Electronic Research Archive

http://www.aimspress.com/journal/era

ERA, 32(10): 5914–5925.
DOI: 10.3934/era.2024273
Received: 26 June 2024
Revised: 10 October 2024
Accepted: 22 October 2024
Published: 30 October 2024

Research article

Non-lightlike framed rectifying curves in Minkowski 3-space

Jiaqi Huang, Kaixin Yao and Donghe Pei*

School of Mathematics and Statistics, Northeast Normal University, Changchun, China

* **Correspondence:** Email: peidh340@nenu.edu.cn.

Abstract: In this paper, we define non-lightlike framed rectifying curves. They may have singularities. We give equivalent definitions and a construction method for non-lightlike framed rectifying curves. Moreover, we also study the relationship between the non-lightlike framed rectifying curves and the non-lightlike framed helices, as well as the properties of the centrode of non-lightlike framed rectifying curves.

Keywords: non-lightlike framed rectifying curves; singularities; non-lightlike framed curves; helices; centrodes

1. Introduction

Rectifying curves have been studied a lot in three-dimensional Euclidean space. Rectifying curve whose definition and equivalent definitions are provided in [1]. Chen and Dillen revealed the relationship between the center point of the spatial curve and the rectifying curve in [2]. Rectifying curves have many properties in Euclidean space [3]. In four-dimensional Euclidean space, İşbilir and Tosun [4] studied rectifying curves. Many scholars studied the properties of multiple curves in three-dimensional Minkowski space [5, 6]. There have also been studies about rectifying curves, such as three-dimensional Minkowski space [7, 8], three-dimensional hyperbolic space [9], and three-dimensional spheres [10]. There is a new article about rectifying curves [11]. These are all valuable geometric information obtained by analyzing the curvature and the torsion of the regular rectifying curve and the Frenet-Serret formula. If the curve has singularities, then other methods need to be used for research. The definition of framed curves has been given in [12]. Framed curves are spatial curves that have moving frames. The framed base curve may have singularities. Next, the rectifying curve was studied by the adapted frame in [13].

Inspired by the above work, we study non-lightlike framed rectifying curves. We define the non-lightlike framed rectifying curves, study the construction of the non-lightlike framed rectifying curves, and obtain valuable geometric information.

In Section 2, we review the basic knowledge of non-lightlike framed curves. In Section 3, the non-lightlike framed rectifying curves are defined and their equivalent definitions are given. In Section 4, a method for constructing non-lightlike framed rectifying curves is provided, and examples of regular curves and singular curves are also provided. In Section 5, we define non-lightlike framed helices to obtain the relationship between them and non-lightlike framed rectifying curves. The centrodes of non-lightlike framed rectifying curves are also studied.

All maps and manifolds considered here are differentiable of class C^∞.

2. Preliminary

Let $\mathbb{R}^3_1$ be Minkowski 3-space with the pseudo scalar product $\langle , \rangle$, the pseudo vector product $\wedge$, and the norm $\|\ \ \|$. The pseudo scalar product is equipped with the signature $(-, +, +)$.

For any nonzero vector $\boldsymbol{a} \in \mathbb{R}^3_1$, it is called spacelike, timelike, or lightlike if $\langle \boldsymbol{a}, \boldsymbol{a} \rangle$ is positive, negative, or zero, respectively. We say the regular curve $\gamma : I \to \mathbb{R}^3_1$ is spacelike, timelike, or lightlike if the vector $\gamma'(t)$ is spacelike, timelike or lightlike for all $t \in I$, respectively. For $\boldsymbol{n} \in \mathbb{R}^3_1 \backslash \{\boldsymbol{0}\}$, define a set $P = \{\boldsymbol{a} \in \mathbb{R}^3_1 | \langle \boldsymbol{a}, \boldsymbol{n} \rangle = 0\}$. It is obvious that P is a plane in $\mathbb{R}^3_1$. The vector $\boldsymbol{n}$ is called the pseudo normal vector of the plane P. The plane P is called spacelike, timelike, or lightlike if the vector $\boldsymbol{n}$ is timelike, spacelike, or lightlike, respectively.

There are three pseudo spheres in $\mathbb{R}^3_1$:

$$S^2_1 = \{\boldsymbol{a} \in \mathbb{R}^3_1 | \langle \boldsymbol{a}, \boldsymbol{a} \rangle = 1\},$$

$$LC^* = \{\boldsymbol{a} \in \mathbb{R}^3_1 \backslash \{\boldsymbol{0}\} | \langle \boldsymbol{a}, \boldsymbol{a} \rangle = 0\}$$

and

$$H^2_0 = \{\boldsymbol{a} \in \mathbb{R}^3_1 | \langle \boldsymbol{a}, \boldsymbol{a} \rangle = -1\}.$$

We call them de Sitter 2-space, (open) lightcone, and hyperbolic 2-space, respectively. Let $\Delta = \{(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) \in \mathbb{R}^3_1 \times \mathbb{R}^3_1 | \langle \boldsymbol{\beta}_1, \boldsymbol{\beta}_2 \rangle = 0, \|\boldsymbol{\beta}_1\| = 1, \|\boldsymbol{\beta}_2\| = 1\}$ and $\gamma : I \to \mathbb{R}^3_1$ be a non-lightlike curve.

