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Research article

Numerical and analytical approach to the Chandrasekhar quadratic functional integral equation using Picard and Adomian decomposition methods

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Abstract: This research aimed to find numerical solutions to a type of nonlinear initial value problem (IVP) for hybrid fractional differential equations. Using the Adomian decomposition method (ADM) and the Picard method (PM), we studied the Chandrasekhar quadratic integral equation (QIE). Furthermore, we investigated existence and uniqueness results using measures of weak noncompactness. Through a set of examples and numerical simulations, a comparison was made between the results of the AMD and PM.

Keywords: hybrid integral equation; measure of noncompactness; Adomian decomposition method; Picard method; fractional derivatives

1. Introduction

Studying the theory of fractional differential equations (FDEs) and fractional integral equations (FIEs) is crucial because they are used in many modeling applications. Fractional equations are essential for many areas of fundamental analysis and their applications in economics, physics, and other disciplines. Quadratic integral equations (QIE), in particular, tend to be helpful in describing a wide range of everyday issues, such as theory of radiative transfer, the theory of neutron transport, the kinetic theory of gases, the queuing theory, and the traffic theory (see, for example, [1–4]).

One of the studied QIEs is called the hybrid integral equation (HIE); see [5]. This issue has received great attention in the last few years; see [6–9]. As we see, the hybrid fixed point theory is used to develop the existing solution of the hybrid equations; see [10–13]. Other researchers focused on the analysis of QIEs in Orlicz spaces [14], equations of QIEs with fractional order arising in the queuing theory and biology [15], and the analysis of QIEs depending on both Schauder and Schauder–Tychonoff fixed point principles [16].

1.1. Chandrasekhar quadratic integral equation

The Chandrasekhar quadratic integral equation (CQIE) occurs in the theory of radiative transfer in a plane-parallel atmosphere [2]. The radiative transfer process and the integral equation for the scattering function and transmitted functions were developed by Chandrasekhar's work in the 1950s; see [14]. This work quickly turned into a major scientific topic in both mathematics and astrophysics, see [15, 16]. The radiative transfer process and the simultaneous integral equation for the transmitted and scattering functions were developed in Chandrasekhar's seminal work from 1960. In [17], the simultaneous integral equation of Chandrasekhar was presented, along with the iteration scheme for the transmitted and scattering functions.

CQIE takes the form

$$\begin{cases} \Phi(\varrho) = \varphi(\varrho) + \varrho\Phi(\varrho)\int_0^{\varrho} \dfrac{\varrho}{\varrho + s}\hat{g}_1\left(s, \Phi\left(s\right)\right)\mathrm{d}s + f_1\left(\varrho, \Phi(\varrho)\right)\ {}^{RL}J^{\mu}\hat{g}_2\left(\varrho, \Phi(\varrho)\right), \\ \qquad\qquad \varrho \in \hat{I},\ \hat{I} = [0, b]\,,\ \mu \in (0, 1)\,, \end{cases} \tag{1.1}$$

where ${}^{RL}J^{\mu}$ denotes the Riemann-Liouville fractional integral (RLFI) of order μ, $f_1\left(\varrho, \Phi(\varrho)\right) \in \mathbb{C}\left(\hat{I} \times \mathbb{R}, \mathbb{R} - \{0\}\right)$, and $\hat{g}_{\kappa}\left(\varrho, \Phi(\varrho)\right) \in \mathbb{C}\left(\hat{I} \times \mathbb{R}, \mathbb{R}\right)$, for $\kappa = 1, 2$.

As a result of the applications of these equations, researchers were interested in studying them, and as a result of the difficulty of finding exact solutions, CQIEs are solved using the ADM and the PM. The ADM provides many advantages, including the ability to solve a variety of linear and nonlinear equations in deterministic or stochastic fields effectively and present an analytical solution for all of these equation types without requiring linearization or discretization. Additionally, it is reliable and provides faster convergence than other classical methods. Moreover, we use measures of weak noncompactness to study existence and uniqueness results. These results demonstrate that the two solutions provide nearly equal accuracy; however, when comparing the time required in each case, the ADM is found to take less time than the PM.

It can be summarized as follows: first, the second section introduces the basic concepts of the measure of noncompactness and the hypothesis. In the third section, we show that if the solution exists, then it will be unique and convergent. After that, we solve some nonlinear Chandraseker QIEs with fractional orders with a comparison between ADM and PM techniques. Finally, graphs are also constructed to illustrate the effectiveness of these two approaches and to compare them.

2. Measure of noncompactness (MNC)

In the complement of this work, the classical Banach space $\mathbb{C}\left(\hat{I}\right) = \mathbb{C}\left[0, b\right]$ is used, which contains all real continuous functions defined on $\hat{I}$ having the norm

$$\|\Phi\| = \max\left\{|\Phi(\varrho)| : \varrho \in \hat{I}\right\}. \tag{2.1}$$

Let us recall the MNC definition in $\mathbb{C}(\hat{I})$ which is used in this investigation and fix a bounded nonempty subset Ω of $\mathbb{C}(\hat{I})$. For $\eta \in \Omega$ and $\epsilon > 0$, the modulus of continuity of the function η on the interval $\hat{I}$ is defined by

$$\omega(\eta, \epsilon) = \sup\left\{|\eta(\varrho) - \eta\left(s\right)| : \varrho,\ s \in \hat{I}, |\varrho - s| \leq \epsilon\right\} \tag{2.2}$$

and

$$\omega(\Omega, \epsilon) = \sup\left\{\omega\left(\eta, \epsilon\right) : \eta \in \Omega\right\}, \omega_0\left(\Omega\right) = \lim_{\epsilon \to 0} \omega\left(\Omega, \epsilon\right). \tag{2.3}$$

It is well-known that $\omega_0\left(\Omega\right)$ is a measure of noncompactness in $\mathbb{C}(\hat{I})$ such that the Hausdorff measure χ may be expressed by the formula

$$\chi\left(\Omega\right) = \frac{1}{2}\omega_0\left(\Omega\right), \tag{2.4}$$

see [18]. We use the following theorem to prove our investigation.

Theorem 1. *[18] Let $\mathbb{Q}$ be a bounded, nonempty, and closed convex subset in the space E. Also, let $\hat{H} : \mathbb{Q} \to \mathbb{Q}$ be a continuous operator such that $\chi(\hat{H}\Omega) \leq l\,\chi(\Omega)$ to any nonempty subset Ω of $\mathbb{Q}$, as $l \in [0, 1)$ is constant, so $\hat{H}$ has a fixed point in the set $\mathbb{Q}$.*

Considering the hypothesis:

i. $\varphi : \hat{I} \to \mathbb{R}$ is continuous.
ii. $f_1 : \hat{I} \times \mathbb{R} \to \mathbb{R}$ is continuous and the function $f_1\left(\varrho, 0\right) \in \mathbb{C}(\mathbb{R}), \mathbb{R}$ is the space of all bounded continuous functions, and there exists a positive constant $M = \sup_{\varrho \in \hat{I}} |f_1\left(\varrho, 0\right)|$.
iii. $|f_1\left(\varrho, \Phi\right) - f_1\left(\varrho, \zeta\right)| \leq L\,|\Phi - \zeta|$ for any $\varrho \in \hat{I}$ for all $\Phi, \zeta \in \mathbb{R}$.
iv. $\hat{g}_\kappa : \hat{I} \times \mathbb{R} \to \mathbb{R}, \kappa = 1, 2$ satisfies the Carathéodory condition (CC) so, it is measurable in ϱ for all $\Phi \in \mathbb{R}$ and continuous in Φ for all $\varrho \in \hat{I}$, and there exist functions m_κ, $\kappa = 1, 2 \in L_1$ such that:

$$|\hat{g}_\kappa\left(\varrho, \Phi\right)| \leq m_\kappa(\varrho) \text{ for all } (\varrho, \Phi) \in \hat{I} \times \mathbb{R}. \tag{2.5}$$

v. ${}^{RL}J^\gamma m_2\left(\varrho\right) < M_2, \gamma \leq \mu, \mathbb{C} \geq 0$, and $\int_0^b \frac{1}{\varrho + s} m_1\left(s\right) \mathrm{d}s \leq M_1$.
vi. There exists a number $\mathbb{R}_0 > 0$, such that

$$\mathbb{R}_0 = \left[\|\varphi\| + \frac{M_2 L b^{\mu-\gamma}}{\Gamma(\mu - \gamma + 1)} + \frac{M_2 M b^{\mu-\gamma}}{\Gamma(\mu - \gamma + 1)}\right]\left[1 - b^2 M_1\right]^{-1}. \tag{2.6}$$

3. Existence and uniqueness theorems

Define the ball $B_{\mathbb{R}_0}$ as

$$B_{\mathbb{R}_0} = \left\{\Phi \in \mathbb{C}\left(\hat{I}\right) : \|\Phi\| \leq \mathbb{R}_0\right\}. \tag{3.1}$$

Theorem 2. *Using the hypotheses* $(i)-(vi)$, *if* $\left[\frac{M_2 L b^{\mu-\gamma}}{\Gamma(\mu-\gamma+1)} + M_1 b^2\right] < 1$, *then there exists at least a solution* $\Phi \in \mathbb{C}\left(\hat{I}\right)$ *for HIE (1.1).*

Proof. Let the operator $\hat{H}$ defined on $\mathbb{C}\left(\hat{I}\right)$ be

$$\left(\hat{H}\Phi\right)(\varrho) = \varphi(\varrho) + \varrho\Phi(\varrho)\int_0^{\varrho} \frac{\varrho}{\varrho + s}\hat{g}_1\left(s,\ \Phi\left(s\right)\right)\mathrm{d}s$$
$$+ f_1\left(\varrho, \Phi(\varrho)\right)\int_0^{\varrho} \frac{(\varrho - s)^{\mu-1}}{\Gamma(\mu)}\hat{g}_2\left(s, \Phi\left(s\right)\right)ds, \varrho \in \hat{I}.$$

