|^4)](y)u_0(x-y)dy + t\int_{\mathbb{R}^n} \mathcal{F}^{-1}_{\xi\to y}[E_{\alpha,2}(-t^\alpha|\xi|^4)](y)u_1(x-y)dy\\
&+ \int_0^t \int_{\mathbb{R}^n} \mathcal{F}^{-1}_{\xi\to y}[(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-(t-\tau)^\alpha|\xi|^4)](t-\tau,y)J_{0|\tau}^{1-\gamma}(e^{u(\tau,x-y)})dyd\tau. \qquad (4.1)
\end{aligned}$$

Similar to the representation of mild solutions in [28], we can rewrite (4.1) as

$$u(t,x) = X_{\alpha,1}(t)u_0(x) + \bar{X}_{\alpha,2}(t)u_1(x) + \int_0^t X_{\alpha,2}(t-\tau)J_{0|\tau}^{1-\gamma}(e^{u(\tau)})d\tau.$$

Readers can refer [29, 30] for more details.

In the following proof, we will use the Banach fixed-point theorem to find the desired solution. Moreover, we also use a decomposition argument, which is used in [13, 15, 16]. The concrete idea is that in view of the density of $C_0^\infty(\mathbb{R}^n)$, we can respectively split the initial data $u_0, u_1 \in \exp L_0^p(\mathbb{R}^n)$ into a small part in $\exp L^p(\mathbb{R}^n)$ and a smooth part in $C_0^\infty(\mathbb{R}^n)$.

Let $u_0, u_1 \in \exp L_0^p(\mathbb{R}^n)$. Then, for every $\epsilon > 0$ there exists $v_0, v_1 \in C_0^\infty(\mathbb{R}^n)$ such that $\|\omega_0\|_{\exp L^p(\mathbb{R}^n)} \leq \epsilon$, $\|\omega_1\|_{\exp L^p(\mathbb{R}^n)} \leq \epsilon$, where $\omega_0 = u_0 - v_0,\ \omega_1 = u_1 - v_1$. Now, let us split (1.1). One is:

$$\begin{cases} \partial_{0|t}^\alpha v + \Delta^2 v = J_{0|t}^{1-\gamma}(e^v), & x \in \mathbb{R}^n, \quad t > 0, \\ v(0) = v_0 \in C_0^\infty(\mathbb{R}^n), & x \in \mathbb{R}^n, \\ v_t(0) = v_1 \in C_0^\infty(\mathbb{R}^n), & x \in \mathbb{R}^n. \end{cases} \tag{4.2}$$

The other one is:

$$\begin{cases} \partial_{0|t}^\alpha \omega + \Delta^2 \omega = J_{0|t}^{1-\gamma}(e^{\omega+v}) - J_{0|t}^{1-\gamma}(e^v), & x \in \mathbb{R}^n, \quad t > 0, \\ \omega(0) = \omega_0, \|\omega_0\|_{\exp L^p} \leq \epsilon, & x \in \mathbb{R}^n, \\ \omega_t(0) = \omega_1, \|\omega_1\|_{\exp L^p} \leq \epsilon, & x \in \mathbb{R}^n. \end{cases} \tag{4.3}$$

After comparing the above two problems with problem (1.1), we can easily find that $u = v + \omega$ is a mild solution of (1.1) if v is a mild solution of (4.2) and ω is a mild solution of (4.3). We now prove the local existence results concerning (4.2) and (4.3), which are necessary to establish the essential result of the section.

Lemma 4.1. *Let* $1 < \alpha < 2$, $0 < \gamma < 1$, $p > 1$, *and* $v_0, v_1 \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *Then, there exists a* $T = T(v_0, v_1) > 0$ *such that* (4.2) *has a mild solution* $v \in C([0,T]; \exp L_0^p(\mathbb{R}^n)) \cap L^\infty(0,T; L^\infty(\mathbb{R}^n))$.

Proof. We first define the following space

$$E_T = \{v \in L^\infty(0,T; L^\infty(\mathbb{R}^n)) \cap C([0,T]; \exp L_0^p(\mathbb{R}^n)) \mid \|v\|_{E_T} \leq 2\|v_0\|_{L^p \cap L^\infty}\},$$

where $\|v\|_{L^p \cap L^\infty} = \|v\|_{L^p} + \|v\|_{L^\infty}$ and $\|v\|_{E_T} = \|v\|_{L^\infty(0,T;L^p)} + \|v\|_{L^\infty(0,T;L^\infty)}$. For $v \in E_T$, we define a mapping Φ on E_T as

$$\Phi(v) = X_{\alpha,1}(t)v_0 + \bar{X}_{\alpha,2}(t)v_1 + \int_0^t X_{\alpha,2}(t-\tau)J_{0|\tau}^{1-\gamma}(e^{v(\tau)})d\tau.$$

We will prove that Φ is a contraction from E_T into itself if $T > 0$ is small enough.

