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*Research article*

# A Schwarz lemma of harmonic maps into metric spaces

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**Abstract:** We established a Schwarz lemma for harmonic maps from Riemannian manifolds to metric spaces of curvature bounded above in the sense of Alexandrov. We adopted the gradient estimate technique which was based on Zhang-Zhu's maximum principle. In particular, when the domain manifold was a hyperbolic surface, the energy of any conformal harmonic maps into CAT($-1$) spaces were bounded from above uniformly.

**Keywords:** harmonic maps; singular spaces; Schwarz lemma; maximum principle; Alexandrov curveture bound

## 1. Introduction

The classical Schwarz-Pick lemma states that any holomorphic map from unit disk in $\mathbb{C}^2$ into itself decreases the Poincaré-Bergman metric. Ahlfors [1], Chern et al. [2], and Lu [3] extended this result to more general domains and targets with curvature conditions.

In 1978, Yau [4] used the so-called Omori-Yau maximum principle to prove the distance decreasing property for holomorphic maps from Kähler manifolds to Hermitian manifolds under suitable curvature conditions. Later, in the Riemannian settings, the Schwarz type lemma was studied extensively; see, e.g., [5, 6]. There are also many generalizations for generalized harmonic maps; readers can refer to [7–11] and references therein.

There is also growing interest on harmonic maps with singular targets. Gromov and Schoen [12] first developed a theory of harmonic maps in which the target spaces can be taken as metric spaces of curvature bounded from above. This theory has further generalized in Korevaar and Schoen [13], and also Jost [14–16]. These investigations have deeply revealed the structure of harmonic maps with singular targets. Thus, it is natural to consider establishing a Schwarz type lemma for harmonic maps in this broader context.

In this note, the target spaces $(N, d)$ we considered are CAT($-k$) ($k > 0$) spaces, which is a class of metric spaces with curvature bounded above by $-k$ and the curvature condition is given by Toponogov's triangle comparison.

A map $u : M \longrightarrow N$ is called *harmonic* if $u$ is a local minimizer of the energy functional in the sense of Korevaar and Schoen. For a detailed definition and its properties, we refer to Section 2 below.

Our main result states the following.

**Theorem 1.1.** *Let $u : M \longrightarrow N$ be a harmonic map from an m-dimensional complete Riemannian manifold with $\mathrm{Ric}_M \geq -A$ into a $\mathrm{CAT}(-k)$ space $N$, where $A \geq 0$ and $k > 0$ are both constants. Suppose u has generalized dilatation of order β, then*

$$\pi \leq \frac{Am^2\beta^4}{2k(1+\beta^2)}g,$$

*where π is the pull-back inner product (cf. Section 2).*

**Remark 1.2.** The notion of harmonic maps with bounded dilatation was originated from Shen [6], where he proved a related Schwarz type lemma. Our result can be viewed as a generalization in the singular targets setting.

**Corollary 1.3.** *If $\mathrm{Ric}_M \geq 0$, then any harmonic map of generalized dilatation from M into a $\mathrm{CAT}(-k)$ ($k > 0$) space is constant.*

**Corollary 1.4.** *If M is a hyperbolic surface and N is a $\mathrm{CAT}(-1)$ space, then the energy of conformal harmonic map $u : M \longrightarrow N$ satisfies*

$$|\nabla u|^2 \leq 2.$$

**Remark 1.5.** Freidin [17] proved Corollary 1.4 under the additional assumption that Σ is closed. Our result improves his.

Owing to the lack of smoothness, one cannot employ the usual argument of the maximum principle directly in this setting. Instead, we will make use of an approximating version of the maximum principle established by Zhang and Zhu [18]. The similar idea has been successfully applied by Zhang et al. [19] to obtain gradient estimates of harmonic maps in the setting of singular targets.

The rest is organized as follows: In Section 2, we recall some basic and known results on $\mathrm{CAT}(\kappa)$ spaces and harmonic maps. In Section 3, we prove the main results.

