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Research article

Sobolev estimates and inverse Hölder estimates on a class of non-divergence variation-inequality problem arising in American option pricing

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Abstract: We studied the Sobolev estimates and inverse Hölder estimates for a class of variational inequality problems involving divergence-type parabolic operator structures. These problems arise from the valuation analysis of American contingent claim problems. First, we analyzed the uniform continuity of the spatially averaged operator with respect to time in a spherical region and the Sobolev estimates for solutions of the variational inequality. Second, by using spatial and temporal truncation, we obtained the Caccioppoli estimate for the variational inequality and consequently derived the inverse Hölder estimate for the solutions.

Keywords: variation-inequality problems; non-divergence parabolic operator; Sobolev estimate; Inverse Hölder estimate

1. Introduction

This study investigates a specific variational inequality problem described by

$$\begin{cases} \min\{-Lu, u-u_0\} = 0, & (x,t) \in \Omega_T, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u(x,t) = \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T) \end{cases} \tag{1}$$

with a non-divergence parabolic operator

$$Lu = u_t - u\mathrm{div}(|\nabla u|^{p-2}\nabla u) - \gamma u|\nabla u|^p. \tag{2}$$

Here, Ω denotes a bounded and open subset of $\mathbb{R}^n$. We consider the case where $p \geq 2$, $\gamma \in (0,1)$ and $T > 0$ are positive constants, $\Omega_T = \Omega \times (0,T)$, and u_0 satisfies

$$u_0 \in C(\bar{\Omega}) \cap W_0^{1,p}(\Omega).$$

From (1), we can infer that $u > u_0$ and $Lu \leq 0$ in Ω_T. It is easily observed that when $u > u_0$, then $Lu = 0$ in Ω_T; conversely, when $u = u_0$, it follows that $Lu \leq 0$. Therefore, in some literature [1–3], the variational inequality (1) is often stated in the following manner:

$$\begin{cases} Lu \leq 0, & (x,t) \in \Omega_T, \\ u \geq u_0, & (x,t) \in \Omega_T, \\ Lu \times (u - u_0) = 0, & (x,t) \in \Omega_T, \\ u(0,x) = u_0(x), & x \in \Omega, \\ u(t,x) = \frac{\partial u}{\partial \nu} = 0, & (x,t) \in \partial\Omega \times (0,T). \end{cases}$$

It is evident that the above formulation is not as concise as the model presented in (1). This paper adopts the statement of the model in (1) for clarity and simplicity.

The study of variational inequality problems of the form (1) originated from the pricing problems of American contingent claims with the inclusion of early exercise provisions [1]. The inclusion of early exercise provisions results in a variational inequality model that is characterized by Eq (1). Further studies on this aspect can be found in [2, 3], and necessary explanations are provided in Section 2. Therefore, we will not repeat them here.

In recent years, there has been an increasing amount of theoretical research on variational inequalities under the framework of linear and quasilinear parabolic operators. In [4], the solvability and regularity of quasilinear parabolic obstacle problems were studied using a symmetric dual-wind discontinuous Galerkin (DG) method. Reference [5] investigated a new class of constrained abstract evolutionary variational inequalities in three-dimensional space. By utilizing mathematical analysis of the unsteady Oseen model for generalized Newtonian incompressible fluids, sufficient conditions for the existence of weak solutions were obtained. Reference [6] focused on studying the existence and stability of weak solutions to variational inequalities under fuzzy parameters. By introducing two parameters into the mappings and constraint sets involved, [6] established the existence results for weak solutions of parameter fuzzy fractional differential variational inequalities (PFFDVI) and further analyzed the compactness and continuous dependence on the initial values of PFFDVI. For more results on the existence of solutions, please refer to [7–9].

There have been some novel results in theoretical research on variational inequalities as well. Reference [10] established the local upper bounds, Harnack inequalities, and Hölder continuity up to the boundary for solutions of variational equations defined by degenerate elliptic operators. Studies on the Hölder continuity of solutions to variational inequalities under parabolic operator structures and other regularity results can be found in [11, 12]. Reference [13] applied regularization and penalization operator methods to prove the existence of solutions to nonlinear degenerate pseudo-parabolic variational inequalities defined in regions with microstructures, and derived a priori estimates for solutions to the microscale problem.

Inspired by [10–12], this study investigates the inverse Hölder estimate for solutions of the variational inequality (1), which has not yet been addressed in the literature. First, we define the integral mean operator $I(t)$ on a spherical region and analyze its uniform continuity with respect to the time variable. Second, using the integral mean operator $I(t)$ and other inequality amplification techniques, we obtain a Sobolev estimate for the variational inequality (1). Then, by combining the Caccioppoli inequality for the variational inequality (1), we derive the inverse Hölder estimate for the gradient of solutions, which allows us to estimate the higher-order norms of the gradient of solutions using lower-order L^p norms. Such results play a key role in many regularity studies.

2. Statement of the problem and its background

The valuation of American options ultimately boils down to a well-posed problem of variational inequality similar to Eq (1). An American call option gives the investor the right to purchase an underlying asset at a predetermined price K at any time within the investment horizon $[0, T]$. It is known that the value of an American option on the maturity date T is given by

$$C(S, T) = \max\{S - K, 0\}.$$

American options only grant the investor the right to exercise their option within the investment horizon $[0, T]$ without imposing any obligations. Thus, we have

$$C(S, t) \geq \max\{S - K, 0\}, \ \forall\, t \in [0, T].$$

If the value of the option $C(S, t)$ at time t exceeds $\max\{S - K, 0\}$,the investor may consider exercising the option, thereby forfeiting the opportunity for higher past returns. In this case, it follows that

$$L_1 C \triangleq \partial_t C + \frac{1}{2}\sigma^2 S^2 \partial_{SS} C + rS\partial_S C - rC = 0 \tag{3}$$

as stated in [1]. Here, σ represents the volatility of the underlying asset linked to the American option, and r denotes the risk-free rate of return in the financial market. On the other hand, if the value $C(S, t)$ of the option at time t is $\max\{S - K, 0\}$, the investor can retain the option to capture higher returns, leading to

$$L_1 C \geq 0 \tag{4}$$

as indicated in [1].