From the hypotheses $(i)-(vi)$, the function $\hat{H}\Phi$ is continuous on $\hat{I}$ for any $\Phi \in B_{\mathbb{R}_0}$. Further, applying the given hypothesis, we derive the following estimate:

$$\begin{aligned}
\left|\hat{H}\Phi(\varrho)\right| &\leq |\varphi(\varrho)| + |\varrho\Phi(\varrho)|\int_0^{\varrho} \frac{\varrho}{\varrho + s}m_1\left(s\right)\mathrm{d}s + \left|f_1\left(\varrho, \Phi\right)\right|\int_0^{\varrho} \frac{(\varrho - s)^{\mu-1}}{\Gamma\left(\mu\right)}m_2\left(s\right)\mathrm{d}s \\
&\leq \|\varphi\| + |\varrho\Phi(\varrho)|\int_0^{\varrho} \frac{\varrho}{\varrho + s}m_1\left(s\right)\mathrm{d}s \\
&\quad + \left[\ \left|f_1\left(\varrho, \Phi\right) - f_1\left(\varrho, 0\right)\right| + \left|f_1\left(\varrho, 0\right)\right|\right]\int_0^{\varrho} \frac{(\varrho - s)^{\mu-1}}{\Gamma\left(\mu\right)}m_2\left(s\right)\mathrm{d}s \\
&\leq \|\varphi\| + \left|b^2\Phi(\varrho)\right|\int_0^{b} \frac{1}{\varrho + s}m_1\left(s\right)\mathrm{d}s \\
&\quad + \left[\ \left|f_1\left(\varrho, \Phi\right) - f_1\left(\varrho, 0\right)\right| + \left|f_1\left(\varrho, 0\right)\right|\right]\ {}^{RL}J^{\mu-\gamma}\ {}^{RL}J^{\gamma}m_2\left(\varrho\right) \\
&\leq \|\varphi\| + b^2\mathbb{R}_0 M_1 + \frac{M_2 L\mathbb{R}_0 b^{\mu-\gamma}}{\Gamma\left(\mu - \gamma + 1\right)} + \frac{M_2 M b^{\mu-\gamma}}{\Gamma\left(\mu - \gamma + 1\right)}.
\end{aligned} \tag{3.2}$$

So $\hat{H}\Phi$ is bounded on the interval $\hat{I}$. Also, we get

$$\left\|\hat{H}\Phi\right\| \leq \|\varphi\| + b^2\mathbb{R}_0 M_1 + \frac{M_2 L\mathbb{R}_0 b^{\mu-\gamma}}{\Gamma\left(\mu - \gamma + 1\right)} + \frac{M_2 M b^{\mu-\gamma}}{\Gamma\left(\mu - \gamma + 1\right)},$$

for the operator $\hat{H}$, which transforms the ball $B_{\mathbb{R}_0}$ into itself, and $\mathbb{R}_0$ is:

$$\mathbb{R}_0 = \left[\|\varphi\| + \frac{M_2 M b^{\mu-\gamma}}{\Gamma\left(\mu - \gamma + 1\right)}\right]\left[1 - b^2 M_1 + \frac{M_2 L b^{\mu-\gamma}}{\Gamma\left(\mu - \gamma + 1\right)}\right]^{-1} > 0.$$

Now, we are going to show that the operator $\hat{H}$ is continuous on the ball $B_{\mathbb{R}_0}$. So, we need to prove that the operator G_2 defined by

$$G_2\Phi(\varrho) = \int_0^{\varrho} \frac{(\varrho - s)^{\mu-1}}{\Gamma\left(\mu\right)}g_2\left(s, \Phi\left(s\right)\right)\mathrm{d}s, \varrho \in \hat{I},$$

is continuous on $B_{\mathbb{R}_0}$. To do this, fix $\epsilon > 0$, let $\Phi_0 \in B_{\mathbb{R}_0}$, and from hypothesis (ii), we find $\delta > 0$ such that $\|\Phi - \Phi_0\| \leq \delta$, and then we have $|\hat{g}_2(s, \Phi) - \hat{g}_2(s, \Phi_0)| \leq \epsilon$ for $s \in \hat{I}$, where Φ is any arbitrary element in $B_{\mathbb{R}_0}$. For arbitrary fixed $\varrho \in \hat{I}$, we get

$$\begin{aligned} |G_2\Phi(\varrho) - G_2\Phi_0(\varrho)| &= \frac{1}{\Gamma(\mu)} \int_0^{\varrho} (\varrho - s)^{\mu-1} |\hat{g}_2(s, \Phi(s)) - \hat{g}_2(s, \Phi_0(s))| \, \mathrm{d}s \\ &\leq \frac{\epsilon}{\Gamma(\mu)} \int_0^{\varrho} (\varrho - s)^{\mu-1} \, \mathrm{d}s \\ &\leq \frac{\epsilon}{\Gamma(\mu+1)}. \end{aligned}$$

The operator $f_1\Phi(\varrho) = f_1(\varrho, \Phi(\varrho))$ is continuous, and then the operator $f_1.G_2$ is continuous on $B_{\mathbb{R}_0}$, and similarly, we can prove that the operator

$$G_1\Phi(\varrho) = \varrho\Phi(\varrho) \int_0^{\varrho} \frac{\varrho}{\varrho + s} \hat{g}_1(s, \Phi(s)) \, \mathrm{d}s, \varrho \in \hat{I},$$

is continuous on $B_{\mathbb{R}_0}$. This shows that $\hat{H}$ is continuous on $B_{\mathbb{R}_0}$. Let χ be a nonempty subset of $B_{\mathbb{R}_0}$. Fix $\epsilon > 0$, and choose $\Phi \in \chi$ and $\varrho_1, \varrho_2 \in \hat{I}$ such that $|\varrho_2 - \varrho_1| \leq \epsilon$. Let $\varrho_1 \leq \varrho_2$, and then

$$\begin{aligned} |(\hat{H}\Phi)(\varrho_2) - (\hat{H}\Phi)(\varrho_1)| = | \, & \varphi(\varrho_2) - \varphi(\varrho_1) \\ & + \varrho_2\Phi(\varrho_2) \int_0^{\varrho_2} \frac{\varrho_2}{\varrho_2 + s} \hat{g}_1(s, \Phi(s)) \, ds \; - \; \varrho_1\Phi(\varrho_1) \int_0^{\varrho_1} \frac{\varrho_1}{\varrho_1 + s} \hat{g}_1(s, \Phi(s)) \, ds \\ & + \varrho_2\Phi(\varrho_2) \int_0^{\varrho_1} \frac{\varrho_1}{\varrho_1 + s} \hat{g}_1(s, \Phi(s)) \, ds \; - \; \varrho_2\Phi(\varrho_2) \int_0^{\varrho_1} \frac{\varrho_1}{\varrho_1 + s} \hat{g}_1(s, \Phi(s)) \, ds \\ & + f_1(\varrho_2, \Phi(\varrho_2)) \, {}^{RL}J^{\mu} \, \hat{g}_2(\varrho_2, \Phi(\varrho_2)) \; - \; f_1(\varrho_1, \Phi(\varrho_1)) \, {}^{RL}J^{\mu} \, \hat{g}_2(\varrho_1, \Phi(\varrho_1)) \\ & + f_1(\varrho_1, \Phi(\varrho_1)) \, {}^{RL}J^{\mu} \, \hat{g}_2(\varrho_2, \Phi(\varrho_2)) \; - \; f_1(\varrho_1, \Phi(\varrho_1)) \, {}^{RL}J^{\mu} \, \hat{g}_2(\varrho_2, \Phi(\varrho_2))|. \end{aligned}$$

Hence,

$$\begin{aligned} \left|\left(\hat{H}\Phi\right)(\varrho_2) - \left(\hat{H}\Phi\right)(\varrho_1)\right| \leq \; & |\varphi(\varrho_2) \; - \; \varphi(\varrho_1)| \\ & + |\varrho_2\Phi(\varrho_2)| \int_0^{\varrho_2} \left|\frac{\varrho_2}{\varrho_2 + s} - \frac{\varrho_1}{\varrho_1 + s}\right| m_1(s) ds \\ & + |\varrho_2\Phi(\varrho_2) - \varrho_1\Phi(\varrho_1)| \int_0^{\varrho_1} \frac{\varrho_1}{\varrho_1 + s} \, m_1(s) ds \\ & + |\varrho_2\Phi(\varrho_2)| \int_{\varrho_1}^{\varrho_2} \left|\frac{\varrho_2}{\varrho_2 + s}\right| m_1(s) ds \\ & + [f_1(\varrho_2, \Phi(\varrho_2)) \; - \; f_1(\varrho_2, \Phi(\varrho_1))] \, {}^{RL}J^{\mu} \, \hat{g}_2(\varrho_2, \Phi(\varrho_2)) \\ & + [f_1(\varrho_2, \Phi(\varrho_1)) \; - \; f_1(\varrho_1, \Phi(\varrho_1))] \, {}^{RL}J^{\mu} \, \hat{g}_2(\varrho_2, \Phi(\varrho_2)) \\ & + f_1(\varrho_1, \Phi(\varrho_1)) \, [\, {}^{RL}J^{\mu} \, \hat{g}_2(\varrho_2, \Phi(\varrho_2)) \; - \; {}^{RL}J^{\mu} \, \hat{g}_2(\varrho_1, \Phi(\varrho_1))], \end{aligned}$$

and so

$$
\begin{aligned}
& \left| {}^{RL}J^{\mu}\hat{g}_2\left(\varrho_2, \Phi\left(\varrho_2\right)\right) - {}^{RL}J^{\mu}\hat{g}_2\left(\varrho_1, \Phi\left(\varrho_1\right)\right)\right| \\
= & \left| \int_0^{\varrho_1} \frac{\left(\varrho_2 - s\right)^{\mu-1}}{\Gamma\left(\mu\right)} \hat{g}_2\left(s, \Phi\left(s\right)\right) \mathrm{d}s \right. \\
& \left. + \int_{\varrho_1}^{\varrho_2} \frac{\left(\varrho_2 - s\right)^{\mu-1}}{\Gamma\left(\mu\right)} \hat{g}_2\left(s, \Phi\left(s\right)\right) \mathrm{d}s - \int_0^{\varrho_1} \frac{\left(\varrho_1 - s\right)^{\mu-1}}{\Gamma\left(\mu\right)} \hat{g}_2\left(s, \Phi\left(s\right)\right) \mathrm{d}s \right| \\
\leq & \left| \int_0^{\varrho_1} \frac{\left(\varrho_2 - s\right)^{\mu-1}}{\Gamma\left(\mu\right)} \hat{g}_2\left(s, \Phi\left(s\right)\right) \mathrm{d}s + \int_{\varrho_1}^{\varrho_2} \frac{\left(\varrho_2 - s\right)^{\mu-1}}{\Gamma\left(\mu\right)} \hat{g}_2\left(s, \Phi\left(s\right)\right) \mathrm{d}s \right. \\
& \left. - \int_0^{\varrho_1} \frac{\left(\varrho_2 - s\right)^{\mu-1}}{\Gamma\left(\mu\right)} \hat{g}_2\left(s, \Phi\left(s\right)\right) \mathrm{d}s \right| \\
\leq & \left| \int_{\varrho_1}^{\varrho_2} \frac{\left(\varrho_2 - s\right)^{\mu-1}}{\Gamma\left(\mu\right)} \hat{g}_2\left(s, \Phi\left(s\right)\right) \mathrm{d}s \right|.
\end{aligned}
$$