## 2. Preliminaries

### 2.1. Alexandrov curvature bound

Give a metric space $(N, d)$. We assume our metric spaces to be *length spaces*, i.e., for each $P, Q \in N$, there is a curve, which we denote $[P, Q]$, such that the length of $[P, Q]$ is exactly $d(P, Q)$. We call $[P, Q]$ a *geodesic* between $P$ and $Q$. We say $N$ is a CAT(0) space (see [20]) if any geodesic triangles in $N$ are thinner than their comparison triangles in $\mathbb{R}^2$. In other words, for every $P, Q, R \in N$, and corresponding points $\bar{P}, \bar{Q}, \bar{R} \in \mathbb{R}^2$ with

$$d(P, Q) = |\bar{P}\bar{Q}|, \quad d(R, Q) = |\bar{R}\bar{Q}|, \quad d(P, R) = |\bar{P}\bar{R}|,$$

we have

$$d(P, Q_t) \leq |\bar{P}\bar{Q}_t|,$$

where $Q_t$ and $\bar{Q}_t$ lie a fraction $t$ of the way along the geodesic segment from $P$ to $Q$ and $\bar{P}$ to $\bar{Q}$, respectively.

A CAT($-1$) space $N$, or $N$ having curvature bounded from above by -1 in the sense of Alexandrov, is simply a length space with a stronger comparison principle. Instead of constructing comparison triangles in $\mathbb{R}^2$, one constructs them in $\mathbb{H}^2$, and the CAT($-1$) space has the same comparison inequality.

### *2.2. Harmonic maps into metric spaces*

In this subsection, we define harmonic maps from an $m$-dimensional complete Riemannian manifold $(M, g)$ to a general metric space $(N, d)$. Let $\Omega \subset M$ be a relatively compact domain. We denote by $L^2(\Omega, N)$ the space of all Borel maps $u : \Omega \longrightarrow N$, (i.e., measurable with respect to $dV_g$) such that for some and, thus, every $p \in N$, we have

$$\int_\Omega d^2(u(x), p) dV_g(x) < \infty.$$

The (Korevaar-Schoen) energy of $u \in L^2(\Omega, N)$ is defined as follows. For $\varepsilon > 0$, we set

$$e_\varepsilon(x) := m \cdot \fint_{S(x,\varepsilon)} \frac{d^2\left(u(x), u\left(x'\right)\right)}{\varepsilon^2} d\Sigma(x')$$

whenever $x \in \Omega$ satisfies $d(x, \partial\Omega) > \varepsilon$ and $e_\varepsilon(x) = 0$ otherwise. Here, $S(x, \varepsilon)$ is the sphere centered at $x$ with radius $\varepsilon$. A map $u$ is said to be in $W^{1,2}(\Omega, N)$ if its *energy*, defined by

$$E(u) := \sup_{\substack{\eta \in C_0(\Omega) \\ 0 \le \eta \le 1}} \left( \limsup_{\varepsilon \to 0} \int_\Omega \eta(x) e_\varepsilon(x) dV_g(x) \right), \tag{2.1}$$

is finite. If $u \in W^{1,2}(\Omega, N)$, then there exists a function $e(u) \in L^1(\Omega)$, called the *energy density* function of $u$, such that $e_\varepsilon dV_g \rightharpoonup e(u) dV_g$ as $\varepsilon \to 0$ and $E(u) = \int_\Omega e(u) dV_g$; see [13]. We often write $|\nabla u|^2(x)$ in place of $e(u)(x)$. In the case that $N$ is a Riemannian manifold and $u$ is smooth, the energy defined in (2.1) coincides with the usual energy.

**Definition 2.1.** A map $u \in W^{1,2}(\Omega, N)$ is said to be *energy minimizing harmonic* if $E(u) \le E(v)$ for all $v \in W^{1,2}(\Omega, N)$. A map $u : M \to N$ is called *energy minimizing harmonic* if its restriction to every relatively compact domain is energy minimizing harmonic.