Definition 2.1. We call $(\gamma, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2) : I \to \mathbb{R}^3_1 \times \Delta$ a non-lightlike framed curve if $\langle \gamma'(t), \boldsymbol{\beta}_1(t) \rangle = 0$, $\langle \gamma'(t), \boldsymbol{\beta}_2(t) \rangle = 0$ for any $t \in I$. We call $\gamma : I \to \mathbb{R}^3_1$ a non-lightlike framed base curve if there exists $(\boldsymbol{\beta}_1, \boldsymbol{\beta}_2) : I \to \Delta$ such that $(\gamma, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$ is a non-lightlike framed curve.

Define $\boldsymbol{\mu}(t) = \boldsymbol{\beta}_1(t) \wedge \boldsymbol{\beta}_2(t)$. There exists a function $\alpha : I \to \mathbb{R}$ satisfying $\gamma'(t) = \alpha(t)\boldsymbol{\mu}(t)$. $\{\boldsymbol{\beta}_1(t), \boldsymbol{\beta}_2(t), \boldsymbol{\mu}(t)\}$ is a moving frame along γ. Frenet-type formulas are

$$\begin{pmatrix} \boldsymbol{\beta}'_1(t) \\ \boldsymbol{\beta}'_2(t) \\ \boldsymbol{\mu}'(t) \end{pmatrix} = \begin{pmatrix} 0 & l_1(t) & l_2(t) \\ \sigma l_1(t) & 0 & l_3(t) \\ -\sigma\delta l_2(t) & \delta l_3(t) & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\beta}_1(t) \\ \boldsymbol{\beta}_2(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},$$

$$\gamma'(t) = \alpha(t)\boldsymbol{\mu}(t),$$

where

$$\sigma=\langle\boldsymbol{\mu}(t),\boldsymbol{\mu}(t)\rangle,\delta=\langle\boldsymbol{\beta}_1(t),\boldsymbol{\beta}_1(t)\rangle,l_1(t)=-\sigma\delta\langle\boldsymbol{\beta}_1'(t),\boldsymbol{\beta}_2(t)\rangle,$$
$$l_2(t)=\sigma\langle\boldsymbol{\beta}_1'(t),\boldsymbol{\mu}(t)\rangle,l_3(t)=\sigma\langle\boldsymbol{\beta}_2'(t),\boldsymbol{\mu}(t)\rangle,\alpha(t)=\sigma\langle\boldsymbol{\gamma}'(t),\boldsymbol{\mu}(t)\rangle.$$

$(l_1,l_2,l_3,\alpha):I\to\mathbb{R}^4$ is called the curvature of $(\gamma,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2)$. If $\boldsymbol{\mu}(t)$ is spacelike (timelike) , we call γ a spacelike (timelike) framed base curve.

We know t_0 is a singular point of γ if and only if $\alpha(t_0)=0$.

Proposition 2.2. $\gamma:I\to\mathbb{R}^3_1$ is a non-lightlike regular curve, and $(\gamma,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2):I\to\mathbb{R}^3_1\times\Delta$ is a non-lightlike framed curve. The relations between the curvature (l_1,l_2,l_3,α) of $(\gamma,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2)$ and the curvature κ and the torsion τ of γ are

$$(|\alpha|\kappa)(t)=\sqrt{|l_2^2-\sigma l_3^2|(t)},$$
$$(-\delta\alpha(l_2^2-\sigma l_3^2)\tau)(t)=(l_2'l_3-l_3'l_2+\sigma l_1l_2^2-l_1l_3^2)(t).$$

We assume $l_2^2(t)\neq\sigma l_3^2(t)$ and denote $\varepsilon=\mathrm{sgn}(l_2^2-\sigma l_3^2)(t)$.

Definition 2.3. $(\gamma,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2):I\to\mathbb{R}^3_1\times\Delta$ is a non-lightlike framed curve, and its curvature is (l_1,l_2,l_3,α). Let

$$\begin{pmatrix}\overline{\boldsymbol{\beta}}_1(t)\\ \overline{\boldsymbol{\beta}}_2(t)\end{pmatrix}=\frac{1}{\sqrt{\varepsilon(l_2^2-\sigma l_3^2)(t)}}\begin{pmatrix}\varepsilon l_2(t) & -\varepsilon\sigma l_3(t)\\ -l_3(t) & l_2(t)\end{pmatrix}\begin{pmatrix}\boldsymbol{\beta}_1(t)\\ \boldsymbol{\beta}_2(t)\end{pmatrix}.$$

We call $\overline{\boldsymbol{\beta}}_1$ direction the principal normal direction of $(\gamma,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2)$ and $\overline{\boldsymbol{\beta}}_2$ direction the binormal direction of $(\gamma,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2)$.