This is a backward differential inequality. Let us define $\tau = T - t$ and $x = \ln S$. By combining Eqs (3) and (4), we have

$$\begin{cases} \min\{-L_2 C, C - \max\{e^x - K, 0\}\} = 0, & (x, t) \in (0, B) \times (0, T), \\ C(x, 0) = \max\{e^x - K, 0\}, & x \in (0, B), \\ C(0, t) = u(B, t) = 0, & t \in (0, T), \end{cases} \tag{5}$$

where

$$L_2 C = \partial_\tau C + \frac{1}{2}\sigma^2 \partial_{xx} C + (r - \frac{1}{2}\sigma^2)\partial_x C - rC.$$

It is worth noting for the reader that x is a one-dimensional variable here, as in this financial example, the American option is linked to only one risky asset. Model (5) represents a specific financial case of the main problem studied in model (1), and thus, in model (1), we set x as an n-dimensional variable. Additionally, the variational inequality suitable for American options shows a high degree of structural similarity with Eq (1).

On the other hand, transaction costs are often associated with the exercise of options, which necessitates a modification of the volatility σ. For instance, Pars and Avellaneda provided a transaction cost model where the volatility σ satisfies [13]

$$\sigma^2 = \sigma_0{}^2(1 + \psi \operatorname{sign}(\partial_x(|\partial_x C|^{p-2}\partial_x C))).$$

Here, $\sigma_0{}^2$ represents the long-term volatility level, and the constant ψ is determined by the trading frequency and cost ratio. This adjustment is also consistent with the parabolic operator structure of model (1).

Lastly, the spatial gradient of solutions to the variational inequality (5) applicable to American options not only measures the change in option value with respect to the underlying asset, but it also allows Black and Scholes to construct a risk-free portfolio to hedge against risk.

Based on this, we examine more general cases than variational inequality (5). This article mainly analyzes the Sobolev estimation of the solution to variational inequality (1) and the inverse Hölder estimation of the spatial gradient. Before that, let us give a few useful symbols.

Expanding upon this framework, we broaden our analysis to encompass wider scenarios than those addressed in variational inequality (5). The main objective of this paper is to conduct a thorough investigation into the Sobolev estimation for solving variational inequality (1), along with exploring the inverse Hölder estimation of the spatial gradient. Before delving into these aspects, we introduce a set of relevant mathematical symbols. For a given non-negative constant ρ, and any $(x_0, t_0) \in \Omega_T$, we define

$$D_\rho = D_\rho(x_0) = \{y \in \mathrm{R}_n : |y - x_0| < \rho\}$$

to denote the ball in space R_n that is also contained within the bounded region Ω. Similarly, we use

$$Q_{\rho,s} = Q_{\rho,s}(x_0, t_0) = D_\rho(x_0) \times (t_0 - s, t_0 + s)$$

to denote the cylinder in space R_{n+1} that is also contained within Ω_T. Lastly, let $|D_\rho|$ represent the Lebesgue measure of the set D_ρ in space R_n, then the averaging operator of u on D_ρ is defined as $I_\rho(t)$, given by

$$I_\rho(t) = \fint_{D_\rho} u \,\mathrm{d}x = \frac{1}{|D_\rho|} \int_{D_\rho} u \,\mathrm{d}x.$$

With the help of the maximal monotone operator,

$$\xi(x) = \begin{cases} 0, & x > 0, \\ M_0, & x = 0, \end{cases}$$

References [8, 11, 12] analyzed the existence of generalized solutions, whose definition is as follows:

Definition 2.1 A pair (u, ξ) is defined as a generalized solution of the variational inequality (1) if it satisfies the following conditions:

(a) $u \in L^\infty(0, T, W^{1,p}(\Omega))$ and $\partial_t u \in L^\infty(0, T, L^2(\Omega))$,

(b) $u(x, t) \geq u_0(x),\ u(x, 0) = u_0(x)$ for any $(x, t) \in \Omega_T$,

(c) For every test function $\varphi \in C^1(\bar{\Omega}_T)$ and for each $t \in [0, T]$, the following equality holds:

$$\int\int_{\Omega_t} \partial_t u \cdot \varphi + u|\nabla u|^{p-2}\nabla u \nabla\varphi \mathrm{d}x\mathrm{d}t + (1-\gamma)\int\int_{\Omega_t} |\nabla u|^p \varphi \mathrm{d}x\mathrm{d}t = \int\int_{\Omega_t} \xi \cdot \varphi \mathrm{d}x\mathrm{d}t.$$

In order to ensure the solvability of the problem, we still impose the restriction $\gamma \in (0, 1)$ in this study.

Lemma 2.2 The solution u of variational inequality (1) is uniformly bounded in Ω_T. That is, for any $(x,t) \leq \Omega_T$, there exists a constant ϑ_0, independent of (x,t), such that

$$|u| \leq \vartheta_0.$$

Proof Note that from (1), we obtain $Lu \leq 0$ for any $(x,t) \in \Omega_T$. By multiplying both sides of $Lu \leq 0$ by $\phi = (u-\vartheta_0)_+$, we have

$$\begin{aligned}&\int_\Omega \partial_t(u-\vartheta_0)\cdot(u-\vartheta_0)_+\mathrm{d}x + \int_\Omega u|\nabla(u-\vartheta_0)|^{p-2}\nabla(u-\vartheta_0)\cdot\nabla(u-\vartheta_0)_+\mathrm{d}x\\ &+(1-\gamma)\int_\Omega u|\nabla(u-\vartheta_0)|^p(u-\vartheta_0)_+\mathrm{d}x \leq 0.\end{aligned} \tag{6}$$

Note that when $u \leq \vartheta_0$, $\partial_t(u-\vartheta_0)_+ = 0$ and $\nabla(u-\vartheta_0)_+ = 0$; when $u \leq \vartheta_0$, $\partial_t(u-\vartheta_0)_+ = \partial_t u$ and $\nabla(u-\vartheta_0)_+ = \nabla u$, thus

$$\int_\Omega u|\nabla(u-\vartheta_0)|^{p-2}\nabla(u-\vartheta_0)\cdot\nabla(u-\vartheta_0)_+\mathrm{d}x = \int_\Omega u|\nabla(u-\vartheta_0)|^p\mathrm{d}x \geq 0. \tag{7}$$

By further removing the non-negative terms $\int_\Omega u|\nabla(u-\vartheta_0)|^p\mathrm{d}x$ and $\int_\Omega u|\nabla(u-\vartheta_0)|^p(u-\vartheta_0)_+\mathrm{d}x$ from (6), we obtain

$$\int_\Omega \partial_t(u-\vartheta_0)_+^2\mathrm{d}x \leq 0.$$

Clearly, when ϑ_0 is sufficiently large, $\int_\Omega(u_0-\vartheta_0)_+^2\mathrm{d}x = 0$ holds. Therefore, for any $t \in (0,\ T)$,

$$\frac{1}{2}\int_\Omega(u-\vartheta_0)_+^2\mathrm{d}x \leq 0.$$

This demonstrates $u \leq \vartheta_0$ a.e. in Ω_T □.