Let $\overline{x}(x, t)$ be the flow generated by a unit vector field $\omega$ on $M$, that is,

$$\overline{x}(x, 0) = x, \qquad \frac{d}{dt} \overline{x}(x, t) = \omega.$$

The directional energy density $|u_*(\omega)|^2$ is defined by

$$|u_*(\omega)|^2(x) := \lim_{\varepsilon \to 0} \frac{d^2\left(u(\overline{x}(x, \varepsilon)), u(x)\right)}{\varepsilon^2}, \qquad a.e.\ x \in M,$$

and the energy density satisfies

$$|\nabla u|^2(x) = \int_{\mathbb{S}^{n-1}} |u_*(\omega)|^2 d\sigma(\omega),$$

where $\mathbb{S}^{n-1} \subset T_xM$.

The CAT($-k$) hypothesis implies that we can make sense of the notion of the *pull-back inner product*.

**Definition 2.2** ( [13]). The pull-back inner product $\pi : \Gamma(TM) \times \Gamma(TM) \longrightarrow L^1(M)$ is defined by

$$\pi(Z, W) := \frac{1}{4}|u_*(Z+W)|^2 - \frac{1}{4}|u_*(Z-W)|^2.$$

**Proposition 2.3** (Theorem 2.3.2 in [13]). *For the operator $\pi$ defined above, we have*

*(1) $\pi$ is continuous, symmetric, bilinear, nonnegative and tensorial.*
*(2) If we write $\pi_{ij} = \pi(e_i, e_j)$ for a local frame $\{e_i\}$ on $M$, then for $Z = Z^i e_i$ and $W = W^j e_j$, we have $\pi(Z, W) = \pi_{ij} Z^i W^j$.*
*(3) The energy density is the trace of $\pi$ with respect to $g$, i.e., $|\nabla u|^2 = g^{ij}\pi_{ij} = \mathrm{tr}_g(\pi)$.*

## 3. The Schwarz lemma

Let $(M^m, g)$ be a complete Riemannian manifold with $\mathrm{Ric}_M \geq -A$, where $A \geq 0$ is a constant, and suppose that $u : M \longrightarrow N$ is a harmonic map into a CAT($-k$) ($k > 0$) space $N$. Under a local frame $\{e_i\}$ on $M$, the pull-back tensor $\pi$ can be expressed as a matrix

$$\pi = (\pi_{ij}).$$

From Proposition 2.3, it is clear that the eigenvalus of $\pi$ are nonnegative, say,

$$\lambda_1(x) \geq \lambda_2(x) \geq \cdots \geq \lambda_m(x) \geq 0.$$

Hence, we can introduce the notion of *generalized dilatation* in this context.

**Definition 3.1.** Let $(M, g)$ be a Riemannian manifold and $N$ be a CAT(0) space. A map $u \in W^{1,2}(M, N)$ is said to have *generalized dilatation* of order $\beta$, if there is a positive number $\beta$ such that $\lambda_1(x) \leq \beta^2(\lambda_2(x) + \cdots + \lambda_m(x))$ for *a.e.* $x \in M$.

Let us recall the proof of Schwarz type lemma in the smooth context. We refer to [7] and [8]. There are two main ingredients: the Bochner formula and a maximum principle. When the target space is of CAT($-1$) type, we have the following Bochner inequality.

**Lemma 3.2** (Theorem 1 in [17]). *For a harmonic map $u : M \longrightarrow N$ from a Riemannian manifold $M$ into a* CAT($-k$) ($k > 0$) *metric space, $|\nabla u|^2$ satisfies*

$$\frac{1}{2}\Delta|\nabla u|^2 \geq \langle \mathrm{Ric}_M, \pi \rangle + k(|\nabla u|^4 - |\pi|^2) \tag{3.1}$$

*in the sense of distributions. Here, $\langle \mathrm{Ric}_M, \pi \rangle$ denotes the inner product on symmetric 2-tensors and $|\pi|^2 = \langle \pi, \pi \rangle$.*

We remark that the energy density $|\nabla u|^2$ is not smooth generally. Moreover, it even may not be continuous. That presents a problem to carry the gradient estimates argument in [7] or Omori-Yau maximum principle in [8] directly due to the lack of smoothness. This can be overcome by making use of the following Zhang-Zhu's maximum principle.