We have $\boldsymbol{\mu}(t)=\overline{\boldsymbol{\beta}}_1(t)\wedge\overline{\boldsymbol{\beta}}_2(t)$. $\{\overline{\boldsymbol{\beta}}_1(t),\overline{\boldsymbol{\beta}}_2(t),\boldsymbol{\mu}(t)\}$ is called the Frenet-type frame along γ. Frenet-type formulas are

$$\begin{pmatrix}\overline{\boldsymbol{\beta}}_1'(t)\\ \overline{\boldsymbol{\beta}}_2'(t)\\ \boldsymbol{\mu}'(t)\end{pmatrix}=\begin{pmatrix}0 & L_1(t) & L_2(t)\\ \sigma L_1(t) & 0 & 0\\ -\sigma\varepsilon\delta L_2(t) & 0 & 0\end{pmatrix}\begin{pmatrix}\overline{\boldsymbol{\beta}}_1(t)\\ \overline{\boldsymbol{\beta}}_2(t)\\ \boldsymbol{\mu}(t)\end{pmatrix},$$

$$\gamma'(t)=\alpha(t)\boldsymbol{\mu}(t),$$

where

$$L_1(t)=\varepsilon\left(\frac{l_2'l_3-l_3'l_2}{l_2^2-\sigma l_3^2}(t)+\sigma l_1(t)\right),$$
$$L_2(t)=\sqrt{\varepsilon(l_2^2(t)-\sigma l_3^2(t))}.$$

Then $(L_1,L_2,0,\alpha)$ is the curvature of $(\gamma,\overline{\boldsymbol{\beta}}_1,\overline{\boldsymbol{\beta}}_2)$.

Remark 2.4. $(\gamma,\boldsymbol{\beta}_1,\boldsymbol{\beta}_2):I\to\mathbb{R}^3_1\times\Delta$ is a non-lightlike framed curve, and its curvature is $(l_1,l_2,0,\alpha)$. If $l_2(t)>0$, then $\overline{\boldsymbol{\beta}}_1(t)=\boldsymbol{\beta}_1(t)$ and $\overline{\boldsymbol{\beta}}_2(t)=\boldsymbol{\beta}_2(t)$. If $l_2(t)<0$, then $\overline{\boldsymbol{\beta}}_1(t)=-\boldsymbol{\beta}_1(t)$ and $\overline{\boldsymbol{\beta}}_2(t)=-\boldsymbol{\beta}_2(t)$.

In this article, we only study the non-lightlike framed curve $(\gamma,\overline{\boldsymbol{\beta}}_1,\overline{\boldsymbol{\beta}}_2)$ and its frame is the Frenet-type frame $\{\overline{\boldsymbol{\beta}}_1(t),\overline{\boldsymbol{\beta}}_2(t),\boldsymbol{\mu}(t)\}$.

Remark 2.5. $\gamma : I \to \mathbb{R}^3_1$ is a non-lightlike regular curve and $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2) : I \to \mathbb{R}^3_1 \times \Delta$ is a non-lightlike framed curve. Let $l_3 = 0$ be in Proposition 2.2. We have the relations among the curvature κ, the torsion τ of γ and the curvature $(L_1, L_2, 0, \alpha)$ of $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2)$ are

$$\kappa(t) = \frac{L_2}{|\alpha|}(t), \tau(t) = -\sigma\delta\frac{L_1}{\alpha}(t)$$

For a non-lightlike framed curve $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2) : I \to \mathbb{R}^3_1 \times \Delta$, the rectifying plane of γ at t_0 is the plane through $\gamma(t_0)$ and spanned by $\overline{\boldsymbol{\beta}}_2(t_0)$ and $\boldsymbol{\mu}(t_0)$.

3. Non-lightlike framed rectifying curves

Definition 3.1. $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2) : I \to \mathbb{R}^3_1 \times \Delta$ is a non-lightlike framed curve. We call $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2)$ a *non-lightlike framed rectifying curve* if γ satisfies

$$\gamma(t) = (\psi\boldsymbol{\mu} + \phi\overline{\boldsymbol{\beta}}_2)(t)$$

for two functions $\psi(t), \phi(t) : I \to \mathbb{R}$. γ is called a base curve of a *non-lightlike framed rectifying curve* (Figure 1).

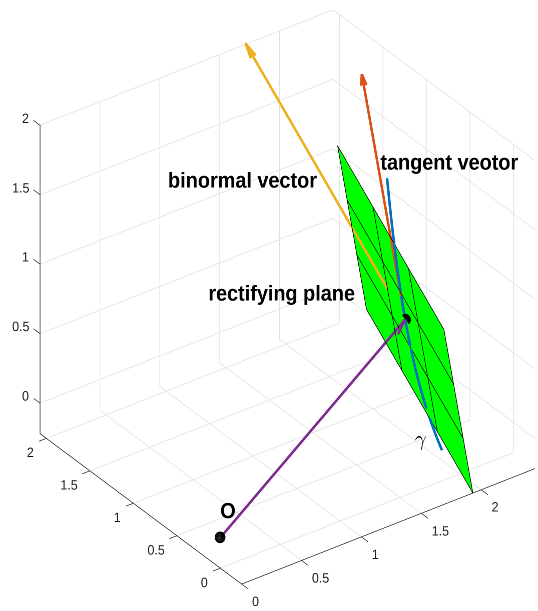

Figure 1. γ is a base curve of a non-lightlike framed rectifying curve.

We call $f(t) = \langle\gamma(t), \gamma(t)\rangle$ the distance squared function of a non-lightlike framed curve $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2)$.

Theorem 3.2. $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2) : I \to \mathbb{R}^3_1 \times \Delta$ is a non-lightlike framed curve. The following statements are equivalent.

(1) $\langle\gamma(t), \boldsymbol{\mu}(t)\rangle' = \sigma\alpha(t)$.