Note that from (1), it is easy to see that $Lu \leq 0$ in $Q_{2\rho,2s}$. By choosing the test function $\phi = \psi^2(u-\lambda)_+$, and then integrating $\phi Lu \leq 0$ over $Q_{2\rho,2s}$,

$$\int_{Q_{2\rho,2s}}\partial_t u\cdot u\mathrm{d}x\mathrm{d}t + \int_{Q_{2\rho,2s}}|\nabla u|^{p-2}\nabla u\cdot\nabla u\mathrm{d}x\mathrm{d}t + (1-\gamma)\int_{Q_{2\rho,2s}}u|\nabla u|^p\mathrm{d}x\mathrm{d}t \leq 0.$$

From Eq (13) in [11] or Theorem 2.1 in [12], it is easy to see that when $\gamma \in (0,\ 1)$, for any $t \in (0,T)$,

$$\nabla u \in L^p(\Omega). \tag{8}$$

3. Sobolev estimates under lower-order W_1^p norm

This section examines the Sobolev estimates for the solution u. A Sobolev estimate on a local spherical region $Q_{\rho,s}(x,t)$ is constructed using the lower-order W_1^p norm of the solution u. Initially, we investigate the time continuity results of an operator $I_\rho(t)$.

Lemma 3.1 For any $Q_{4\rho,s} \in \Omega_T$, there exists a constant C, which depends solely on p and ϑ_0, such that

$$|I_\rho(t_1) - I_\rho(t_2)| \leq \frac{Cs}{\rho}\oint_{Q_{2\rho,s}}|\nabla u|^{p-1}\mathrm{d}x\mathrm{d}t.$$

Proof Assume $t_1 < t_2$, choose η sufficiently small, and suppose the function $\psi_1 \in C_0^\infty(t_1-\eta, t_2+\eta)$ satisfies

$$\psi_1 = 1 \text{ in } (t_1,\ t_2) \text{ and } 0 \le \psi_1 \le 1 \text{ in } (t_1-\eta, t_2+\eta).$$

By applying integration by parts, it is easy to obtain

$$\int_\Omega u\,\mathrm{div}(|\nabla u|^{p-2}\nabla u)\,dx = \int_{\partial\Omega} u|\nabla u|^{p-2}\nabla u\,dx - \int_\Omega |\nabla u|^p\,dx.$$

Furthermore, due to $|D_{\bar\rho}| \times \left(I_{\bar\rho}(t_1) - I_{\bar\rho}(t_2)\right) = \int\int_{D_{\bar\rho}\times(t_1,t_2)} \partial_t u \mathrm{d}x\mathrm{d}t$, we have

$$|D_{\bar\rho}| \times \left(I_{\bar\rho}(t_1) - I_{\bar\rho}(t_2)\right) \le \int\int_{\partial D_{\bar\rho}\times(t_1,t_2)} u|\nabla u|^{p-2}\nabla u \mathrm{d}x\mathrm{d}t - (1-\gamma)\int\int_{D_{\bar\rho}\times(t_1,t_2)} |\nabla u|^p \mathrm{d}x\mathrm{d}t. \tag{9}$$

Considering $(1-\gamma)\int\int_{D_{\bar\rho}\times(t_1,t_2)} |\nabla u|^p\mathrm{d}x\mathrm{d}t$ is non-negative, by Lemma 2.2, we obtain

$$|D_{\bar\rho}| \times \left(I_{\bar\rho}(t_1) - I_{\bar\rho}(t_2)\right) \le \vartheta_0 \int\int_{\partial D_{\bar\rho}\times(t_1,t_2)} |\nabla u|^{p-2}\nabla u \mathrm{d}x\mathrm{d}t. \tag{10}$$

Here, we choose $\bar\rho$ to satisfy $\rho < \bar\rho < 2\rho$. On the other hand, according to [14, p. 5, line 3] and Lemma 4.4 of [15], there exists a constant C that depends only on n and p such that

$$\int\int_{\partial D_{\bar\rho}\times(t_1,t_2)} |\nabla u|^{p-1}\mathrm{d}x\mathrm{d}t \le \frac{C}{\rho}\int\int_{D_{2\rho}\times(t_1,t_2)} |\nabla u|^{p-1}\mathrm{d}x\mathrm{d}t. \tag{11}$$

By combining (10) and (11) and substituting the result into (9), the proposition is established. □

Theorem 3.1 Assume u is a solution to the variational inequality (1). If $u \in L^\alpha(\tau-2s, \tau+2s;\ W^{1,p}(D_{2\rho}(z)))$, then for any $\alpha \in (1,\infty)$, we have

$$\int\int_{Q_{\rho,s}} |u(x,t) - I_\rho(t)|^{\alpha(1+2/n)}\mathrm{d}x\mathrm{d}t \le C\left(\int\int_{Q_{\rho,s}} |\nabla u|^\alpha \mathrm{d}x\mathrm{d}t\right)\left(\operatorname*{ess\,sup}_{t\in(\tau-2s,\ \tau+2s)} \int_{D_{2\rho}} |u - I_{\hat\rho}(t)|^2\mathrm{d}x\right)^{\frac{\alpha}{n}}.$$