**Theorem 3.3** (Theorem 1.3 in [18]). *Let $\Omega$ be a bounded domain in a Riemannian manifold $(M^m, g)$ with $\mathrm{Ric}_M \geq -A$ for some constant $A \geq 0$. Let $f \in W^{1,2}_{loc}(\Omega) \cap L^\infty_{loc}(\Omega)$ satisfy $\Delta f$ as a signed Radon measure with $\Delta^{\mathrm{sing}} f \geq 0$, where $\Delta f = \Delta^{\mathrm{ac}} f \cdot dV_g + \Delta^{\mathrm{sing}} f$ is the Radon-Nikodym decomposition with respect to $dV_g$. Assume $f$ attains its strict maximum in $\Omega$ in the following sense: there is a neighborhood $U \subset\subset \Omega$ such that*

$$\sup_U f > \sup_{\Omega \setminus U} f.$$

*Then, for any function $w \in W^{1,2}(\Omega) \cap L^\infty(\Omega)$, there is a sequence $\{x_j\}_{j\in\mathbb{N}} \subset U$ satisfying that they are approximate continuity points of $\Delta^{\mathrm{ac}} f$ and $\langle \nabla f, \nabla w \rangle$, with the following properties:*

$$f(x_j) \geq \sup_\Omega f - \frac{1}{j}, \qquad \Delta^{\mathrm{ac}} f(x_j) + \langle \nabla f, \nabla w \rangle (x_j) \leq \frac{1}{j}.$$

*In the following, $\sup_U f$ means $\text{ess-sup}_U f$.*

We prove the main result.

**Theorem 3.4.** *Let $u : M \longrightarrow N$ be a harmonic map from a complete Riemannian manifold with $\mathrm{Ric}_M \geq -A$ into a $\mathrm{CAT}(-k)$ space $N$, where $A \geq 0$ and $k > 0$ are both constants. Suppose $u$ has generalized dilatation of order $\beta$, then*

$$\pi \leq \frac{Am^2\beta^4}{2k(1+\beta^2)} g,$$

*where $\pi$ is the pull-back inner product.*

*Proof.* For simplicity, we assume $k = 1$. By the curvature condition and the Bochner inequality (3.1), we have

$$\begin{aligned} \frac{1}{2}\Delta|\nabla u|^2 &\geq \langle \mathrm{Ric}, \pi \rangle + |\nabla u|^4 - |\pi|^2 \\ &\geq -A|\nabla u|^2 + |\nabla u|^4 - |\pi|^2. \end{aligned} \tag{3.2}$$

Note that $u$ is of bounded dilatation, thus

$$\begin{aligned} |\nabla u|^4 - |\pi|^2 &= \left[\mathrm{tr}_g(\pi)\right]^2 - |\pi|^2 = \left(\sum_{i=1}^m \lambda_i\right)^2 - \left(\sum_{i=1}^m \lambda_i^2\right) \\ &= 2\sum_{1\leq i<j\leq m} \lambda_i\lambda_j \geq 2\lambda_1 \sum_{j=2}^m \lambda_j \geq \frac{2}{\beta^2}\lambda_1^2 \\ &\geq \frac{2}{\beta^2}\left(\frac{\lambda_1 + \ldots + \lambda_m}{m}\right)^2. \end{aligned} \tag{3.3}$$

On the other hand,

$$|\nabla u|^2 = \mathrm{tr}_g(\pi) = \sum_{i=1}^m \lambda_i. \tag{3.4}$$

Combining (3.2), (3.3), and (3.4), we conclude that $|\nabla u|^2$ satisfies

$$\Delta|\nabla u|^2 \geq -2A|\nabla u|^2 + \frac{4}{\beta^2 m^2}|\nabla u|^4, \tag{3.5}$$

in the weak sense. For simplicity, let $F := |\nabla u|^2$.