First, we show Φ maps E_T into itself. Let $v \in E_T$. For $q = p$ or ∞, we have

$$\begin{aligned} \|J_{0|t}^{1-\gamma}(e^{v(t)})\|_{L^q} &= \frac{1}{\Gamma(1-\gamma)} \left\| \int_0^t (t-s)^{-\gamma} e^{v(s)} ds \right\|_{L^q} \\ &\leq \frac{T^{1-\gamma}}{\Gamma(1-\gamma)} e^{\|v\|_{E_T}} \\ &\leq \frac{T^{1-\gamma}}{\Gamma(1-\gamma)} e^{2\|v_0\|_{L^p \cap L^\infty}}. \end{aligned} \tag{4.4}$$

Therefore, $J_{0|t}^{1-\gamma}(e^{v(t)}) \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. Since $v_0, v_1, J_{0|t}^{1-\gamma}(e^{v(t)}) \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, by Lemma 2.1, we can obtain $v_0,\ v_1, J_{0|t}^{1-\gamma}(e^{v(t)}) \in \exp L_0^p(\mathbb{R}^n)$. Then, combining Propositions 3.2 and 3.3, we deduce that $\Phi(v) \in C([0,T]; \exp L_0^p(\mathbb{R}^n))$. Moreover, using (3.5), (3.6) and (4.4), we can obtain

$$\begin{aligned}\|\Phi(v)\|_{E_T} &\le \|v_0\|_{L^p\cap L^\infty} + T\|v_1\|_{L^p\cap L^\infty} + \frac{2T^{1-\gamma}}{\Gamma(\alpha)\Gamma(1-\gamma)} e^{2\|v_0\|_{L^p\cap L^\infty}} \int_0^t (t-\tau)^{\alpha-1} d\tau \\ &\le \|v_0\|_{L^p\cap L^\infty} + T\|v_1\|_{L^p\cap L^\infty} + \frac{2T^{1-\gamma+\alpha}}{\Gamma(\alpha)\Gamma(1-\gamma)} e^{2\|v_0\|_{L^p\cap L^\infty}} \\ &\le 2\|v_0\|_{L^p\cap L^\infty},\end{aligned}$$

Choose $T > 0$ sufficiently small satisfying $T\|v_1\|_{L^p\cap L^\infty} + \frac{2T^{1-\gamma+\alpha}}{\Gamma(\alpha)\Gamma(1-\gamma)} e^{2\|v_0\|_{L^p\cap L^\infty}} \le \|v_0\|_{L^p\cap L^\infty}$. This proves $\Phi(v) \in E_T$.

Let $v_2, v_3 \in E_T$. For $q = p$ or $q = \infty$, we have

$$\begin{aligned}\|J_{0|t}^{1-\gamma}(e^{v_2}) - J_{0|t}^{1-\gamma}(e^{v_3})\|_{L^q} &\le \frac{1}{\Gamma(1-\gamma)} \int_0^t (t-s)^{-\gamma} \|e^{v_2(s)} - e^{v_3(s)}\|_{L^q} ds \\ &\le \frac{T^{1-\gamma}}{\Gamma(1-\gamma)} \|e^{v_2(s)} - e^{v_3(s)}\|_{L^\infty([0,T];L^q)} \\ &\le \frac{T^{1-\gamma}}{\Gamma(1-\gamma)} e^{\|\lambda v_2(s)+\mu v_3(s)\|_{L^\infty([0,T];L^q)}} \|v_2 - v_3\|_{L^\infty([0,T];L^q)} \\ &\le \frac{T^{1-\gamma}}{\Gamma(1-\gamma)} e^{2\|v_0\|_{L^p\cap L^\infty}} \|v_2 - v_3\|_{E_T},\end{aligned} \tag{4.5}$$

where we have used the following equality

$$|e^{u(s)} - e^{v(s)}| = e^{au(s)+bv(s)}|u(s) - v(s)|,\ 0 < a, b < 1,\ a + b = 1. \tag{4.6}$$