Then

$$
\begin{aligned}
\left| {}^{RL}J^{\mu}\, \hat{g}_2\left(\varrho_2, \Phi\left(\varrho_2\right)\right) - {}^{RL}J^{\mu}\, \hat{g}_2\left(\varrho_1, \Phi\left(\varrho_1\right)\right)\right| & \leq {}^{RL}J^{\mu}_{\varrho_1} \left|\hat{g}_2\left(\varrho_2, \Phi\left(\varrho_2\right)\right)\right| \\
& \leq {}^{RL}J^{\mu}_{\varrho_1}\, m_2\left(\varrho_2\right) \\
& \leq {}^{RL}J^{\mu-\gamma}_{\varrho_1}\; {}^{RL}J^{\gamma}_{\varrho_1}\, m_2\left(\varrho_2\right) \\
& \leq M_2 \frac{\left(\varrho_2 - \varrho_1\right)^{\mu-\gamma}}{\Gamma\left(\mu - \gamma + 1\right)}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\left|\left(\hat{H}\Phi\right)\left(\varrho_2\right) - \left(\hat{H}\Phi\right)\left(\varrho_1\right)\right| \leq & \left|\varphi\left(\varrho_2\right) - \varphi\left(\varrho_1\right)\right| + b^2\, \mathbb{R}_0 \int_0^{\varrho_1} \frac{\left|\varrho_2 - \varrho_1\right|}{\varrho_1 + s} (\varrho_2 + s) m_1\left(s\right) \mathrm{d}s \\
& + b^2 \mathbb{R}_0 \int_{\varrho_1}^{\varrho_2} \frac{1}{\varrho_2 + s} m_1\left(s\right) \mathrm{d}s \\
& + b\left[\left|\varrho_2 \Phi\left(\varrho_2\right) - \varrho_2 \Phi\left(\varrho_1\right)\right| + \left|\varrho_2 \Phi\left(\varrho_1\right) - \varrho_1 \Phi\left(\varrho_1\right)\right|\right] \int_0^{b} \frac{1}{\varrho_1 + s} m_1\left(s\right) \mathrm{d}s \\
& + \left|f_1\left(\varrho_2, \Phi\left(\varrho_2\right)\right) - f_1\left(\varrho_2, \Phi\left(\varrho_1\right)\right)\right| \, {}^{RL}J^{\mu-\gamma}\, {}^{RL}J^{\gamma}\, m_2\left(\varrho_2\right) \\
& + \left|f_1\left(\varrho_2, \Phi\left(\varrho_1\right)\right) - f_1\left(\varrho_1, \Phi\left(\varrho_1\right)\right)\right| \, {}^{RL}J^{\mu-\gamma}\, {}^{RL}J^{\gamma}\, m_2\left(\varrho_2\right) \\
& + \left|f_1\left(\varrho_1, \Phi\left(\varrho_1\right)\right) - f_1\left(\varrho_1, 0\right)\right| M_2 \frac{\left(\varrho_2 - \varrho_1\right)^{\mu-\gamma}}{\Gamma\left(\mu - \gamma + 1\right)} \\
& + \left|f_1(\varrho_1, 0)\right| M_2 \frac{\left(\varrho_2 - \varrho_1\right)^{\mu-\gamma}}{\Gamma\left(\mu - \gamma + 1\right)}.
\end{aligned}
$$

We get

$$\left|\left(\hat{H}\Phi\right)(\varrho_2)-\left(\hat{H}\Phi\right)(\varrho_1)\right| \leq \left|\varphi(\varrho_2)-\varphi(\varrho_1)\right| + b\mathbb{R}_0\left|\varrho_2-\varrho_1\right|\int_0^b \frac{1}{\varrho_1+s}m_1(s)\,\mathrm{d}s$$
$$+b^2\mathbb{R}_0\int_{\varrho_1}^{\varrho_2} m_1(s)\,\mathrm{d}s + M_1b^2\left|\Phi(\varrho_2)-\Phi(\varrho_1)\right| + b\mathbb{R}_0\left|\varrho_2-\varrho_1\right|$$
$$+L\left|\Phi(\varrho_2)-\Phi(\varrho_1)\right| \, {}^{RL}J^{\mu}m_2(\varrho_2) + \eta(f_1,\epsilon)\, {}^{RL}J^{\mu}m_2(\varrho_2)$$
$$+L|\Phi(\varrho_1)|M\frac{\left|\varrho_2-\varrho_1\right|^{\mu-\eta}}{\Gamma(\mu-\gamma+1)} + M_2M\frac{\left|\varrho_2-\varrho_1\right|^{\mu-\gamma}}{\Gamma(\mu-\gamma+1)},$$

so,

$$\left|\left(\hat{H}\Phi\right)(\varrho_2)-\left(\hat{H}\Phi\right)(\varrho_1)\right| \leq \left|\varphi(\varrho_2)-\varphi(\varrho_1)\right| + b\mathbb{R}_0M_1\left|\varrho_2-\varrho_1\right| + b^2\mathbb{R}_0\int_{\varrho_1}^{\varrho_2} m_1(s)\,\mathrm{d}s$$
$$+M_1b^2\left|\Phi(\varrho_2)-\Phi(\varrho_1)\right| + b\mathbb{R}_0M_1\left|\varrho_2-\varrho_1\right|$$
$$+\left|\Phi(\varrho_2)-\Phi(\varrho_1)\right|\frac{M_2Lb^{\mu-\gamma}}{\Gamma(\mu-\gamma+1)}$$
$$+\frac{M_2\Psi(f_1,\epsilon)\,b^{\mu-\gamma}}{\Gamma(\mu-\gamma+1)} + L\left|\Phi(\varrho_1)\right|M\frac{\left|\varrho_2-\varrho_1\right|^{\mu-\gamma}}{\Gamma(\mu-\gamma+1)}$$
$$+M_2M\frac{\left|\varrho_2-\varrho_1\right|^{\mu-\eta}}{\Gamma(\mu-\gamma+1)},$$

where $\Psi(f_1,\epsilon) = \sup\left\{|f_1(\varrho_2,\Phi(\varrho_1) - f_1(\varrho_1,\Phi(\varrho_1))| : \varrho_1,\varrho_2 \in \hat{I}, \left|\varrho_2-\varrho_1\right| \leq \epsilon, \Phi \in B_{\mathbb{R}_0}\right\}$.

Knowing that f_1 is uniformly continuous on the set $\hat{I} \times B_{\mathbb{R}_0}$, we derive the inequality

$$\omega_0\left(\hat{H}\Phi\right) \leq \left[\frac{M_2Lb^{\mu-\gamma}}{\Gamma(\mu-\gamma+1)} + M_1b^2\right]\omega_0(\Phi).$$

Then

$$\omega_0\left(\hat{H}\Phi\right) \leq \left[\frac{M_2Lb^{\mu-\gamma}}{\Gamma(\mu-\gamma+1)} + M_1b^2\right]\omega_0(\Phi).$$

From (2.4), this inequality leads to

$$\chi\left(\hat{H}\Phi\right) \leq \left[\frac{M_2Lb^{\mu-\gamma}}{\Gamma(\mu-\gamma+1)} + M_1b^2\right]\chi(\Phi).$$

If $\left[\frac{M_2Lb^{\mu-\gamma}}{\Gamma(\mu-\gamma+1)} + M_1b^2\right] < 1$ and using Theorem 1, there exists at least a solution $\Phi \in \mathbb{C}\left(\hat{I}\right)$ for HIE (1.1).

The following hypotheses are satisfied if there exist $\hat{g}_\kappa$, $\kappa = 1,2$, such that $(iv)^*$ $\hat{g}_\kappa : \hat{I} \times \mathbb{R} \to \mathbb{R}$, $\kappa = 1,2$, satisfies CC, so it can be measurable in ϱ for all $\Phi \in \mathbb{R}$ and continuous in Φ for all $\varrho \in \hat{I}$. Then

$$|\hat{g}_\kappa(\varrho,\Phi) - \hat{g}_\kappa(\varrho,\zeta)| \leq L_\kappa|\Phi-\zeta|, \quad \kappa = 1,2, \tag{3.3}$$

for all $\varrho \in \hat{I}$ and $\Phi,\zeta \in \mathbb{R}$. Let Φ_1 *and* Φ_2 be two solutions for the HIE (1.1), and hence