(2) The distance squared function satisfies

$$f(t) = \sigma\langle\gamma(t), \boldsymbol{\mu}(t)\rangle^2 - \phi^2\sigma\varepsilon\delta.$$

(3) $\langle \gamma(t), \overline{\boldsymbol{\beta}}_2(t)\rangle = \phi\varepsilon\delta$, $\phi \in \mathbb{R}\backslash\{0\}$.

(4) $\gamma(t)$ is a base curve of a non-lightlike framed rectifying curve.

Proof. Let $\gamma(t)$ be a base curve of a non-lightlike framed rectifying curve. We know there exist two functions $\psi(t)$ and $\phi(t)$ such that

$$\gamma(t) = (\psi\boldsymbol{\mu} + \phi\overline{\boldsymbol{\beta}}_2)(t). \tag{3.1}$$

According to the Frenet-type formulas and deriving (3.1), we have

$$(\alpha\boldsymbol{\mu})(t) = (\psi'\boldsymbol{\mu} + (-\sigma\varepsilon\delta\psi L_2 + \sigma\phi L_1)\overline{\boldsymbol{\beta}}_1 + \phi'\overline{\boldsymbol{\beta}}_2)(t).$$

Then

$$\psi'(t) = \alpha(t), (\sigma\varepsilon\delta\psi L_2)(t) = (\sigma\phi L_1)(t), \phi'(t) = 0. \tag{3.2}$$

From the first equation of (3.2), we obtain $\langle \gamma, \boldsymbol{\mu}\rangle'(t) = \sigma\psi'(t) = \sigma\alpha(t)$. This proves the statement (1).

By (3.1) and (3.2), we can obtain that

$$\langle \gamma, \gamma\rangle(t) = (\sigma\psi^2 - \phi^2\sigma\varepsilon\delta)(t) = (\sigma\langle \gamma, \boldsymbol{\mu}\rangle^2 - \phi^2\sigma\varepsilon\delta)(t),$$

If $\phi = 0$, then $\psi(t) = 0$ and $\gamma(t)$ is a point. So $\phi \neq 0$. This proves statements (2) and (3).

Conversely, we assume the statement (1) holds , then

$$\langle \gamma, \boldsymbol{\mu}\rangle'(t) = (\langle \alpha\boldsymbol{\mu}, \boldsymbol{\mu}\rangle + \langle \gamma, -\sigma\varepsilon\delta L_2\overline{\boldsymbol{\beta}}_1\rangle)(t) = \sigma\alpha(t).$$

By assumption, we obtain $\langle \gamma(t), \overline{\boldsymbol{\beta}}_1(t)\rangle = 0$. So $\gamma(t)$ is a base curve of a non-lightlike framed rectifying curve.

If the statement (2) holds, then

$$\langle \gamma, \gamma\rangle(t) = (\sigma\langle \gamma, \boldsymbol{\mu}\rangle^2 - \phi^2\sigma\varepsilon\delta)(t).$$

Then,

$$2\langle \gamma, \alpha\boldsymbol{\mu}\rangle(t) = (2\sigma\langle \gamma, \boldsymbol{\mu}\rangle\sigma\alpha + \langle \gamma, -\sigma\varepsilon\delta L_2\overline{\boldsymbol{\beta}}_1\rangle)(t).$$

So we get $\langle \gamma(t), \overline{\boldsymbol{\beta}}_1(t)\rangle = 0$. $\gamma(t)$ is a base curve of a non-lightlike framed rectifying curve.

If the statement (3) holds, $\langle \gamma(t), \overline{\boldsymbol{\beta}}_2(t)\rangle = \phi\sigma\varepsilon\delta$. By taking the derivative, we have

$$(\langle \alpha\boldsymbol{\mu}, \overline{\boldsymbol{\beta}}_2\rangle + \langle \gamma, \sigma L_1\overline{\boldsymbol{\beta}}_1\rangle)(t) = 0.$$

So $\langle \gamma(t), \overline{\boldsymbol{\beta}}_1(t)\rangle = 0$. $\gamma(t)$ is a base curve of a non-lightlike framed rectifying curve. □

Remark 3.3. $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2)$ is a non-lightlike framed rectifying curve. If the base curve of a non-lightlike framed rectifying curve γ is singular at t_0, then from Eq (3.2) and the statement (2) in Theorem 3.2, we have

$$\frac{L_1}{L_2}(t) = \frac{\sigma\delta\psi}{\phi}(t), \left(\frac{L_1}{L_2}(t)\right)' = \frac{\sigma\delta\alpha}{\phi}(t).$$

So $\left(\dfrac{L_1(t_0)}{L_2(t_0)}\right)' = 0$. Moreover, we know

$$f'(t) = (2\alpha\langle \gamma, \boldsymbol{\mu}\rangle)(t).$$

So $f'(t_0) = 0$.