Proof By selecting $\tau - 2s < t < \tau + 2s$ and $1 < \alpha < \infty$, we analyze the Sobolev-type estimates for the solution to the variational inequality problem (1) under the condition

$$u \in L^\alpha(\tau-2s, \tau+2s; W^{1,p}(D_{2\rho}(z))\ ).$$

Define

$$v(x,t) = |u(x,t) - I_\rho(t)|\psi(x,t), \tag{12}$$

where $\psi(x,t)$ is a cut-off function on $Q_{2\rho,2s}$,

$$\psi(x,t) = 1 \text{ in } Q_{\rho,s}, 0 \le \psi(x,t) \le 1 \text{ in } Q_{2\rho,2s},\ |\nabla\psi(x,t)| \le C\frac{1}{\rho}. \tag{13}$$

Evidently, when $t \notin (\tau-s, \tau-s)$, for any $x \in D_{2\rho}(z)$, we have $\psi(x,\tau) = 0$. When $t \in (\tau-s, \tau-s)$, for any $x \in D_{2\rho}(z)$, we have

$$\psi(x,t) \ge 0 \text{ and } |\partial_t\psi(x,t)| \le \frac{C}{s} \tag{14}$$

For convenience, let $\hat{\rho} = 2\rho$, and define

$$J = \int_{D_{\hat{\rho}}} v^{\alpha(1+2/n)} \mathrm{d}x = \int_{D_{\hat{\rho}}} v^{2/n} v^{\alpha(1+2/n)-2/n} \mathrm{d}x. \tag{15}$$

Thus, by the Hölder inequality,

$$J \leq \left(\int_{D_{\hat{\rho}}} v^2 \mathrm{d}x \right)^{1/n} \left(\int_{D_{\hat{\rho}}} v^{[\alpha+(2/n)(\alpha-1)]n/(n-1)} \mathrm{d}x \right)^{(n-1)/n}. \tag{16}$$

Further applying the Sobolev inequality to $\left(\int_{D_{\hat{\rho}}} v^{[\alpha+(2/n)(\alpha-1)]n/(n-1)} \mathrm{d}x \right)^{(n-1)/n}$, we get

$$\left(\int_{D_{\hat{\rho}}} v^{[\alpha+(2/n)(\alpha-1)]n/(n-1)} \mathrm{d}x \right)^{(n-1)/n} \leq C(n) \int_{D_{\hat{\rho}}} |\nabla v^{\alpha+(2/n)(\alpha-1)}| \mathrm{d}x \leq C(n) \int_{D_{\hat{\rho}}} v^{(\alpha-1)(1+2/n)} \nabla v \mathrm{d}x. \tag{17}$$

The final inequality sign in the above expression holds because $(\alpha - 1)(1 + \frac{2}{n}) = \alpha + \frac{2(\alpha-1)}{n} - 1$. Using the Hölder inequality again,

$$\int_{D_{\hat{\rho}}} v^{(\alpha-1)(1+2/n)} |\nabla v| \mathrm{d}x \leq \left(\int_{D_{\hat{\rho}}} |\nabla v|^\alpha \mathrm{d}x \right)^{\frac{1}{\alpha}} \left(\int_{D_{\hat{\rho}}} v^{\alpha(1+2/n)} |\nabla v| \mathrm{d}x \right)^{\frac{\alpha-1}{\alpha}}. \tag{18}$$

Therefore, by combining inequalities (16)–(18), we obtain

$$J \leq C(n) J^{\frac{\alpha-1}{\alpha}} \left(\int_{D_{\hat{\rho}}} v^2 \mathrm{d}x \right)^{1/n} \left(\int_{D_{\hat{\rho}}} |\nabla v|^\alpha \mathrm{d}x \right)^{\frac{1}{\alpha}}. \tag{19}$$

Thus, to estimate J, it suffices to analyze $\left(\int_{D_{\hat{\rho}}} |\nabla v|^\alpha \mathrm{d}x \right)^{\frac{1}{\alpha}}$ and $\left(\int_{D_{\hat{\rho}}} v^2 \mathrm{d}x \right)^{1/n}$ and obtain their upper bounds with respect to u. Notice that the cut-off function $\psi(x, t)$ satisfies $0 \leq \psi(x, t) \leq 1$ *in* $Q_{2\rho,2s}$ and $|\nabla \psi(x, t)| \leq C \frac{1}{\rho}$, thus

$$\left(\int_{D_{\hat{\rho}}} |\nabla v|^\alpha \mathrm{d}x \right)^{\frac{1}{\alpha}} \leq \frac{C}{\hat{\rho}} \left(\int_{D_{\hat{\rho}}} |u - I_\rho(t)|^\alpha \mathrm{d}x \right)^{\frac{1}{\alpha}} + \left(\int_{D_{\hat{\rho}}} |\nabla u|^\alpha \mathrm{d}x \right)^{\frac{1}{\alpha}}. \tag{20}$$

Applying the Minkowski inequality again, we get

$$\begin{aligned} &\left(\int_{D_{\hat{\rho}}} |u - I_\rho(t)|^\alpha \mathrm{d}x \right)^{\frac{1}{\alpha}} \\ &\leq \left(\int_{D_{\hat{\rho}}} |u - I_{\hat{\rho}}(t)|^\alpha \mathrm{d}x \right)^{\frac{1}{\alpha}} + \left(\int_{D_{\hat{\rho}}} |I_{\hat{\rho}}(t) - I_\rho(t)|^\alpha \mathrm{d}x \right)^{\frac{1}{\alpha}} \leq \left(\int_{D_{\hat{\rho}}} |u - I_{\hat{\rho}}(t)|^\alpha \mathrm{d}x \right)^{\frac{1}{\alpha}} + |I_{\hat{\rho}}(t) - I_\rho(t)| \times |D_{\hat{\rho}}|^{\frac{1}{\alpha}}. \end{aligned} \tag{21}$$

Further analyzing $|I_{\hat{\rho}}(t) - I_\rho(t)| \times |D_{\hat{\rho}}|^{\frac{1}{\alpha}}$ in (21), by $I_\rho(t)$, we have

$$|I_{\hat{\rho}}(t) - I_\rho(t)| \times |D_{\hat{\rho}}|^{\frac{1}{\alpha}} \leq |D_{\hat{\rho}}|^{\frac{1}{\alpha}} |D_\rho|^{-1} \times \int_{D_\rho} |u - I_{\hat{\rho}}(t)| \mathrm{d}x \leq |D_{\hat{\rho}}|^{\frac{1}{\alpha}} |D_\rho|^{-1} \times \int_{D_{\hat{\rho}}} |u - I_{\hat{\rho}}(t)| \mathrm{d}x. \tag{22}$$