$$
\begin{aligned}
\left|\Phi_1(\varrho) - \Phi_2(\varrho)\right| &= \Big|\varrho\, \Phi_1(\varrho) \int_0^{\varrho} \frac{\varrho}{\varrho + s}\, \hat{g}_1(s, \Phi_1(s))\mathrm{d}s - \varrho\Phi_2(\varrho) \int_0^{\varrho} \frac{\varrho}{\varrho + s}\, \hat{g}_1(s, \Phi_2(s))\mathrm{d}s \\
&+ \varrho\Phi_2(\varrho) \int_0^{\varrho} \frac{\varrho}{\varrho + s}\, \hat{g}_1(s, \Phi_1(s))\mathrm{d}s - \varrho\Phi_2(\varrho) \int_0^{\varrho} \frac{\varrho}{\varrho + s}\, \hat{g}_1(s, \Phi_1(s))\mathrm{d}s \\
&+ f_1(\varrho, \Phi_1(\varrho)) \int_0^{\varrho} \frac{(\varrho - s)^{\mu-1}}{\Gamma(\mu)} \hat{g}_2(s, \Phi_1(s))\mathrm{d}s - f_1(\varrho, \Phi_2(\varrho)) \int_0^{\varrho} \frac{(\varrho - s)^{\mu-1}}{\Gamma(\mu)} \hat{g}_2(s, \Phi_2(s))\mathrm{d}s \\
&+ f_1(\varrho, \Phi_1(\varrho)) \int_0^{\varrho} \frac{(\varrho - s)^{\mu-1}}{\Gamma(\mu)} \hat{g}_2(s, \Phi_2(s))\mathrm{d}s - f_1(\varrho, \Phi_1(\varrho)) \int_0^{\varrho} \frac{(\varrho - s)^{\mu-1}}{\Gamma(\mu)} \hat{g}_2(s, \Phi_2(s))\mathrm{d}s\Big| \qquad (3.4) \\
&\leq \varrho\left|\Phi_1(\varrho) - \Phi_2(\varrho)\right| \int_0^{\varrho} \frac{\varrho}{\varrho + s} m_1(s)\,\mathrm{d}s + \varrho\left|\Phi_2(\varrho)\right| \int_0^{\varrho} \frac{\varrho}{\varrho + s} L_1 \left|\Phi_1(s) - \Phi_2\right| \mathrm{d}s \\
&\quad + \left|f_1(\varrho, \Phi_1(\varrho)) - f_1(\varrho, \Phi_2(\varrho))\right| \int_0^{\varrho} \frac{(\varrho - s)^{\mu-1}}{\Gamma(\mu)} \hat{g}_2(s, \Phi_2(s))\,\mathrm{d}s \\
&\quad + \left|f_1(\varrho, \Phi_1(\varrho))\right| \int_0^{\varrho} \frac{(\varrho - s)^{\mu-1}}{\Gamma(\mu)} \hat{g}_2(s, \Phi_2(s)) - \hat{g}_2(s, \Phi_1(s)))\,\mathrm{d}s \\
&\leq \left(b^2 M_1 + L_1 b^2 \mathbb{R}_0 + L M_2 \frac{b^{\mu-\gamma}}{\Gamma(\mu - \gamma + 1)} + \frac{L_2 b^{\mu}(L\mathbb{R}_0 + M)}{\Gamma(\mu + 1)}\right) \left|\Phi_1(\varrho) - \Phi_2(\varrho)\right| \\
&\leq \Upsilon \left|\Phi_1(\varrho) - \Phi_2(\varrho)\right|, \qquad (3.5)
\end{aligned}
$$

where $\Upsilon = \left(b^2 M_1 + L_1 b^2 \mathbb{R}_0 + L M_2 \frac{b^{\mu-\gamma}}{\Gamma(\mu-\gamma+1)} + \frac{L_2 b^{\mu}(L\mathbb{R}_0+M)}{\Gamma(\mu+1)}\right)$. Then we get the theorem:

Theorem 3. *Assume that the hypotheses (i)–(vi) are satisfied, and* $\Upsilon < 1$. *Then, the solution* $\Phi \in \mathbb{C}(\hat{I})$ *of (1.1) is unique.*

4. Methods for the solutions

4.1. Adomian decomposition method (ADM)

In the 1980s, Adomian presented the ADM [19–21], which is an analytical method used to solve a lot of different equations such as DEs, IEs, integro-differential equations, and partial DEs [22–25]. The obtained solution is an infinite series that converges to the exact solution. An important benefit of the ADM is that there is no linearization or perturbation that can change the main problem that has been solved, which is serious. A lot of researchers are interested in using the ADM, as it is successfully applied to many applications that appear in applied sciences [26–28]. In this research, the ADM is used as the first method to solve the HIE (1.1).

Applying the ADM to (1.1), the ADM solution algorithm is

$$
\Phi_0(\varrho) = \varphi(\varrho), \qquad (4.1)
$$

$$
\Phi_{\kappa}(\varrho) = \varrho \hat{A}_{\kappa-1}(\varrho) + \check{D}_{\kappa-1}(\varrho), \qquad (4.2)
$$

where $\hat{A}_{\kappa}$, and $\check{D}_{\kappa}$ are Adomian polynomials of the nonlinear terms $\hat{g}_1(\varrho, \Phi)$, $f_1(\varrho, \Phi)$, and $\hat{g}_2(\varrho, \Phi)$ which take the forms

$$
\hat{A}_{\kappa} = \frac{1}{\kappa!}\left[\frac{d^{\kappa}}{d\lambda^{\kappa}}\left(\sum_{\kappa=0}^{\infty} \lambda^{\kappa}\Phi_{\kappa} \int_0^{\varrho} \frac{\varrho}{\varrho + s} \hat{g}_1\left(s, \sum_{\kappa=0}^{\infty} \lambda^{\kappa}\Phi_{\kappa}(s)\right)\mathrm{d}s\right)\right]_{\lambda=0}, \qquad (4.3)
$$

$$\check{D}_\kappa = \frac{1}{\kappa!}\left[\frac{d^{\,\kappa}}{d\lambda^\kappa}\left(f_1\left(\varrho,\sum_{\kappa=0}^{\infty}\lambda^\kappa\Phi_\kappa\right)\,{}^{RL}J^{\mu}\hat{g}_2\left(\varrho,\sum_{\kappa=0}^{\infty}\lambda^\kappa\Phi_\kappa(\varrho)\right)\right)\right]_{\lambda=0}. \tag{4.4}$$

Finally, the ADM solution is

$$\Phi(\varrho)=\sum_{\kappa=0}^{\infty}\Phi_\kappa(\varrho). \tag{4.5}$$

4.1.1. Convergence analysis

Theorem 4. *If hypotheses* $(i)-(vi)$ *are satisfied,* $\Upsilon_2<1$*, and* $|\Phi_1(\varrho)|<\mathbb{k}$*, where* $\mathbb{k}$ *is a positive constant, then the series solution (4.5) of (1.1) using the ADM is convergent.*

Proof. Define the sequence $\left\{\hat{S}_\rho\right\}$ such that $\hat{S}_\rho=\sum_{\kappa=0}^{\rho}\Phi_\kappa(\varrho)$ is a sequence of partial sums taken from the series (4.5), and

$$\begin{aligned}\Phi(\varrho)\int_0^{\varrho}\frac{\varrho}{\varrho+s}\hat{g}_1(s,\Phi(s))\,\mathrm{d}s &= \sum_{\kappa=0}^{\infty}\hat{A}_\kappa,\\ f_1(\varrho,\Phi(\varrho))\,{}^{RL}J^{\mu}\hat{g}_2(\varrho,\Phi(\varrho)) &= \sum_{\kappa=0}^{\infty}\check{D}_\kappa.\end{aligned}$$

Let $\hat{S}_\rho$ and $\hat{S}_\theta$ be two partial sums of the ADM series solution such that $\rho>\theta$. We want to prove that $\left\{\hat{S}_\rho\right\}$ is a Cauchy sequence (CS) in this Banach space (Bs).

$$\begin{aligned}\hat{S}_\rho-\hat{S}_\theta &= \sum_{\kappa=0}^{\rho}\Phi_\kappa-\sum_{\kappa=0}^{\theta}\Phi_\kappa\\ &= \varrho\sum_{\kappa=0}^{\rho}\hat{A}_{\kappa-1}(\varrho)+\sum_{\kappa=0}^{\rho}\check{D}_{\kappa-1}(\varrho)\\ &\quad -\varrho\sum_{\kappa=0}^{\theta}\hat{A}_{\kappa-1}(\varrho)-\sum_{\kappa=0}^{\theta}\check{D}_{\kappa-1}(\varrho),\end{aligned}$$

hence,

$$\begin{aligned}\hat{S}_\rho-\hat{S}_\theta &= \left[\varrho\left(\sum_{\kappa=0}^{\rho}\hat{A}_{\kappa-1}(\varrho)-\sum_{\kappa=0}^{\theta}\hat{A}_{\kappa-1}(\varrho)\right)\right]\\ &\quad+\left[\sum_{\kappa=0}^{\rho}\check{D}_{\kappa-1}(\varrho)-\sum_{\kappa=0}^{\theta}\check{D}_{\kappa-1}(\varrho)\right]\\ &= \left[\varrho\left(\sum_{\kappa=\theta+1}^{\rho}\hat{A}_{\kappa-1}(\varrho)\right)\right]+\left[\sum_{\kappa=\theta+1}^{\rho}\check{D}_{\kappa-1}(\varrho)\right].\end{aligned}$$

Thus, by applying $|\cdot|$ to both sides, we find

$$\left|\hat{S}_\rho-\hat{S}_\theta\right|=\left|\left[\varrho\left(\sum_{\kappa=\theta+1}^{\rho}\hat{A}_{\kappa-1}(\varrho)\right)\right]+\left[\sum_{\kappa=\theta+1}^{\rho}\check{D}_{\kappa-1}(\varrho)\right]\right|$$

$$
\begin{aligned}
&\leq \left|\varrho \sum_{\kappa=\theta}^{\rho-1} \hat{A}_{\kappa}\,(\varrho)\right| + \left|\sum_{\kappa=\theta}^{\rho-1} \check{D}_{\kappa}\,(\varrho)\right| \\
&\leq \left|\varrho\left(\hat{S}_{\rho-1} - \hat{S}_{\theta-1}\right)\right| \int_0^{\varrho} \left|\frac{\varrho}{\varrho+s}\left[\hat{g}_1\left(s,\left(\sum_{\kappa=\theta}^{\rho-1}\Phi_{\kappa}\right)\right)\right]\right| \mathrm{d}s \\
&\quad + \left|f_1\left(\varrho,\left(\sum_{\kappa=\theta}^{\rho-1}\Phi_{\kappa}\right)\right)\, {}^{RL}J^{\mu}\hat{g}_2\left(\varrho,\left(\sum_{\kappa=\theta}^{\rho-1}\Phi_{\kappa}\right)\right)\right| \\
&\leq b^2\left|\left(\hat{S}_{\rho-1} - \hat{S}_{\theta-1}\right)\right| \int_0^{b} \frac{1}{\varrho+s}\left|\hat{g}_1\,(s,\Phi)\right| \mathrm{d}s \\
&\quad + \left|f_1\left(\varrho,\hat{S}_{\rho-1}\right) - f_1\left(\varrho,\hat{S}_{\theta-1}\right)\right|\, {}^{RL}J^{\mu}\left|\hat{g}_2\,(\varrho,\Phi)\right|,
\end{aligned}
$$

$$
\begin{aligned}
\left|\hat{S}_{\rho} - \hat{S}_{\theta}\right| &\leq b^2\left|\left(\hat{S}_{\rho-1} - \hat{S}_{\theta-1}\right)\right| \int_0^{b} \frac{1}{\varrho+s}\left|\hat{g}_1\,(s,\Phi)\right| \mathrm{d}s + L\left|\hat{S}_{\rho-1} - \hat{S}_{\theta-1}\right|\, {}^{RL}J^{\mu-\gamma}\, {}^{RL}J^{\gamma} m_2(\varrho) \\
&\leq b^2\left|\left(\hat{S}_{\rho-1} - \hat{S}_{\theta-1}\right)\right| M_1 + LM_2\left|\hat{S}_{\rho-1} - \hat{S}_{\theta-1}\right|\, {}^{RL}J^{\mu-\gamma}\,(1)
\end{aligned}
$$

$$
\begin{aligned}
\left\|\hat{S}_{\rho} - \hat{S}_{\theta}\right\| &\leq \left[b^2 M_1 + LM_2\frac{(b)^{\mu-\gamma}}{\Gamma(\mu-\gamma+1)}\right]\left\|\hat{S}_{\rho-1} - \hat{S}_{\theta-1}\right\| \\
&\leq \Upsilon_2\left\|\hat{S}_{\rho-1} - \hat{S}_{\theta-1}\right\|,
\end{aligned}
$$

where $\Upsilon_2 = \left[b^2 M_1 + LM_2\frac{(b)^{\mu-\gamma}}{\Gamma(\mu-\gamma+1)}\right]$. Let $\rho = \theta + 1$, and we get

$$
\left\|\hat{S}_{\theta+1} - \hat{S}_{\theta}\right\| \leq \Upsilon_2\left\|\hat{S}_{\theta} - \hat{S}_{\theta-1}\right\| \leq \Upsilon_2^2\left\|\hat{S}_{\theta-1} - \hat{S}_{\theta-2}\right\| \leq \cdots \leq \Upsilon_2^{\theta}\left\|\hat{S}_{1} - \hat{S}_{0}\right\|.
$$