Then, using (3.6) and (4.5), we have

$$\begin{aligned}\|\Phi(v_2) - \Phi(v_3)\|_{E_T} &\le \int_0^t \|X_{\alpha,2}(t-\tau)(J_{0|\tau}^{1-\gamma}(e^{v_2(\tau)}) - J_{0|\tau}^{1-\gamma}(e^{v_3(\tau)}))\|_{E_T} d\tau \\ &\le \frac{2}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \|J_{0|\tau}^{1-\gamma}(e^{v_2(\tau)}) - J_{0|\tau}^{1-\gamma}(e^{v_3(\tau)})\|_{L^q} d\tau \\ &\le \frac{2T^{1-\gamma+\alpha}}{\Gamma(\alpha)\Gamma(1-\gamma)} e^{2\|v_0\|_{L^p\cap L^\infty}} \|v_2 - v_3\|_{E_T} \\ &\le \frac{1}{2}\|v_2 - v_3\|_{E_T},\end{aligned}$$

by choosing $T > 0$ sufficiently small satisfying $\frac{2T^{1-\gamma+\alpha}}{\Gamma(\alpha)\Gamma(1-\gamma)} e^{2\|v_0\|_{L^p\cap L^\infty}} \le \frac{1}{2}$. Therefore, according to the contraction mapping principle, we conclude that Φ has a unique fixed point $v \in E_T$. □

Lemma 4.2. *Let $1 < \alpha < 2$, $0 < \gamma < 1$, $n \ge 1$, $p > 1$, $\frac{n}{p} < 8$ and $\omega_0, \omega_1 \in \exp L_0^p(\mathbb{R}^n)$. Suppose $v \in L^\infty(0,T; L^\infty(\mathbb{R}^n))$ be obtained in Lemma 4.1. Then, for $\|\omega_0\|_{\exp L^p} \le \epsilon$, $\|\omega_1\|_{\exp L^p} \le \epsilon$ with $\epsilon << 1$ sufficiently small, there exists a $T_1 = T_1(\omega_0, \omega_1, v, \epsilon) > 0$ and a mild solution $\omega \in C([0,T_1]; \exp L_0^p(\mathbb{R}^n))$ to problem* (4.3).

Proof. For $T_1 > 0$, we define

$$\Omega_{T_1} = \{\omega \in C([0,T_1]; \exp L_0^p(\mathbb{R}^n)) \mid \|\omega\|_{L^\infty(0,T_1;\exp L^p)} \le 4\epsilon\}.$$

For $\omega \in \Omega_{T_1}$, we define a mapping G on Ω_{T_1} as

$$G(\omega) = X_{\alpha,1}(t)\omega_0 + \bar{X}_{\alpha,2}(t)\omega_1 + \int_0^t X_{\alpha,2}(t-\tau)(J_{0|\tau}^{1-\gamma}(e^{\omega(\tau)+v(\tau)}) - J_{0|\tau}^{1-\gamma}(e^{v(\tau)}))d\tau.$$

We will prove that G is a contraction map from Ω_{T_1} into itself if ϵ and T_1 are small enough.

First, we show that G is a contraction. Let $\omega_2, \omega_3 \in \Omega_{T_1}$. Using (2.1), we have

$$\|G(\omega_2) - G(\omega_3)\|_{\exp L^p} \le \frac{1}{(\ln 2)^{\frac{1}{p}}}(\|G(\omega_2) - G(\omega_3)\|_{L^p} + \|G(\omega_2) - G(\omega_3)\|_{L^\infty}). \tag{4.7}$$

Then by (2.2),(3.2) and (4.6), we obtain

$$\begin{aligned}
\|G(\omega_2) - G(\omega_3)\|_{L^\infty} &\le \int_0^t \|X_{\alpha,2}(t-\tau)(J_{0|\tau}^{1-\gamma}(e^{\omega_2(\tau)+v(\tau)}) - J_{0|\tau}^{1-\gamma}(e^{\omega_3(\tau)+v(\tau)}))\|_{L^\infty} d\tau \\
&\le C\int_0^t (t-\tau)^{\alpha-1-\frac{\alpha n}{4p}} \|J_{0|\tau}^{1-\gamma}(e^{\omega_2(\tau)+v(\tau)}) - J_{0|\tau}^{1-\gamma}(e^{\omega_3(\tau)+v(\tau)})\|_{L^p} d\tau \\
&\le C\int_0^t (t-\tau)^{\alpha-1-\frac{\alpha n}{4p}} \|J_{0|\tau}^{1-\gamma}(e^{\omega_2(\tau)+v(\tau)}) - J_{0|\tau}^{1-\gamma}(e^{\omega_3(\tau)+v(\tau)})\|_{\exp L^p} d\tau \\
&\le C\int_0^t (t-\tau)^{\alpha-1-\frac{\alpha n}{4p}} \int_0^\tau (\tau-s)^{-\gamma} \|e^{\omega_2(s)+v(s)} - e^{\omega_3(s)+v(s)}\|_{\exp L^p} ds d\tau \\
&\le C\int_0^t (t-\tau)^{\alpha-1-\frac{\alpha n}{4p}} \tau^{1-\gamma} d\tau \|e^{\omega_2+v} - e^{\omega_3+v}\|_{L^\infty([0,T_1];\exp L^p)} \\
&\le Ce^{4\epsilon+\|v\|_{L^\infty}} T_1^{1-\gamma+\alpha-\frac{\alpha n}{4p}} \|\omega_2 - \omega_3\|_{L^\infty([0,T_1];\exp L^p)}.
\end{aligned} \tag{4.8}$$