Note that we previously assumed $u \in L^{\alpha}(\tau - 2s, \tau + 2s; W^{1,p}(D_{2\rho}(z)))$. Thus, combining inequalities (21) and (22), and using the Sobolev inequality, we obtain

$$\left(\int_{D_{\hat{\rho}}} |u - I_{\rho}(t)|^{\alpha} \mathrm{d}x\right)^{\frac{1}{\alpha}} \le C(1 + \rho^{n/\alpha}\hat{\rho}^{-n})\left(\int_{D_{\hat{\rho}}} |u - I_{\hat{\rho}}(t)|^{\alpha} \mathrm{d}x\right)^{\frac{1}{\alpha}} \le C\hat{\rho}\left(\int_{D_{\hat{\rho}}} |\nabla u|^{\alpha} \mathrm{d}x\right)^{\frac{1}{\alpha}}. \tag{23}$$

Here, we have used the conditions $\alpha > 1$ and $\hat{\rho} > \rho > 0$. Substituting (23) into (20), we obtain an estimate for $\left(\int_{D_{\hat{\rho}}} |\nabla v|^{\alpha} \mathrm{d}x\right)^{\frac{1}{\alpha}}$ as

$$\left(\int_{D_{\hat{\rho}}} |\nabla v|^{\alpha} \mathrm{d}x\right)^{\frac{1}{\alpha}} \le C\left(\int_{D_{\hat{\rho}}} |\nabla u|^{\alpha} \mathrm{d}x\right)^{\frac{1}{\alpha}}. \tag{24}$$

First, we analyze $\left(\int_{D_{\hat{\rho}}} v^2 \mathrm{d}x\right)^{\frac{1}{n}}$. By estimating $\left(\int_{D_{\hat{\rho}}} |u - I_{\rho}(t)|^2 \mathrm{d}x\right)^{\frac{1}{n}}$ and applying Minkowski's inequality, we can derive

$$\left(\int_{D_{\hat{\rho}}} |u - I_{\rho}(t)|^2 \mathrm{d}x\right)^{\frac{1}{n}} \le \left(\left(\int_{D_{\hat{\rho}}} |u - I_{\hat{\rho}}(t)|^2 \mathrm{d}x\right)^2 + \left(\int_{D_{\hat{\rho}}} |I_{\rho}(t) - I_{\hat{\rho}}(t)|^2 \mathrm{d}x\right)^2\right)^{\frac{1}{2n}}. \tag{25}$$

By utilizing the inequality $(a + b)^{\frac{1}{2n}} \le (2n)^{2n}(a^{\frac{1}{2n}} + b^{\frac{1}{2n}})$, the estimate for $\left(\int_{D_{\hat{\rho}}} |u - I_{\rho}(t)|^2 \mathrm{d}x\right)^{\frac{1}{n}}$ can be reformulated as

$$\left(\int_{D_{\hat{\rho}}} |u - I_{\rho}(t)|^2 \mathrm{d}x\right)^{\frac{1}{n}} \le (2n)^{2n}\left(\left(\int_{D_{\hat{\rho}}} |u - I_{\hat{\rho}}(t)|^2 \mathrm{d}x\right)^{\frac{1}{n}} + \left(\int_{D_{\hat{\rho}}} |I_{\rho}(t) - I_{\hat{\rho}}(t)|^2 \mathrm{d}x\right)^{\frac{1}{n}}\right). \tag{26}$$

Furthermore, by the definition of $I_{\rho}(t)$, we obtain

$$\left(\int_{D_{\hat{\rho}}} |I_{\rho}(t) - I_{\hat{\rho}}(t)|^2 \mathrm{d}x\right)^{\frac{1}{n}} = |I_{\hat{\rho}}(t) - I_{\rho}(t)|^{\frac{2}{n}} \times |D_{\hat{\rho}}|^{\frac{1}{n}} \le C\rho^{n/2}\hat{\rho}^{-n}\left(\int_{D_{\hat{\rho}}} |u - I_{\hat{\rho}}(t)|^2 \mathrm{d}x\right)^{\frac{1}{n}}. \tag{27}$$

Thus, by combining inequalities (17) and (18), we derive

$$\left(\int_{D_{\hat{\rho}}} |u - I_{\rho}(t)|^2 \mathrm{d}x\right)^{\frac{1}{n}} \le C\rho^{n/2}\hat{\rho}^{-n}\left(\int_{D_{\hat{\rho}}} |u - I_{\hat{\rho}}(t)|^2 \mathrm{d}x\right)^{\frac{1}{n}}. \tag{28}$$

The truncation function $\psi(x, t)$ fulfills $0 \le \psi(x, t) \le 1$ in $Q_{2\rho,2s}$, thereby yielding an estimate for $\left(\int_{D_{\hat{\rho}}} v^2 \mathrm{d}x\right)^{\frac{1}{n}}$ denoted as

$$\left(\int_{D_{\hat{\rho}}} v^2 \mathrm{d}x\right)^{\frac{1}{n}} \le \left(\int_{D_{\hat{\rho}}} |u - I_{\rho}(t)|^2 \mathrm{d}x\right)^{\frac{1}{n}} \le C\rho^{n/2}\hat{\rho}^{-n}\left(\int_{D_{\hat{\rho}}} |u - I_{\hat{\rho}}(t)|^2 \mathrm{d}x\right)^{\frac{1}{n}}. \tag{29}$$

In summary, by substituting the results from (24) and (29) into Eq (19), we obtain

$$J \le C(n)J^{\frac{\alpha-1}{\alpha}} C\rho^{n/2}\hat{\rho}^{-n}\left(\int_{D_{\hat{\rho}}} |u - I_{\hat{\rho}}(t)|^2 \mathrm{d}x\right)^{\frac{1}{n}} C\left(\int_{D_{\hat{\rho}}} |\nabla u|^{\alpha} \mathrm{d}x\right)^{\frac{1}{\alpha}}.$$

It is readily seen that $\int_{D_{\rho}} |u(x, t) - I_{\rho}(t)|^{\alpha(1+2/n)} \mathrm{d}x \le J$, and thus we obtain the result of Theorem 3.1. □

4. Inverse Hölder inequality

This section presents an inverse Hölder inequality result. Before that, we introduce a Caccioppoli inequality, which is utilized in the proof.