4. The construction of non-lightlike framed rectifying curves

Theorem 4.1. $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2) : I \to \mathbb{R}^3_1 \times \Delta$ is a non-lightlike framed rectifying curve. $\gamma(t)$ is a base curve of a non-lightlike framed rectifying curve if and only if $\gamma(t)$ can be expressed as one of the following two equations

$$\gamma(t) = \rho(\sec(\int \|\boldsymbol{y}'(t)\|\mathrm{d}t + M))\boldsymbol{y}(t),$$

where M is a constant, $\rho \in \mathbb{R}\backslash\{0\}$ and $\boldsymbol{y}(t)$ is a spacelike framed base curve on S^2_1. Or

$$\gamma(t) = 2\phi\frac{\mathrm{e}^{\int \|\boldsymbol{y}'(t)\|\mathrm{d}t + \frac{1}{2}M}}{|1 - \mathrm{e}^{2\int \|\boldsymbol{y}'(t)\|\mathrm{d}t + M}|}\boldsymbol{y}(t),$$

where M is a constant, $\phi \in \mathbb{R}\backslash\{0\}$ and $\boldsymbol{y}(t)$ is a spacelike (timelike) framed base curve on H^2_0 (S^2_1).

Proof. First, we prove the first equation. Let $\gamma(t)$ be a base curve of a spacelike framed rectifying curve, which has a spacelike rectifying plane. So $\langle\gamma, \gamma\rangle(t) = (\psi^2(t) + \rho^2)(t)$, where $\rho \in \mathbb{R}\backslash\{0\}$. Let $\boldsymbol{y}(t) = \left(\frac{1}{(\psi^2 + \rho^2)^{\frac{1}{2}}}\gamma\right)(t)$ be a spacelike framed base curve on S^2_1. We have

$$\gamma'(t) = \left(\frac{\psi\alpha}{(\psi^2 + \rho^2)^{\frac{1}{2}}}\boldsymbol{y} + (\psi^2 + \rho^2)^{\frac{1}{2}}\boldsymbol{y}'\right)(t).$$

Since $\gamma'(t) = \alpha(t)\boldsymbol{\mu}(t)$ and $\boldsymbol{y}'(t)$ is orthogonal to $\boldsymbol{y}(t)$, we can obtain

$$\langle\gamma', \gamma'\rangle(t) = \left(\frac{\psi^2\alpha^2}{\psi^2 + \rho^2} + (\psi^2 + \rho^2)\langle\boldsymbol{y}', \boldsymbol{y}'\rangle\right)(t).$$

So

$$\|\boldsymbol{y}'(t)\| = \left(\frac{|\rho\alpha|}{\psi^2 + \rho^2}\right)(t).$$

We only consider $\rho\alpha(t) \geq 0$, and it is similar for $\rho\alpha(t) \leq 0$. Then

$$\int \|\boldsymbol{y}'(t)\|\mathrm{d}t + M = \arctan\frac{\psi(t)}{\rho}.$$

That is

$$\psi(t) = \rho\tan(\int \|\boldsymbol{y}'(t)\|\mathrm{d}t + M).$$

So

$$\gamma(t) = \rho(\sec(\int \|\boldsymbol{y}'(t)\|\mathrm{d}t + M))\boldsymbol{y}(t).$$

Conversely, let $(\boldsymbol{y}, \boldsymbol{\beta}_{y_1}, \boldsymbol{\beta}_{y_2})$ be a spacelike framed curve and $\gamma(t)$ be defined by

$$\gamma(t) = \rho(\sec(\int \|\boldsymbol{y}'(t)\|\mathrm{d}t + M))\boldsymbol{y}(t).$$

Let $\overline{\psi}(t) = \rho(\tan^2(\int \|\boldsymbol{y}'(t)\|\mathrm{d}t + M))$ and $\overline{\alpha}(t) = \overline{\psi}'(t)$. Then

$$\gamma(t) = \left((\overline{\psi}^2 + \rho^2)^{\frac{1}{2}}\boldsymbol{y}\right)(t),$$

$$\gamma'(t) = \left(\frac{\overline{\psi}\overline{\alpha}}{(\overline{\psi}^2 + \rho^2)^{\frac{1}{2}}}\boldsymbol{y} + (\overline{\psi}^2 + \rho^2)^{\frac{1}{2}}\boldsymbol{y}'\right)(t).$$

Since $\boldsymbol{y}(t)$ is also a spacelike framed curve, we define that $\boldsymbol{y}'(t) = \psi(t)\boldsymbol{\mu}_y(t)$, where $\boldsymbol{\mu}_y(t) = \boldsymbol{\beta}_{y_1}(t) \wedge \boldsymbol{\beta}_{y_2}(t)$. We can obtain

$$\int \|\boldsymbol{y}'(t)\|\mathrm{d}t + M = \arctan\frac{\overline{\psi}(t)}{\rho}$$

and

$$\|\boldsymbol{y}'(t)\| = \left(\frac{|\rho\overline{\alpha}|}{\overline{\psi}^2 + \rho^2}\right)(t).$$

Therefore, we denote that $\boldsymbol{y}'(t) = \left(\frac{\rho\overline{\alpha}}{\overline{\psi}^2 + \rho^2}\boldsymbol{\mu}_y\right)(t)$. That is $y(t) = \left(\frac{\rho\overline{\alpha}}{\overline{\psi}^2(t) + \rho^2}\right)(t)$. Then we have

$$\gamma'(t) = \left(\overline{\alpha}\frac{\overline{\psi}}{(\overline{\psi}^2 + \rho^2)^{\frac{1}{2}}}\boldsymbol{y} + \frac{\rho}{(\overline{\psi}^2 + \rho^2)^{\frac{1}{2}}}\boldsymbol{\mu}_y\right)(t) = (\overline{\alpha}\boldsymbol{\mu})(t).$$

Hence, we can calculate that $\langle\gamma, \boldsymbol{\mu}\rangle^2(t) = \overline{\psi}(t)$. Since $\langle\gamma, \gamma\rangle(t) = (\overline{\psi}^2 + \rho^2)(t)$, we have

$$\langle\gamma, \gamma\rangle(t) = (\langle\gamma, \boldsymbol{\mu}\rangle^2 + \rho^2)(t).$$

It indicates that the function satisfies the statement (2) in Theorem 3.2. So $\gamma(t)$ is a base curve of a spacelike framed rectifying curve.