On the other hand, applying the same estimate above, we can easily obtain

$$\|G(\omega_2) - G(\omega_3)\|_{L^p} \le Ce^{4\epsilon+\|v\|_{L^\infty}} T_1^{1-\gamma+\alpha} \|\omega_2 - \omega_3\|_{L^\infty([0,T_1];\exp L^p)}. \tag{4.9}$$

Using (4.8) and (4.9) into (4.7), we finally obtain

$$\begin{aligned}
\|G(\omega_2) - G(\omega_3)\|_{\exp L^p} &\le Ce^{4\epsilon+\|v\|_{L^\infty}} (T_1^{1-\gamma+\alpha} + T_1^{1-\gamma+\alpha-\frac{\alpha n}{4p}}) \|\omega_2 - \omega_3\|_{L^\infty([0,T_1];\exp L^p)} \\
&\le \frac{1}{2}\|\omega_2 - \omega_3\|_{L^\infty([0,T_1];\exp L^p)},
\end{aligned} \tag{4.10}$$

where $T_1 << 1$ is chosen sufficiently small such that $Ce^{4\epsilon+\|v\|_{L^\infty}}(T_1^{1-\gamma+\alpha} + T_1^{1-\gamma+\alpha-\frac{\alpha n}{4p}}) \le \frac{1}{2}$.

Now, we prove G maps Ω_{T_1} into itself. Let $\omega \in \Omega_{T_1}$. Then, using the similar proof as in Lemma 4.1, we conclude that $G(\omega) \in C([0,T_1]; \exp L_0^p(\mathbb{R}^n))$. Moreover, by using (3.7), (3.8), and (4.10) with $\omega_2 = \omega, \omega_3 = 0$ for $T_1 << 1$, we have

$$\|G(\omega)\|_{\Omega_{T_1}} \le \|\omega_0\|_{\exp L^p} + T_1\|\omega_1\|_{\exp L^p} + \frac{1}{2}\|\omega\|_{L^\infty([0,T_1];\exp L^p)} \le \epsilon + \epsilon + \frac{1}{2}\cdot 4\epsilon = 4\epsilon.$$

This proves that $G(\omega) \in \Omega_{T_1}$. □

With the above two lemmas, we are able to prove the local existence and uniqueness to mild solutions of the problem (1.1).

Theorem 4.1. *Let* $1 < \alpha < 2$, $0 < \gamma < 1$, $n \geq 1$, $p > 1$ *and* $\frac{n}{p} < 8$. *Suppose that* $u_0, u_1 \in \exp L_0^p(\mathbb{R}^n)$. *Then, there exists a* $T > 0$ *such that the problem* (1.1) *has a unique mild solution* $u \in C([0,T]; \exp L_0^p(\mathbb{R}^n))$.

Proof. First, we prove the existence. Let $\frac{n}{p} < 8$ and $0 < \epsilon << 1$. Then, we split the initial data $u_0 = v_0 + \omega_0$, $u_1 = v_1 + \omega_1$ with $v_0, v_1 \in C_0^\infty(\mathbb{R}^n)$ and $\|\omega_0\|_{\exp L^p} \leq \epsilon$, $\|\omega_1\|_{\exp L^p} \leq \epsilon$. By Lemma 4.1, there exists a time $0 < T_2 = T_2(v_0, v_1) << 1$ and a mild solution $v \in C([0,T_2]; \exp L_0^p(\mathbb{R}^n)) \cap L^\infty(0, T_2; L^\infty(\mathbb{R}^n))$ such that $\|v\|_{L^\infty(0,T_2;L^p\cap L^\infty)} \leq 2\|v_0\|_{L^p\cap L^\infty}$. By choosing $T_1 > 0$ small enough satisfying $T_1 < T_2$ and

$$Ce^{4\epsilon+2\|v_0\|_{L^p\cap L^\infty}}(T_1^{1-\gamma+\alpha} + T_1^{1-\gamma+\alpha-\frac{\alpha n}{4p}}) \leq \frac{1}{2},$$

and using Lemma 4.2, there exists a mild solution $\omega \in C([0,T_1]; \exp L_0^p(\mathbb{R}^n))$ to problem (4.3). Then, we can draw the conclusion that $u = v + \omega$ is a mild solution of problem (1.1) in $C([0,T_1]; \exp L_0^p(\mathbb{R}^n))$.