Lemma 4.1 (Caccioppoli's Inequality) Suppose u is a solution to the variational inequality (1). Then, for any non-negative constant λ , we have

$$\sup_{t\in I_\rho}\int_{D_\rho}(u-\lambda)^2\mathrm{d}x+\int\int_{Q_{\rho,s}}u|\nabla u|^p\mathrm{d}x\mathrm{d}t\leq\frac{C}{s}\int\int_{Q_{2\rho,2s}}(u-\lambda)^2\mathrm{d}x\mathrm{d}t+\frac{C}{\rho^p}\int\int_{Q_{2\rho,2s}}(u-\lambda)^p\mathrm{d}x\mathrm{d}t.$$

Proof Note that from (1), it can be easily deduced that $Lu\leq 0$ in $Q_{2\rho,2s}$. Let us choose a test function $\phi=\psi^2\times(u-\lambda)_+$, and then integrate $\phi\times Lu\leq 0$ over $Q_{2\rho,2s}$, resulting in

$$\begin{aligned}&\int\int_{Q_{2\rho,2s}}\partial_t u\cdot\psi^2(u-\lambda)\mathrm{d}x\mathrm{d}t+\int\int_{Q_{2\rho,2s}}|\nabla u|^{p-2}\nabla u\cdot\nabla[\psi^2(u-\lambda)]\mathrm{d}x\mathrm{d}t\\&+(1-\gamma)\int\int_{Q_{2\rho,2s}}\psi^2(u-\lambda)_+|\nabla u|^p\mathrm{d}x\mathrm{d}t=0.\end{aligned}\tag{30}$$

First, let us analyze $\int\int_{Q_{2\rho,2s}}\partial_t u\times\psi^2(u-\lambda)_+\mathrm{d}x\mathrm{d}t$ by employing the method of integration by parts, yielding

$$\int\int_{Q_{2\rho,2s}}\partial_t[\psi^2(u-\lambda)_+{}^2]\mathrm{d}x\mathrm{d}t=\int\int_{Q_{2\rho,2s}}\partial_t u\cdot\psi^2(u-\lambda)_+\mathrm{d}x\mathrm{d}t+2\int\int_{Q_{2\rho,2s}}\psi\psi'(u-\lambda)_+{}^2\mathrm{d}x\mathrm{d}t,\tag{31}$$

and

$$\begin{aligned}&\int\int_{Q_{2\rho,2s}}u|\nabla u|^{p-2}\nabla u\cdot\nabla[\psi^2(u-\lambda)_+]\mathrm{d}x\mathrm{d}t\\&=\int\int_{Q_{2\rho,2s}}\psi^2u|\nabla u|^p\mathrm{d}x\mathrm{d}t+2\int\int_{Q_{2\rho,2s}}\psi\nabla\psi\times u|\nabla u|^{p-2}\nabla u\times(u-\lambda)_+\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{32}$$

Substituting Eqs (31) and (32) into Eq (30), we obtain

$$\begin{aligned}&\int\int_{Q_R^s}\partial_t[\psi^2(u-\lambda)_+{}^2]\mathrm{d}x\mathrm{d}t-2\int\int_{Q_R^s}\psi\psi'(u-\lambda)_+{}^2\mathrm{d}x\mathrm{d}t\\&+\int\int_{Q_R^s}\psi^2|\nabla u|^p\mathrm{d}x\mathrm{d}t+2\int\int_{Q_R^s}\psi\nabla\psi\times|\nabla u|^{p-2}\nabla u\times(u-\lambda)_+\mathrm{d}x\mathrm{d}t=0,\end{aligned}$$

that is

$$\begin{aligned}&\int_{B_R}\psi^2(u-\lambda)_+{}^2\mathrm{d}x|_{t=s}+\int\int_{Q_R^s}\psi^2|\nabla u|^p\mathrm{d}x\mathrm{d}t\\&=2\int\int_{Q_R^s}\psi\psi'(u-\lambda)_+{}^2\mathrm{d}x\mathrm{d}t-2\int\int_{Q_R^s}\psi\nabla\psi\times|\nabla u|^{p-2}\nabla u\times(u-\lambda)_+\mathrm{d}x\mathrm{d}t.\end{aligned}\tag{33}$$

Based on [11, 12], by applying the Hölder's inequality and Young's inequality to Eq (4), we have

$$\sup_{t\in I_\rho}\int_{D_\rho}(u-\lambda)_+{}^2\mathrm{d}x+\int\int_{Q_{\rho,s}}u|\nabla u|^p\mathrm{d}x\mathrm{d}t\leq\frac{C}{s}\int\int_{Q_{2\rho,2s}}(u-\lambda)^2\mathrm{d}x\mathrm{d}t+\frac{C}{\rho^p}\int\int_{Q_{2\rho,2s}}(u-\lambda)^p\mathrm{d}x\mathrm{d}t.\tag{34}$$

Moreover, by selecting $\phi=\psi^2(u-\lambda)_-$ and repeating the proof process of Eqs (30)–(34), we can easily obtain

$$\sup_{t\in I_\rho}\int_{D_\rho}(u-\lambda)_+{}^2\mathrm{d}x+\int\int_{Q_{\rho,s}}u|\nabla u|^p\mathrm{d}x\mathrm{d}t\leq\frac{C}{s}\int\int_{Q_{2\rho,2s}}(u-\lambda)^2\mathrm{d}x\mathrm{d}t+\frac{C}{\rho^p}\int\int_{Q_{2\rho,2s}}(u-\lambda)^p\mathrm{d}x\mathrm{d}t.\tag{35}$$