Next we prove the second equation. Let $\gamma(t)$ be the base curve of a spacelike framed rectifying curve, which has a timelike rectifying plane and a spacelike position vector. (We only prove this case, and the proof for other cases is similar to it.) So $\langle\gamma, \gamma\rangle(t) = (\psi^2 - \phi^2)(t)$, where $\phi \in \mathbb{R}\backslash\{0\}$. Let $\boldsymbol{y}(t) = \left(\frac{1}{(\psi^2 - \phi^2)^{\frac{1}{2}}}\gamma\right)(t)$ be a spacelike framed base curve on S^2_1. We have

$$\gamma'(t) = \left(\frac{-\psi\alpha}{(\psi^2 - \phi^2)^{\frac{1}{2}}}\boldsymbol{y} + (\psi^2 - \phi^2)^{\frac{1}{2}}\boldsymbol{y}'\right)(t),$$

Since $\gamma'(t) = (\alpha\boldsymbol{\mu})(t)$ and $\boldsymbol{y}'(t)$ is orthogonal to $\boldsymbol{y}(t)$,

$$\langle\gamma', \gamma'\rangle(t) = \alpha^2(t) = \left(\frac{-\psi^2\alpha^2}{\psi^2 - \phi^2} + (\psi^2 - \phi^2)\langle\boldsymbol{y}', \boldsymbol{y}'\rangle\right)(t).$$

So

$$\|\boldsymbol{y}'(t)\| = \left|\frac{\phi\alpha}{\psi(t)^2 - \phi^2}\right|(t)$$

and

$$\int \|\boldsymbol{y}'(t)\|\mathrm{d}t + M = \frac{1}{2}\ln\left|\frac{\psi(t)-\phi}{\psi(t)+\phi}\right|.$$

Then,

$$\psi(t) = \phi\frac{1+\mathrm{e}^{2\int\|\boldsymbol{y}'(t)\|\mathrm{d}t+M}}{1-\mathrm{e}^{2\int\|\boldsymbol{y}'(t)\|\mathrm{d}t+M}}.$$

So

$$\gamma(t) = 2\phi\frac{\mathrm{e}^{\int\|\boldsymbol{y}'(t)\|\mathrm{d}t+\frac{1}{2}M}}{|1-\mathrm{e}^{2\int\|\boldsymbol{y}'(t)\|\mathrm{d}t+M}|}\boldsymbol{y}(t).$$

Conversely, we can obtain the proof of this section by referring to the proof of the first equation. □

Remark 4.2. If $\gamma(t)$ is a base curve of a spacelike framed rectifying curve, which has a timelike rectifying plane and a lightlike position vector, then $\langle\gamma(t),\gamma(t)\rangle = 0$. That means $\psi^2(t) = \phi^2$, $\alpha(t) = 0$, then $\gamma(t)$ is a point. So $\gamma(t)$ does not exist.

Example 4.3. Let $\boldsymbol{y}_1(t) = \left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{7}}{2}\cos 2t, \frac{\sqrt{7}}{2}\sin 2t\right), t \in \left(-\frac{\pi}{2\sqrt{3}}, \frac{\pi}{2\sqrt{3}}\right)$ be a curve on S^2_1. We have $\|\boldsymbol{y}_1'(t)\| = \sqrt{7}$. Let $\rho = 1$ and $M = 0$. We have the curve

$$\gamma_1(t) = (\sec\sqrt{7}t)\left(-\frac{\sqrt{3}}{2}, \frac{\sqrt{7}}{2}\cos 2t, \frac{\sqrt{7}}{2}\sin 2t\right)$$

is a base curve of a non-lightlike framed rectifying curve in $\mathbb{R}^3_1$ (Figure 2).

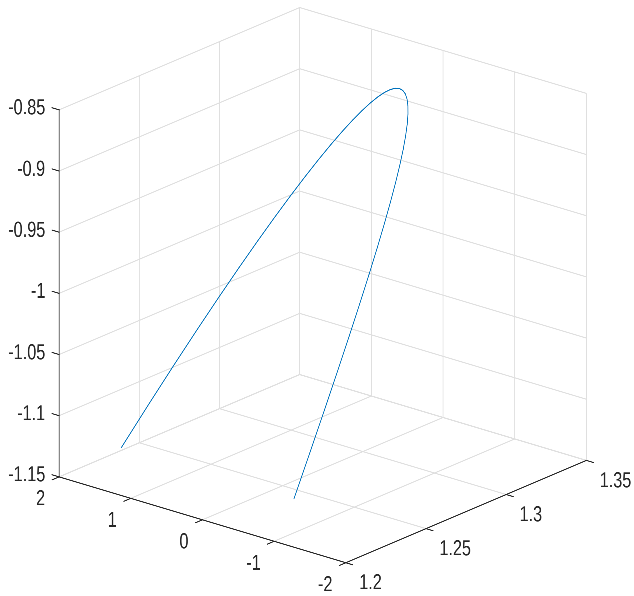

Figure 2. The curve $\gamma_1(t)$ is a non-lightlike framed rectifying curve.