Using the triangle inequality, we arrive at

$$
\begin{aligned}
\left\|\hat{S}_{\rho} - \hat{S}_{\theta}\right\| &\leq \left\|\hat{S}_{\theta+1} - \hat{S}_{\theta}\right\| + \left\|\hat{S}_{\theta+2} - \hat{S}_{\theta+1}\right\| + \cdots + \left\|\hat{S}_{\rho} - \hat{S}_{\rho-1}\right\| \\
&\leq \left[\Upsilon_2^{\theta} + \Upsilon_2^{\theta+1} + \cdots + \Upsilon_2^{\rho-1}\right]\left\|\hat{S}_{1} - \hat{S}_{0}\right\| \\
&\leq \Upsilon_2^{\theta}\left[1 + \Upsilon_2 + \cdots + \Upsilon_2^{\rho-\theta-1}\right]\left\|\hat{S}_{1} - \hat{S}_{0}\right\| \\
&\leq \Upsilon_2\left[\frac{1-\Upsilon_2^{\rho-\theta}}{1-\Upsilon_2}\right]\left\|\Phi_1\right\|.
\end{aligned}
$$

If $0 < \Upsilon_2 < 1$ and $\rho > \theta$, this leads to $\left(1-\Upsilon_2^{\rho-\theta}\right) \leq 1$. Then

$$\begin{aligned}\left\|\hat{S}_\rho - \hat{S}_\theta\right\| &\leq \frac{\Upsilon_2^\theta}{1-\Upsilon_2}\|\Phi_1\| \\ &\leq \frac{\Upsilon_2^\theta}{1-\Upsilon_2}\max_{\varrho\in\hat{I}}|\Phi_1(\varrho)|.\end{aligned}$$

If $|\Phi_1(\varrho)| < \Bbbk$, $\theta \to \infty$, then $\left\|\hat{S}_\rho - \hat{S}_\theta\right\| \to 0$, which leads to $\left\{\hat{S}_\rho\right\}$ being a CS in this BS and the series (4.5) is convergent.

4.1.2. Error analysis

Theorem 5. *The maximum absolute error of the ADM series solution (4.5) is*

$$\max_{\varrho\in\hat{I}}\left|\Phi(\varrho) - \sum_{\kappa=0}^{\theta}\Phi_\kappa(\varrho)\right| \leq \frac{\Upsilon_2^\theta}{1-\Upsilon_2}\max_{\varrho\in\hat{I}}|\Phi_1(\varrho)|. \tag{4.6}$$

Proof. In Theorem 2, we see that

$$\left\|\hat{S}_\rho - \hat{S}_\theta\right\| \leq \frac{\Upsilon_2^\theta}{1-\Upsilon_2}\max_{\varrho\in\hat{I}}|\Phi_1(\varrho)|,$$

and $\hat{S}_\rho = \sum_{\kappa=0}^{\rho}\Phi_\kappa(\varrho)$, $\rho \to \infty$. Then, $\hat{S}_\rho \to \Phi(\varrho)$, and hence

$$\left\|\Phi(\varrho) - \hat{S}_\theta\right\| \leq \frac{\Upsilon_2^\theta}{1-\Upsilon_2}\max_{\varrho\in\hat{I}}|\Phi_1(\varrho)|,$$

and the maximum absolute error is written as

$$\max_{\varrho\in\hat{I}}\left|\Phi(\varrho) - \sum_{\kappa=0}^{\theta}\Phi_\kappa(\varrho)\right| \leq \frac{\Upsilon_2^\theta}{1-\Upsilon_2}\max_{\varrho\in\hat{I}}|\Phi_1(\varrho)|.$$

4.2. Picard method (PM)

The method of successive approximations (PM) was presented by Emile Picard in 1891. PM and ADM methods were first compared by Rach and Bellomo in 1987 [26,29]. In 1999, Golberg deduced that these two methods were equivalent for linear differential equations [30]. But this equivalence is not achieved in the nonlinear case. In 2010, El-Sayed et al. used them to solve QIE [31]. In 2012, El-Sayed et al. used them to solve a coupled system of fractional QIEs [32]. In 2014, El-Sayed et al. used them to solve FQIE [33]. In 2024, Ziada used them to solve a nonlinear FDE system containing the Atangana–Baleanu derivative [34]. In this research, we use them to get the solution for a nonlinear HDE and compare their results.

Applying the PM to the QIE (1.1), the solution is a sequence constructed by

$$\Phi_0(\varrho) = \varphi(\varrho),$$

$$\begin{aligned}\Phi_{\kappa}(\varrho) &= \Phi_0(\varrho)+\varrho\Phi_{\kappa-1}(\varrho)\int_0^{\varrho}\frac{\varrho}{\varrho+s}\hat{g}_1\left(s,\Phi_{\kappa-1}\left(s\right)\right)\mathrm{d}s\\ &\quad +f_1\left(\varrho,\Phi_{\kappa-1}(\varrho)\right)\ {}^{RL}J^{\mu}\hat{g}_2\left(\varrho,\Phi_{\kappa-1}(\varrho)\right).\end{aligned} \tag{4.7}$$

All the functions $\Phi_{\kappa}(\varrho)$ are continuous functions and Φ_{κ} are the sum of successive differences

$$\Phi_{\kappa}(\varrho)=\Phi_0(\varrho)+\sum_{\kappa=1}^{\infty}\left(\Phi_{\kappa}-\Phi_{\kappa-1}\right). \tag{4.8}$$

Therefore, the sequence Φ_{κ} convergence is the same as the infinite series $\sum\left(\Phi_{\kappa}-\Phi_{\kappa-1}\right)$ convergence. The final PM solution takes the form

$$\Phi(\varrho)=\lim_{\kappa\to\infty}\Phi_{\kappa}(\varrho). \tag{4.9}$$

From the above relations, we can deduce that if the series $\sum\left(\Phi_{\kappa}-\Phi_{\kappa-1}\right)$ is convergent, then the sequence $\Phi_{\kappa}(\varrho)$ is convergent to $\Phi(\varrho)$. To prove that the sequence $\{\Phi_{\kappa}(\varrho)\}$ is informally convergent, consider the associated series

$$\sum_{\kappa=1}^{\infty}\left[\Phi_{\kappa}(\varrho)-\Phi_{\kappa-1}(\varrho)\right]. \tag{4.10}$$

From (4.7) for $\kappa=1$, we get

$$\Phi_1(\varrho)-\Phi_0(\varrho)=\varrho\Phi_0(\varrho)\int_0^{\varrho}\frac{\varrho}{\varrho+s}\hat{g}_1\left(s,\Phi_0\left(s\right)\right)\mathrm{d}s+f_1\left(\varrho,\Phi_0(\varrho)\right)\ {}^{RL}J^{\mu}\hat{g}_2\left(\varrho,\Phi_0(\varrho)\right). \tag{4.11}$$

So, we have

$$\begin{aligned}\left|\Phi_1(\varrho)-\Phi_0(\varrho)\right| &= \left|\varrho\Phi_0(\varrho)\int_0^{\varrho}\frac{\varrho}{\varrho+s}\hat{g}_1\left(s,\Phi_0\left(s\right)\right)\mathrm{d}s+f_1\left(\varrho,\Phi_0(\varrho)\right)\ {}^{RL}J^{\mu}\ \hat{g}_2\left(\varrho,\Phi_0(\varrho)\right)\right|\\ &\le |\varrho|\ \left|\Phi_0(\varrho)\right|\int_0^{\varrho}\left|\frac{\varrho}{\varrho+s}\right|\left|\hat{g}_1\left(s,\Phi_0\left(s\right)\right)\right|\mathrm{d}s\\ &\quad +\left|f_1\left(\varrho,\Phi_0(\varrho)\right)\right|\ {}^{RL}J^{\mu}\hat{g}_2\left(\varrho,\Phi_0(\varrho)\right).\end{aligned} \tag{4.12}$$

Thus,

$$\begin{aligned}\left|\Phi_1(\varrho)-\Phi_0(\varrho)\right| &\le b^2\left|\Phi_0(\varrho)\right|\int_0^{b}\frac{1}{\varrho+s}m_1\left(s\right)\mathrm{d}s\\ &\quad +\left[\left|f_1\left(\varrho,\Phi_0(\varrho)\right)\right)-f_1\left(\varrho,0\right)+f_1\left(\varrho,0\right)\right|\right]\ {}^{RL}J^{\mu}\ {}^{RL}J^{\eta}m_2(\varrho)\\ &\le \left[b^2\mathbb{R}_0M_1+\frac{M_2b^{\mu-\eta}\left(L\mathbb{R}_0+M\right)}{\Gamma\left(\mu-\eta+1\right)}\right]:=\psi.\end{aligned} \tag{4.13}$$