By combining Eqs (34) and (35), the theorem is proven.□

Theorem 4.1 Define $q = \max\{p-1, pn/(n+2)\}$ and let u be a solution to the variational inequality (1). Then, we have

$$\int\int_{Q_{\rho,s}} u|\nabla u|^p \mathrm{d}x\mathrm{d}t \le \left(\int\int_{Q_{2\rho,2s}} |\nabla u|^q \mathrm{d}x\mathrm{d}t\right)^{\frac{q}{p}}.$$

Proof It is important to note that $I_{2\rho}(t)$ is not a constant over $Q_{2\rho,2s}$, and thus, we choose

$$\lambda = a(Q_{2\rho,2s}) = \frac{1}{|Q_{2\rho,2s}|}\int\int_{Q_{2\rho,2s}} u \,\mathrm{d}x\mathrm{d}t$$

in the Caccioppoli inequality, resulting in

$$\int\int_{Q_{\rho,s}} u|\nabla u|^p \mathrm{d}x\mathrm{d}t \le \frac{C}{s}\int\int_{Q_{2\rho,2s}} |u-a(Q_{2\rho,2s})|^2 \mathrm{d}x\mathrm{d}t + \frac{C}{\rho^p}\int\int_{Q_{2\rho,2s}} |u-a(Q_{2\rho,2s})|^p \mathrm{d}x\mathrm{d}t. \tag{36}$$

By utilizing the Hölder and Young inequalities, we can obtain

$$\begin{aligned} &\tfrac{C}{s}\int\int_{Q_{2\rho,2s}} |u-a(Q_{2\rho,2s})|^2 \mathrm{d}x\mathrm{d}t \\ &\le \tfrac{C}{s}|Q_{2\rho,2s}|^{\frac{p-2}{p}}\left(\int\int_{Q_{2\rho,2s}} |u-a(Q_{2\rho,2s})|^p \mathrm{d}x\mathrm{d}t\right)^{\frac{2}{p}} \le C(p)|Q_R^s|\left(\tfrac{\rho^2}{s}\right)^{\frac{p}{p-2}} + \tfrac{1}{\rho^p}\int\int_{Q_{2\rho,2s}} |u-a(Q_{2\rho,2s})|^p \mathrm{d}x\mathrm{d}t. \end{aligned} \tag{37}$$

Combining Eqs (36) and (37), we can estimate $\int\int_{Q_{\rho,s}} u|\nabla u|^p \mathrm{d}x\mathrm{d}t$ by analyzing only $\int\int_{Q_{2\rho,2s}} |u-a(Q_{2\rho,2s})|^p \mathrm{d}x\mathrm{d}t$. Using the Minkowski inequality, we have

$$\begin{aligned} &\tfrac{C}{p\rho^p}\int\int_{Q_{2\rho,2s}} |u-a(Q_{2\rho,2s})|^p \mathrm{d}x\mathrm{d}t \\ &\le C\rho^{-p}\int\int_{Q_{2\rho,2s}} |u-I_{2\rho}(t)|^p \mathrm{d}x\mathrm{d}t + C\rho^{-p}|Q_{2\rho}| \mathop{ess\,\sup}_{t\in(\tau-s,\tau+s)} |I_{2\rho}(t)-a(Q_{2\rho,2s})|^p. \end{aligned} \tag{38}$$

Next, we analyze the $\int\int_{Q_{2\rho,2s}} |u-I_{2\rho}(t)|^p \mathrm{d}x\mathrm{d}t$ and $|I_{2\rho}(t)-a(Q_{2\rho,2s})|^p$ in Eq (38). By the definition of $I_\rho(t)$ and Lemma 3.1, we have

$$|I_{2\rho}(t)-a(Q_{2\rho,2s})|^p \le (4s)^{-p}\left(\int_{\tau-2s}^{\tau+2s} |I_{2\rho}(t)-I_{2\rho}(\xi)|\mathrm{d}\xi\right)^p \le C\rho^{-p}\left(\oint\oint_{Q_{2\rho,2s}} |\nabla u|^{p-1}\mathrm{d}x\mathrm{d}t\right)^p. \tag{39}$$

Therefore, by utilizing the Hölder and Young inequalities, we can estimate the second term on the right-hand side of Eq (38) as follows:

$$|I_{2\rho}(t)-a(Q_{2\rho,2s})|^p \le C\rho^{-p}|Q_{2\rho,2s}|^{\frac{1}{p}}\left(\oint\oint_{Q_{2\rho,2s}} |\nabla u|^p \mathrm{d}x\mathrm{d}t\right)^{\frac{p-1}{p}}. \tag{40}$$

Now, let us estimate the first term on the right-hand side of Eq (38). By considering Theorem 3.1, we have

$$\int\int_{Q_{2\rho,2s}} |u(x,t)-I_{2\rho}(t)|^p \mathrm{d}x\mathrm{d}t \le C\left(\int\int_{Q_{2\rho,2s}} |\nabla u|^q \mathrm{d}x\mathrm{d}t\right)\left(\mathop{\mathrm{ess\,sup}}_{t\in(\tau-4s,\,\tau+4s)}\int_{D_{4\rho}} |u-I_{4\rho}(t)|^2 \mathrm{d}x\right)^{\frac{q}{n}}. \tag{41}$$

Furthermore, by utilizing the Sobolev inequality on $D_{4\rho}$, we obtain

$$\begin{aligned} &\int_{D_{4\rho}} |u-I_{4\rho}(t)|^2 \mathrm{d}x \le C\rho^2\int_{D_{4\rho}} |\nabla u|^2 \mathrm{d}x \\ &\le C\rho^2\left(\int_{D_{4\rho}} |\nabla u|^p \mathrm{d}x\right)^{2/p}|D_{4\rho}|^{(p-2)/p} = C\rho^2\left(\oint_{D_{4\rho}} |\nabla u|^p \mathrm{d}x\right)^{2/p}|D_{4\rho}|. \end{aligned} \tag{42}$$

By substituting (42) into (41) and combining it with (40), we obtain

$$
\begin{aligned}
&\int\int_{Q_{\rho,s}} u|\nabla u|^p \mathrm{d}x\mathrm{d}t \\
&\leq C(p)|Q_R^s|\Big(\frac{\rho^2}{s}\Big)^{\frac{p}{p-2}} + C\rho^{-p}\Big(c\rho^2|D_{4\rho}|\Big)^{\frac{q}{n}}\Big(\int\int_{Q_{2\rho,2s}} |\nabla u|^q \mathrm{d}x\mathrm{d}t\Big) + C\rho^{-2p}|\mathrm{Q}_{2\rho,2s}|^{1-p}\Big(\int\int_{\mathrm{Q}_{2\rho,2s}} |\nabla u|^{p-1}\mathrm{d}x\mathrm{d}t\Big)^p.
\end{aligned}
$$