Fix a constant $\delta > 0$ to be sufficiently small and let $\eta(x) := \eta(r(x))$ be a function of the distance $r(x, x_0)$, where $x_0$ is a fixed point in $M$, such that

$$\delta \le \eta \le 1 \quad \text{on} \quad B_R(x_0), \qquad \eta = \begin{cases} 1, & \text{on} \quad B_{\frac{R}{2}}(x_0), \\ \delta, & \text{on} \quad B_R(x_0) \setminus B_{\frac{3R}{4}}(x_0), \end{cases}$$

and

$$-\frac{C_1}{R}\eta^{\frac{1}{2}} \le \eta' \le 0, \qquad |\eta''| \le \frac{C_1}{R^2}, \quad \forall r \in \left(0, \frac{3R}{4}\right),$$

where $C_1$ is a constant independent of $m, K, R$. We remark that the function $\eta$ can be chosen in the following ways: one first takes a function $\phi$ such that $0 \le \phi \le 1$, $\phi = 1$ on $(0, \frac{R}{2})$, $\phi = 0$ on $(\frac{3R}{4}, R)$, $-\frac{C}{R} \le \phi' \le 0$, and $|\phi''| \le \frac{C}{R^2}$, and then sets $\eta := \left(\frac{\varepsilon+\phi}{\varepsilon+1}\right)^2$, where $\frac{\varepsilon}{\varepsilon+1} = \sqrt{\delta}$. One can verify that $\eta$ satisfies all of the requirements by direct calculations.

Thus,

$$|\nabla\eta| = |\eta'||\nabla r| \le \frac{C_1}{R}\eta^{\frac{1}{2}}, \quad \text{on} \quad B_{\frac{3R}{4}}(x_0). \tag{3.6}$$

By the Laplacian comparison theorem, we also have that

$$\begin{aligned} \Delta\eta &\ge -\frac{C_1}{R}\left(\sqrt{(m-1)A}\coth\left(r\sqrt{\frac{A}{m-1}}\right)\right) - \frac{C_1}{R^2} \\ &\ge -\frac{C_1}{R}\left(\sqrt{(m-1)A} + \frac{m-1}{R}\right) - \frac{C_1}{R^2} \\ &\ge -\frac{C_2}{R^2}, \quad \text{on} \quad B_{\frac{3R}{4}}(x_0) \end{aligned} \tag{3.7}$$

in the sense of distributions. Here, $C_2 := C_1\left(R\sqrt{(m-1)A} + m\right)$.

Next, we set $G := F \cdot \eta$. It is obvious that $G$ is in $W^{1,2}(B_{\frac{3R}{4}}(x_0)) \cap L^\infty(B_{\frac{3R}{4}}(x_0))$ and $G$ achieves one of its strict maxima in $B_{\frac{R}{2}}(x_0)$ in the sense of Theorem 3.3. Then,

$$\begin{aligned} \Delta G &= 2\left\langle\nabla\eta, \nabla\left(\frac{G}{\eta}\right)\right\rangle + \eta\cdot\Delta F + \Delta\eta\cdot F \\ &\ge \Delta\eta\cdot\frac{G}{\eta} + 2\left\langle\nabla\log\eta, \nabla G\right\rangle - 2\frac{|\nabla\eta|^2}{\eta}\cdot\frac{G}{\eta} + \eta\cdot\Delta F \\ &\ge -\frac{G}{\eta}\cdot\frac{C_2}{R^2} + 2\left\langle\nabla\log\eta, \nabla G\right\rangle - \frac{G}{\eta}\cdot\frac{2C_1^2}{R^2} + \eta\left(-2AF + \frac{4}{\beta^2 m^2}F^2\right). \end{aligned}$$