Example 4.4. Let $\boldsymbol{y}_2(t) = (\sinh t^2, \cosh t^2, 0), t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ be a curve on S_1^2. We have $\|\boldsymbol{y}_2'(t)\| = 2|t|$. Let $\rho = 1$ and $M = 0$. We have the curve

$$\gamma_2(t) = \sec t^2(\sinh t^2, \cosh t^2, 0)$$

is a base curve of a non-lightlike framed rectifying curve with a singular point $\mathbb{R}_1^3$ (Figure 3).

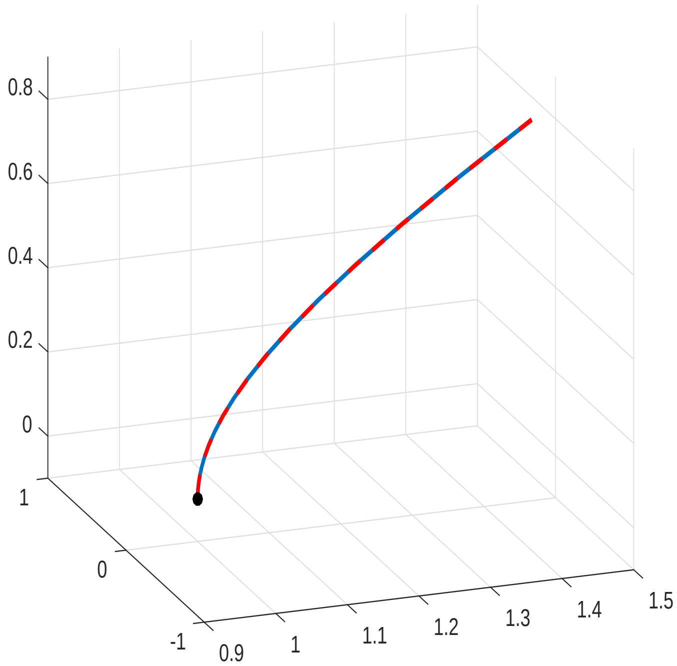

Figure 3. The curve $\gamma_2(t)$ is a base curve of a non-lightlike framed rectifying curve with a singular point. The black point is the singular point of the curve, whose two segments coincide.

5. Non-lightlike framed rectifying curves versus non-lightlike framed helices and centrodes

Definition 5.1. $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2) : I \to \mathbb{R}_1^3 \times \Delta$ is a non-lightlike framed curve. γ is called a *non-lightlike framed helix* if there exists a fixed unit vector $\boldsymbol{\eta}$ satisfying

$$\langle \boldsymbol{\mu}(t), \boldsymbol{\eta} \rangle = p,$$

where $p \in \mathbb{R} \backslash \{0\}$.

Remark 5.2. For a non-lightlike framed curve $(\gamma, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2)$, we can also call γ a non-lightlike framed helix if there exists a fixed unit vector $\boldsymbol{\eta}$ satisfying

$$\langle \boldsymbol{\mu}(t), \boldsymbol{\eta} \rangle = p,$$

where $p \in \mathbb{R} \backslash \{0\}$.

$(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2) : I \to \mathbb{R}^3_1 \times \Delta$ is a non-lightlike framed curve with the curvature $(L_1, L_2, 0, \alpha)$. γ is a non-lightlike framed helix. We consider the ratio $\dfrac{L_1(t)}{L_2(t)}$.

Since

$$\langle \boldsymbol{\mu}, \boldsymbol{\eta} \rangle^{'}(t) = (\langle -\sigma\varepsilon\delta L_2 \overline{\boldsymbol{\beta}}_1, \boldsymbol{\eta} \rangle(t) = 0.$$

Then

$$\langle \overline{\boldsymbol{\beta}}_1(t), \boldsymbol{\eta} \rangle = 0. \tag{5.1}$$

$\boldsymbol{\eta}$ is located in the plane, and the plane has basis vectors $\boldsymbol{\mu}(t)$ and $\overline{\boldsymbol{\beta}}_2(t)$. Since $\langle \boldsymbol{\mu}(t), \boldsymbol{\eta} \rangle = p$, we have $\langle \overline{\boldsymbol{\beta}}_2(t), \boldsymbol{\eta} \rangle$ is a constant, denoted by p_1. If $p_1 = 0$, then $\boldsymbol{\mu}(t) = \dfrac{\sigma}{p}\boldsymbol{\eta}$. At this point, γ is a segment of a straight line. So we always assume $p_1 \neq 0$. We take the derivative of (5.1), so

$$\langle L_2 \boldsymbol{\mu} + L_1 \overline{\boldsymbol{\beta}}_2, \boldsymbol{\eta} \rangle(t) = 0.$$

Then

$$\frac{L_1}{L_2}(t) = -\frac{p}{p_1}.$$

By Theorem 3.2, we can obtain $\gamma(t)$ is a base curve of the non-lightlike framed rectifying curve if and only if

$$\frac{L_1}{L_2}(t) = c_1 \int \alpha(t) \mathrm{d}t + c_2,$$

where $c_1, c_2 \in \mathbb{R}$, $c_1 \neq 0$.