Now, we get an estimate for $\Phi_{\kappa}(\varrho)-\Phi_{\kappa-1}(\varrho)$, $\kappa\geqslant 2$:

$$\begin{aligned}\Phi_\kappa(\varrho)-\Phi_{\kappa-1}(\varrho) &= \varrho\Phi_{\kappa-1}(\varrho)\int_0^\varrho \frac{\varrho}{\varrho+s}\hat{g}_1\left(s,\Phi_{\kappa-1}(s)\right)\mathrm{d}s + f_1\left(\varrho,\Phi_{\kappa-1}(\varrho)\right)\ {}^{RL}J^\mu\ \hat{g}_2\left(\varrho,\Phi_{\kappa-1}(\varrho)\right)\\ &\quad -\varrho\Phi_{\kappa-2}(\varrho)\int_0^\varrho \frac{\varrho}{\varrho+s}\hat{g}_1\left(s,\Phi_{\kappa-2}(s)\right)\mathrm{d}s + f_1\left(\varrho,\Phi_{\kappa-2}(\varrho)\right)\ {}^{RL}J^\mu\ \hat{g}_2\left(\varrho,\Phi_{\kappa-2}(\varrho)\right)\\ &= \varrho\Phi_{\kappa-1}(\varrho)\int_0^\varrho \frac{\varrho}{\varrho+s}\hat{g}_1\left(s,\Phi_{\kappa-1}(s)\right)\mathrm{d}s + f_1\left(\varrho,\Phi_{\kappa-1}(\varrho)\right)\ {}^{RL}J^\mu\ \hat{g}_2\left(\varrho,\Phi_{\kappa-1}(\varrho)\right)\\ &\quad -\varrho\Phi_{\kappa-2}(\varrho)\int_0^\varrho \frac{\varrho}{\varrho+s}\hat{g}_1\left(s,\Phi_{\kappa-2}(s)\right)\mathrm{d}s\\ &\quad -f_1\left(\varrho,\Phi_{\kappa-2}(\varrho)\right)\ {}^{RL}J^\mu\ \hat{g}_2\left(\varrho,\Phi_{\kappa-2}(\varrho)\right),\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\Phi_\kappa(\varrho)-\Phi_{\kappa-1}(\varrho) &= \varrho\Phi_{\kappa-1}(\varrho)\int_0^\varrho \frac{\varrho}{\varrho+s}\hat{g}_1\left(s,\Phi_{\kappa-1}(s)\right)\mathrm{d}s + \varrho\Phi_{\kappa-1}(\varrho)\int_0^\varrho \frac{\varrho}{\varrho+s}\hat{g}_1\left(s,\Phi_{\kappa-2}(s)\right)\mathrm{d}s\\ &\quad -\varrho\Phi_{\kappa-1}(\varrho)\int_0^\varrho \frac{\varrho}{\varrho+s}\hat{g}_1\left(s,\Phi_{\kappa-2}(s)\right)\mathrm{d}s - \varrho\Phi_{\kappa-2}(\varrho)\int_0^\varrho \frac{\varrho}{\varrho+s}\hat{g}_1\left(s,\Phi_{\kappa-2}(s)\right)\mathrm{d}s\\ &\quad +f_1\left(\varrho,\Phi_{\kappa-1}(\varrho)\right)\ {}^{RL}J^\mu\ \hat{g}_2\left(\varrho,\Phi_{\kappa-1}(\varrho)\right) + f_1\left(\varrho,\Phi_{\kappa-1}(\varrho)\right)\ {}^{RL}J^\mu\ \hat{g}_2\left(\varrho,\Phi_{\kappa-2}(\varrho)\right)\\ &\quad -f_1\left(\varrho,\Phi_{\kappa-1}(\varrho)\right)\ {}^{RL}J^\mu\ \hat{g}_2\left(\varrho,\Phi_{\kappa-2}(\varrho)\right)\\ &\quad -f_1\left(\varrho,\Phi_{\kappa-2}(\varrho)\right)\ {}^{RL}J^\mu\ \hat{g}_2\left(\varrho,\Phi_{\kappa-2}(\varrho)\right).\end{aligned} \tag{4.15}$$

Thus,

$$\begin{aligned}\Phi_\kappa(\varrho)-\Phi_{\kappa-1}(\varrho) &= \varrho\Phi_{\kappa-1}(\varrho)\int_0^\varrho \frac{\varrho}{\varrho+s}\left[\hat{g}_1\left(s,\Phi_{\kappa-1}(s)\right)-\hat{g}_1\left(s,\Phi_{\kappa-2}(s)\right)\right]\mathrm{d}s\\ &\quad +\varrho\left[\Phi_{\kappa-1}(\varrho)-\Phi_{\kappa-2}(\varrho)\right]\int_0^\varrho \frac{\varrho}{\varrho+s}\hat{g}_1\left(s,\Phi_{\kappa-2}(s)\right)\mathrm{d}s\\ &\quad +f_1\left(\varrho,\Phi_{\kappa-1}(\varrho)\right)\ {}^{RL}J^\mu\ \left[\hat{g}_2\left(\varrho,\Phi_{\kappa-1}(\varrho)\right)-\hat{g}_2\left(\varrho,\Phi_{\kappa-2}(\varrho)\right)\right]\\ &\quad +\left[f_1\left(\varrho,\Phi_{\kappa-1}(\varrho)\right)-f_1\left(\varrho,\Phi_{\kappa-2}(\varrho)\right)\right]\ {}^{RL}J^\mu\ \hat{g}_2\left(\varrho,\Phi_{\kappa-2}(\varrho)\right).\end{aligned} \tag{4.16}$$

From the hypotheses (ii) and (iii), we have

$$\begin{aligned}\left|\Phi_\kappa(\varrho)-\Phi_{\kappa-1}(\varrho)\right| &\leq \left|\varrho\Phi_{\kappa-1}(\varrho)\right|\int_0^\varrho \frac{\varrho}{\varrho+s}\left|\hat{g}_1\left(s,\Phi_{\kappa-1}(s)\right)-\hat{g}_1\left(s,\Phi_{\kappa-2}(s)\right)\right|\mathrm{d}s\\ &\quad +\varrho\left|\Phi_{\kappa-1}(\varrho)-\Phi_{\kappa-2}(\varrho)\right|\int_0^\varrho \frac{\varrho}{\varrho+s}\left|\hat{g}_1\left(s,\Phi_{\kappa-2}(s)\right)\right|\mathrm{d}s\\ &\quad +\left|f_1\left(\varrho,\Phi_{\kappa-1}(\varrho)\right)\right|\ {}^{RL}J^\mu\left|\hat{g}_2\left(\varrho,\Phi_{\kappa-1}(\varrho)\right)-\hat{g}_2\left(\varrho,\Phi_{\kappa-2}(\varrho)\right)\right|\\ &\quad +\left|f_1\left(\varrho,\Phi_{\kappa-1}(\varrho)\right)-f_1\left(\varrho,\Phi_{\kappa-2}(\varrho)\right)\right|\ {}^{RL}J^\mu\ \left|\hat{g}_2\left(\varrho,\Phi_{\kappa-2}(\varrho)\right)\right|,\end{aligned} \tag{4.17}$$

so,

$$
\begin{aligned}
|\Phi_\kappa(\varrho)\ -\Phi_{\kappa-1}(\varrho)| &\leq b^2\mathbb{R}_0 L_1\left|\Phi_{\kappa-1}(\varrho)-\Phi_{\kappa-2}(\varrho)\right|\int_0^b \frac{1}{\varrho+s}\mathrm{d}s \\
&\quad + b^2\left|\Phi_{\kappa-1}(\varrho)-\Phi_{\kappa-2}(\varrho)\right|\int_0^b \frac{1}{\varrho+s}m_1(s)\,\mathrm{d}s \\
&\quad + \left|f_1(\varrho,\Phi_{\kappa-1}(\varrho))-f_1(\varrho,0)+f_1(\varrho,0)\right| \\
&\quad + L_2\left|\Phi_{\kappa-1}(\varrho)-\Phi_{\kappa-2}(\varrho)\right|\ {}^{RL}J^{\mu}\ (1) \\
&\quad + L\left|\Phi_{\kappa-1}(\varrho)-\Phi_{\kappa-2}(\varrho)\right|\ {}^{RL}J^{\mu}\ {}^{RL}J^{\eta}m_2(\varrho) \\
&\leq \left[b^2\mathbb{R}_0 L_1 + b^2 M_1 + \frac{L_2 b^{\mu}(L\mathbb{R}_0+M)}{\Gamma(\mu+1)} + \frac{LM_2 b^{\mu-\eta}}{\Gamma(\mu-\eta+1)}\right]\left|\Phi_{\kappa-1}(\varrho)-\Phi_{\kappa-2}(\varrho)\right| \\
&\leq \left[b^2\mathbb{R}_0 M_1 + \frac{M_2 b^{\mu-\eta}(L\mathbb{R}_0+M)}{\Gamma(\mu-\eta+1)}\right]\left|\Phi_{\kappa-1}(\varrho)-\Phi_{\kappa-2}(\varrho)\right| \\
&\leq \Upsilon_1\left|\Phi_{\kappa-1}(\varrho)-\Phi_{\kappa-2}(\varrho)\right|, \qquad (4.18)
\end{aligned}
$$

where $\Upsilon_1 = \left[b^2\mathbb{R}_0 M_1 + \frac{M_2 b^{\mu-\eta}(L\mathbb{R}_0+M)}{\Gamma(\mu-\eta+1)}\right]$.