Next is the proof of uniqueness. Let $u, v \in C([0,T]; \exp L_0^p(\mathbb{R}^n))$ be two mild solutions of (1.1) with $u(0) = v(0) = u_0$, $u_t(0) = v_t(0) = u_1$. Then, using (3.8) and (4.6), we have

$$\begin{aligned}
\|u(t) - v(t)\|_{\exp L^p} &\leq \int_0^t \|X_{\alpha,2}(t-\tau)(J_{0|\tau}^{1-\gamma}(e^{u(\tau)}) - J_{0|\tau}^{1-\gamma}(e^{v(\tau)}))\|_{\exp L^p} d\tau \\
&\leq C\int_0^t (t-\tau)^{\alpha-1}\|(J_{0|\tau}^{1-\gamma}(e^{u(\tau)}) - J_{0|\tau}^{1-\gamma}(e^{v(\tau)}))\|_{\exp L^p} d\tau \\
&\leq C\int_0^t (t-\tau)^{\alpha-1}\int_0^\tau (\tau-s)^{-\gamma}\|e^{u(s)} - e^{v(s)}\|_{\exp L^p} ds d\tau \\
&\leq Ce^{4\epsilon+2\|v_0\|_{L^p\cap L^\infty}}\int_0^t (t-\tau)^{\alpha-1}\int_0^\tau (\tau-s)^{-\gamma}\|u(s) - v(s)\|_{\exp L^p} ds d\tau \\
&\leq Ce^{4\epsilon+2\|v_0\|_{L^p\cap L^\infty}}\int_0^t (t-\tau)^{\alpha-1}\tau^{1-\gamma}\|u(\tau) - v(\tau)\|_{\exp L^p} d\tau.
\end{aligned}$$

Hence, according to Gronwall's inequality, we conclude that $u = v$. □

5. Blow-up of solutions

In this section, we prove the blow-up results of (1.1) by using the test function method. First, we give the definition of weak solution of (1.1).

Definition 5.1. *Let* $1 < \alpha < 2$, $0 < \gamma < 1$, $p \geq 1$ *and* $T > 0$. *For* $u_0, u_1 \in L_{loc}^p(\mathbb{R}^n)$ *and* $T > 0$, *we call* $u \in L^p((0,T), L_{loc}^p(\mathbb{R}^n))$ *is a weak solution of* (1.1) *if*

$$\int_0^T\int_{\mathbb{R}^n}(u_0 + tu_1)\partial_{t|T}^\alpha\psi(t,x)dxdt + \int_0^T\int_{\mathbb{R}^n}J_{0|t}^{1-\gamma}(e^u)\psi(t,x)dxdt$$

$$= \int_0^T\int_{\mathbb{R}^n}u(t,x)\Delta^2\psi(t,x)dxdt + \int_0^T\int_{\mathbb{R}^n}u(t,x)\partial_{t|T}^\alpha\psi(t,x)dxdt,$$

for every test function $\psi \in C^{2,2}_{t,x}([0,T]\times\mathbb{R}^n)$ *and* $\psi_t \in C^{0,2}_{t,x}([0,T]\times\mathbb{R}^n)$ *with* $supp_x\psi \subset\subset \mathbb{R}^n$ *and* $\psi(T,x)=\psi_t(T,x)=0$, *where*

$$C^{2,2}_{t,x}([0,T]\times\mathbb{R}^n)=\{f(t,x)\mid f,f_{x_i},f_{x_ix_i},f_t,f_{tt}\in C([0,T],\mathbb{R}^n),i=1,2,\ldots,n\},$$
$$C^{0,2}_{t,x}([0,T]\times\mathbb{R}^n)=\{f(t,x)\mid f,f_{x_i},f_{x_ix_i}\in C([0,T],\mathbb{R}^n),i=1,2,\ldots,n\}.$$

Lemma 5.1. *Let* $T>0$ *and* $u_0,u_1\in\exp L^p_0(\mathbb{R}^n)$. *If* $u\in C([0,T];\exp L^p_0(\mathbb{R}^n))$ *is a mild solution of* (1.1), *then* u *is also a weak solution of* (1.1).