Proposition 5.3. $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2) : I \to \mathbb{R}^3_1 \times \Delta$ is a non-lightlike framed curve and its curvature is $(L_1, L_2, 0, \alpha)$. The curvature satisfies $\left(\dfrac{L_1(t)}{L_2(t)}\right)^{'} = c_1 \alpha(t)$.

(1) If $c_1 = 0$, then $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2)$ is a non-lightlike framed helix.

(2) If $c_1 \neq 0$, then $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2)$ is a non-lightlike framed rectifying curve.

Definition 5.4. $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2) : I \to \mathbb{R}^3_1 \times \Delta$ is a non-lightlike framed curve and its curvature is $(L_1, L_2, 0, \alpha)$. We call $\boldsymbol{d}(t)$ the *centrode* of $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2)$ if

$$\boldsymbol{d}(t) = (L_1 \boldsymbol{\mu} + L_2 \overline{\boldsymbol{\beta}}_2)(t).$$

Proposition 5.5. $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2) : I \to \mathbb{R}^3_1 \times \Delta$ is a non-lightlike framed curve, and its curvature is $(L_1, L_2, 0, \alpha)$. Where $L_1(t)$ is a nonzero constant and $L_2(t)$ is a nonconstant function.

(1) Let $\boldsymbol{d}(t) = (L_1 \boldsymbol{\mu} + L_2 \overline{\boldsymbol{\beta}}_2)(t)$ be the centrode of $(\gamma, \overline{\boldsymbol{\beta}}_1, \overline{\boldsymbol{\beta}}_2)$. Then $\boldsymbol{d}(t)$ is a base curve of a non-lightlike framed rectifying curve.

(2) The base curve of any non-lightlike framed rectifying curve in $\mathbb{R}^3_1$ is the centrode of some non-lightlike framed curve.

6. Conclusions

Future research could extend the concept of non-lightlike framed rectifying curves to high-dimensional Minkowski space or it could study lightlike framed rectifying curves. This provides assistance in studying the properties and classification of higher-dimensional non-lightlike framed rectifying curves.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant No. 11671070).

Conflict of interest

The authors declare that there is no conflict of interest.

References

1. B. Chen, When does the position vector of a space curve always lie in its rectifying plane?, *Am. Math. Mon.*, **110** (2003), 147–152. https://doi.org/10.2307/3647775

2. B. Chen, F. Dillen, Rectifying curves as centrodes and extremal curves, *Bull. Inst. Math. Acad. Sin.*, **33** (2005), 77–90.

3. S. Deshmukh, B. Chen, S. H. Alshammari, On rectifying curves in Euclidean 3-space, *Turkish J. Math.*, **42** (2018), 609–620. https://doi.org/10.3906/mat-1701-52

4. Z. İşbilir, M. Tosun, A new insight on rectifying-type curves in Euclidean 4-space, *Int. Electron. J. Geom.*, **16** (2023), 644–652. https://doi.org/10.36890/iejg.1291893

5. H. A. Erdem, A. Uçum, K. İlarslan, C. Camcı, New approach to timelike Bertrand curves in 3-dimensional Minkowski space, *Carpathian Math. Publ.*, **15** (2023), 482–494. https://doi.org/10.15330/cmp.15.2.482-494

6. Y. Li, J. Qian, Geometric structure of lightlike growth surface in Minkowski space, *Acta Math. Sin. (Chin. Ser.)*, **66** (2023), 1045–1056. https://doi.org/10.12386/A20220058

7. K. Ilarslan, E. Nesovic, Some characterizations of null, pseudo null and partially null rectifying curves in Minkowski space-time, *Taiwan. J. Math.*, **12** (2008), 1035–1044. https://doi.org/10.11650/twjm/1500574245

8. K. Ilarslan, E. Nesovic, On rectifying curves as centrodes and extremal curves in the Minkowski 3-space, *Novi Sad J. Math*, **37** (2007), 53–64.

9. P. Lucas, J. Ortega-Yagües, Rectifying curves in the three-dimensional hyperbolic space, *Mediterr. J. Math.*, **13** (2016), 2199–2214. https://doi.org/10.1007/s00009-015-0615-2

10. P. Lucas, J. Ortega-Yagües, Rectifying curves in the three-dimensional sphere, *J. Math. Anal. Appl.*, **421** (2015), 1855–1868. https://doi.org/10.1016/j.jmaa.2014.07.079

11. M. Aydin, A. Has, B. Yilmaz, A non-newtonian approach in differential geometry of curves: Multiplicative rectifying curves, *Bull. Korean Math. Soc.*, **61** (2024), 849–866. https://doi.org/10.4134/BKMS.b230491

12. S. Honda, M. Takahashi, Framed curves in the Euclidean space, *Adv. Geom.*, **16** (2016), 265–276. https://doi.org/10.1515/advgeom-2015-0035

13. Y. Wang, D. Pei, R. Gao, Generic properties of framed rectifying curves, *Mathematics*, **7** (2019), 37. https://doi.org/10.3390/math7010037