In the above relation, if we put $\kappa = 2$ and use (4.13), we get

$$
\begin{aligned}
|\Phi_2(\varrho)-\Phi_1(\varrho)| &\leq \Upsilon_1\left|\Phi_1(\varrho)-\Phi_0(\varrho)\right| \\
|\Phi_2-\Phi_1| &\leq \Upsilon_1\,\psi. \qquad (4.19)
\end{aligned}
$$

Doing the same for $\kappa = 3, 4, \cdots$ gives us

$$
\begin{aligned}
|\Phi_3-\Phi_2| &\leq \Upsilon_1\left|\Phi_2(\varrho)-\Phi_1(\varrho)\right| \\
&\leq \Upsilon_1^2\,\psi \\
|\Phi_4-\Phi_3| &\leq \Upsilon_1\left|\Phi_3(\varrho)-\Phi_2(\varrho)\right| \\
&\leq \Upsilon_1^3\,\psi \\
&\vdots
\end{aligned}
$$

Then the general form of this relation is

$$
|\Phi_\kappa-\Phi_{\kappa-1}| \leq \Upsilon_1^{\kappa-1}\,\psi. \qquad (4.20)
$$

Since $\Upsilon_1 < 1$, then the series

$$
\sum_{\kappa=1}^{\infty}\left[\Phi_\kappa(\varrho)-\Phi_{\kappa-1}(\varrho)\right] \qquad (4.21)
$$

is uniformly convergent. Hence, the sequence $\{\Phi_\kappa(\varrho)\}$ is uniformly convergent. Since $\hat{g}_1(\varrho,\Phi(\varrho)), \hat{g}_2(\varrho,\Phi(\varrho))$, and $f_1(\varrho,\Phi(\varrho))$ are continuous in Φ, then

$$
\begin{aligned}
\Phi(\varrho) &= \lim_{\kappa\to\infty}\varrho\Phi_\kappa(\varrho)\int_0^{\varrho}\frac{\varrho}{\varrho+s}\hat{g}_1(s,\Phi_\kappa(s))\,\mathrm{d}s + f_1(\varrho,\Phi_\kappa(\varrho))\ {}^{RL}J^{\mu}\hat{g}_2(\varrho,\Phi_\kappa(\varrho)) \\
&= \varrho\Phi(\varrho)\int_0^{\varrho}\frac{\varrho}{\varrho+s}\hat{g}_1(s,\Phi(s))\,\mathrm{d}s + f_1(\varrho,\Phi(\varrho))\ {}^{RL}J^{\mu}\hat{g}_2(\varrho,\Phi(\varrho)). \qquad (4.22)
\end{aligned}
$$

Hence, the solution exists.

5. Numerical examples

Example 1. *For the HIE of Chandraseker type:*

$$\Phi(\xi) = \frac{1}{50}\xi\Phi(\xi)\int_0^{\xi}\frac{\xi}{\xi+s}\Phi^2(\xi)\,\mathrm{d}s + \frac{1}{20}\Phi(\xi)\,{}^{RL}J^{\mu}\,\Phi^3(\xi),\quad \xi(0)=0, \tag{5.1}$$

where

$$\varphi(\xi) = \left[\xi^2 - \left(\frac{-\frac{7}{12}+\ln(2)}{50}\right)\varrho^7 - \frac{\Gamma(7)}{20\Gamma(7+\mu)}\xi^{8+\mu}\right],$$

and its exact solution is $\Phi(\xi) = \xi^2$.

Applying the ADM to Eq (5.1), we get

$$\Phi_0(\xi) = \varphi(\xi), \tag{5.2}$$

$$\Phi_\kappa(\xi) = \frac{1}{50}\xi\hat{A}_{\kappa-1}(\xi) + \frac{1}{20}\check{D}_{\kappa-1}(\xi), \kappa \geq 1. \tag{5.3}$$

Using the PM in Eq (5.1), the solution algorithm is

$$\Phi_0(\xi) = \varphi(\xi), \tag{5.4}$$

$$\Phi_\kappa(\xi) = \Phi_0(\xi) + \frac{1}{50}\xi\Phi_{\kappa-1}(\xi)\int_0^{\xi}\frac{\xi}{\xi+s}\Phi^2_{\kappa-1}(s)\,\mathrm{d}s + \frac{1}{20}\Phi(\xi)\,{}^{RL}J^{\mu}\,\Phi^3_{\kappa-1}(\xi), \kappa \geq 1. \tag{5.5}$$

Figure 1 shows ADM solutions at different values of μ *(*$\mu = 0.5, 0.6, 0.7, 0.8, 0.9, 1$*), and Figure 2 shows PM solutions at the same values.*

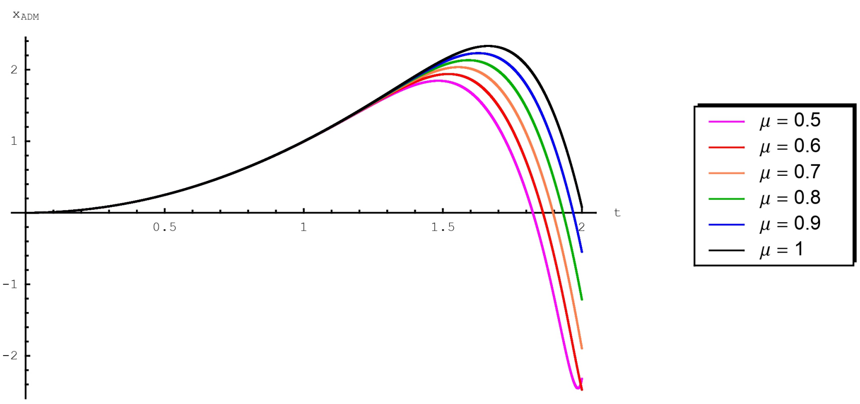


Figure 1. ADM solutions at different values of μ.

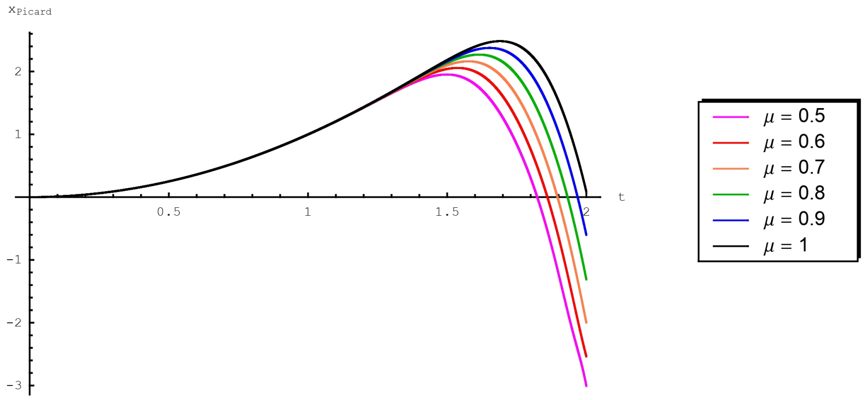
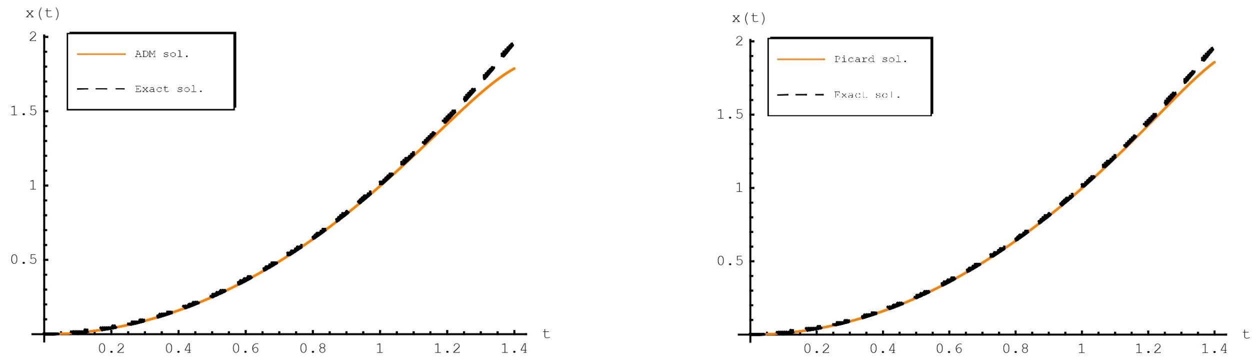


Figure 2. PM solutions at different values of μ.

Remark 1. *A comparison between the absolute relative error (ARE) of ADM and PM solutions with the exact solution (where $\mu = 0.5$) is given in Table 1. It is clear from these results that the two solutions nearly give the same accuracy, but when a comparison is made between the time used in these two cases, it is found that the ADM takes less time than the PM (ADM time = 22 sec., PM time = 319.188 sec.). Figure 3(a) shows the ADM and the exact solution, while Figure 3(b) shows the PM with the exact solution.*

Table 1. ARE of (ADM, PM) for Example 1.

ϱ	$\left\lvert\frac{\Phi_{ES}-\Phi_{ADM}}{\Phi_{ES}}\right\rvert$	$\left\lvert\frac{\Phi_{ES}-\Phi_{PM}}{\Phi_{ES}}\right\rvert$
0.1	2.48379×10^{-9}	2.19628×10^{-8}
0.2	2.47045×10^{-7}	7.0281×10^{-7}
0.3	2.04146×10^{-6}	5.33709×10^{-6}
0.4	6.32111×10^{-6}	0.0000224935
0.5	5.3704×10^{-6}	0.0000686815
0.6	0.0000360447	0.000171175
0.7	0.000215013	0.000371487
0.8	0.000750286	0.000731218
0.9	0.0020904	0.00134688
1	0.00511876	0.00240123
1.1	0.0115464	0.00436134
1.2	0.024631	0.00870185
1.3	0.0503602	0.0201514
1.4	0.0990462	0.0519358
1.5	0.186618	0.132278



(a) Exact and ADM solutions

(b) Exact and Picard solutions

Figure 3. The exact solution versus ADM and Picard solutions to (5.1).

Example 2. *For the HIE of Chandraseker type:*

$$\Phi(\xi) = \varphi(\xi) + \frac{1}{10}\xi\Phi(\xi)\int_0^{\xi}\frac{\xi}{\xi+s}\sqrt{\Phi(s)}ds + \frac{\Phi^2(\xi)}{50}\,{}^{RL}J^{\mu}\frac{\xi^3}{20}\left(5+\Phi^4(\xi)\right),\quad \xi(0)=0, \tag{5.6}$$

where

$$\varphi(\xi) = \frac{2\xi^3}{15},$$

and its exact solution is $\Phi(\xi) = \xi$.