Proof. According to the embedding $\exp L^p(\mathbb{R}^n)\hookrightarrow L^p(\mathbb{R}^n)$ for $1\le p<\infty$, we can use similar proof of Lemma 5.2 in [29] to deduce that u is also a weak solution of (1.1). □

Then, we present some auxiliary functions that are needed for the blow-up result of solutions.
Let $\varphi\in C^{2,2}_{t,x}([0,T]\times\mathbb{R}^n)$ and $\varphi_t\in C^{0,2}_{t,x}([0,T]\times\mathbb{R}^n)$ such that

$$\varphi(t,x)=\varphi_1(t)\varphi_2^l(x),\quad l>>1,$$

where

$$\varphi_1(t)=(1-\frac{t}{T})^\eta_+,\quad \eta>>1,$$
$$\varphi_2(x)=\xi(\frac{|x|}{T^{\frac{\alpha}{4}}}),$$

and ξ is a regular function such that

$$\xi(x)=\begin{cases}1, & x\le 1,\\ 0, & x\ge 2,\end{cases}$$

and monotonically decreasing if $1\le x\le 2$.

Remark 5.1. *From [31] Lemma 14, we know that if we make a slight modification to the independent variable of* ξ, *then for* $l>4$, *the following estimate holds by direct calculation*

$$|\Delta^2\varphi_2^l|\le C_0T^{-\alpha}\varphi_2^{l-4}, \tag{5.1}$$

for some $C_0=C_0(l)>0$.

Next, we present the main blow-up result in the space $\exp L^p_0(\mathbb{R}^n)$.

Theorem 5.1. *Let* $u_0\in\exp L^p_0(\mathbb{R}^n)$ *and* $u_0\ge 0$, $u_0\not\equiv 0$, $u_1\equiv 0$. *If* $\frac{\alpha n}{4}-\alpha+\gamma\ge 0$, *then the solutions of* (1.1) *blow up in a finite time.*

Proof. First, let us assume that u is a global mild solution of (1.1). Then, $u\in C([0,T];\exp L^p_0(\mathbb{R}^n))$ for all $T>>1$ such that $|u(t)|>0$ for all $t\in[0,T]$.

Let $\psi(t,x)=\partial^{1-\gamma}_{t|T}\varphi(t,x)$. Then, by Definition 5.1, we have

$$\int_0^T\int_{\mathbb{R}^n}u_0(x)D^{\alpha+1-\gamma}_{t|T}\varphi(t,x)dxdt+\int_0^T\int_{\mathbb{R}^n}e^u\varphi(t,x)dxdt$$
$$=\int_0^T\int_{\mathbb{R}^n}u(t,x)\Delta^2D^{1-\gamma}_{t|T}\varphi(t,x)dxdt+\int_0^T\int_{\mathbb{R}^n}u(t,x)D^{\alpha+1-\gamma}_{t|T}\varphi(t,x)dxdt.$$

We notice that if we set $\Omega = \{x \in \mathbb{R}^n \mid |x| \leq 2T^{\frac{\alpha}{4}}\}$ and let $\Omega_T = [0, T] \times \Omega$, we can obtain the following equation:

$$\int_{\Omega} u_0(x)\varphi_2^l(x)dx \int_0^T D_{t|T}^{\alpha+1-\gamma}\varphi_1(t)dt + \int_{\Omega_T} e^u \varphi(t,x)dxdt$$
$$= \int_{\Omega_T} u(t,x)\Delta^2\varphi_2^l(x)D_{t|T}^{1-\gamma}\varphi_1(t)dxdt + \int_{\Omega_T} u(t,x)\varphi_2^l(x)D_{t|T}^{\alpha+1-\gamma}\varphi_1(t)dxdt.$$

Using (2.4), we have

$$C_1 T^{\gamma-\alpha}\int_{\Omega} u_0(x)\varphi_2^l(x) + \int_{\Omega_T} e^{u(t,x)}\varphi(t,x)$$
$$= \int_{\Omega_T} u(t,x)\Delta^2\varphi_2^l(x)D_{t|T}^{1-\gamma}\varphi_1(t) + \int_{\Omega_T} u(t,x)\varphi_2^l(x)D_{t|T}^{\alpha+1-\gamma}\varphi_1(t),$$