Here, we make use of the fundamental result $\nabla u \in L^p(\Omega)$, which is detailed in (8). With this, the proof of Theorem 4.1 is complete. □

5. Discussion and conclusions

This study considers a type of variational inequality problem involving a non-divergence parabolic operator, as shown in (1) and (2). In other words, the Sobolev estimates and inverse Hölder estimates are examined for the solutions of variational inequality (1). First, we define the averaging operator of the variational inequality (1) on the local spatial region D_ρ as $I_\rho(t)$ and prove the uniform continuity of the mean inequality $I_\rho(t)$ with respect to time t. Second, we establish a Sobolev inequality for the averaging operator $I_\rho(t)$, which serves as the cornerstone for proving the inverse Hölder estimates, as stated in Theorem 3.1. Finally, we examine the inverse Hölder estimate problem in local cylindrical regions. The proof relies on Lemma 4.1 and Theorem 3.1, as well as commonly used amplification techniques such as Minkowski inequality, Young's inequality, and Hölder's inequality.

There are still some areas for improvement in the proofs presented in this paper. It is important to note that we make use of the condition $\gamma \in (0,\ 1)$, as when $\gamma > 1$ holds, Eqs (6), (9), and (30) cannot be used as they are in this paper. In such cases, $1-\gamma < 0$, and we cannot eliminate the non-negative terms containing $1-\gamma$. Furthermore, in [8], the existence of weak solutions to similar problems is discussed under the condition $\gamma \in (0, 1)$, which we also continue to adopt here. On the other hand, in order to employ Young's inequality and Hölder's inequality, we also restrict $p \geq 2$. It is worth noting that in the study of regularity theory for parabolic equations, the literature has considered the case $p \in (1, 2)$, and we also aim to explore this limitation in our future research.

Use of AI tools declaration

The author declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

The author declares there is no conflict of interest.

References

1. M. Moradipour, S. A. Yousefi, Using spectral element method to solve variational inequalities with applications in finance, *Chaos Solitons Fractals*, **81** (2015), 208–217. https://doi.org/10.1016/j.chaos.2015.09.006
2. J. Shen, W. Huang, J. Ma, An efficient and provable sequential quadratic programming method for American and swing option pricing, *Eur. J. Oper. Res.*, **316** (2024), 19–35. https://doi.org/10.1016/j.ejor.2023.11.012
3. H. Sultan, A. Hifsa, N. Muhammad, A. A. Pantelous, Pricing American options under Azzalini Ito-McKean skew Brownian motions, *Appl. Math. Comput.*, **451** (2023), 128040. https://doi.org/10.1016/j.amc.2023.128040
4. S. B. Boyana, T. Lewis, A. Rapp, Y. Zhang, Convergence analysis of a symmetric dual-wind discontinuous Galerkin method for a parabolic variational inequality, *J. Comput. Appl. Math.*, **422** (2023), 114922. https://doi.org/10.1016/j.cam.2022.114922
5. S. Migorski, S. Dudek, Constrained evolutionary variational-hemivariational inequalities with application to fluid flow model, *Commun. Nonlinear Sci. Numer. Simul.*, **127** (2023), 107555. https://doi.org/10.1016/j.cnsns.2023.107555
6. Z. Wu, W. Li, Q. Zhang, Y. Xiao, New existence and stability results of mild solutions for fuzzy fractional differential variational inequalities, *J. Comput. Appl. Math.*, **448** (2024), 115926. https://doi.org/10.1016/j.cam.2024.115926
7. X. Wang, S. Chen, H. Qi, A class of delay differential variational inequalities with boundary conditions, *Commun. Nonlinear Sci. Numer. Simul.*, **129** (2024), 107684. https://doi.org/10.1016/j.cnsns.2023.107684
8. Y. Bai, N. Costea, S. Zeng, Existence results for variational-hemivariational inequality systems with nonlinear couplings, *Commun. Nonlinear Sci. Numer. Simul.*, **134** (2024), 108026. https://doi.org/10.1016/j.cnsns.2024.108026
9. J. Zhao, J. Chen, Z. Liu, Second order evolutionary problems driven by mixed quasi-variational-hemivariational inequalities, *Commun. Nonlinear Sci. Numer. Simul.*, **120** (2023), 107192. https://doi.org/10.1016/j.cnsns.2023.107192
10. P. M. N. Feehan, C. A. Pop, Boundary-degenerate elliptic operators and Hölder continuity for solutions to variational equations and inequalities, *Ann. Inst. H. Poincaré C Anal. Non Lineairé,* **34** (2017), 1075–1129. https://doi.org/10.1016/j.anihpc.2016.07.005
11. Y. Sun, T. Wu, Hölder and Schauder estimates for weak solutions of a certain class of non-divergent variation inequality problems in finance, *AIMS Math.*, **8** (2023), 18995–19003. https://doi.org/10.3934/math.2023968
12. J. Li, Z. Tong, Local Hölder continuity of inverse variation-inequality problem constructed by non-Newtonian polytropic operators in finance, *AIMS Math.*, **8** (2023), 28753–28765. https://doi.org/10.3934/math.20231472
13. M. Ptashnyk, Homogenization of some degenerate pseudoparabolic variational inequalities, *J. Math. Anal. Appl.*, **469** (2019), 44–75. https://doi.org/10.1016/j.jmaa.2018.08.047

14. J. Kinnunen, J. L. Lewis, Higher integrability for parabolic systems of p-Laplacian type, *Duke Math. J.*, **102** (2000), 253–271. https://doi.org/10.1215/S0012-7094-00-10223-2

15. P. Baroni, Marcinkiewicz estimates for degenerate parabolic equations with measure data, *J. Funct. Anal.*, **267** (2014), 3397–3426. https://doi.org/10.1016/j.jfa.2014.08.017