Applying the ADM to Eq (5.6), we get

$$\Phi_0(\xi) = \varphi(\xi), \tag{5.7}$$

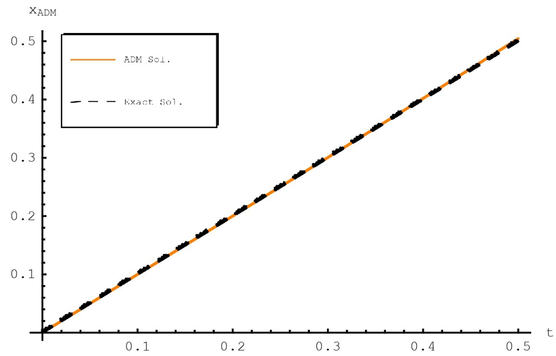
$$\Phi_{\kappa}(\xi) = \frac{1}{10}\xi\hat{A}_{\kappa-1}(\xi) + \frac{1}{50}\check{D}_{\kappa-1}(\xi),\ \kappa\geq 1. \tag{5.8}$$

Using the PM in Eq (5.6), we have

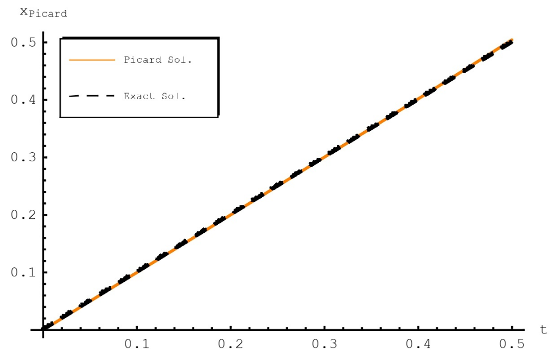
$$\Phi_0(\xi) = \varphi(\xi), \tag{5.9}$$

$$\Phi_{\kappa}(\xi) = \Phi_0(\xi) + \frac{1}{10}\xi\Phi_{\kappa-1}(\xi)\int_0^{\xi}\frac{\xi}{\xi+s}\sqrt{\Phi_{\kappa-1}(s)}ds + \frac{1}{50}\Phi_{\kappa-1}^2(\xi)\,{}^{RL}J^{\mu}\frac{\xi^3}{20}\left(5+\Phi_{\kappa-1}^4(\xi)\right),\kappa\geq 1. \tag{5.10}$$

Figure 4(a) shows the ADM and exact solution, while Figure 4(b) shows the PM bwith the exact solution.



(**a**) Exact and ADM solutions



(**b**) Exact and Picard solutions

Figure 4. The exact solution versus ADM and Picard solutions to (5.6).

Remark 2. *The absolute difference (AD) between ADM and PM solutions (where* $\mu = 0.9$*) is* $|\Phi_{PM} - \Phi_{ADM}| = 0$ *for* $\xi = 0.2, 0.4, ..., 2$*. It is clear from these results that the two solutions are nearly the same, but when a comparison is made between the time used in these two cases, it is found that the ADM takes less time than the PM (ADM time = 42 sec., PM time = 253.2 sec.). Figure 5 shows ADM and PM solutions at (*$\mu = 0.9$*).*

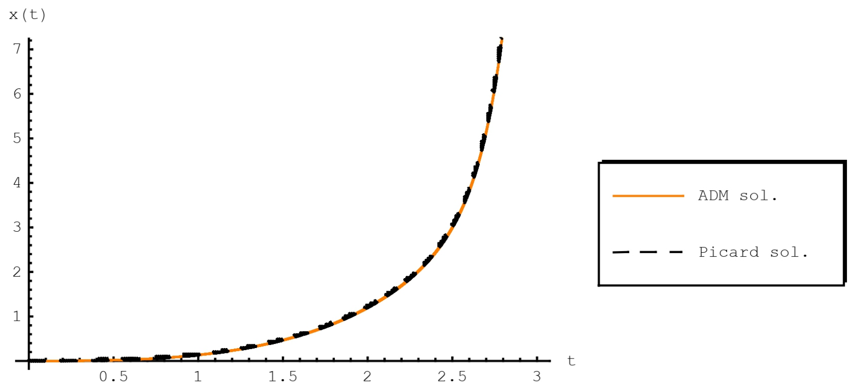


Figure 5. PM and ADM solutions at $\mu = 0.9$.

Example 3. *For the HIE of Chandraseker type:*

$$\Phi(\xi) = \varphi(\xi) + \frac{1}{10}\xi\Phi(\xi)\int_0^{\xi}\frac{\xi}{\xi+s}\Phi(s)\; e^s ds + \frac{1}{10}\frac{\Phi(\xi)}{1+\Phi(\xi)}\;{}^{RL}J^{0.5}\xi^2\left(1+\Phi(\xi)\right),\quad \xi(0)=0, \tag{5.11}$$

where

$$\varphi(\xi) = \left[\xi + \frac{1}{10}\frac{-\xi}{1+\xi}(0.601802\xi^{2.5} + 0.51583\xi^{3.5}) - \frac{1}{10}\frac{\xi^3(1+\; e^{\xi}(\xi-1))}{\xi+2}\right],$$

and its exact solution is $\Phi(\xi) = \xi$.

Applying the ADM to Eq (5.11), we have

$$\Phi_0(\xi) = \varphi(\xi), \tag{5.12}$$

$$\Phi_{\kappa}(\xi) = \frac{1}{10}\xi\hat{A}_{\kappa-1}(\xi) + \frac{1}{10}\check{D}_{\kappa-1}(\xi), \ \kappa \geq 1. \tag{5.13}$$

Using the PM in Eq (5.11), we get

$$\Phi_0(\xi) = \varphi(\xi), \tag{5.14}$$

$$\Phi_{\kappa}(\xi) = \Phi_0(\xi) + \frac{1}{10}\xi\Phi_{\kappa-1}(\xi)\int_0^{\xi}\frac{\xi}{\xi+s}\Phi_{\kappa-1}(s)e^{s}\mathrm{d}s + \frac{1}{10}\frac{\Phi_{\kappa-1}(\xi)}{1+\Phi_{\kappa-1}(\xi)}\,{}^{RL}J^{0.5}\xi^{2}(1+\Phi_{\kappa-1}(\xi)), \kappa \geq 1. \tag{5.15}$$

Remark 3. *A comparison between the ARE of ADM and PM solutions with the exact solution is given in Table 2. It is clear from these results that the two solutions nearly give the same accuracy, but when a comparison is made between the time used in these two cases, it is found that the ADM takes less time than the PM (ADM time = 69.124 sec., PM time = 70.875 sec.). Figure 6 shows ADM solutions at different values of* μ *(*$\mu = 0.5, 0.6, 0.7, 0.8, 0.9, 1$*), and Figure 7 shows PM solutions at the same values.*

Table 2. ARE of (ADM, PM) for Example 3.

ϱ	$\left\lvert\frac{\chi_{ES}-\chi_{ADM}}{\chi_{ES}}\right\rvert$	$\left\lvert\frac{\chi_{ES}-\chi_{PM}}{\chi_{ES}}\right\rvert$
0.1	$1.0964*10^{-6}$	0.0000724208
0.2	0.0000268761	0.00060454
0.3	0.000179457	0.00212539
0.4	0.000701695	0.0052377
0.5	0.00204302	0.0106117
0.6	0.00493079	0.0189737
0.7	0.0104472	0.0310892
0.8	0.0201116	0.0477376
0.9	0.0359674	0.0696801
1	0.0606745	0.0976178

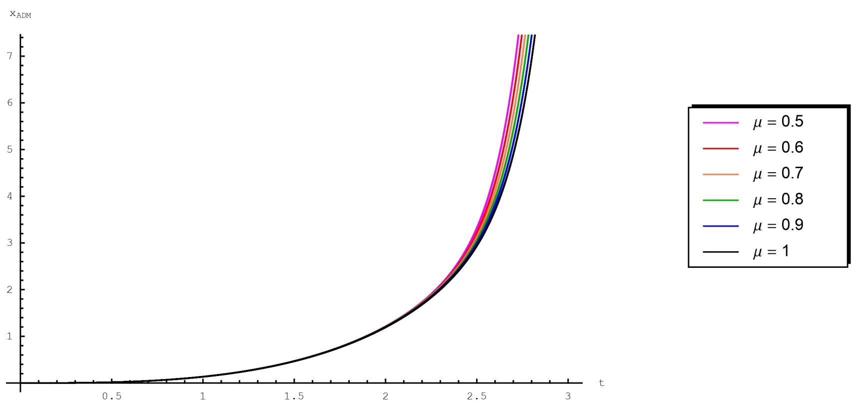


Figure 6. ADM solutions at different values of μ.

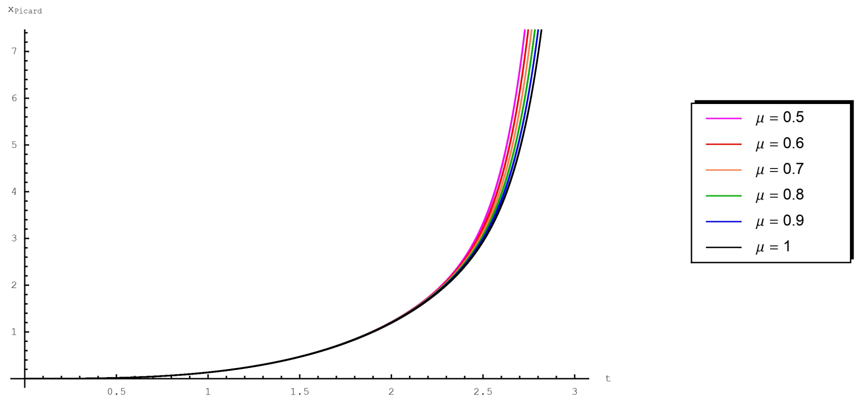


Figure 7. PM solutions at different values of μ.

6. Conclusions

In this research, two analytical methods (ADM and PM) are used to solve the fractional CQIE that was found in the nonlinear analysis and its applications. The existence of a unique solution and its convergence to the two methods are proved (see Theorems 2, 4, and 5). This article focused on making a comparison between them with the exact solution (see the results in Tables 1 and 2). It is observed from the obtained results that the difference between their accuracy is too small to consider, but when we compare their used time, it was clear that the ADM takes less time than the PM (it is more clear in Example 1). These results showed that the two methods satisfied certain criteria that were provided by the solutions.

Table 3. Abbreviations.

IVP	Initial value problem
ADM	Adomian decomposition method
PM	Picard method
FDEs	Fractional differential equations
HDE	Hybrid differential equation
CQIE	Chandrasekhar quadratic integral equation
RLFI	Riemann–Liouville fractional integral
MNC	Measure of noncompactness
BS	Banach space
CS	Cauchy sequence
ARE	Absolute relative error

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Author contributions

Conceptualization, Eman A. A. Ziada and Hind Hashem; Formal analysis, Hind Hashem; Funding acquisition, Asma Al-Jaser; Investigation, Asma Al-Jaser and Osama Moaaz; Methodology, Eman A. A. Ziada and Monica Botros; Software, Asma Al-Jaser and Monica Botros; Writing—original draft, Eman A. A. Ziada and Monica Botros; Writing—review and editing, Osama Moaaz. All authors have read and agreed to the published version of the manuscript.

Conflict of interest

The authors declare there is no conflict of interest.

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