where

$$C_1 = \frac{\Gamma(\eta+1)}{\Gamma(\eta-\alpha+\gamma+1)}.$$

Then, using (5.1), we obtain

$$\begin{aligned}
&C_1 T^{\gamma-\alpha}\int_{\Omega} u_0(x)\varphi_2^l(x) + \int_{\Omega_T} e^{u(t,x)}\varphi(t,x)\\
\leq&\left|\int_{\Omega_T} u(t,x)\Delta^2\varphi_2^l(x)D_{t|T}^{1-\gamma}\varphi_1(t)\right| + \left|\int_{\Omega_T} u(t,x)\varphi_2^l(x)D_{t|T}^{\alpha+1-\gamma}\varphi_1(t)\right|\\
\leq&C_0 T^{-\alpha}\int_{\Omega_T} |u(t,x)|\varphi_2^{l-4}(x)D_{t|T}^{1-\gamma}\varphi_1(t) + \int_{\Omega_T} |u(t,x)|\varphi_2^l(x)D_{t|T}^{\alpha+1-\gamma}\varphi_1(t)\\
\leq&C_0 T^{-\alpha}\int_{\Omega_T} |u(t,x)|D_{t|T}^{1-\gamma}\varphi_1(t) + \int_{\Omega_T} |u(t,x)|D_{t|T}^{\alpha+1-\gamma}\varphi_1(t)\\
=&C_0 T^{-\alpha}I_1 + I_2.
\end{aligned} \tag{5.2}$$

Next, by using Young's inequality

$$ab \leq \epsilon e^a + b\ln\frac{b}{e\epsilon}, \quad for\ a, b > 0, \quad \epsilon > 0,$$

with $\epsilon = \frac{\epsilon_0 T^\alpha}{4C_0}\varphi(t,x)$, $a = |u(t,x)|$ and $b = D_{t|T}^{1-\gamma}\varphi_1(t)$ in I_1, where ϵ_0 is an appropriately small positive constant satisfying $e^u - \frac{\epsilon_0}{2}e^{|u|} > 0$, then we have

$$I_1 \leq \int_{\Omega_T} D_{t|T}^{1-\gamma}\varphi_1(t)\ln\left(\frac{4C_0T^{-\alpha}D_{t|T}^{1-\gamma}\varphi_1(t)}{e\epsilon_0\varphi_2^l(x)\varphi_1(t)}\right) + \frac{\epsilon_0 T^\alpha}{4C_0}\int_{\Omega_T} e^{|u(t,x)|}\varphi(t,x).$$

For I_2 with $\epsilon = \frac{\epsilon_0}{4}\varphi(t,x)$, $a = |u(t,x)|$ and $b = D_{t|T}^{\alpha+1-\gamma}\varphi_1(t)$, we obtain

$$I_2 \leq \int_{\Omega_T} D_{t|T}^{\alpha+1-\gamma}\varphi_1(t)\ln\left(\frac{4D_{t|T}^{\alpha+1-\gamma}\varphi_1(t)}{e\epsilon_0\varphi_2^l(x)\varphi_1(t)}\right) + \frac{\epsilon_0}{4}\int_{\Omega_T} e^{|u(t,x)|}\varphi(t,x).$$

Using (2.3), we obtain

$$I_1 \le \int_{\Omega_T} D_{t|T}^{1-\gamma}\varphi_1(t)\ln\Big(\frac{C_2T^{\gamma-\alpha-1}(1-\frac{t}{T})_+^{\eta+\gamma-1}}{\varphi_2^l(x)(1-\frac{t}{T})_+^{\eta}}\Big)+\frac{\epsilon_0T^\alpha}{4C_0}\int_{\Omega_T} e^{|u(t,x)|}\varphi(t,x),$$

and

$$I_2 \le \int_{\Omega_T} D_{t|T}^{\alpha+1-\gamma}\varphi_1(t)\ln\Big(\frac{C_3T^{\gamma-\alpha-1}(1-\frac{t}{T})_+^{\eta-\alpha+\gamma-1}}{\varphi_2^l(x)(1-\frac{t}{T})_+^{\eta}}\Big)+\frac{\epsilon_0}{4}\int_{\Omega_T} e^{|u(t,x)|}\varphi(t,x),$$

where

$$C_2=\frac{4C_0\Gamma(\eta+1)}{e\epsilon_0\Gamma(\eta+\gamma)}\ ,\ \ C_3=\frac{4\Gamma(\eta+1)}{e\epsilon_0\Gamma(\eta-\alpha+\gamma)},$$

then

$$I_1 \le \int_{\Omega_T} D_{t|T}^{1-\gamma}\varphi_1(t)\ln\Big(\frac{C_2T^{\gamma-\alpha-1}(1-\frac{t}{T})_+^{\gamma-1}}{\varphi_2^l(x)}\Big)+\frac{\epsilon_0T^\alpha}{4C_0}\int_{\Omega_T} e^{|u(t,x)|}\varphi(t,x), \tag{5.3}$$

and

$$I_2 \le \int_{\Omega_T} D_{t|T}^{\alpha+1-\gamma}\varphi_1(t)\ln\Big(\frac{C_3T^{\gamma-\alpha-1}(1-\frac{t}{T})_+^{\gamma-\alpha-1}}{\varphi_2^l(x)}\Big)+\frac{\epsilon_0}{4}\int_{\Omega_T} e^{|u(t,x)|}\varphi(t,x). \tag{5.4}$$