Let $w := -2\log\eta$, then $w \in W^{1,2}(B_{\frac{3R}{4}}(x_0)) \cap L^\infty(B_{\frac{3R}{4}}(x_0))$. The above inequality reads as

$$\begin{aligned}\Delta_w G &= \Delta G + \langle \nabla w, \nabla G\rangle \\ &\geq -\frac{G}{\eta}\cdot\frac{C_2}{R^2} - \frac{G}{\eta}\cdot\frac{2C_1^2}{R^2} + \eta\cdot\left(\frac{4}{\beta^2 m^2}\left(\frac{G}{\eta}\right)^2 - 2A\left(\frac{G}{\eta}\right)\right) \\ &\geq -\frac{G}{\eta}\cdot\frac{C_2}{R^2} - \frac{G}{\eta}\cdot\frac{2C_1^2}{R^2} + \frac{4}{\beta^2 m^2}\frac{G^2}{\eta} - 2A\frac{G}{\eta} \\ &\geq \frac{G}{\eta}\left[-\frac{C_2}{R^2} - \frac{2C_1^2}{R^2} + \frac{4}{\beta^2 m^2}G - 2A\right] \\ &= \frac{G}{\eta}\left[-\frac{C_3}{R^2} + \frac{4}{\beta^2 m^2}G - 2A\right],\end{aligned}$$

where we have used $G \geq 0$, $1 \leq \frac{1}{\eta}$, and $C_3 := C_2 + 2C_1^2$. That is, we have

$$\Delta G + \langle \nabla w, \nabla G\rangle \geq \frac{G}{\eta}\left[-\frac{C_3}{R^2} + \frac{4}{\beta^2 m^2}G - 2A\right] \tag{3.8}$$

in the weak sense. By Theorem 3.3, we can conclude that there exists a sequence of points $\{x_j\}$ such that for each $j \in \mathbb{N}$,

$$G_j := G(x_j) \geq \sup_{\frac{3R}{4}} G - \frac{1}{j}$$

and

$$\frac{G_j}{\eta(x_j)}\left[-\frac{C_3}{R^2} + \frac{4}{\beta^2 m^2}G_j - 2A\right] \leq \frac{1}{j}.$$

Since $\eta(x_j) \geq \delta > 0$, we can take $j \to \infty$ to obtain

$$\sup_{\frac{3R}{4}} G = \lim_{j\to\infty} G_j \leq \frac{A\beta^2 m^2}{2} + \frac{C_3\beta^2 m^2}{4R^2},$$

which implies

$$\sup_{\frac{R}{2}} |\nabla u|^2 \leq \sup_{\frac{3R}{4}} G \leq \frac{A\beta^2 m^2}{2} + \frac{C_3\beta^2 m^2}{4R^2}.$$

Letting $R \to \infty$, it follows that

$$|\nabla u|^2 \leq \frac{A\beta^2 m^2}{2}. \tag{3.9}$$

As $|\nabla u|^2 = \mathrm{tr}_g(\pi) = \sum_{i=1}^m \lambda_i$, we have

$$\lambda_1 + \frac{1}{\beta^2}\lambda_1 \leq \sum_{i=1}^m \lambda_i \leq \frac{A\beta^2 m^2}{2}.$$

This yields $\lambda_1 \leq \frac{A\beta^4 m^2}{2(1+\beta^2)}$, and we have finished the proof. □

*Proof of Corollary 1.4.* Note that a mapping $u : M \longrightarrow N$ is called *conformal* if $\pi$ satisfies $\pi = \lambda g$ for some nonnegative function $\lambda$. Then, the corollary follows from Theorem 1.1 immediately. □

## Use of AI tools declaration

The author declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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## Conflict of interest

The author declare there is no conflict of interest.

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