Applying (5.3) and (5.4) into (5.2), we deduce that

$$\begin{aligned}&C_1T^{\gamma-\alpha}\int_\Omega u_0(x)\varphi_2^l(x)+\int_{\Omega_T}(e^{u(t,x)}-\frac{\epsilon_0}{2}e^{|u(t,x)|})\varphi(t,x)\\ \le& C_0T^{-\alpha}\int_{\Omega_T} D_{t|T}^{1-\gamma}\varphi_1(t)\ln\Big(\frac{C_2T^{\gamma-\alpha-1}(1-\frac{t}{T})_+^{\gamma-1}}{\varphi_2^l(x)}\Big)\\ &+\int_{\Omega_T} D_{t|T}^{\alpha+1-\gamma}\varphi_1(t)\ln\Big(\frac{C_3T^{\gamma-\alpha-1}(1-\frac{t}{T})_+^{\gamma-\alpha-1}}{\varphi_2^l(x)}\Big).\end{aligned} \tag{5.5}$$

Then, if we let $\tau=\frac{t}{T}$ and $y=\frac{x}{T^{\frac{\alpha}{4}}}$, $T>>1$, we have

$$dxdt=T^{\frac{\alpha n}{4}+1}dyd\tau,$$

$$D_{t|T}^{1-\gamma}\varphi_1(t)=C_4T^{\gamma-1}(1-\tau)_+^{\eta+\gamma-1},$$

and

$$D_{t|T}^{\alpha+1-\gamma}\varphi_1(t)=C_5T^{\gamma-\alpha-1}(1-\tau)_+^{\eta-\alpha+\gamma-1},$$

where

$$C_4=\frac{\Gamma(\eta+1)}{\Gamma(\eta+\gamma)},C_5=\frac{\Gamma(\eta+1)}{\Gamma(\eta-\alpha+\gamma)}.$$

Let $\Omega_2=[0,1]\times\{y\in\mathbb{R}^n,\ \ |y|\le 2\}$. Then we obtain

$$\begin{aligned}&\int_{\Omega_T}(e^{u(t,x)}-\frac{\epsilon_0}{2}e^{|u(t,x)|})\varphi(t,x)\\ \le& C_0C_4T^{\frac{\alpha n}{4}-\alpha+\gamma}\int_{\Omega_2}(1-\tau)_+^{\eta+\gamma-1}\ln\Big(\frac{C_2T^{\gamma-\alpha-1}(1-\tau)_+^{\gamma-1}}{\varphi_2^l(T^{\frac{\alpha}{4}}y)}\Big)\end{aligned} \tag{5.6}$$

$$+ C_5 T^{\frac{\alpha n}{4}-\alpha+\gamma} \int_{\Omega_2} (1-\tau)_+^{\eta-\alpha+\gamma-1} \ln\Big(\frac{C_3 T^{\gamma-\alpha-1}(1-\tau)_+^{\gamma-\alpha-1}}{\varphi_2^l(T^{\frac{\alpha}{4}} y)}\Big) - C_1 T^{\gamma-\alpha} \int_\Omega u_0(x)\varphi_2^l(x).$$

By the definition of φ_2, we have a bounded function φ_2 in Ω_2 and

$$\varphi_2 \to 1 \;\; as \;\; T \to +\infty.$$

Finally, according to the Lebesgue's dominated convergence theorem, we can get that if $\frac{\alpha n}{4} - \alpha + \gamma \geq 0$, then the right side of (5.6) will diverge to $-\infty$ if $T \to +\infty$, while the left is positive. This is a contradiction, and we prove the theorem. □

6. Conclusions

In this paper, we study the local existence and blow-up of solutions of the Cauchy problem to a time fractional biharmonic equation with exponentional nonlinear memory. We first establish a L^p-L^q estimate for solution operators and obtain the bilinear estimates for mild solutions. Then, based on the contraction mapping principle, we prove the local existence and uniqueness of mild solutions in $\exp L_0^p(\mathbb{R}^n)$. Finally, with some conditions on the initial data and parameters, a blow-up result is derived.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares no conflicts of interest.

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